NONLINEAR ELLIPTIC EQUATIONS WITH FRACTIONAL DIFFUSION

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Contents

1. Introduction

These notes will be helpful for my mini-course in the

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Nonlinear elliptic an parabolic equations with fractional diffusion is a hot topic nowadays, involving a very large number of researchers in PDEs, Nonlinear Analysis, and the Calculus of Variations. Equations with fractional diffusion are integro-differential equations. The fractional Laplacians are the simplest linear operators within the class and are the generators of Lévy flights or jump processes. These are diffusions that go far beyond the classical Brownian process. Except for few results, nonlinear theory for these equations has started to be developed only in the last decade. Its interest and applications go from Probability theory, Potential theory, and Fluid Dynamics (which are at its origins), to Conformal Geometry and Mathematical Finance. In addition, fractional diffusions have been important in Physics for many decades (well before the mathematical developments), and more recently also in Biology. We are still very far from a complete mathematical understanding of the field.

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This mini-course starts explaining basic ideas concerning fractional Laplacians and the essential tools to treat nonlinear equations involving these operators. Next we present results on minimizers of semilinear elliptic fractional equations, and their applications towards a fractional version of a famous conjecture of De Giorgi. The conjecture is still largely open in the fractional case. These results correspond to papers [CSo, CSi1, CSi2, CC1, CC2] in collaboration with J. Solà-Morales, Y. Sire, and E. Cinti.

Depending on the audience interest and time availability we will next focus on results from [CR1, CR2, CCR], in collaboration with J.-M. Roquejoffre and A.-C. Coulon, on front propagation for Fisher-KPP equations with fractional diffusion, both in homogeneous and in periodic media. I will then explain a result with N. Consul and J.V. Mandé [CCM] on traveling fronts for a fractional-diffusion type problem: the classical homogeneous heat equation in a half-plane with a nonlinear Neumann boundary condition.

The course finishes explaining an important new notion: nonlocal minimal surfaces, as introduced by Caffarelli, Roquejoffre and Savin [CRS]. Besides the basics, I will also describe recent results from [CFSW, CFW] (with M.M. Fall, J. Solà-Morales, and T. Weth) about curves and surfaces with constant nonlocal mean curvature — a geometric quasilinear fractional equation describing minimizers of the fractional perimeter functional under a volume constraint.

- All the previous papers can be found in arXiv, here: `http://arxiv.org/find/math/1/au:+Cabre_X/0/1/0/all/0/1` or in my web page: `http://www.pagines.ma1.upc.edu/~cabre`

2. Basic facts about fractional Laplacians

The fractional Laplacian $(-\Delta)^\alpha$ is a nonlocal operator defined, for $0 < \alpha < 1$, as follows. If $u \in C^2_{\rm loc}(\mathbb{R}^n)$ has sufficiently slow growth at infinity —for instance $|u(x)| \le C(1+|x|^\gamma)$ with $\gamma < 2\alpha$— then

$$(-\Delta)^\alpha u(x) = C_{n,\alpha}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}\, dy,$$

where $P.V.$ stands for principal value and the constant $C_{n,\alpha}$ is adjusted for the symbol of $(-\Delta)^\alpha$ to be $|\xi|^{2\alpha}$ (see [CSi1, NPV] for the value of the constant). Note that $u \in C^2_{\rm loc}(\mathbb{R}^n)$ ensures the integrability at $y = x$ in the principal value sense. The statement above about its symbol says that the operator corresponds to the α-fraction of the Laplacian, in the sense that $(-\Delta)^\beta(-\Delta)^\alpha = (-\Delta)^{\alpha+\beta}$.

Good references for this introductory part of the course are:

- Hitchhiker's guide to the fractional Sobolev spaces, by E. Di Nezza, G. Palatucci, and E. Valdinoci [NPV], `http://arxiv.org/abs/1104.4345`
- Nonlocal diffusion and applications, by C. Bucur and E. Valdinoci [BV], `http://arxiv.org/abs/1504.08292`

- Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates, by X. Cabré and Y. Sire [CSi1], `http://arxiv.org/abs/1012.0867`

We will be concerned with fractional diffusion equations for a function $u = u(t, x)$ depending also on time t. The simplest one is the homogeneous linear heat equation with fractional diffusion:

$$u_t + (-\Delta)^\alpha u = 0. \tag{2.1}$$

It will be useful to have some probabilistic intuition about the fractional Laplacian. Solutions to the standard heat equation for the classical Laplacian correspond to the probability density (or concentration) of a substance which diffuses under Brownian motion (see for instance the Textbook in PDEs by Sandro Salsa). The same fact holds for the fractional Laplacian when instead of the Brownian process one considers Lévy processes. These are pure jump process; particles may jump to far away positions. More precisely, consider a random walk on the lattice $h\mathbb{Z}^n$. We suppose that at any unit of time τ, a particle jumps from any point of $h\mathbb{Z}^n$ to any other point. The probability for which a particle jumps from the point $hk \in h\mathbb{Z}^n$ to the point $h\widetilde{k}$ is taken to be $K(k - \widetilde{k}) = K(\widetilde{k} - k)$. Note that, differently from the standard random walk, in this process the particle may experience arbitrarily long jumps, though with a small probability. We call $u(t, x)$ the probability that our particle lies at $x \in h\mathbb{Z}^n$ at time $t \in \tau\mathbb{Z}$. Of course, we have

$$u(t + \tau, x) = \sum_{k \in \mathbb{Z}^n} K(k) u(t, x + hk).$$

It is easy to see that from this relation, and choosing $K(k) = |y|^{-(n+2\alpha)}$, τ and $h^{2\alpha}$ proportional and tending to zero, one obtains in the limit the fractional diffusion equation (2.1). See the short article [V1] by E. Valdinoci for details.

An important tool in some works on nonlinear equations with fractional Laplacians is the following extension problem. The fractional Laplacian can be realized in a local manner through the boundary value problem

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x, 0) = u(x) & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{2.2}$$

where $\mathbb{R}^{n+1}_+ = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$ is the upper half-space and $\partial\mathbb{R}^{n+1}_+ = \{y = 0\}$. The parameter a belongs to $(-1, 1)$ and is related to the power α of the fractional Laplacian $(-\Delta)^\alpha$ by the formula $a = 1 - 2\alpha \in (-1, 1)$. Defining

$$\frac{\partial U}{\partial \nu^a} = -\lim_{y \downarrow 0} y^a \partial_y U,$$

Caffarelli and Silvestre [CS] have proved the following formula relating the fractional Laplacian $(-\Delta)^\alpha$ to the Dirichlet-to-Neumann operator for (2.2):

$$(-\Delta)^\alpha u = d_\alpha \frac{\partial U}{\partial \nu^a} \qquad \text{in } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+,$$

where d_α is a positive constant depending only on α (see [CSi1] for its value).

To have some feeling about this fact, consider the case $\alpha = 1/2$. Then, $a = 0$ and the first equation in (2.2) becomes $\Delta U = 0$. It is now easy to see that, if we call $Tu := \partial_\nu U(\cdot, 0)$ (that is, T is the Dirichlet-to-Neumann operator), then we have $T^2 u = -\Delta_x u = -\Delta u$. One just checks that, by doing T twice, one obtains the minus Laplacian —since the harmonic extension of $-U_y(\cdot, 0)$ is $-U_y$ and $U_{yy} = -\Delta_x U = -\Delta u$.

3. Local minimizers of nonlinear elliptic fractional equations

Here we will explain the concept of minimizer for a nonlinear fractional elliptic equation (posed, for instance, in all of $\mathbb{R}^n$). We will present sharp energy estimates for minimizers, as well as the Hamiltonian structure of semilinear equations involving the fractional Laplacian. The Hamiltonian structure of these equations was discovered in [CSo], in collaboration with J. Solà-Morales, for the square root of the Laplacian. It was extended later in [CSi1] (with Yannick Sire) to all fractions $\alpha \in (0, 1)$ of the Laplacian. The Hamiltonian structure has had an important application in the proof of the first result on uniqueness and nondegeneracy of ground states for fractional Laplacians. This is paper [FLS] by R. Frank, E. Lenzmann, and L. Silvestre.

The sharp energy estimates were proved first using the extension problem in [CSo, CC1, CC2]. They can be proved also working only "downstairs" (without the extension). This was done by O. Savin and E. Valdinoci (see [BV, V2]).

The sharp energy estimates are the main tool to establish the only known results on the fractional version of a conjecture of De Giorgi. It concerns minimizers of the fractional Allen-Cahn equation,

$$(-\Delta)^\alpha u = u - u^3 \qquad \text{in } \mathbb{R}^n, \tag{3.1}$$

and also solutions of this equation which are monotone in one direction. The setting and the known results ([CSo, CSi2, CC1, CC2]) are well explained in

- Section 2 of [V2], A fractional framework for perimeters and phase transitions, by E. Valdinoci, `http://arxiv.org/abs/1210.5612`
- Section 5 of [BV], Nonlocal diffusion and applications, by C. Bucur and E. Valdinoci, `http://arxiv.org/abs/1504.08292`

In the local case ($\alpha = 1$) the conjecture has been settled not long ago in very important papers (see the two references above for more information); here 8 is a threshold dimension. Instead, for $\alpha \in (0, 1)$ results are only known up to dimension 3.

The motivation for the fractional de Giorgi conjecture comes from its connection with fractional perimeters and nonlocal minimal surfaces. These concepts will be treated at the end of the course (see Section 7 below), and are also explained in the two references papers displayed above.

4. Fisher-KPP equations with fractional diffusion in homogeneous media

Here we will describe in more detail our results on front propagation with fractional diffusion. Let f be a KPP-type nonlinearity. By this here we mean that $f \in C^1([0, 1])$

is concave, $f(0) = f(1) = 0$, and $f'(1) < 0 < f'(0)$. We may take for instance $f(u) = u(1-u)$. We are interested in the large time behavior of solutions $u = u(t,x)$ to the Cauchy problem

$$\left\{ \begin{array}{rcll} u_t + Au & = & f(u) & \text{in } (0,+\infty) \times \mathbb{R}^n, \\ u(0,\cdot) & = & u_0 & \text{in } \mathbb{R}^n, \qquad 0 \leqslant u_0 \leqslant 1, \end{array} \right. \tag{4.1}$$

where A is the infinitesimal generator of a Feller semigroup. By this we mean that we are given a continuous function $p = p(t,x)$, with $t > 0$ and $x \in \mathbb{R}^n$, such that

- $0 < p \in C((0,+\infty) \times \mathbb{R}^n)$ and $\displaystyle\int_{\mathbb{R}^n} p(t,x)\,dx = 1$ for all $t > 0$, (4.2)
- $p(t,\cdot) * p(s,\cdot) = p(t+s,\cdot)$ for all $(s,t) \in (0,\infty)^2$, (4.3)

and such that, given a function $u_0 \in L^\infty(\mathbb{R}^n)$ and $t > 0$, the function

$$u(t,x) = T_t u_0(x) := \left(p(t,\cdot) * u_0\right)(x) = \int_{\mathbb{R}^n} p(t,y) u_0(x-y)\,dy$$

is the solution of the homogeneous problem

$$\left\{ \begin{array}{rcll} u_t + Au & = & 0 & \text{in } (0,+\infty) \times \mathbb{R}^n, \\ u(0,\cdot) & = & u_0 & \text{in } \mathbb{R}^n. \end{array} \right. \tag{4.4}$$

The function p is called the kernel of the semigroup $\{T_t\}$; it is also called the transition probability function. The operator A is said to be the infinitesimal generator of a Feller semigroup —since $0 \leqslant u_0 \leqslant 1$ leads to $0 \leqslant T_t u_0 \leqslant 1$. In addition, the operator $-A$ can be recovered from the semigroup by the expression

$$-Au = \lim_{t \downarrow 0} \frac{T_t u - u}{t}.$$

Important examples are $A = -\Delta$ (the classical Laplacian) and $A = (-\Delta)^\alpha$ with $\alpha \in (0,1)$ (the fractional Laplacian).

The reaction-diffusion problem (4.1) models de density of a population (or of a substance) that diffuses (for instance to look for food, and this is modeled by the operator A) and, at the same time, reacts —this is modeled by the nonlinearity $f(u)$, which represents birth (f is increasing when the density is small) ans death (f is decreasing when the density is close to the saturation point $u = 1$). Given an initial density u_0 (compactly supported in space, for instance) and $\lambda \in (0,1)$, we want to describe how the level sets $\{x \in \mathbb{R}^n : u(t,x) = \lambda\}$ spread as time goes to $+\infty$.

When $A = -\Delta$ is the standard Laplacian (that corresponds to Brownian diffusion), the following classical result describes the evolution of compactly supported data.

Theorem 4.1 (Aronson–Weinberger)**.** *Assume that $A = -\Delta$. Let u be a solution of (4.1) with $u(0,\cdot) \not\equiv 0$ compactly supported in $\mathbb{R}^n$ and satisfying $0 \leqslant u(0,\cdot) \leqslant 1$. Let $c_* = 2\sqrt{f'(0)}$. Then,*

a) *if $c > c_*$, then $u(t,x) \to 0$ uniformly in $\{|x| \geqslant ct\}$ as $t \to +\infty$.*
b) *if $c < c_*$, then $u(t,x) \to 1$ uniformly in $\{|x| \leqslant ct\}$ as $t \to +\infty$.*

In addition, when $A = -\Delta$, (4.1) admits *planar traveling wave solutions* connecting 0 and 1, that is, solutions of the form $u(t,x) = \phi(x \cdot e + ct)$ with

$$-\phi'' + c\phi' = f(\phi) \quad \text{in } \mathbb{R}, \quad \phi(-\infty) = 0,\ \phi(+\infty) = 1. \tag{4.5}$$

The constant c_* in Theorem 4.1 is the smallest possible speed c in (4.5) for a planar traveling wave to exist. In addition, Komogorov, Petrovskii, and Piskunov showed that the solution of (4.1) for $n = 1$, $A = -\Delta$, and with initial datum the Heaviside function $H(x) = \chi_{(0,\infty)}(x)$ converges as $t \to +\infty$ to a traveling wave with speed $c = c_*$.

Recall now that the fractional Laplacian is the generator of a stable Lévy process —a jump process. It is reasonable to expect that, when $A = (-\Delta)^\alpha$ with $\alpha \in (0,1)$, the existence of jumps (or flights) in the diffusion process will accelerate the invasion of the unstable state $u = 0$ by the stable one, $u = 1$. This has been sustained in the Physics literature (see references in [CR2]) through the linearization of the equation at the leading edge of the front (i.e., $u = 0$), as well as through numerical simulations. As we will see below, these heuristics predict that the front position will be exponential in time —in contrast with the classical case where it is linear in time by Theorem 4.1. The purpose of [CR1, CR2] is to provide a rigorous mathematical justification of this fact.

The key difference between Brownian and Lévy diffusions can be seen, analytically, in the behaviour for $|x|$ large of the heat kernel $p(t,x)$ above. While $p(t,\cdot)$ is a Gaussian distribution when $A = -\Delta$, for $A = (-\Delta)^\alpha$ and $\alpha \in (0,1)$ the kernel p has power tails in x, in the sense that it satisfies:

- There exist $\alpha \in (0,1)$ and $B > 1$ such that, for $t > 0$ and $x \in \mathbb{R}^n$,
$$\frac{B^{-1}}{t^{\frac{n}{2\alpha}}(1 + |t^{-\frac{1}{2\alpha}}x|^{n+2\alpha})} \leqslant p(t,x) \leqslant \frac{B}{t^{\frac{n}{2\alpha}}(1 + |t^{-\frac{1}{2\alpha}}x|^{n+2\alpha})}. \tag{4.6}$$

This power decay assumption for p is crucial for the results of [CR1, CR2] to hold. It is satisfied by $A = (-\Delta)^\alpha$ when $\alpha \in (0,1)$. For instance, as an exercise, one can verify that, when $\alpha = 1/2$, $p_{1/2}$ admits the explicit expression

$$p_{1/2}(t,x) = B_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}} = \frac{B_n}{t^n(1 + |t^{-1}x|^2)^{(n+1)/2}},$$

where $B_n = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}$ is chosen to ensure property (4.2) above. That is, $p_{1/2}$ is the Poisson kernel which reproduces harmonic functions in $\mathbb{R}^{n+1}_+$ given their trace on $\mathbb{R}^n$. To verify this, one can use the extension property described in section 2 above to check that solving $\Delta_{t,x}U = \Delta_x U + U_{tt} = 0$ in $\mathbb{R}^{n+1}_+ = \{(t,x) \ :\ t > 0\}$, the function $U(t,x)$ is the solution of the evolution problem (4.4) when $A = (-\Delta)^{1/2}$ and $u_0 = U(0,\cdot)$. Alternatively, consider the composition $(\partial_t - A)(\partial_t + A)$.

Our first result concerns a class of initial data in $\mathbb{R}^n$, possibly discontinuous, which includes compactly supported functions. We show that the position of all level sets moves exponentially fast in time.

Theorem 4.2 ([CR2]). *Let $n \geqslant 1$, $\alpha \in (0,1)$, and p be a kernel satisfying* (4.2), (4.3), *and* (4.6).

Let $\sigma_* = \frac{f'(0)}{n+2\alpha}$. Let u be a solution of (4.1), *where $u_0 \not\equiv 0$, $0 \leqslant u_0 \leqslant 1$ is measurable, and*

$$u_0(x) \leqslant C|x|^{-n-2\alpha} \quad \text{for all } x \in \mathbb{R}^n$$

and for some constant C. Then,

a) *if $\sigma > \sigma_*$, then $u(t,x) \to 0$ uniformly in $\{|x| \geqslant e^{\sigma t}\}$ as $t \to +\infty$.*
b) *if $\sigma < \sigma_*$, then $u(t,x) \to 1$ uniformly in $\{|x| \leqslant e^{\sigma t}\}$ as $t \to +\infty$.*

Part b) on convergence towards 1 is the most delicate part of the theorem. The following is an easy consequence on non-existence of traveling waves.

Corollary 4.3 ([CR2]). *Let $n \geqslant 1$, $\alpha \in (0,1)$, and p be a kernel satisfying* (4.2), (4.3), *and* (4.6). *Then, there exists no nonconstant traveling wave solution of* (4.1). *That is, all solutions of* (4.1) *taking values in $[0,1]$ and of the form $u(t,x) = \varphi(x+te)$, for some vector $e \in \mathbb{R}^n$, are identically 0 or 1. Equivalently, the only solutions $\varphi : \mathbb{R}^n \to [0,1]$ of $A\varphi + e \cdot \nabla\varphi = f(\varphi)$ in $\mathbb{R}^n$ are $\varphi \equiv 0$ and $\varphi \equiv 1$.*

Let us now explain some heuristics that predict the speed of propagation of fronts both in the case of Brownian and fractional diffusions. We linearize the problem around the leading edge of the front, that is, at $u = 0$. In fact, since f is concave, the solution $\overline{u}$ of

$$\overline{u}_t - \Delta\overline{u} = f'(0)\overline{u} \quad \text{and} \quad \overline{u}(0,\cdot) = u(0,\cdot) \quad \text{in } \mathbb{R}^n$$

is a supersolution of (4.1). Looking at the particular case $\overline{u}(0,\cdot) = \delta_0$, the Dirac mass at 0, we obtain $\overline{u}(t,x) = (4\pi t)^{-\frac{n}{2}} e^{f'(0)t - \frac{|x|2}{4t}}$. Thus, $\overline{u} = \lambda$ if $|x| = 2\sqrt{f'(0)}t + \mathrm{o}(t)$.

Heuristic arguments. Let us now make the same heuristic argument when $0 < \alpha < 1$ and (4.6) holds. Now the solution $\overline{u}$ of $\overline{u}_t + A\overline{u} = f'(0)\overline{u}$ and $\overline{u}(0,\cdot) = \delta_0$ in $\mathbb{R}^n$ is given by

$$\overline{u}(t,x) = e^{f'(0)t} p(t,x).$$

Estimate (4.6) gives that $\overline{u} = \lambda$ if $|t^{-\frac{1}{2\alpha}} x|^{n+2\alpha} = t^{-\frac{n}{2\alpha}} e^{f'(0)t}\,\mathrm{O}(1)$, that is, if

$$|x| = t^{\frac{1}{n+2\alpha}}\, e^{\sigma_* t}\,\mathrm{O}(1), \quad \text{where } \sigma_* = \frac{f'(0)}{n+2\alpha}$$

is the same exponent as in Theorem 4.2. However, in next section we will improve the previous theorem and we will see that linearizing at the front edge is not as accurate in the presence of fractional diffusion as it is for Brownian diffusion. Indeed, the factor $t^{\frac{1}{n+2\alpha}}$ will not appear in the correct expression for the position of the front. In next section we will give a more accurate description of the level sets of the solution, both in the present setting and in the case of heterogeneous media.

Nondecreasing initial data. Let us mention that in one space dimension it is also of interest to understand the dynamics of nondecreasing initial data. As mentioned before, for the standard Laplacian the level sets of u travel with the speed c_*, provided that $u(0,\cdot)$ decays sufficiently fast at $-\infty$. In the fractional case, the mass at $+\infty$ has an effect and what happens is not a mere copy of the result of Theorem 4.2 for compactly supported data. The mass at $+\infty$ makes the front travel faster to the left, indeed it travels exponentially fast but with a larger exponent σ_{**} than σ_*. The new exponent (see [CR2]), is given by

$$\sigma_{**} = \frac{f'(0)}{2\alpha} > \frac{f'(0)}{1+2\alpha} = \sigma_*.$$

Proofs of our results. Let us briefly discuss the main ideas in the proofs of our results. The supersolutions obtained by solving $\overline{u}_t + A\overline{u} = f'(0)\overline{u}$ give an upper bound for the position (in norm) of the level sets. This leads immediately to part a) of Theorem 4.2.

Part b) on convergence towards 1 is the delicate point and it is done in two steps. The first one consists of showing that, for every $\sigma < \sigma_*$ there exists $\varepsilon \in (0,1)$ and $\underline{t} > 0$ such that

$$u(t,x) \geqslant \varepsilon \quad \text{for all } t \geqslant \underline{t} \text{ and } |x| \leqslant e^{\sigma t}.$$

This is accomplished by constructing solutions of the equation

$$\underline{v}_t + A\underline{v} = (f(\delta)/\delta)\underline{v} \tag{4.7}$$

which take values in $(0,\delta)$ —and, as a consequence of the concavity of f, are subsolutions of (4.1). This is done truncating an initial datum v_0 at a level ε, where $\varepsilon < \delta$, i.e., considering $\min(v_0, \varepsilon)$. We then solve the linear equation (4.7) for $\underline{v}$ with this new datum, up to the time T where $\underline{v}$ takes the value δ. At this point we compute how the level sets have propagated. We then truncate $\underline{v}(T,\cdot)$ at the level ε as before, and we iterate this procedure.

The convergence towards 1 is shown using (4.7) and a subsolution taking values in $[\varepsilon, 1]$ built through the linear equation

$$\underline{w}_t + A\underline{w} = (f(\varepsilon')/(1-\varepsilon'))(1-\underline{w})$$

for some $0 < \varepsilon' < \varepsilon$ and an initial condition involving $|x|^\gamma$, with $\gamma \in (0, 2\alpha)$. Here again we use the concavity of f to ensure that $f(\varepsilon')(1-\varepsilon')^{-1}(1-\underline{w}) \leqslant f(\underline{w})$ for $\underline{w} \in [\varepsilon, 1]$.

5. Fisher-KPP type equations with fractional diffusion in periodic media

We are interested in the time asymptotic location of the level sets of solutions to the equation

$$u_t + (-\Delta)^\alpha u = \mu(x)u - u^2, \qquad t > 0,\ x \in \mathbb{R}^n, \tag{5.1}$$

with initial condition $u(\cdot, 0) = u_0$, where $\alpha \in (0,1)$, μ is periodic in each x_i-variable and satisfies $0 < \min \mu \leqslant \mu(x)$, and $(-\Delta)^\alpha$ is the fractional Laplacian. The nonlinearity $\mu(x)u - u^2$ is often referred to as a Fisher-KPP type nonlinearity. When $\mu \equiv 1$ we recover the problem studied in the previous section.

Let λ_1 be the principal periodic eigenvalue of the operator $(-\Delta)^\alpha - \mu(x)I$. From [BRR] one knows that if $\lambda_1 \geqslant 0$, every solution to (5.1) starting with a bounded nonnegative initial condition tends to 0 as $t \to +\infty$ (there is extinction of the population). Thus we assume $\lambda_1 < 0$. Then, by [BRR], the solution to (5.1) tends, as

$t \to +\infty$, to a bounded positive steady solution to (5.1), denoted by $u_+ = u_+(x)$, which can be proved to be the unique bounded positive steady solution. By uniqueness, u_+ is periodic. The convergence holds on every compact set. Hence, the level sets of u spread to infinity for large times, and we wish to understand how fast. To do it, we look for a function $R_e(t)$ going to $+\infty$ as t tends to $+\infty$ such that, for every direction $e \in S^{d-1}$ and every constant $c \in (0,1)$,

$$\liminf_{t\to+\infty}\left(\inf_{\{x=\rho e, 0\leqslant\rho\leqslant R_e(ct)\}} u(t,x)\right) > 0 \quad \text{and} \quad \limsup_{t\to+\infty}\left(\sup_{\{x=\rho e,\rho\geqslant R_e(c^{-1}t)\}} u(t,x)\right) = 0. \tag{5.2}$$

The case $\alpha = 1$ corresponding to homogeneous media ($\mu \equiv 1$) has been well studied. If u_0 is compactly supported, we saw in the previous section that we may choose $R_e(t) = 2t$ regardless of the direction e of propagation. In space periodic media (i.e., when μ is periodic in each variable), starting from a compactly supported initial data, Freidlin and Gärtner have characterized $R_e(t)$ by

$$R_e(t) = w^*(e)t, \quad w^*(e) = \min_{e'\in S^{d-1}, e'\cdot e>0} \frac{c^*(e')}{e'\cdot e}, \tag{5.3}$$

where $c^*(e')$ is the minimal speed of pulsating traveling fronts in the direction e'.

For $\alpha \in (0,1)$ and $\mu \equiv 1$ in (5.1), we saw in the previous section that propagation is exponential in time and that we may take $R_e(t) = e^{\frac{t}{n+2\alpha}}$ in this case.

In the paper [CCR] we prove the following result. It gives a precise description of the location of the level sets of the solution.

Theorem 5.1 ([CCR]). *Assume that $\lambda_1 < 0$. Let u be the solution to* (5.1) *with u_0 piecewise continuous, nonnegative, $u_0 \not\equiv 0$, and $u_0(x) = \mathrm{O}(|x|^{-(n+2\alpha)})$ as $|x| \to \infty$.*

Then, for every $\lambda \in (0, \min\mu)$, there exist $c_\lambda > 0$ and a time $t_\lambda > 0$ (all depending on λ and u_0) such that, for all $t \geqslant t_\lambda$,

$$\{x \in \mathbb{R}^n \ : \ u(t,x) = \lambda\} \subset \{x \in \mathbb{R}^n \ : \ c_\lambda e^{\frac{|\lambda_1|}{n+2\alpha}t} \leqslant |x| \leqslant c_\lambda^{-1} e^{\frac{|\lambda_1|}{n+2\alpha}t}\}. \tag{5.4}$$

Theorem 5.1 gives that (5.2) holds with $R_e(t) = e^{\frac{|\lambda_1|}{n+2\alpha}t}$. Thus, spreading does not depend on the direction of propagation, and this is in contrast with (5.3) for the standard Laplacian. Note that when $\lambda \geqslant \min u_+$, (5.4) can not hold since $u(t,x) \to u_+(x)$ as $t \to +\infty$.

Moreover, in the homogeneous case $\mu \equiv 1$, we have $\lambda_1 = -1$ and the estimate in the previous theorem is much sharper than the results of the previous section. Indeed, to guarantee the limits in (5.2), the results of [CR2] described in the previous section needed to assume $|x| \leqslant Ce^{\sigma_1 t}$ (respectively $|x| \geqslant Ce^{\sigma_2 t}$) with $\sigma_1 < \frac{1}{n+2\alpha} < \sigma_2$.

The proof of Theorem 5.1 is quite simple: it relies on the construction of explicit subsolutions and supersolutions, which are themselves based on a nonlinear transport equation, (5.5), satisfied asymptotically by a correctly rescaled version of the solution u. More precisely, recall that $\lambda_1 < 0$ denotes the principal periodic eigenvalue of the operator $(-\Delta)^\alpha - \mu(x)I$ and that the corresponding periodic eigenfunction is denoted by ϕ_1.

We write
$$u(t,x) = \phi_1(x)v(t,x)$$
and define
$$w(t,y) = v(t,r(t)y) \qquad \text{where } r(t) = e^{\frac{|\lambda_1|t}{n+2\alpha}}.$$
Thus, for $t > 0$ and $y \in \mathbb{R}^n$, w solves
$$w_t - \frac{|\lambda_1|}{n+2\alpha} y \cdot w_y + e^{\frac{-2\alpha|\lambda_1|t}{n+2\alpha}} \left\{ (-\Delta)^\alpha w - \frac{Kw}{\phi_1(r(t)y)} \right\} = |\lambda_1| w - \phi_1(r(t)y)\ w^2,$$
where we have used $\lambda_1 < 0$ and we have defined
$$Kw(y) = C_{n,\alpha}\ P.V. \int_{\mathbb{R}^n} \frac{\phi_1(r(t)y) - \phi_1(r(t)\overline{y})}{|y-\overline{y}|^{n+2\alpha}} (w(y) - w(\overline{y})) d\overline{y}.$$
If we formally neglect the term $e^{\frac{-2\alpha|\lambda_1|t}{n+2\alpha}} \left\{ (-\Delta)^\alpha w - \frac{Kw}{\phi_1(r(t)y)} \right\}$ which should go to 0 as $t \to +\infty$, we get the transport equation
$$\widetilde{w}_t - \frac{|\lambda_1|}{n+2\alpha} y \cdot \widetilde{w}_y = |\lambda_1| \widetilde{w} - \phi_1(r(t)y)\ \widetilde{w}^2, \qquad t > 0, y \in \mathbb{R}^n. \tag{5.5}$$
Equation (5.5), completed by an initial datum $\widetilde{w_0}$, is solved by
$$\widetilde{w}(t,y) = \left\{ \phi_1(r(t)y)|\lambda_1|^{-1}(1 - e^{-|\lambda_1|t}) + \widetilde{w_0}(r(t)y)^{-1} e^{-|\lambda_1|t} \right\}^{-1}.$$
Taking into account (see [CR2]) that $|x|^{n+2\alpha}u(x,t)$ is uniformly bounded from above and below (but of course not uniformly in t), it is natural to consider the initial datum $\widetilde{w_0}(y) = (1 + |y|^{n+2\alpha})^{-1}$. In this case we have
$$\widetilde{w}(y,t) = \left\{ \phi_1(r(t)y)|\lambda_1|^{-1}(1 - e^{-|\lambda_1|t}) + e^{-|\lambda_1|t} + |y|^{n+2\alpha} \right\}^{-1}.$$

Since ϕ_1 is bounded above and below and t tends to $+\infty$, coming back to the function $v(t,x) = w(t, r(t)^{-1}x)$, the idea is to consider the following family of functions modeled by $\widetilde{w}$:
$$\widetilde{v}(t,x) = \frac{a}{|\lambda_1|^{-1} + b(t)|x|^{n+2\alpha}}, \qquad \widetilde{u}(t,x) = \phi_1(x)\widetilde{v}(t,x). \tag{5.6}$$

It is then possible to adjust $a > 0$ and $b(t)$ (asymptotically proportional to $e^{-|\lambda_1|t}$ and solving certain ODEs) so that the function $\widetilde{u}(x,t)$ serves as a subsolution (respectively, a supersolution) to (5.1). To be able to place above (respectively, below) the initial datum, one must first let the equation run for some time and also use some results in [CR2] from the previous section. See [CCR] for all the details of the proof.

6. Traveling wave solutions in a half-space for boundary reactions

The article [CCM] concerns the problem
$$\begin{cases} v_t - \Delta v = 0 & \text{in } (0,\infty) \times \mathbb{R}^2_+ \\ \dfrac{\partial v}{\partial \nu} = f(v) & \text{on } (0,\infty) \times \partial\mathbb{R}^2_+ \end{cases} \tag{6.1}$$
for the homogeneous heat equation in a half-plane with a nonlinear Neumann boundary condition. To study the propagation of fronts given an initial condition, it is

important to understand first the existence and properties of *traveling fronts* —or traveling waves— for (6.1). Taking $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \,:\, y > 0\}$, these are solutions of the form $v(t,x,y) := u(x-ct,y)$ for some speed $c \in \mathbb{R}$. Thus, the pair (c,u) must solve the elliptic problem

$$\begin{cases} \Delta u + cu_x = 0 & \text{in } \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \,:\, y > 0\} \\ \dfrac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{6.2}$$

where $\partial u/\partial\nu = -u_y$ is the exterior normal derivative of u on $\partial\mathbb{R}^2_+ = \{y=0\}$, u is real valued, and $c \in \mathbb{R}$. We look for solutions u with $0 < u < 1$ and having the limits

$$\lim_{x\to-\infty} u(x,0) = 1 \quad \text{and} \quad \lim_{x\to+\infty} u(x,0) = 0. \tag{6.3}$$

Note that u is a solution of (6.2) if and only if its trace $w(x) := u(x,0)$ solves the fractional diffusion equation

$$(-\partial_{xx} - c\partial_x)^{1/2} w = f(w) \ \text{ in } \mathbb{R}, \quad \text{for } w(x) := u(x,0). \tag{6.4}$$

This follows from two facts. First, if u solves the first equation in (6.2), then so does $-u_y$. Second, we have $(-\partial_y)^2 u = \partial_{yy} u = (-\partial_{xx} - c\partial_x)u$.

In [CCM] we study nonlinearities f of non-balanced bistable type or of combustion type, as defined next. Let f satisfy

$$f(0) = f(1) = 0 \qquad \text{and} \qquad f' \leqslant 0 \ \text{ in } (0,\delta) \cup (1-\delta,1) \tag{6.5}$$

for some $\delta \in (0,1/2)$. We say that f is of *positively-balanced bistable type* if, in addition to (6.5), f has a unique zero —named α— in $(0,1)$ and f is “positively-balanced” in the sense that $\int_0^1 f(s)ds > 0$. Instead, we say that f is of *combustion type* if, in addition to (6.5), there exists $0 < \beta < 1$ (called the *ignition temperature*) such that $f \equiv 0$ in $(0,\beta)$ and $f > 0$ in $(\beta,1)$.

In problem (6.2) one must find not only the solution u but also the speed c, which is apriori unknown. Our results are collected in the following result.

Theorem 6.1 ([CCM]). *Let f be of positively-balanced bistable type or of combustion type. We have:*

(i) *There exists a solution pair (c,u) to problem* (6.2)*, where $c > 0$, $0 < u < 1$, and u has the limits* (6.3)*. The solution u lies in the weighted Sobolev space*

$$H^1_c(\mathbb{R}^2_+) := \{w \in H^1_{\rm loc}(\mathbb{R}^2_+) \,:\, ||w||_c := \int_{\mathbb{R}^2_+} e^{cx}\{w^2 + |\nabla w|^2\} dxdy < \infty\}.$$

(ii) *Up to translations in the x variable, (c,u) is the unique solution pair to problem* (6.2) *among all constants $c \in \mathbb{R}$ and solutions u satisfying $0 \leqslant u \leqslant 1$ and the limits* (6.3).

(iii) *For all $y \geqslant 0$, u is decreasing in the x variable, and has limits $u(-\infty,y) = 1$ and $u(+\infty,y) = 0$. Besides, $\lim_{y\to+\infty} u(x,y) = 0$ for all $x \in \mathbb{R}$. If f is of combustion type then we have, in addition, $u_y \leqslant 0$ in $\mathbb{R}^2_+$.*

(iv) *If f_1 is of positively-balanced bistable type or of combustion type, if the same holds for another nonlinearity f_2, and if we have that $f_1 \geqslant f_2$ and $f_1 \not\equiv f_2$, then their corresponding speeds satisfy $c_1 > c_2$.*

(v) *Assume that f is of positively-balanced bistable type and that $f'(0) < 0$ and $f'(1) < 0$. Then, there exists a constant $b > 1$ such that*

$$\begin{aligned} \frac{1}{b}\,\frac{e^{-cx}}{x^{3/2}} \leqslant u(x,0) \leqslant b\,\frac{e^{-cx}}{x^{3/2}} \qquad & \textit{for } x > 1, \quad \textit{and} \\ \frac{1}{b}\,\frac{1}{(-x)^{1/2}} \leqslant 1 - u(x,0) \leqslant b\,\frac{1}{(-x)^{1/2}} \qquad & \textit{for } x < -1. \end{aligned}$$

In the case of combustion nonlinearities, problem (6.2) in a half-plane has also been studied by Caffarelli, Mellet, and Sire [CMS]. They establish the existence of a speed admitting a monotone front. As mentioned in [CMS], our approaches towards the existence result are different. Their work, in contrast with ours, does not use minimization methods. In addition, [CMS] establishes the following precise behavior of the combustion front at the side of the invaded state $u = 0$ —a different behavior than ours. For some constant $\mu_0 > 0$,

$$u(x,0) = \mu_0\,\frac{e^{-cx}}{x^{1/2}} + \mathrm{O}\left(\frac{e^{-cx}}{x^{3/2}}\right) \qquad \text{as } x \to +\infty.$$

Note that Theorem 6.1 states that there is a unique $c \in \mathbb{R}$ for which the fractional equation (6.4), that is

$$(-\partial_{xx} - c\partial_x)^{1/2} w = f(w)$$

admits a solution connecting 1 and 0. Instead, the existence of traveling fronts for the fractional diffusion equation (the same of previous sections)

$$\partial_t v + (-\partial_{xx})^{\alpha} v = f(v) \quad \text{in } \mathbb{R} \tag{6.6}$$

has been established in [MRS] when $\alpha \in (1/2, 1)$ and f is a combustion nonlinearity. This article also shows that v tends to 0 at $+\infty$ at the power rate $1/|x|^{2\alpha-1}$. Note that the equation for traveling fronts of (6.6) is

$$\{(-\partial_{xx})^{\alpha} - c\partial_x\}\, w = f(w) \quad \text{in } \mathbb{R},$$

which should be compared with (6.4). In the case of bistable nonlinearities, [GZ] establishes that (6.6) admits a unique traveling front and a unique speed for any $\alpha \in (0,1)$. In contrast with the decay in [MRS] for combustion nonlinearities, in the bistable case [GZ] shows that the front reaches its two limiting values at the rate $1/|x|^{2\alpha}$ —as in [CSo, CSi2] for balanced bistable nonlinearities.

Our result on the existence of the traveling front will be proved using a variational method introduced by Steffen Heinze [H] to study problem (6.2) in infinite cylinders of $\mathbb{R}^n$ instead of half-spaces. For these domains and for both bistable and combustion nonlinearities, he showed the existence of a traveling front.

We first extend f linearly in C^1 fashion to $(-\infty, 0)$ and to $(1, +\infty)$. Consider now the potential $G \in C^2(\mathbb{R})$ defined by $G(s) := -\int_0^s f(\sigma)d\sigma$ for $s \in \mathbb{R}$. The following are important properties of G. We have $G(1) < G(0) = 0$, $G'(0) = -f(0) = 0$, and $G \geqslant 0$ in $[0, \beta]$.

For $a > 0$, consider the weighted Sobolev space $(H^1_a(\mathbb{R}^2_+), || \, ||_a)$ defined by

$$H^1_a(\mathbb{R}^2_+) = \{w \in H^1_{\text{loc}}(\mathbb{R}^2_+) \, : \, ||w||_a < \infty\},$$

where the norm $|| \, ||_a$ is defined by $||w||_a^2 = \int_{\mathbb{R}^2_+} e^{ax}\{w^2 + |\nabla w|^2\} dxdy$. In both the bistable and combustion cases, the traveling front u will be constructed from *a minimizer* $\underline{u}$ *to the constraint problem* $E_a(\underline{u}) = \inf_{w \in B_a} E_a(w) =: I_a$, after scaling its independent variables x and y. Here, the energy functional is

$$E_a(w) = \frac{1}{2} \int_{\mathbb{R}^2_+} e^{ax} |\nabla w|^2 dxdy + \int_{\partial \mathbb{R}^2_+} e^{ax} G(w(x,0)) dx$$

and it is minimized over the submanifold

$$B_a = \{w \in H^1_a(\mathbb{R}^2_+) : \Gamma_a(w) = 1\}, \qquad \text{where} \quad \Gamma_a(w) = \int_{\mathbb{R}^2_+} e^{ax} |\nabla w|^2 dxdy.$$

The shape of the potential G will lead (for $a > 0$ small) to the existence of functions u in $H^1_a(\mathbb{R}^2_+)$ with negative energy $E_a(u) < 0$. This will be essential in order to prove that our variational problem attains its infimum. In addition, the constraint will introduce a Lagrange multiplier and, through it, the apriori unknown speed c of the traveling front.

A *rearrangement technique* after making the change of variables $z = e^{ax}/a$ (also used in [H]), produces a monotone front. Its monotonicity will be crucial in order to establish that it has limits 1 and 0 as $x \to \mp\infty$.

Our result on uniqueness of the speed and of the front relies heavily on the powerful *sliding method* of Berestycki and Nirenberg (see [CSo] for an application of the method to problem (6.2) with $c = 0$ and f balanced). Among other things, [CSo] established the existence, uniqueness, and monotonicity of a front for (6.2) when $c = 0$ and f is a balanced bistable nonlinearity. It was shown also there that in the balanced bistable case, the front reaches its limits 1 and 0 at the power rate $1/|x|$.

To prove our decay estimates as $x \to \pm\infty$, we use ideas from [CMS] and [CSi2]. The estimates rely on the construction of a family of explicit fronts for some bistable nonlinearities. These explicit fronts will be based on the fundamental solution for the homogeneous heat equation associated to the fractional operator in (6.4), that is, equation $\partial_t v + (-\partial_{xx} - c\partial_x)^{1/2} v = 0$. The process to find such heat kernel uses an idea from the paper [CMS] by Caffarelli, Mellet, and Sire, to reduce equation $\Delta w + 2w_x = 0$ in $\mathbb{R}^2_+$ to the Helmholtz equation $-\Delta\phi + \phi = 0$ after the change of variables $w = e^{-x}\phi$. The fundamental solution of the Helmholtz equation (a well known modified Bessel function) is then used; see [CCM].

7. Curves and surfaces with constant nonlocal mean curvature

In the final part of the course we will explain some recent results on nonlocal (or fractional) minimal surfaces and on surfaces with constant nonlocal mean curvature (CNMC surfaces). These are the Euler-Lagrange equations associated with the fractional perimeter functional:

$$P_{\overline{\alpha}}(E) := c_{n,\overline{\alpha}} \int_E \int_{E^c} \frac{dx\,dy}{|x-y|^{n+\overline{\alpha}}}. \tag{7.1}$$

Here $\overline{\alpha} := 2\alpha \in (0,1)$ —and hence $\alpha \in (0,1/2)$— and $E \subset \mathbb{R}^n$ is a bounded smooth set.

Its Euler-Lagrange equation can be extended to unbounded sets and leads to the notion of nonlocal mean curvature:

$$H_E(x) = -PV \int_{\mathbb{R}^N} \frac{1_E(y) - 1_{E^c}(y)}{|x-y|^{n+\overline{\alpha}}}\,dy, \tag{7.2}$$

where $E \subset \mathbb{R}^n$ and $x \in \partial E$. An alternative expression is given by

$$H_E(x) = -\frac{2}{\overline{\alpha}} PV \int_{\partial E} \frac{(x-y)\cdot \nu(y)}{|x-y|^{n+\overline{\alpha}}}\,dy. \tag{7.3}$$

Good references for nonlocal minimal surfaces (introductory and with most of the known results; many things are still to be discovered) are the same papers that we used for the fractional Allen-Cahn equation:

- Section 1 now of [V2], A fractional framework for perimeters and phase transitions, by E. Valdinoci, `http://arxiv.org/abs/1210.5612`
- Section 6 now of [BV], Nonlocal diffusion and applications, by C. Bucur and E. Valdinoci, `http://arxiv.org/abs/1504.08292`

When fractional perimeter is minimized under a volume constraint one obtains surfaces with constant nonlocal mean curvature (CNMC surfaces). There are only two papers on this equation. We will explain the results in

- Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay, by X. Cabré, M.M. Fall, J. Solà-Morales, and T. Weth [CFSW], `http://arxiv.org/abs/1503.00469`

In this work we prove the nonlocal analogue of the Alexandrov result characterizing spheres as the only closed embedded hypersurfaces in $\mathbb{R}^n$ with constant mean curvature. Here we use the moving planes method. Our second result establishes the existence of periodic bands or "cylinders" in $\mathbb{R}^2$ with constant nonlocal mean curvature and bifurcating from a straight band $\{(s_1,s_2) \in \mathbb{R}^2 : -\lambda R < s_2 < \lambda R\}$. These are Delaunay type bands in the nonlocal setting. Here we use a Lyapunov-Schmidt procedure for the quasilinear type fractional elliptic equation

$$\int_{\mathbb{R}} \left\{ F\left(\frac{u(s)-u(s-t)}{|t|}\right) - \left\{ F\left(\frac{u(s)+u(s-t)}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right)\right\}\right\} \frac{dt}{|t|^{1+\overline{\alpha}}} = 0,$$

where F is an odd function on $\mathbb{R}$, bounded and concave in $(0,+\infty)$, the unknown function $u : \mathbb{R} \to (0,+\infty)$ is even and 2π-periodic, and the CNMC set will be given by

$$E := \{(s_1,s_2) \in \mathbb{R}^2 : -u(s_1) < s_2 < u(s_1)\}.$$

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