Nonlinear elliptic and parabolic equations with fractional diffusion

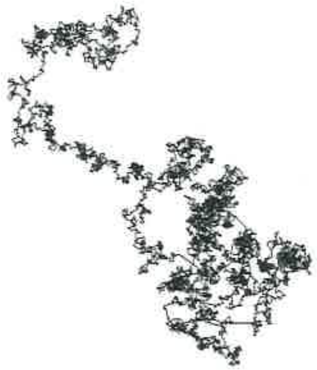
Xavier Cabré

ICREA and UPC, Barcelona

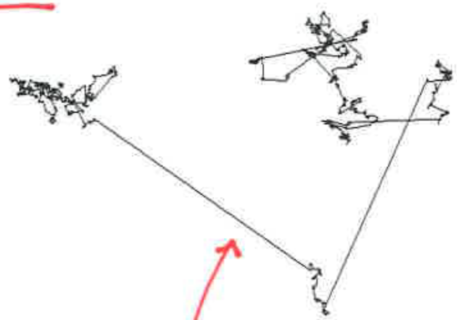
- The fractional Laplacian
- Semilinear equations

Levy processes and fractional Laplacians

$\boxed{-\Delta}$: Brownian motion



$\boxed{(-\Delta)^s, \ \underline{0 < s < 1}}$: Levy processes



PURE
JUMP

$(-\Delta) + (-\Delta)^{1/2}$, e.g.

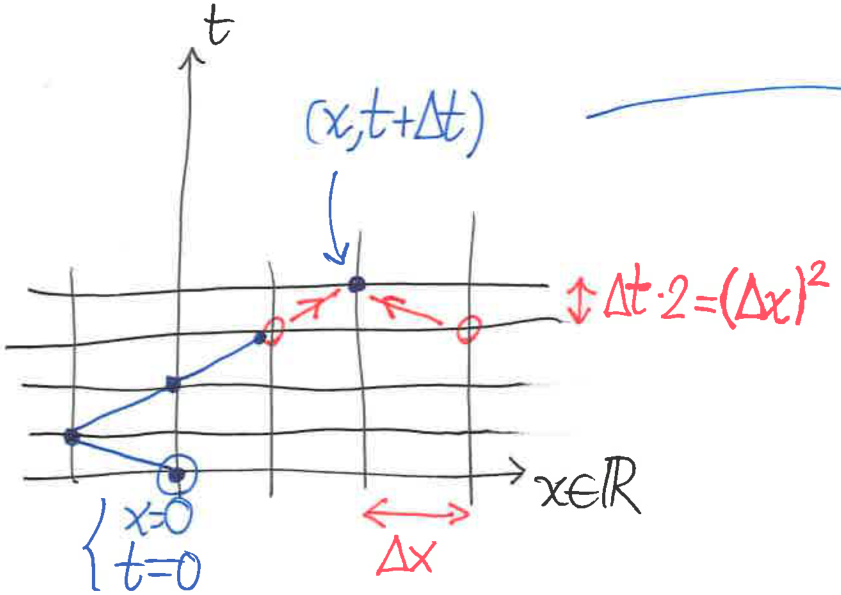
Levy processes & Fractional Laplacians,
type of “anomalous diffusions” in:

- Dislocation of crystals
(boundary reactions: the Peierls-Nabarro Problem)

- Micro-magnetism (thin films)
- Mathematical finance (American options,...)
- Quasigeostrophic equations
- The Signorini problem (“thin obstacle problem”)

- Fluids, biology (front propagation, travelling waves)

The heat equation & the Central Limit Theorem



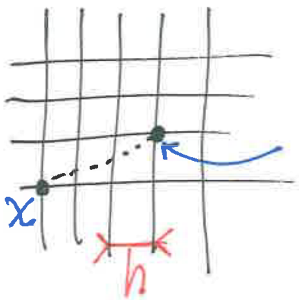
Probability :

$$u(x, t+\Delta t) = \frac{1}{2}\left(u(x-\Delta x, t) + u(x+\Delta x, t) \right)$$

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x-\Delta x,t) + u(x+\Delta x,t) - 2u(x,t)}{|\Delta x|^2}$$

$$\begin{cases} \boxed{\partial_t u = \Delta_x u} \\ u(t=0) = \delta_0 \end{cases}$$

The long jump random walk and the fractional Laplacian



$x+hk$
$k\in\mathbb{Z}^n$

$$\begin{cases} \tau = \text{time step} \\ h = \text{space step} \end{cases}$$

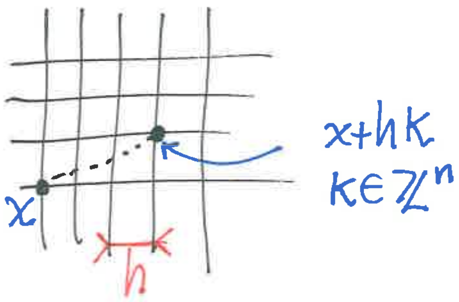
$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^n} \underbrace{K(k)}_{\text{Probab of jump } x\leftrightarrow x+hk} u(x+hk,t)$$

$$\Downarrow$$

$$\left.\begin{aligned} &u(x,t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}^n} K(k)\{u(x+hk,t) - u(x,t)\} \\ &\tau = h^{2s} \ \& \ K(y) = |y|^{-n-2s} \qquad (0<s<1) \end{aligned}\right\} \rightarrow$$

$$\frac{u(x,t+\tau)-u(x)}{\tau} = h^n \sum_{k\in\mathbb{Z}^n} K(hk)\{u(x+hk,t) - u(x,t)\}$$

The long jump random walk and the fractional Laplacian



$$\begin{cases} \tau = \text{time step} \\ h = \text{space step} \end{cases}$$

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^n} \underbrace{K(k)}_{\text{Probab of jump } x \leftrightarrow x+hk} u(x+hk,t)$$

$$\Downarrow$$

$$\left.\begin{aligned} & u(x,t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}^n} K(k)\{u(x+hk,t) - u(x,t)\} \\ & \tau = h^{2s} \ \& \ K(y) = |y|^{-n-2s} \qquad (0<s<1) \end{aligned}\right\} \rightarrow$$

$$\frac{u(x,t+\tau)-u(x)}{\tau} = h^n \sum_{k\in\mathbb{Z}^n} K(hk)\{u(x+hk,t)-u(x,t)\}$$

$$\xrightarrow[\substack{h\downarrow 0 \\ h^{2s}=\tau\downarrow 0}]{} \quad \boxed{\partial_t u = -\underbrace{c_{n,s}}\ \mathrm{P.V.}\int_{\mathbb{R}^n} \underbrace{\frac{u(x,t)-u(x+y,t)}{|y|^{n+2s}}\,dy =: -c_{n,s}(-\Delta)^s u}}$$

$$= \underbrace{-\frac{c_{n,s}}{2}} \int_{\mathbb{R}^n} \underbrace{\frac{2u(x,t)-u(x+y,t)-u(x-y,t)}{|y|^{n+2s}}\,dy}$$

The fractional Laplacian , $0<s<1$

$u: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(-\Delta)^s u(x) := C_{n,s}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(\bar{x})}{|x-\bar{x}|^{n+2s}}\, d\bar{x}$$

$$\int_{\mathbb{R}^n} u \cdot (-\Delta)^s u = \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \approx \|u\|_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} d\bar{x}\, \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+2s}} + \|u\|_{L^2(\mathbb{R}^n)}$$

$$\Updownarrow$$

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\, \hat{u}$$

(Fourier transform)

The half Laplacian (square root of Laplacian)

$u : \mathbb{R}^n \longrightarrow \mathbb{R}$

$(-\Delta)^{1/2} u$: $(-\Delta)^{1/2} \circ (-\Delta)^{1/2} = -\Delta$

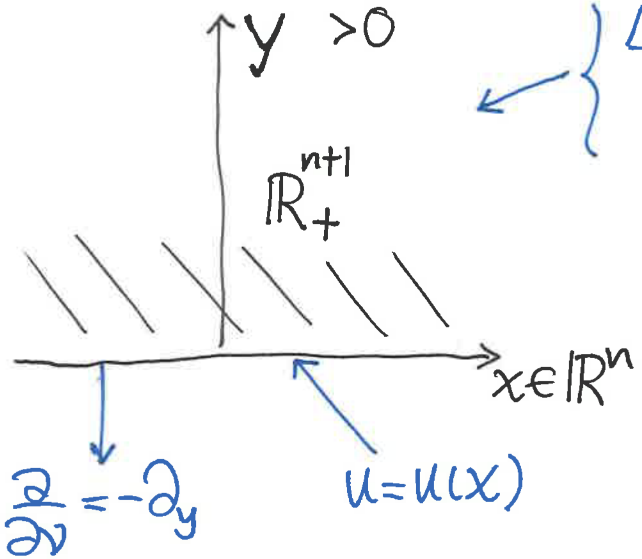
elliptic nonlocal operator of "first order".

Fourier transform: $\widehat{(-\Delta)^{1/2} u} = |\xi|\, \hat{u}$

a local (boundary reaction) representation:

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v = u & \text{on } \{y = 0\} \end{cases} \qquad \Delta_{x,y}$$

$$\rightsquigarrow \boxed{(-\Delta)^{1/2} u\,(x) = -\partial_y \mathbf{v}(x,0)}$$



The half Laplacian (square root of Laplacian)

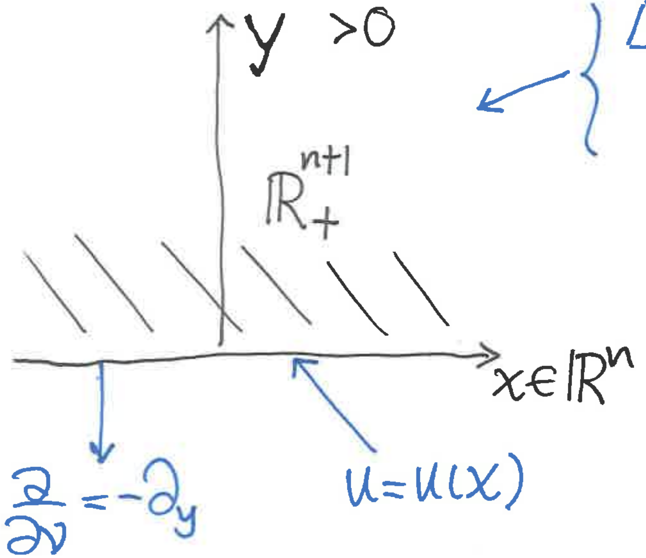
$u : \mathbb{R}^n \longrightarrow \mathbb{R}$

$(-\Delta)^{1/2} u$: $\quad (-\Delta)^{1/2} \circ (-\Delta)^{1/2} = -\Delta$

↑ elliptic nonlocal operator of "first order".

$$\begin{cases} \text{Fourier transform:} \\ \widehat{(-\Delta)^{1/2} u} = |\xi| \, \hat{u} \end{cases}$$

a local (boundary reaction) representation:



$$\begin{cases} \overbrace{\Delta}^{\Delta_{x,y}} v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v = u & \text{on } \{y=0\} \end{cases} \rightsquigarrow \boxed{(-\Delta)^{1/2} u(x) = -\partial_y v(x,0)}$$

since $(-\Delta)^{1/2} \circ (-\Delta)^{1/2} u =$

$= -\partial_y(-\partial_y v) = v_{yy} = -\Delta_x v(x,0)$
$= -\Delta_x u$

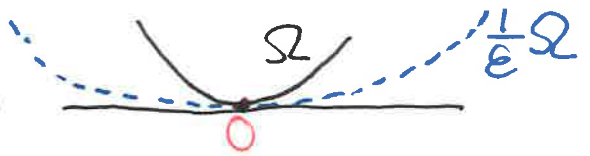
$$(-\Delta)^{1/2} u = h(x) \text{ in } \mathbb{R}^n \iff \begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial v}{\partial \nu} = h(x) & \text{on } \partial \mathbb{R}^{n+1}_+ \end{cases}$$

Phase transitions: boundary reactions

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega} G(u) \longrightarrow \begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon} f(u_\varepsilon) & \text{on } \partial\Omega \end{cases} \qquad (P_\varepsilon)$$

$\varepsilon > 0$, $\Omega \subset \mathbb{R}^n$ bounded

BLOW-UP



$\varepsilon \downarrow 0$

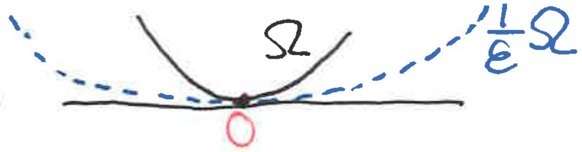
$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \ (= \mathbb{R}^2_+) \\ -u_y = f(u) & \text{on } \{y=0\} \end{cases}$$

Phase transitions: boundary reactions

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega} G(u) \longrightarrow \begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon} f(u_\varepsilon) & \text{on } \partial\Omega \end{cases} \qquad (P_\varepsilon)$$

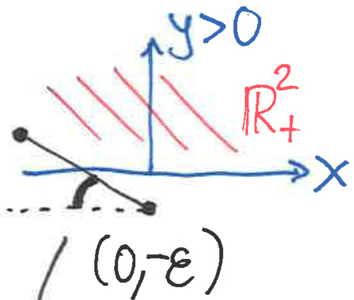
$\varepsilon > 0, \quad \Omega \subset \mathbb{R}^n$ bounded

$\downarrow$ BLOW-UP



$\xleftarrow{\varepsilon \downarrow 0}$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \ (= \mathbb{R}^2_+) \\ -u_y = f(u) & \text{on } \{y=0\} \end{cases}$$



Peierls-Nabarro problem

$f(u) = c \cdot \sin(\pi u)$

$\Updownarrow$

$$\boxed{(-\Delta)^{1/2} u = f(u) \quad \text{in } \mathbb{R}}$$

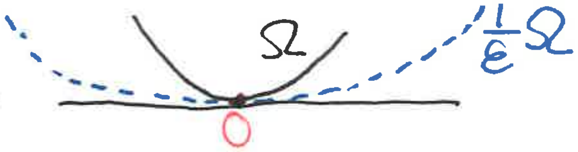
[Cabré, Sola-Morales '05]

Phase transitions: boundary reactions

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega} G(u) \longrightarrow \begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon} f(u_\varepsilon) & \text{on } \partial\Omega \end{cases} \qquad (P_\varepsilon)$$

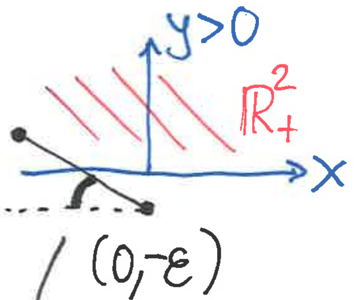
$\varepsilon > 0, \quad \Omega \subset \mathbb{R}^n$ bounded

BLOW-UP



$\varepsilon \downarrow 0$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \ (= \mathbb{R}^2_+) \\ -u_y = f(u) & \text{on } \{y = 0\} \end{cases}$$



Peierls-Nabarro problem

$f(u) = c \cdot \sin(\pi u)$

$\Updownarrow$

$$\boxed{(-\Delta)^{1/2} u = f(u) \quad \text{in } \mathbb{R}}$$

[Cabré, Sola-Morales '05]

Explicit sol'ns $-1 \nearrow 1$ =

= primitive of heat kernel $\approx \displaystyle\int_{-\infty} \frac{1}{|x|^2}$

$\|$

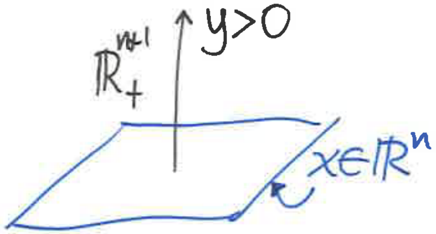
$$\underbrace{u(x,y) = \frac{2}{\pi} \arctan \frac{x}{y+\varepsilon}}_{\text{fast transition at } (0,0)}$$

The extension problem [Caffarelli-Silvestre 2007]

$0<s<1$

$u:\mathbb{R}^n\to\mathbb{R}$ $\rightsquigarrow$

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v)=0 & \text{in } \mathbb{R}^{n+1}_+ \\ v(x,0)=u(x) & \text{on } \partial\mathbb{R}^{n+1}_+=\mathbb{R}^n. \end{cases}$$



$v=v(x,y)$.

Thm [Caff-Silv]

$$\boxed{-\lim_{y\downarrow 0} y^{1-2s} v_y = \frac{\partial v}{\partial y^s}(x,0) = \tilde{c}_{n,s}(-\Delta)^s u(x).}$$

NONLOCAL ELLIPTIC EQUATIONS IN BOUNDED DOMAINS: A SURVEY

XAVIER ROS-OTON

The aim of this paper is to survey some results on Dirichlet problems of the form

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.1}$$

Here, Ω is any bounded domain in $\mathbb{R}^n$, and L is an elliptic integro-differential operator of the form

$$Lu(x) = \mathrm{PV} \int_{\mathbb{R}^n} \{u(x) - u(x+y)\} K(y) dy. \tag{1.2}$$

The function $K(y) \geq 0$ is the kernel of the operator[1] [2], and we assume

$$K(y) = K(-y) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \min\{|y|^2, 1\} K(y) dy < \infty.$$

NONLOCAL ELLIPTIC EQUATIONS IN BOUNDED DOMAINS: A SURVEY

XAVIER ROS-OTON

The aim of this paper is to survey some results on Dirichlet problems of the form

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here, Ω is any bounded domain in $\mathbb{R}^n$, and L is an elliptic integro-differential operator of the form

$$Lu(x) = \mathrm{PV} \int_{\mathbb{R}^n} \{u(x) - u(x+y)\} K(y) dy. \tag{1.2}$$

The function $K(y) \geq 0$ is the kernel of the operator[1][2], and we assume

$$K(y) = K(-y) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \min\{|y|^2, 1\} K(y) dy < \infty.$$

$$Lu(x) = \frac{1}{2} \int_{\mathbb{R}^n} \{2u(x) - u(x+y) - u(x-y)\} K(y) dy,$$

with $K(y) = K(-y)$. We will use this last expression for L throughout the paper.

As an example, let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and let us consider a Lévy process X_t, $t \geq 0$, starting at $x \in \Omega$. Let $u(x)$ be the expected first exit time, i.e., the expected time $\mathbb{E}[\tau]$, where $\tau = \inf\{t > 0 : X_t \notin \Omega\}$ is the first time at which the particle exits the domain. Then, $u(x)$ solves

$$\begin{cases} Lu = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $-L$ is the infinitesimal generator of X_t.

The aim of this paper is to survey some results on Dirichlet problems of the form

$$\begin{cases} Lu &= f \quad \text{in } \Omega \\ u &= g \quad \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.1}$$

Here, Ω is any bounded domain in $\mathbb{R}^n$, and L is an elliptic integro-differential operator of the form

$$Lu(x) = \mathrm{PV} \int_{\mathbb{R}^n} \{u(x) - u(x+y)\} K(y) dy. \tag{1.2}$$

The function $K(y) \geq 0$ is the kernel of the operator[1][2], and we assume

$$K(y) = K(-y) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \min\{|y|^2, 1\} K(y) dy < \infty.$$

The energy functional associated to the problem (1.1) is

$$\mathcal{E}(u) = \frac{1}{4} \int\int_{\mathbb{R}^{2n} \backslash (\Omega^c \times \Omega^c)} (u(x) - u(z))^2 K(z-x) dx\, dz - \int_\Omega fu. \tag{3.1}$$

The minimizer of $\mathcal{E}$ among all functions with $u = g$ in $\mathbb{R}^n \setminus \Omega$ will be the unique weak solution of (1.1).

The aim of this paper is to survey some results on Dirichlet problems of the form

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here, Ω is any bounded domain in $\mathbb{R}^n$, and L is an elliptic integro-differential operator of the form

$$Lu(x) = \mathrm{PV} \int_{\mathbb{R}^n} \{u(x) - u(x+y)\} K(y) dy. \tag{1.2}$$

The function $K(y) \geq 0$ is the kernel of the operator[1] [2], and we assume

$$K(y) = K(-y) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \min\{|y|^2, 1\} K(y) dy < \infty.$$

- Interior regularity : $K \in C^\beta(\mathbb{R}^n \setminus \{0\})$, $f \in C^\beta$, $g \in L^\infty \Rightarrow u \in C^{2s+\beta}$ when:

[J. Serra, Calc. Var, to appear '15] $\to$ $\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \qquad 0 < \lambda \leq \Lambda;$

[X. Ros-Oton & J. Serra, arXiv '14] $\to$ $K(y) = \frac{a(y/|y|)}{|y|^{n+2s}}, \qquad a \in L^1(S^{n-1}), \quad a \geq 0.$

$\hookrightarrow$ stable Lévy processes

Interior Hölder regularity results for Fully Nonlinear Integro-Diff. operators ($\inf_\gamma L_\gamma \leftarrow K = K_\gamma$) : [Caffarelli-Silvestre] (several papers)

- Boundary regularity for $\begin{cases} Lu=f & \text{in } \Omega\subset\mathbb{R}^n \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases}$

established in [X. Ros-Oton & J. Serra, ARMA '14]:

$\boxed{\dfrac{u}{\mathrm{dist}(\cdot,\partial\Omega)^s}}$ is Hölder continuous up to the boundary $\partial\Omega$

It appears and it is needed in the

- Pohozaev identity for the fractional Laplacian of [X. Ros-Oton & J. Serra, JMPA '14]

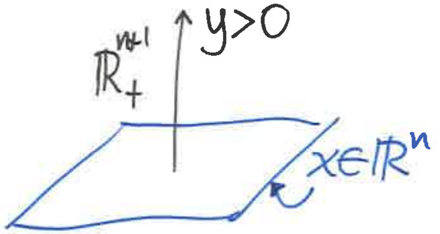
Nonexistence of positive solutions in star-shaped domains for $\begin{cases} (-\Delta)^s u = u^p & \text{in } \Omega\subset\mathbb{R}^n \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega \end{cases}$

if $p > \dfrac{n+2s}{n-2s}$.

The extension problem [Caffarelli-Silvestre 2007]

$0<s<1$

$u:\mathbb{R}^n\to\mathbb{R}$ $\rightsquigarrow$ $\begin{cases} \operatorname{div}(y^{1-2s}\nabla v)=0 & \text{in } \mathbb{R}^{n+1}_+ \\ v(x,0)=u(x) & \text{on } \partial\mathbb{R}^{n+1}_+=\mathbb{R}^n. \end{cases}$



$v=v(x,y).$

Thm [Caff-Silv]

$$-\lim_{y\downarrow 0} y^{1-2s} v_y = \frac{\partial v}{\partial y^s}(x,0) = \tilde{c}_{n,s}\,(-\Delta)^s u\,(x).$$

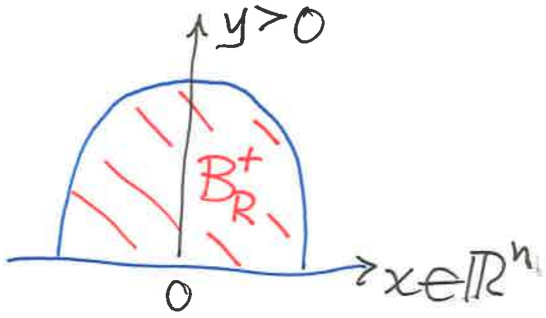
Semilinear pb:

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n$$

$\Updownarrow$

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v)=0 & \text{in } \mathbb{R}^{n+1}_+, \\ -y^{1-2s} v_y\big|_{y=0} = f(v) & \text{on } \partial\mathbb{R}^{n+1}_+=\{y=0\}. \end{cases}$$

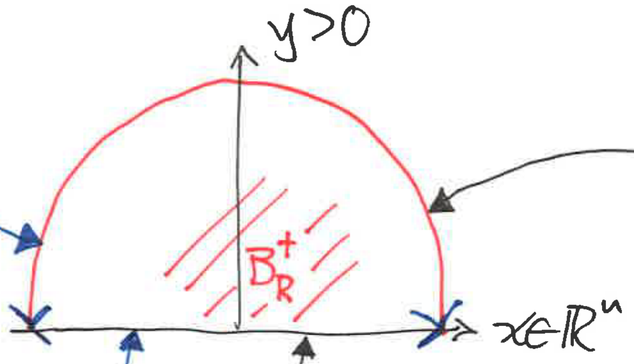
Energy:



$$E_{B_R^+}(v) = \iint_{B_R^+} dx\,dy\, \frac{y^{1-2s}}{2}|\nabla v|^2 + \int_{\{|x|<R\}} dx\; G(v(x,0)).$$

Minimizers v :

$$E_{B_R^+}(v) \leq E_{B_R^+}(w)$$



$w \equiv u$ on

w free on

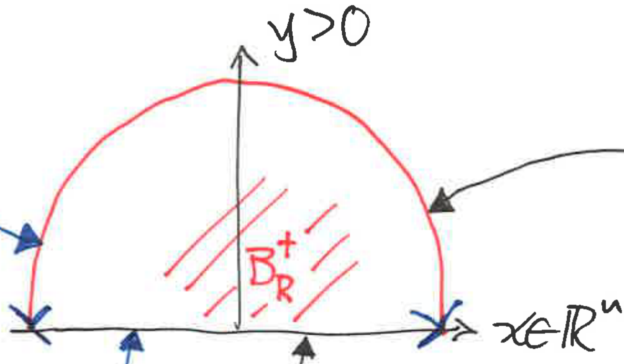
$$\iint_{B_R^+} dx\,dy \quad y^{1-2s} |\nabla v|^2$$

$\approx H^s$-norm of trace

$\approx H^{1/2}$ weighted norm

Minimizers v :

$E_{B_R^+}(v) \leq E_{B_R^+}(w)$



$w \equiv u$ on

w free on

$$\iint_{B_R^+} dx\,dy \; y^{1-2s} |\nabla v|^2$$

$\approx H^s$-norm of trace

$\approx H^{1/2}$ weighted norm

- **Thm** [C.-Cinti 2010]

Sharp energy estimates for minimizers of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$:

$$E_{B_R^+}(v) \approx C \begin{cases} R^{n-2s} & \text{if } 0<s<1/2, \\ R^{n-1}\log R & \text{if } s=1/2, \\ R^{n-1} & \text{if } 1/2<s\leq 1. \end{cases}$$

- <u>Thm</u> $\forall f$, global minimizers of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ are $\boxed{1\text{-}D}$ if
  - $n=2$ & $0<s<1$
  - $n=3$ & $\frac{1}{2} \leq s \leq 1$ [C.-Cinti '10]

Examples of global minimizers: monotone sol'ns ($u_{x_n} > 0$) with $u(x', x_n) \xrightarrow[x_n \to \pm\infty]{} \pm 1$

DENSITY ESTIMATES FOR A VARIATIONAL MODEL DRIVEN BY THE GAGLIARDO NORM

OVIDIU SAVIN AND ENRICO VALDINOCI

→ optimal energy estimates using (not the extension)

We define also

$$\mathscr{K}(u;\Omega) := \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega\int_{\mathscr{C}\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

the Ω contribution in the H^s norm of u

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

i.e we omit the set where $(x,y)\in\mathscr{C}\Omega\times\mathscr{C}\Omega$ since all $u\in X$ are fixed outside Ω.

The energy functional J_ε in Ω is defined as

$$J_\varepsilon(u;\Omega) := \varepsilon^{2s}\mathscr{K}(u;\Omega) + \int_\Omega W(u)\,dx.$$

Throughout the paper we assume that $W:[-1,1]\to[0,\infty)$,

$$W\in C^2([-1,1]),\quad W(\pm1)=0,\quad W>0\quad\text{in } (-1,1) \tag{1.1}$$

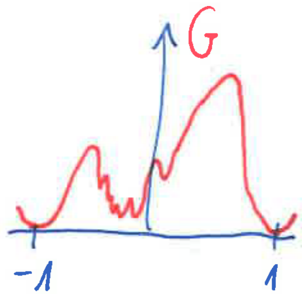
$$W'(\pm1)=0,\quad\text{and}\quad W''(\pm1)>0.$$

We say that u is a minimizer[1] for J_ε in Ω if

$$J_\varepsilon(u;\Omega)\leqslant J_\varepsilon(v;\Omega)$$

for any v which coincides with u in $\mathscr{C}\Omega$.

The equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$, $0<s<1$.



- **Thm** [C. & Solà-Morales, '05, $s=1/2$] [C.-Sire '10, $0<s<1$]

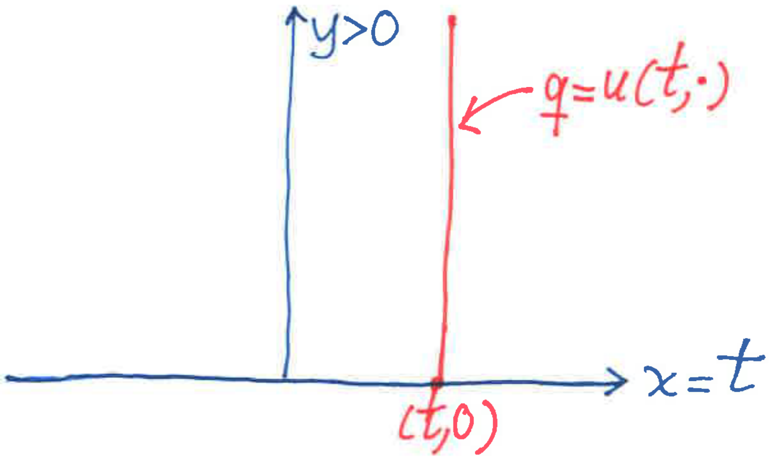
  $\exists$ sol'n $u \nearrow_{-1}^{1}$ in $\mathbb{R}$ $\iff$ $\exists$ such u for $s=1$ $\iff$ $\begin{cases} G'(\pm 1)=0 \text{ \& } \\ G(s)>G(1)=G(-1) \text{ in } (-1,1) \end{cases}$

  $\oplus$ Hamiltonian equalities (in $\mathbb{R}$).

$(-\Delta)^s u = c\cdot f(u)$ in $\mathbb{R}^1$ has HAMILTONIAN STRUCTURE

$$\Updownarrow$$

$$(*)\quad \begin{cases} \operatorname{div}\left(y^{1-2s}\nabla v\right) = 0 & \text{in } \mathbb{R}^2_+ \\ 2(1-s)\displaystyle\lim_{y\downarrow 0} -y^{1-2s} v_y = f(v) & \text{in } \mathbb{R} \end{cases}$$



$$\begin{cases} q = v(x,\cdot) = v(t,\cdot) \\ p = q' = v_x(t,\cdot) \end{cases}$$

Energy $\longrightarrow$ $L(q,p) = \frac{1}{2}\|p\|_s^2 + W(q)$

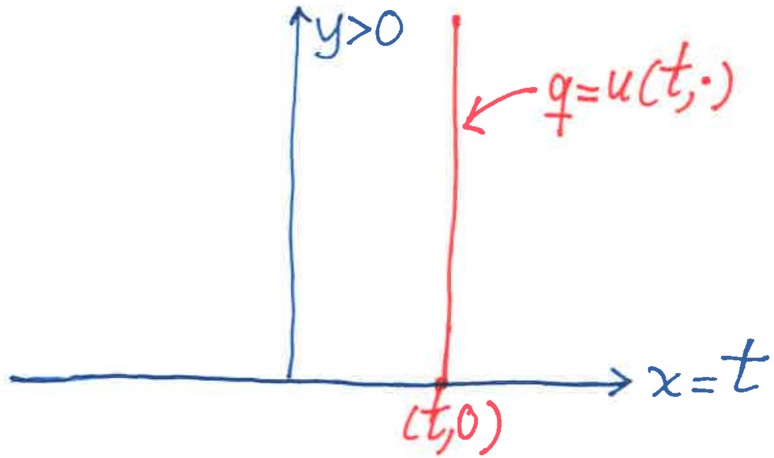
$$W(q) = \frac{1}{2}\|\partial_y q\|_s^2 + \frac{1}{2(1-s)}\, G(q(0))$$

$\longleftarrow \quad \|w\|_s^2 := \int_0^{+\infty} y^{1-2s}|w(y)|^2\,dy$

$(-\Delta)^s u = c\cdot f(u)$ in $\mathbb{R}^1$ has HAMILTONIAN STRUCTURE

$$\Updownarrow$$

$$(*)\quad \begin{cases} \operatorname{div}\left(y^{1-2s}\nabla v\right) = 0 & \text{in } \mathbb{R}^2_+ \\ 2(1-s)\lim\limits_{y\downarrow 0} -y^{1-2s} v_y = f(v) & \text{in } \mathbb{R} \end{cases}$$



$$\begin{cases} q = v(x,\cdot) = v(t,\cdot) \\ p = q' = v_x(t,\cdot) \end{cases}$$

Energy $\longrightarrow$ $L(q,p) = \frac{1}{2}\|p\|_s^2 + W(q)$

$$W(q) = \frac{1}{2}\|\partial_y q\|_s^2 + \frac{1}{2(1-s)}\,G(q(0))$$

$\longleftarrow$ $\|w\|_s^2 := \int_0^{+\infty} y^{1-2s}\,|w(y)|^2\,dy$

Hamiltonian : $H(q,p) = \frac{1}{2}\|p\|_s^2 - W(q)$

$$= \int_0^{+\infty} \frac{y^{1-2s}}{2}\left\{v_x^2(t,y) - v_y^2(t,y)\right\}dy - \frac{1}{2(1-s)}\,G(v(t,0))$$

$$\Longrightarrow \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ W'(q) \end{pmatrix} = \begin{pmatrix} H_p \\ -H_q \end{pmatrix}$$

Hamiltonian identity & estimate

- **Thm** [C.-Sire '10, $0<s<1$]

$n=1$, u layer $(u\nearrow\pm 1)$ soln of $(-\Delta)^s u = f(u)$, $\forall f$

v = s-extension of u. Then:

$$2(1-s)\int_0^{+\infty} \frac{z^{1-2s}}{2}\left\{v_x^2(x,z) - v_y^2(x,z)\right\} dz = G(v(x,0)) - G(1) \quad \forall x\in\mathbb{R}$$

&

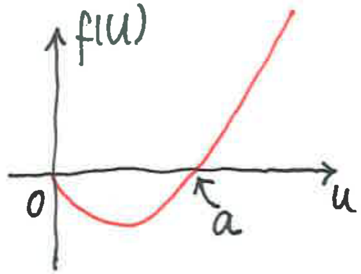
$$2(1-s)\int_0^{y} \frac{z^{1-2s}}{2}\left\{v_x^2(x,z) - v_y^2(x,z)\right\} dz < G(v(x,0)) - G(1) \quad \begin{cases}\forall x\in\mathbb{R}\\ \forall y\geq 0\end{cases}.$$

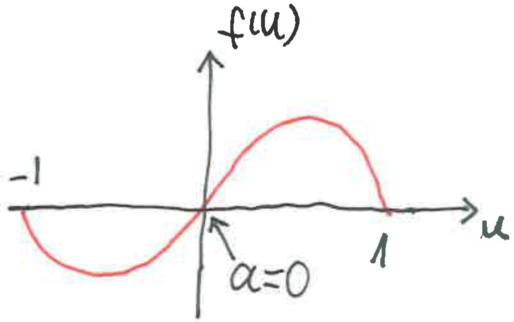
Open pb for $n>1$!

[Cabré & Solà-Morales '15]: same method (Lyapunov-Schmidt reduction) to establish existence of periodic (small) solutions for

$$(-\Delta)^{\alpha} u = f(u) \quad \text{in } \mathbb{R}$$

close to $u \equiv a$ ($=$ ctt) if $f(a)=0$, $f'(a)>0$.





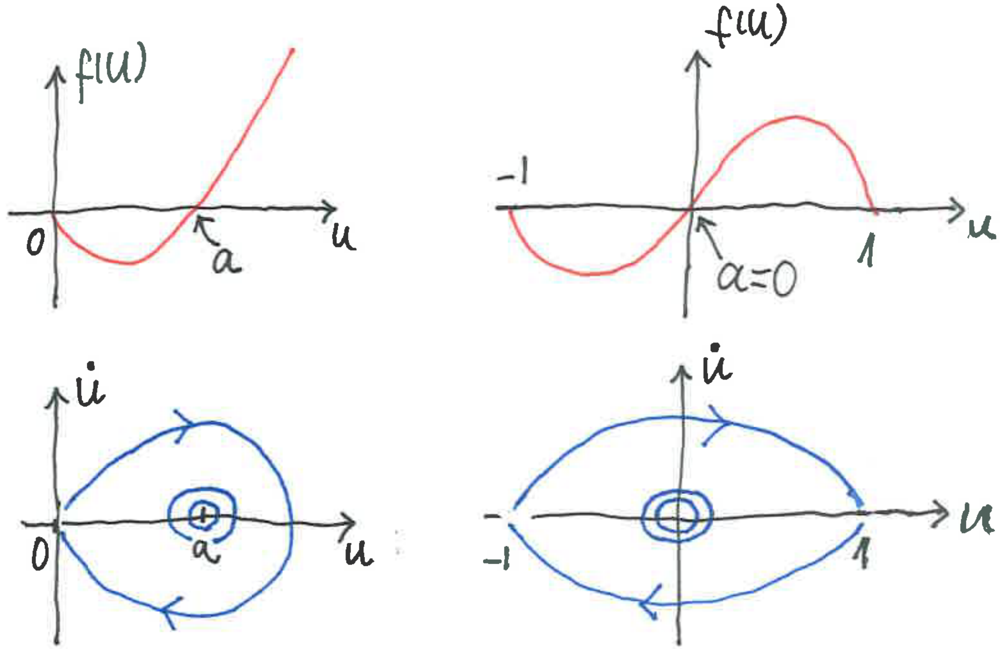
[Cabré & Solà-Morales '15]: same method (Lyapunov-Schmidt reduction) to establish existence of periodic (small) solutions for

$$(-\Delta)^\alpha u = f(u) \quad \text{in } \mathbb{R}$$

close to $u \equiv a$ ($=$ ctt) if $f(a)=0$, $f'(a)>0$.

- Periodic orbits when $\alpha = 1/2$ & $f(u) = -u+u^2$ or $f(u) = \sin(\pi u)$ found by [Amick-Toland, Acta Math 1991] & [Toland, JFA '97]

$(-\Delta)$: $\alpha = 1$ $\rightarrow$

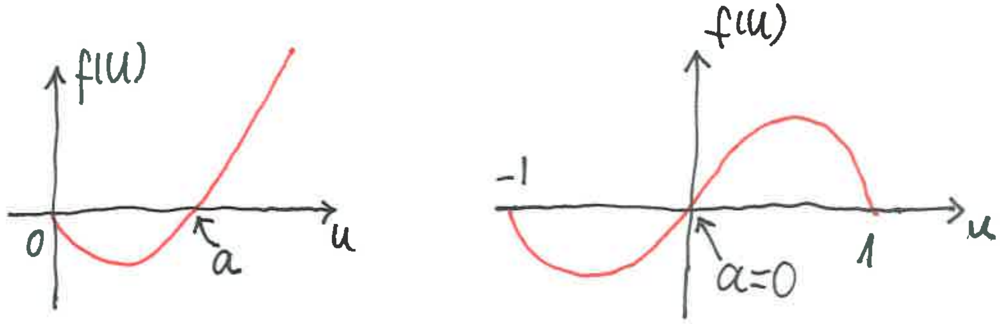


[Cabré & Solà-Morales '15]: Lyapunov-Schmidt reduction

to establish existence of periodic (small) solutions for

$$(-\Delta)^\alpha u = f(u) \quad \text{in } \mathbb{R}$$

close to $u \equiv a$ (= ctt) if $f(a)=0$, $f'(a)>0$.

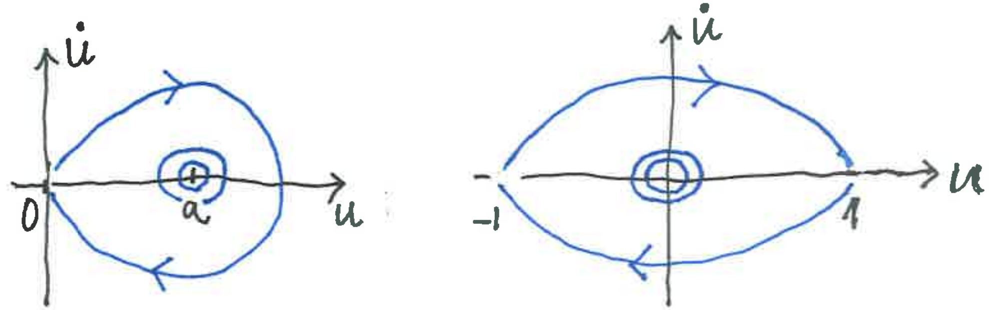


• Periodic orbits when $\alpha = 1/2$ & $f(u) = -u+u^2$ or $f(u)=\sin(\pi u)$ found by [Amick-Toland, Acta Math 1991] & [Toland, JFA '97]

↓

these 2 f's give a completely integrable Hamiltonian system

$(-\Delta): \alpha = 1$ →



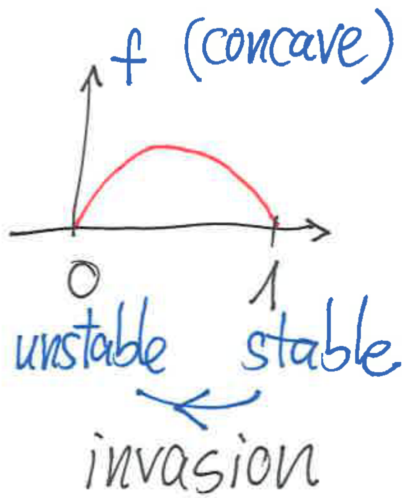
→ That $(-\Delta)^\alpha u = f(u)$ in $\mathbb{R}$ has ($\forall f$) Hamiltonian structure found by [Cabré & Solà-Morales, CPAM '05] ($\alpha = 1/2$) [Cabré & Sire '14] ($\forall \alpha \in (0,1)$)

→ Hamiltonian used by [Frank-Lenzmann-Silvestre, CPAM '15]: ∃! GROUND STATES

- Front propagation under fractional diffusion

- Front propagation: monostable KPP nonlinearities

$$\begin{cases} u_t - \Delta u = f(u) & (= u(1-u) = u - u^2) \quad \text{in } \mathbb{R}\times(0,\infty) \\ u(t=0) = u_0(x) \in [0,1] & \text{on } \mathbb{R} \end{cases}$$

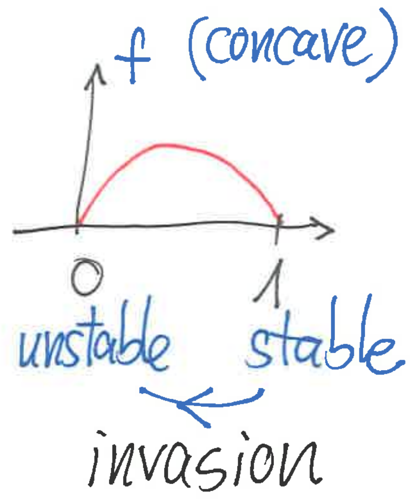


Travelling wave solutions:

$u(x,t) = \phi(x+ct)$ $\exists$ for all $c \geq c^* = 2\sqrt{f'(0)}$

• Front propagation: monostable KPP nonlinearities

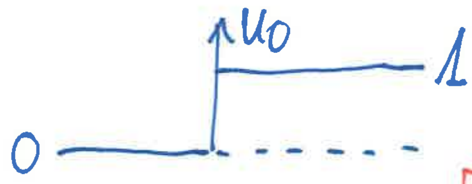
$$\begin{cases} u_t - \Delta u = f(u) \quad (= u(1-u) = u-u^2) & \text{in } \mathbb{R}\times(0,\infty)\\ u(t=0)=u_0(x)\in[0,1] & \text{on } \mathbb{R}\end{cases}$$



Travelling wave solutions:

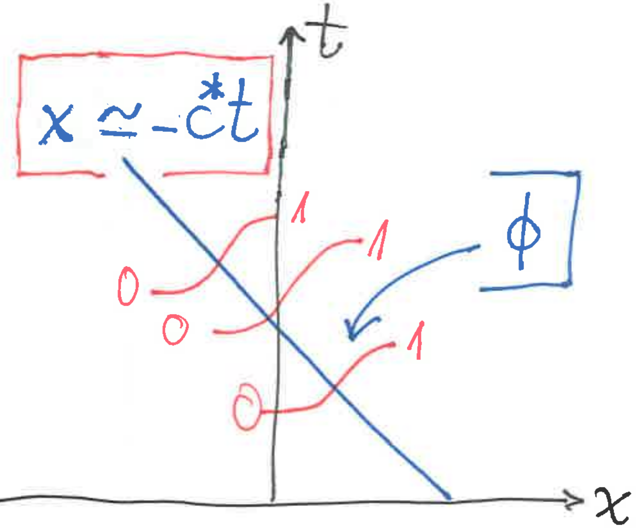
$u(x,t)=\phi(x+ct)$ $\exists$ for all $\boxed{c \geq c^* = 2\sqrt{f'(0)}}$

Initial condition: $u_0(x)$ = Heaviside

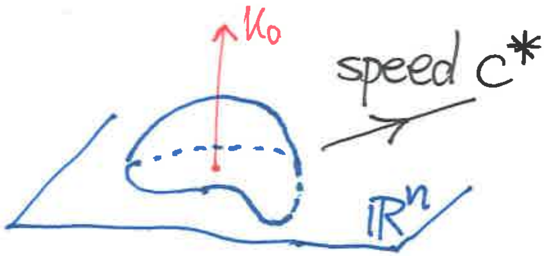


Thm (Kolmogorov-Petrovski-Piskunov '37)

$$\boxed{\lim_{t\to+\infty} u(\underline{-ct},t) = \begin{cases}0 & \text{if } c>c^*\\ 1 & \text{if } c<c^*\end{cases} \quad \forall x\in\mathbb{R}}$$

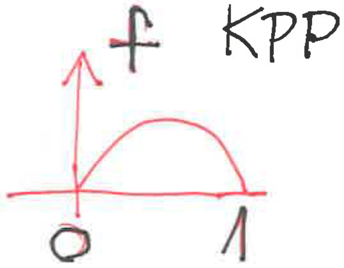


Also,

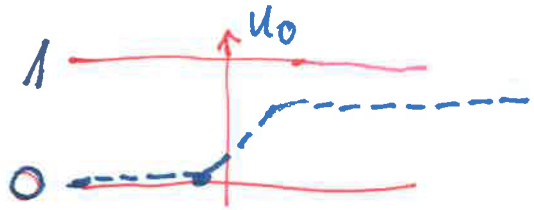


- Front propagation for KPP fractional diffusions

[Cabré - Roquejoffre '09]



$$\begin{cases} \boxed{u_t + (-\Delta)^{\alpha} u = f(u)} \quad \text{in } \mathbb{R} \times (0,\infty), \quad \boxed{0<\alpha<1} \\ u(t=0) \text{ nondecreasing \& } \mathbb{R}_- \cap \operatorname{supp}(u(0,\cdot)) \text{ compact} \end{cases}$$

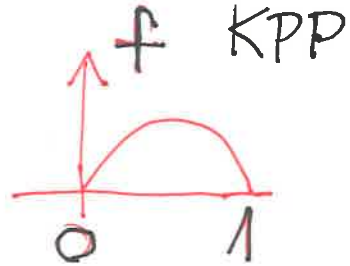


- Thm [C-R '09] $\nexists$ travelling waves & $\forall x$

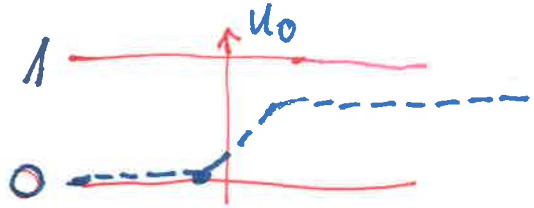
$$\boxed{\lim_{t\to+\infty} u(\underbrace{-e^{\sigma t}}_{\uparrow}, t) = \begin{cases} 0 & \text{if } \sigma > \sigma^{**} = \dfrac{f'(0)}{2\alpha} \\ 1 & \text{if } \sigma < \sigma^{**} \end{cases}}$$

- Front propagation for KPP fractional diffusions

[Cabré–Roquejoffre '09]



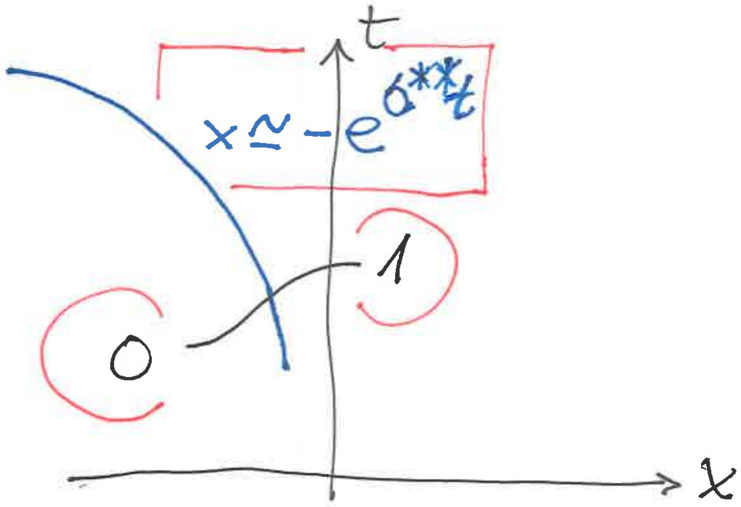
$$\begin{cases} \boxed{u_t + (-\Delta)^\alpha u = f(u)} \quad \text{in } \mathbb{R}\times(0,\infty), \quad \boxed{0<\alpha<1} \\ u(t=0) \text{ nondecreasing } \& \ \mathbb{R}_- \cap \operatorname{supp}(u(0,\cdot)) \text{ compact} \end{cases}$$



- Thm [C-R '09] $\nexists$ travelling waves & $\forall x$

$$\boxed{\lim_{t\to+\infty} u(\underbrace{-e^{\sigma t}}_{\uparrow}, t) = \begin{cases} 0 & \text{if } \sigma > \sigma^{**} = \dfrac{f'(0)}{2\alpha} \\ 1 & \text{if } \sigma < \sigma^{**} \end{cases}}$$

The front travels exponentially fast :



Announced in Physics
(no math proof) by :

[Mancinelli–Vergni–Vulpiani '03]

& [del-Castillo-Negrete, Carreras, Lynch '03]

Initial conditions with compact support in $\mathbb{R}^n$

$$\begin{cases} u_t + (-\Delta)^\alpha u = f(u) & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^n, \; 0 \le u_0 \le 1 \end{cases}$$

- **Thm** [C.-Roquejoffre '09] Let $\boxed{\sigma^* := \frac{f'(0)}{n+2\alpha}}$. Then:

(a) $\sigma > \sigma^* \Rightarrow u(x,t) \to 0$ unif. in $\{|x| \ge e^{\sigma t}\}$ as $t \to +\infty$.

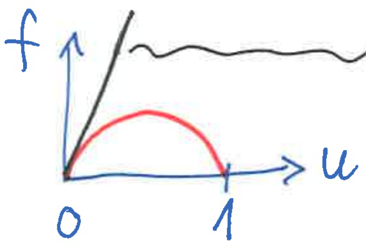
(b) $\sigma < \sigma^* \Rightarrow u(x,t) \to 1$ unif. in $\{|x| \le e^{\sigma t}\}$ as $t \to +\infty$.

- Note: $n=1 \Rightarrow$ $\sigma^* = \frac{f'(0)}{1+2\alpha} < \frac{f'(0)}{2\alpha} = \sigma^{**}$

increasing initial data

compactly supported initial data

• Heuristics :



Linearization at the front:

$$\boxed{v_t + (-\Delta)^\alpha v = f'(0)\, v}$$

$$\downarrow$$

$$\text{Solution} = v(t,x) = e^{f'(0)t} \int_{\mathbb{R}^n} \underbrace{p_\alpha(t,y)}_{\downarrow} u_0(x-y)\,dy$$

Fractional heat Kernel : $\boxed{p_\alpha(t,x) \approx} \dfrac{1}{t^{\frac{n}{2\alpha}}\left(1+\left|\frac{x}{t^{1/2\alpha}}\right|^{n+2\alpha}\right)} \underset{|x| \text{ large}}{\approx} c \cdot \dfrac{t}{|x|^{n+2\alpha}}$

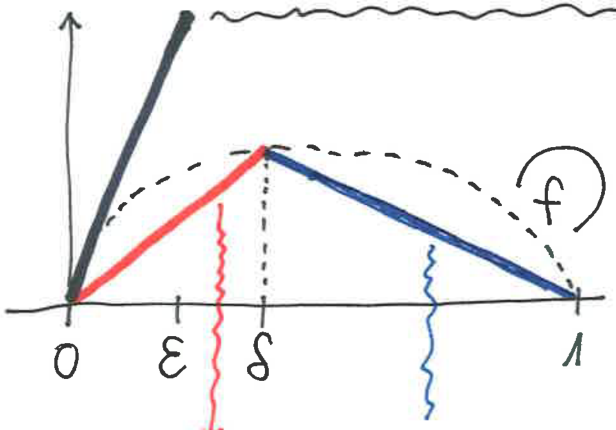
Solution remains bdd, $\in (0,1)$ $\Longrightarrow$ $e^{f'(0)t} \cdot \dfrac{t}{|x|^{n+2\alpha}} \approx 1$

$$\Updownarrow$$

$$\boxed{|x| \simeq \underbrace{t^{\frac{1}{n+2\alpha}}}_{\text{WRONG factor:}} \; e^{\frac{f'(0)}{n+2\alpha} t}}$$

Correct

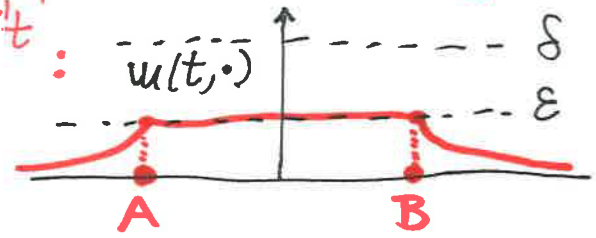
• Proofs (homogeneous media, $\mu \equiv 1$)

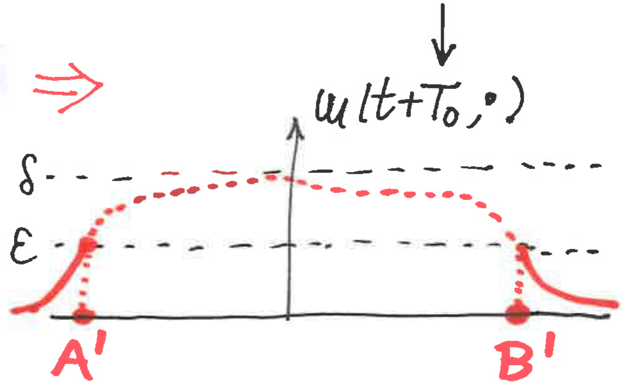


$\ast$ SUPERSOLUTION $\Rightarrow$ $u \to 0$ if $|x| \geq e^{\delta t}$

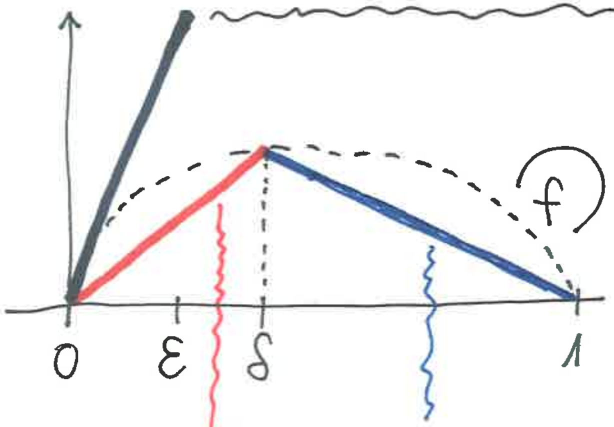
$\ast$ SUBSOLUTION (using linear pb $w_t + (-\Delta)^\alpha w = \frac{f(\delta)}{\delta} w$) $\Rightarrow$

$\Rightarrow u \geq \varepsilon$ for $|x| \geq e^{\delta' t}$:





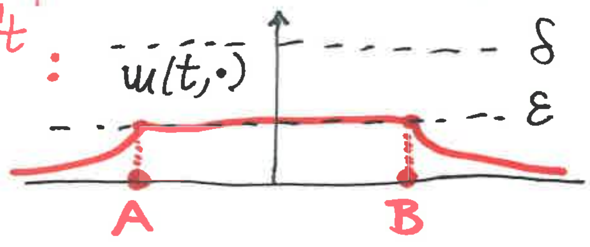
• **Proofs** (homogeneous media, $\mu \equiv 1$)

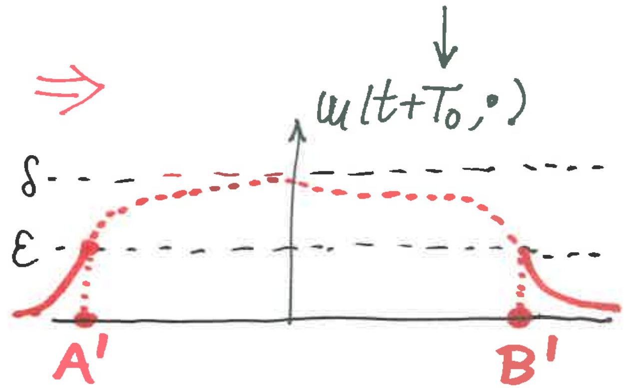


$\ast$ SUPERSOLUTION $\Rightarrow$ $u \to 0$ if $|x| \geq e^{\sigma t}$

$\ast$ SUBSOLUTION (using linear pb $w_t + (-\Delta)^\alpha w = \frac{f(\delta)}{\delta} w$) $\Rightarrow$

$\Rightarrow u \geq \varepsilon$ for $|x| \geq e^{\sigma' t}$:





$\ast$ SUBSOLUTION $\Rightarrow u \to 1$ for $|x| \leq e^{\sigma'' t}$:

linear equation with $\tilde{w}_0 := 1 + c|x|^\gamma$.

$$\boxed{c \cdot \underbrace{e^{-qt}}_{\substack{\downarrow \; t \uparrow \infty \\ 0}} \cdot \tilde{w}_0(x) \geq 1 - u}$$

Fractional diffusion in periodic media

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mu(x)\, u - u^2, \quad x \in \mathbb{R}^n, t > 0 \quad (*) \\ \qquad\qquad\qquad \mu \text{ periodic} \\ \lambda_1 = \text{principal periodic eigenvalue of } (-\Delta)^\alpha - \mu(x) Id \end{cases}$$

$\leftarrow \lambda_1 = -1$ if $\mu \equiv 1$.

- $\lambda_1 \geq 0 \Rightarrow u \xrightarrow[t\to\infty]{} 0 \quad \forall u_0$
- $\lambda_1 < 0 \Rightarrow u \xrightarrow[t\to\infty]{} u_+ =$ the stationary sol'n of $(*)$

$\sigma^* = \frac{1}{n+2\alpha}$

Fractional diffusion in periodic media

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mu(x)\, u - u^2, \quad x \in \mathbb{R}^n, t>0 \quad (*) \\ \quad \mu \text{ periodic} \\ \lambda_1 = \text{principal periodic eigenvalue of } (-\Delta)^\alpha - \mu(x)\, Id \end{cases}$$

$\leftarrow \lambda_1 = -1$ if $\mu \equiv 1$.

$\sigma^* = \frac{1}{n+2\alpha}$

- $\lambda_1 \geq 0 \Rightarrow u \xrightarrow[t\to\infty]{} 0 \quad \forall u_0$
- $\lambda_1 < 0 \Rightarrow u \xrightarrow[t\to\infty]{} u_+$ = the stationary sol'n of $(*)$

- **Thm** [C.-Coulon-Roquejoffre '12] Assume $\boxed{\mu(x) \geq \min \mu > 0}$ ($\Rightarrow \lambda_1 < 0$) $u_0 \geq 0$, $u_0 \not\equiv 0$ with compact support. Then,

$$\forall \lambda \in (0, \min \mu) \quad \{x \in \mathbb{R}^n : u(t,x) = \lambda\} \subset \left\{ c_\lambda\, e^{\frac{|\lambda_1|}{n+2\alpha} t} \leq |x| \leq \frac{1}{c_\lambda} e^{\frac{|\lambda_1|}{n+2\alpha} t} \right\}$$

for t large.

Open pb. $\lambda \in (0, \min u_+)$?

Heuristics predicted wrong

Fractional diffusion in periodic media

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mu(x)\, u - u^2, \quad x \in \mathbb{R}^n, t > 0 \quad (*) \\ \mu \text{ periodic} \\ \lambda_1 = \text{principal periodic eigenvalue of } (-\Delta)^\alpha - \mu(x)\, Id \end{cases}$$

$\lambda_1 = -1$ if $\mu \equiv 1$.

$\sigma^* = \frac{1}{n+2\alpha}$

- $\lambda_1 \geq 0 \Rightarrow u \xrightarrow[t\to\infty]{} 0 \quad \forall u_0$
- $\lambda_1 < 0 \Rightarrow u \xrightarrow[t\to\infty]{} u_+ =$ the stationary sol'n of $(*)$

- **Thm** [C.-Coulon-Roquejoffre '12] Assume $\boxed{\mu(x) \geq \min \mu > 0}$ $(\Rightarrow \lambda_1 < 0)$

$u_0 \geq 0$, $u_0 \not\equiv 0$ with compact support. Then,

$$\forall \lambda \in (0, \min \mu) \quad \{x \in \mathbb{R}^n : u(t,x) = \lambda\} \subset \left\{ c_\lambda\, e^{\frac{|\lambda_1|}{n+2\alpha} t} \leq |x| \leq \frac{1}{c_\lambda} e^{\frac{|\lambda_1|}{n+2\alpha} t} \right\}$$

for t large.

Open pb. $\lambda \in (0, \min u_+)$?

Great difference with $\alpha = 1$, where speed is cst but depends on every direction of the periodic media (Freidlin-Gärtner formula)

Heuristics predicted wrong

• Proof (heterogeneous periodic media, $\mu(x) \geq \min \mu > 0$)

$u(t,x) = \phi_1(x) \underbrace{v(t,x)}$, ϕ_1: periodic 1^{st} eigenfunction

$\left((-\Delta)^\alpha \phi_1 - \mu(x)\phi_1 = \lambda_1 \phi_1 \right)$

$$w(t,y) := v\left(t, e^{\frac{|\lambda_1|}{N+2\alpha} t} y\right)$$

$\hookrightarrow$ w approx. soln of a transport equation

• <u>Proof</u> (heterogeneous periodic media, $\mu(x) \geq \min \mu > 0$)

$$u(t,x) = \phi_1(x)\, \underbrace{v(t,x)}, \qquad \phi_1 : \text{periodic } 1^{st} \text{ eigenfunction}$$

$$\left((-\Delta)^\alpha \phi_1 - \mu(x)\phi_1 = \lambda_1 \phi_1 \right)$$

$$w(t,y) := v\left(t, e^{\frac{|\lambda_1|}{n+2\alpha} t} y\right)$$

$\hookrightarrow$ w approx. soln of a transport equation

ANSATZ :

$$\boxed{\tilde{u}(t,x) := \phi_1(x) \frac{a}{|\lambda_1|^{-1} + b(t)\, |x|^{n+2\alpha}}}$$

$$\tilde{u} \begin{matrix} \text{sub} \\ \text{super} \end{matrix} \text{soln} \Longleftarrow \begin{matrix} b(t) \approx e^{-|\lambda_1| t} \\ \text{solves ODE's} \end{matrix}$$

■

The transport equation:

$u(t,x) = \phi_1(x)\, v(t,x)$; $w(t,y) = v(t, r(t)y)$ with $r(t) = e^{\frac{|\lambda_1|}{n+2\alpha} t}$.

$$w_t - \frac{|\lambda_1|}{n+2\alpha}\, y \cdot w_y + e^{\frac{-2\alpha|\lambda_1|}{n+2\alpha} t} \left\{ (-\Delta)^\alpha w - \frac{K}{\phi_1(r(t)y)} w \right\}$$ neglect $\rightsquigarrow \tilde{w}$

$$= |\lambda_1|\, w - \phi_1(r(t)y)\, w^2.$$

$$\tilde{w}_t - \frac{|\lambda_1|}{n+2\alpha}\, y \cdot \tilde{w}_y = |\lambda_1|\, \tilde{w} - \phi_1(r(t)y)\, \tilde{w}^2.$$

Solution: $\tilde{w}(t,y) = \left\{ \phi_1(r(t)y) |\lambda_1|^{-1} (1 - e^{-|\lambda_1| t}) + e^{-|\lambda_1| t} \frac{1}{\tilde{w}_0(r(t)y)} \right\}^{-1}$

Specialise $\tilde{w}_0(y) = \frac{1}{1+|y|^{n+2\alpha}}$

$$\tilde{w}(t,y) = \left\{ \phi_1(r(t)y) |\lambda_1|^{-1} (1 - e^{-|\lambda_1| t}) + e^{-|\lambda_1| t} + |y|^{d+2\alpha} \right\} \simeq$$

$$\simeq \frac{a}{|\lambda_1|^{-1} + e^{-|\lambda_1| t} |x|^{2n+\alpha}}$$

Ansatz:

$$\boxed{\tilde{u}(t,x) = \phi_1(x) \frac{a}{|\lambda_1|^{-1} + b(t) |x|^{n+2\alpha}}}$$

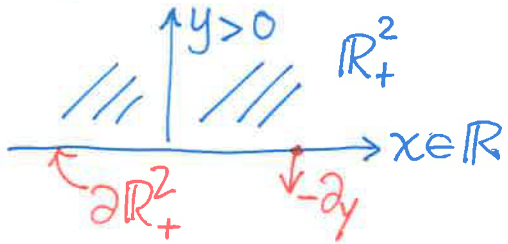
- Traveling fronts for a fractional-type problem

• **Traveling waves for boundary reactions**

[Cabré–Consul–Mandé '14 arXiv]

$v = v(t,x,y)$

$(x,y) \in \mathbb{R}^2_+ = \{y>0\}$



$$\boxed{\begin{cases} v_t - \Delta v = 0 & \text{in } (0,+\infty)\times\mathbb{R}^2_+ \\ -v_y = f(v) & \text{on } (0,+\infty)\times\partial\mathbb{R}^2_+ \end{cases}}$$

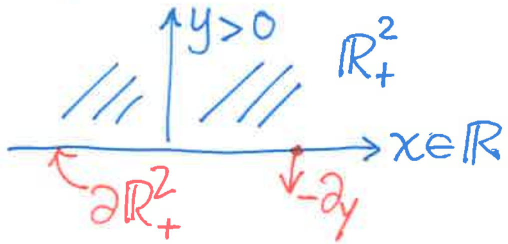
• Traveling waves: $\underline{v(t,x,y) = u(x-ct,y)}$

- Traveling waves for boundary reactions

[Cabré–Consul–Mandé '14 arXiv]

$v = v(t,x,y)$

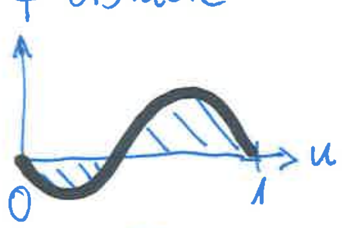
$(x,y) \in \mathbb{R}^2_+ = \{y>0\}$



$$\begin{cases} v_t - \Delta v = 0 & \text{in } (0,+\infty)\times\mathbb{R}^2_+ \\ -v_y = f(v) & \text{on } (0,+\infty)\times\partial\mathbb{R}^2_+ \end{cases}$$

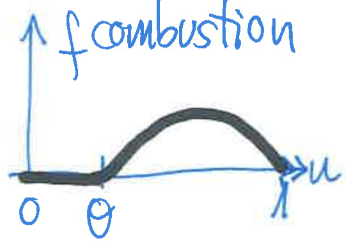
- Traveling waves : $v(t,x,y) = u(x-ct,y)$

f bistable



or

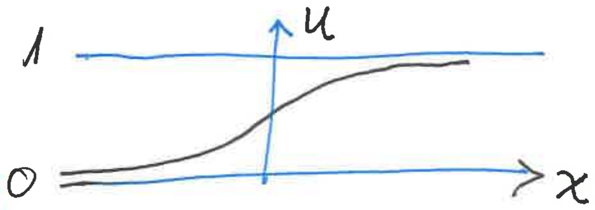
f combustion



$$\begin{cases} \Delta u + c u_x = 0 & \text{in } \mathbb{R}^2_+ \\ -u_y = f(u) & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

$u(-\infty, y=0) = 0$ & $u(+\infty, y=0) = 1$

& $u_x > 0$



u "layer" solution
$c = $ ctt.

SOLUTION PAIR

$$\begin{cases} \Delta u + c u_x = 0 & \text{in } \mathbb{R}^2_+ = \{y>0\} \\ -u_y = f(u) & \text{on } \partial\mathbb{R}^2_+ = \{y=0\} \end{cases}$$

$u(-\infty,0)=0$, $u(+\infty,0)=1$

- $c=0$: [Cabré - Solà Morales, CPAM '05]
- $c=0$ & $\Delta u=0 \rightsquigarrow (-\Delta)^\alpha u=0$

  [Cabré - Sire, AIHP '14 & TrAMS]

  f balanced, i.e., $\int_0^1 f = 0$

$$\begin{cases} \Delta u + c u_x = 0 & \text{in } \mathbb{R}^2_+ = \{y>0\} \\ -u_y = f(u) & \text{on } \partial\mathbb{R}^2_+ = \{y=0\} \end{cases}$$

$u(-\infty,0)=0\ ,\ u(+\infty,0)=1$

- $c=0$: [Cabré – Solà Morales, CPAM '05]
- $c=0$ & $\Delta u = 0 \rightsquigarrow (-\Delta)^\alpha u = 0$

  [Cabré – Sire, AIHP '14 & TrAMS]

  f balanced, i.e., $\int_0^1 f = 0$

- f combustion ($\Rightarrow$ unbalanced)

  [Caffarelli – Mellet – Sire, Adv. Math '12]

  Thm: $\exists\,(c,u)$ & $u(x,0) \approx \mu_0 \, e^{-c|x|}/|x|^{1/2}$ as $x \to -\infty$ $(u \to 0)$

$$\begin{cases} \Delta u + c u_x = 0 & \text{in } \mathbb{R}^2_+ = \{y>0\} \\ -u_y = f(u) & \text{on } \partial\mathbb{R}^2_+ = \{y=0\} \end{cases}$$

$u(-\infty,0)=0$, $u(+\infty,0)=1$

- $c=0$: [Cabré - Solà Morales, CPAM '05]
- $c=0$ & $\Delta u = 0 \rightsquigarrow (-\Delta)^\alpha u = 0$
  [Cabré - Sire, AIHP '14 & TrAMS]

f balanced, i.e., $\int_0^1 f = 0$

- f combustion ($\Rightarrow$ unbalanced)
  [Caffarelli - Mellet - Sire, Adv. Math '12]
  Thm : $\exists\,(c,u)$ & $u(x,0) \approx \mu_0 \dfrac{e^{-c|x|}}{|x|^{1/2}}$ as $x \to -\infty$ $(u \to 0)$

- Thm [C-Consul-Mandé '14] f bistable or combustion $\Rightarrow$
  (a) $\exists!\,(c,u)$ solution pair
  (b) $u_x > 0$ in $\mathbb{R}^2_+$
  (c) $f_1 \geq f_2$ & $f_1 \not\equiv f_2 \Rightarrow c_1 > c_2$.
  (d) f bistable $\Longrightarrow$

$$\begin{cases} b^{-1} \dfrac{e^{-c|x|}}{|x|^{3/2}} \leq u(x,0) \leq b\, \dfrac{e^{-c|x|}}{|x|^{3/2}}, & x<-1 \\ b^{-1} \dfrac{1}{x^{1/2}} \leq 1-u(x,0) \leq b\, \dfrac{1}{x^{1/2}}, & x>1. \end{cases}$$

$1/2 \leftrightarrow 3/2$

$$\left.\begin{aligned} &\Delta u + c u_x = 0 \quad \text{in } \mathbb{R}^2_+ = \{y>0\} \\ &-u_y = f(u) \quad \text{on } \partial\mathbb{R}^2_+ = \{y=0\} \end{aligned}\right\} \iff \boxed{\begin{aligned} &(-\partial_{xx} - c\,\partial_x)^{1/2} v = f(v) \quad \text{in } \mathbb{R} \\ &\text{where} \quad v(x) = u(x,0) \end{aligned}}$$

FRACTIONAL DIFUSSION PB.

Thm [C-C-M] : $\exists!\,(c,v)$ given f

$$\left.\begin{aligned} \Delta u + c u_x &= 0 \quad \text{in } \mathbb{R}^2_+ = \{y>0\} \\ -u_y &= f(u) \quad \text{on } \partial\mathbb{R}^2_+ = \{y=0\} \end{aligned}\right\} \iff \boxed{\begin{aligned} &(-\partial_{xx} - c\partial_x)^{1/2} v = f(v) \quad \text{in } \mathbb{R} \\ &\text{where} \quad v(x) = u(x,0) \end{aligned}}$$

FRACTIONAL DIFUSSION PB.

Thm [C-C-M] : $\exists !\,(c,v)$ given f

- Instead :

Traveling waves for

$$w_t + (-\partial_{xx})^{\alpha} w = f(w) \quad \text{in } \mathbb{R}$$

$$\boxed{(-\partial_{xx})^{\alpha} v - c\,\partial_x v = f(v) \quad \text{in } \mathbb{R}} \quad (*)$$

$\alpha = 1/2$

studied in :

- [Mellet-Roquejoffre-Sire '10] for f combustion :
  $\exists c\ \exists TW$ if $\alpha \in (1/2, 1)$ & $v \to 0$ as $1/|x|^{2\alpha-1}$, $x \to -\infty$.
- [C. Gui-Zhao '14] for f bistable :
  $\exists !\,(c,v)$ & $v \to {0 \atop 1}$ as $1/|x|^{2\alpha}$, $x \to \mp\infty$.

(*)

[Caff.-Mellet-Sire] : $\begin{cases} \bullet \exists \text{ by free boundary approximation.} \\ \bullet \text{Asymptotics: } u = e^{-x}\phi \text{ then } -\Delta\phi + \phi = 0 \quad (c=2) \end{cases}$

Helmoltz equation.

[Caff.-Mellet-Sire] : • $\exists$ by free boundary approximation.
• Asymptotics : $u = e^{-x}\phi$ then $-\Delta\phi + \phi = 0$ $(c=2)$
Helmoltz equation.

[C.-Consul-Mandé]:

• $\exists$ by a variational method (Heinze):

$$\min_{v} \left\{ \int_{\partial \mathbb{R}^2_+} e^{-ax} G(v(x,0))\,dx \;\middle|\; \iint_{\mathbb{R}^2_+} e^{-ax} |\nabla v|^2 \,dx\,dy = 1 \right\}$$

(previous results in cylinders : y bounded : [Heinze'01][Kyed '08])
[Landes '09 '12]

• Uniqueness : sliding method of Berestycki-Nirenberg

[Caff.-Mellet-Sire] : $\left\{\begin{array}{l}\bullet\ \exists \text{ by free boundary approximation.} \\ \bullet \text{ Asymptotics: } u = e^{-x}\phi \text{ then } -\Delta\phi + \phi = 0 \quad (c=2)\end{array}\right.$

Helmoltz equation.

[C.-Consul-Mandé]:

- $\exists$ by a variational method (Heinze):

$$\min_{v} \left\{ \int_{\partial \mathbb{R}^2_+} e^{-ax} G(v(x,0))\,dx \;\middle|\; \iint_{\mathbb{R}^2_+} e^{-ax} |\nabla v|^2\,dx\,dy = 1 \right\}$$

(previous results in cylinders : y bounded : [Heinze'01][Kyed '08])
[Landes '09 '12]

- Uniqueness : sliding method of Berestycki-Nirenberg
- Monotonicity of u : Rearrangement of [Landes '07] : $z = e^{-ax} \in (0, +\infty)$
(decreasing in z)

$\uparrow$ $u=1$ $\uparrow$ $u=0$

both

- $f_1 \overset{\geqslant}{\underset{\not\equiv}{}} f_2 \Rightarrow c_1 > c_2$.
- Asymptotics (f bistable) $\Leftarrow$ max. principle $\oplus$ Construction of explicit layers [Hf.

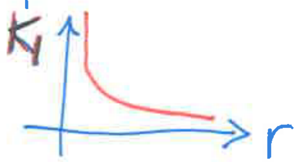
- Asymptotics in [Cabré-Consul-Mandé '14] through:

- Thm [CC-M'14]. $\forall t>0$ $\forall c>0$ $\exists\, f=f^{t,c}$ bistable for which

$$u^t(x,y)=\int_{-\infty}^{x} e^z \frac{y+t}{\pi\sqrt{(y+t)^2+z^2}}\cdot K_1\left(\sqrt{(y+t)^2+z^2}\right)dz$$ is a layer for $f^{t,c}$

modified Bessel fcn of 2^{nd} kind

& $(f^{t,c})'(0)=(f^{t,c})'(1)=\frac{-c}{2t}$.



• Asymptotics in [Cabré-Consul-Mandé '14] through:

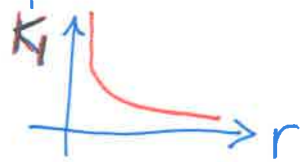
• **Thm [CC-M'14]** $\forall t>0 \ \forall c>0 \ \exists f=f^{t,c}$ bistable for which

$$u^t(x,y)=\int_{-\infty}^{x} e^z \frac{y+t}{\pi\sqrt{(y+t)^2+z^2}}\cdot K_1\left(\sqrt{(y+t)^2+z^2}\right)dz$$

is a layer for $f^{t,c}$

modified Bessel fcn of 2^{nd} kind

& $(f^{t,c})'(0)=(f^{t,c})'(1)=\frac{-c}{2t}$.



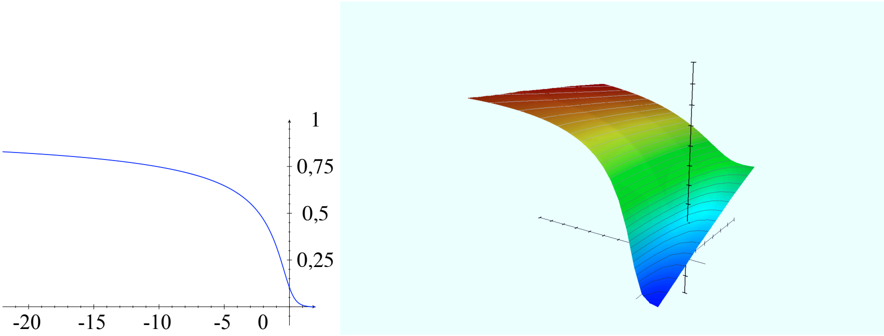
As in [Cabré-Sire] constructed from the heat kernel

for $\begin{cases} \partial_t v+(-\partial_{xx}-c\partial_x)^{1/2}v=0 \\ v(0,x)=v_0(x) \quad \text{in } \mathbb{R} \end{cases}$

$(y=t)$

Poisson kernel for

$\begin{cases} \Delta w+c\,w_x=0 & \text{in } \mathbb{R}^2_+ \\ w(x,0)=w_0(x) & \text{on } \partial\mathbb{R}^2_+ \end{cases}$



(a) $-22 \leq x \leq 2$ and $y = 0$

(b) $-22 \leq x \leq 2$ and $0 \leq y \leq 2$

Figure: The explicit bistable front u^1

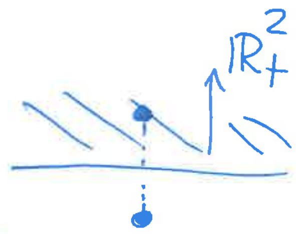
To find the Poisson Kernel for $(c=2)$

$$\begin{cases} \Delta w + 2 w_x = 0 & \text{in } \mathbb{R}^2_+ = \{y>0\} \\ w(\cdot,0) = w_0 & \text{on } \partial\mathbb{R}^2_+ = \{y=0\} \end{cases}$$

we use an idea of [Caff.-Mellet-Sire]:

$$w = e^{-x}\phi \implies \boxed{-\Delta\phi + \phi = 0 \quad \text{in } \mathbb{R}^2_+}$$: Helmoltz eq'n

$$-\Delta\phi + \phi = \delta_0 \quad \text{in } \mathbb{R}^2$$



$\hookrightarrow \phi(r) = \frac{1}{2\pi} K_0(r)$

$+$ reflection $\longrightarrow \ldots K_1(r) \ldots$

& layer $= \int_{-\infty}^{x}$ Poisson kernel, $0 \nearrow 1$. ∎

- Curves and surfaces with constant nonlocal mean curvature

$E \subset \mathbb{R}^n$ bounded (and sufficiently smooth)

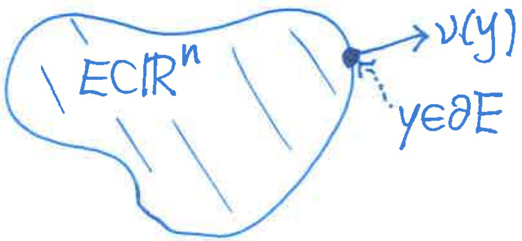
• (Standard) Perimeter :

$W^{1,1}$-seminorm

$$P(E) = \sup_{\|X\|_\infty \leq 1} \int_{\partial E} X(y)\cdot \nu(y)\,dy = \|\nabla 1\!\!1_E\|_{L^1(\mathbb{R}^n)} = [1\!\!1_E]_{W^{1,1}(\mathbb{R}^n)}$$

$$\int_{\partial E} X(y)\cdot \nu(y)\,dy = \int_E (\operatorname{div} X)(y)\,dy = \int_{\mathbb{R}^n} (\operatorname{div} X)\cdot 1\!\!1_E = \int_{\mathbb{R}^n} -X\cdot \nabla 1\!\!1_E$$

$E \subset \mathbb{R}^n$ bounded (and sufficiently smooth)



• (Standard) Perimeter :

$W^{1,1}$-seminorm

$$\boxed{P(E) = \sup_{\|X\|_\infty \leq 1} \int_{\partial E} X(y) \cdot \nu(y)\, dy = \|\nabla 1\!\!1_E\|_{L^1(\mathbb{R}^n)} = [1\!\!1_E]_{W^{1,1}(\mathbb{R}^n)}}$$

$$\int_{\partial E} X(y) \cdot \nu(y)\, dy = \int_E (\operatorname{div} X)(y)\, dy = \int_{\mathbb{R}^n} (\operatorname{div} X) \cdot 1\!\!1_E = \int_{\mathbb{R}^n} -X \cdot \nabla 1\!\!1_E$$

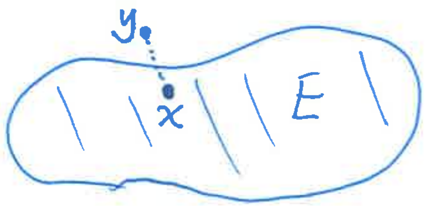
• Fractional Perimeter: $0<\alpha<1$ & $p \geq 1$ $\longrightarrow$ Fractional Sobolev seminorm:

$$[1\!\!1_E]^p_{W^{\frac{\alpha}{p},p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|1\!\!1_E(x) - 1\!\!1_E(y)|^p}{|x-y|^{n+\alpha}}\, dx\, dy = 2 \int_E \int_{E^c} \frac{dx\, dy}{|x-y|^{n+\alpha}}$$

$$\hookrightarrow \boxed{P_\alpha(E) = C_{n,\alpha} \int_E \int_{E^c} \frac{dx\, dy}{|x-y|^{n+\alpha}}}.$$

$0<\alpha<1 \leftarrow \alpha = 2s\,;\ s \in (0, 1/2)$

E bdd



• Fractional isoperimetric inequality : balls minimize fractional perimeter for a given volume [Frank-Seiringer, JAMS 2008]

Quantitative version : $\underbrace{\underbrace{\text{Fusco-Millot-Morini}}_{2011}\text{-Figalli-Maggi}}_{2014}$

• First and second variation of fractional perimeter :

1st variation is NONLOCAL (or fractional) MEAN CURVATURE (NMC):

• Fractional isoperimetric inequality: balls minimize fractional perimeter for a given volume [Frank-Seiringer, JAMS 2008]

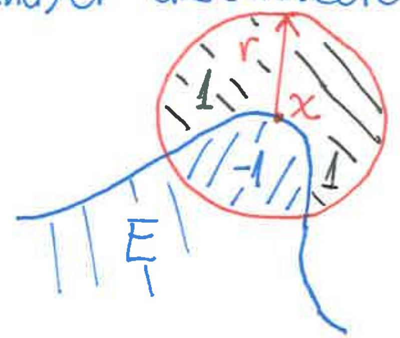
Quantitative version: $\underbrace{\text{Fusco-Millot-Morini}}_{2011}$-Figalli-Maggi (Fusco-Millot-Morini-Figalli-Maggi: 2014)

• First and second variation of fractional perimeter:

1st variation is NONLOCAL (or fractional) MEAN CURVATURE (NMC):

$E \subset \mathbb{R}^n$, $\partial E \in C^2$

(E perhaps unbounded and/or disconnected)



$\rightsquigarrow$ $H_E(x) = H_{\alpha,E}(x) = P.V. \displaystyle\int_{\mathbb{R}^n} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|x-y|^{n+\alpha}}\, dy$

for $x \in \partial E$

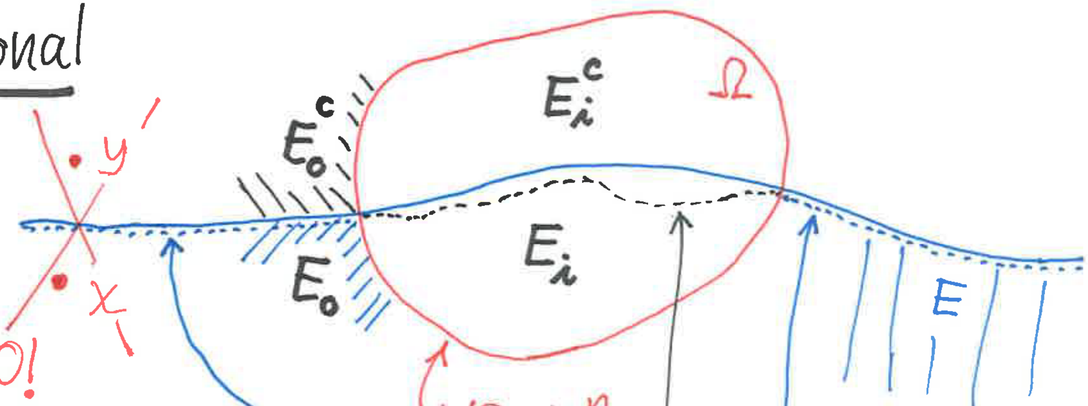
(up to multiplicative ctt)

• Fractional perimeter for E unbounded must be adapted: DIRICHLET pb:

• integrable at ∞

• cancellation at $x=y$

- <u>Fractional perimeter functional</u> <u>for</u> <u>E unbounded</u>



NO!

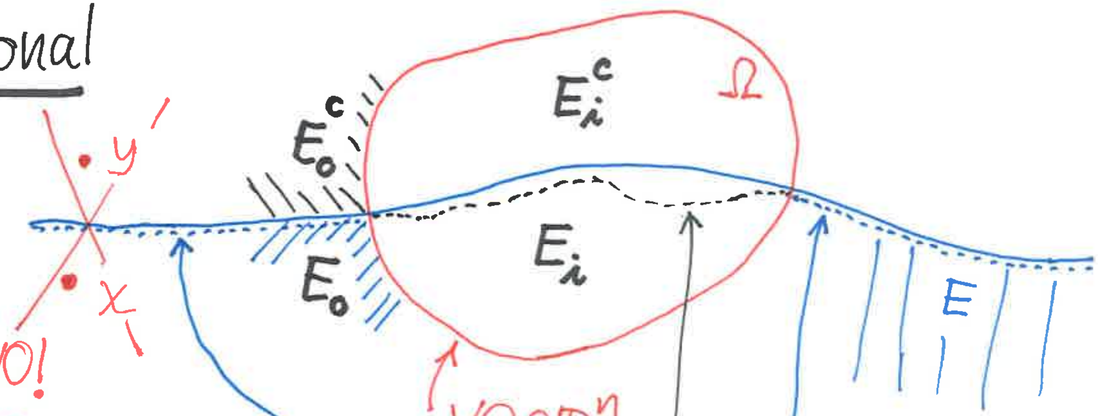
$\forall \Omega \subset \mathbb{R}^n$ bounded

Dirichlet data in $\mathbb{R}^n \setminus \Omega$

Competitor for E in Ω.

$$P_\alpha(E) := \left\{ \int_{E_i}\int_{E_i^c} + \int_{E_i}\int_{E_o^c} + \int_{E_o}\int_{E_i^c} \right\} \frac{dx\,dy}{|x-y|^{n+\alpha}}$$

- **Fractional perimeter functional for E unbounded**



$\forall \Omega \subset \mathbb{R}^n$ bounded

Dirichlet data in $\mathbb{R}^n \setminus \Omega$

Competitor for E in Ω.

$$P_\alpha(E) := \left\{ \int_{E_i}\int_{E_i^c} + \int_{E_i}\int_{E_o^c} + \int_{E_o}\int_{E_i^c} \right\} \frac{dx\,dy}{|x-y|^{n+\alpha}}$$

This and NMC first introduced in

[Caffarelli-Roquejoffre-Savin, CPAM '10]: Nonlocal minimal surfaces ($H_E \equiv 0$)

Euler-Lagrange eq'n is NMC : Nonlocal mean curvature } minimizers ($\forall \Omega$) are

• Motivation for [Caff.-Roquej.-Savin '10] came from:

[Caffarelli-Souganidis 2008]:

↳ "cellular automata" $\xrightarrow[\delta t \downarrow 0]{}$ Motion by $\left\langle \begin{array}{c} \text{classical} \\ \text{nonlocal} \end{array} \right\rangle$ mean curvature

$\mathbb{1}_{E^c} - \mathbb{1}_E$ as initial condition

for the (classical or fractional) (linear) heat equation

(≡ convolution with Gaussian or power decay distribution)

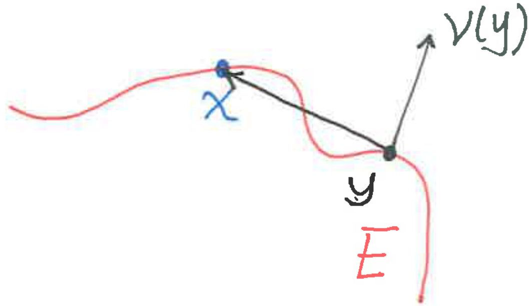
↳ Small time step δt ⟶ New $E = E_{\delta t} = \{u(\cdot, \delta t) < 0\}$

& repeat process

NMC (nonlocal mean curvature): $E \subset \mathbb{R}^n$, $\partial E \in C^2$ (E perhaps unbdd)

$$H_E(x) = \int_{\mathbb{R}^n} \frac{1\!\!1_{E^c}(y) - 1\!\!1_E(y)}{|x-y|^{n+\alpha}}\, dy = -\frac{2}{\alpha} \int_{\partial E} \frac{(x-y)\cdot \nu(y)}{|x-y|^{n+\alpha}}\, d\sigma(y)$$

for $x \in \partial E$



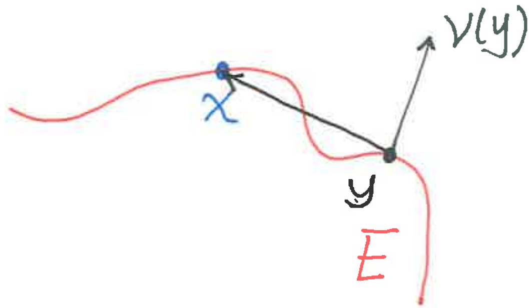
Use:

$$\mathrm{div}_y \frac{x-y}{|x-y|^{n+\alpha}} = \alpha \frac{1}{|x-y|^{n+\alpha}}$$

NMC (nonlocal mean curvature): $E\subset\mathbb{R}^n$, $\partial E\in C^2$ (E perhaps unbdd)

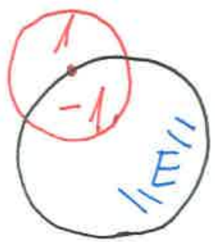
$$H_E(x) = \int_{\mathbb{R}^n} \frac{1_{E^c}(y)-1_E(y)}{|x-y|^{n+\alpha}}\,dy = -\frac{2}{\alpha}\int_{\partial E}\frac{(x-y)\cdot\nu(y)}{|x-y|^{n+\alpha}}\,d\sigma(y)$$

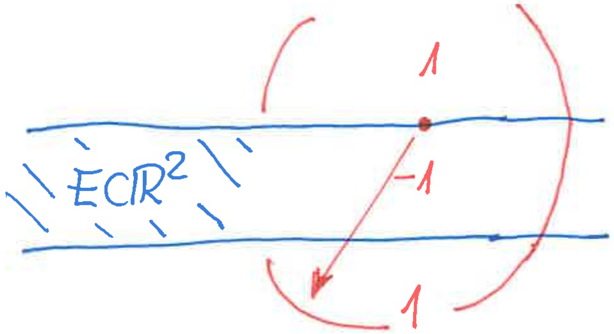
for $x\in\partial E$



Use:

$$\mathrm{div}_y \frac{x-y}{|x-y|^{n+\alpha}} = \alpha\,\frac{1}{|x-y|^{n+\alpha}}$$





- E = hyperplane $\to H_E\equiv 0$
- E = ball in $\mathbb{R}^n \to H_E\equiv ctt>0$
- E = band in $\mathbb{R}^2$ or cylinder in $\mathbb{R}^n \longrightarrow H_E\equiv ctt>0$

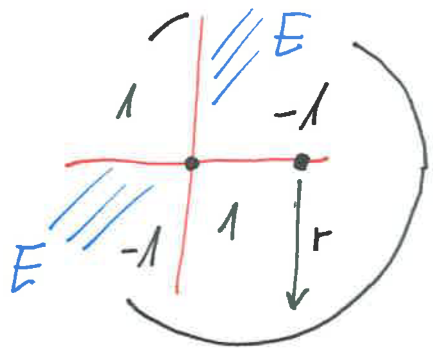
- Nonlocal minimal surfaces : $E \subset \mathbb{R}^n$, $H_E(x) \equiv 0 \quad \forall x \in \partial E$

- Thm [Figalli-Valdinoci '13]

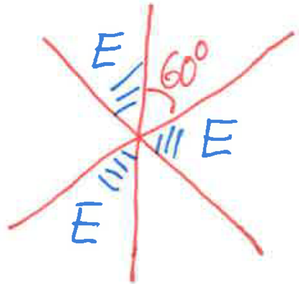
$\mathbb{R}^n \supset E$ minimizing nonlocal minimal set & $\partial E \in Lip$ $\Rightarrow \partial E \in C^\infty$

- Thm [Savin-Valdinoci '12]

$\mathbb{R}^2 \supset E$ minimizing nonl. min. set $\Rightarrow$ E = hyperplane



$H_E \equiv 0$ BUT
are not minimizers



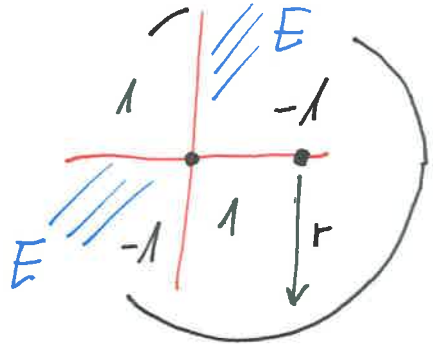
• Nonlocal minimal surfaces: $E \subset \mathbb{R}^n$, $H_E(x) \equiv 0$ $\forall x \in \partial E$

• Thm [Figalli-Valdinoci '13]

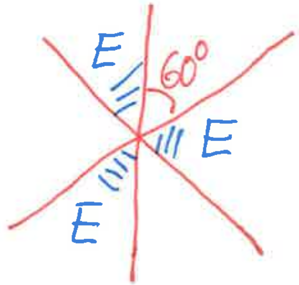
$\mathbb{R}^n \supset E$ minimizing nonlocal minimal set & $\partial E \in Lip$ $\Rightarrow$ $\partial E \in C^\infty$

• Thm [Savin-Valdinoci '12]

$\mathbb{R}^2 \supset E$ minimizing nonl. min. set $\Rightarrow$ E = hyperplane



$H_E \equiv 0$ BUT are not minimizers



• [Davila-del Pino-Wei '14]

Cones in higher dimensions

Nonlocal helicoid

are they stable?

¿ $n \lesseqgtr 7$? Role of $\alpha \in (0,1)$?

- Classical mean curvature :

  CMC surfaces : $H_E \equiv ctt$; are extremals of perimeter for given volume

- Thm [Aleksandrov 1958]

  $E \subset \mathbb{R}^n$ bdd connected, $\partial E \in C^2$, ∂E CMC hypersurface

  $\Rightarrow E$ is a ball

• Classical mean curvature :

CMC surfaces : $H_E \equiv ctt$; are extremals of perimeter for given volume

• Thm [Aleksandrov 1958]

$E \subset \mathbb{R}^n$ bdd connected, $\partial E \in C^2$, ∂E CMC hypersurface

$\Rightarrow$ E is a ball

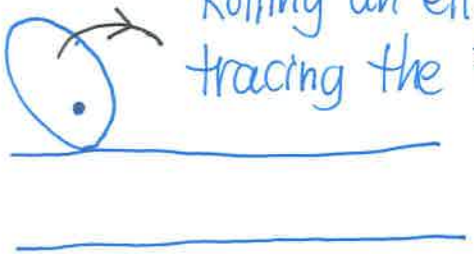
• Thm [Delaunay 1841, JMPA]

In $\mathbb{R}^3$ (also in $\mathbb{R}^n$, $n \geq 3$), $\exists$ periodic CMC cylinders

called UNDULOID

(see them in many http)

(Do NOT exist in $\mathbb{R}^2$)



Rolling an ellipse & tracing the FOCUS $\rightsquigarrow$

$\to$ extremals of fractional perimeter under volume constraint

$\boxed{\text{CNMC sets}}$ (sets with ctt nonlocal mean curvature H_E)

Joint work (arXiv 2015) with M.M. Fall, J. Solà-Morales & T. Weth

→ extremals of fractional perimeter under volume constraint

$\boxed{\text{CNMC sets}}$ (sets with ctt nonlocal mean curvature H_E)

Joint work (arXiv 2015) with M.M. Fall, J. Solà-Morales & T. Weth

- Thm 1 $\emptyset \neq E \subset \mathbb{R}^n$ bdd $C^{2,\beta}$ $(\beta > \alpha)$, $H_E \equiv$ ctt on ∂E
  $\Longrightarrow E$ is a ball.

↑ also proved by [Ciraolo-Figalli-Maggi-Novaga, arXiv 2015]
with a quantitative version: $B_s \subset E \subset B_t$ with $t-s$ small
if $\|H_E\|_{Lip(\partial E)}$ is small

→ extremals of fractional perimeter under volume constraint

$\boxed{\text{CNMC sets}}$ (sets with ctt nonlocal mean curvature H_E)

Joint work (arXiv 2015) with M.M. Fall, J. Solà-Morales & T. Weth

- Thm 1 $\emptyset \neq E \subset \mathbb{R}^n$ bdd $C^{2,\beta}$ $(\beta > \alpha)$, $H_E \equiv$ ctt on ∂E

  $\Longrightarrow$ E is a ball.

  ↑ also proved by [Ciraolo-Figalli-Maggi-Novaga, arXiv 2015]

  with a quantitative version : $B_s \subset E \subset B_t$ with $t-s$ small

  if $\|H_E\|_{Lip(\partial E)}$ is small

- Thm 2 $n=2 \rightarrow \exists R = R(\alpha)$ & $u_a : \mathbb{R} \to \mathbb{R}$ $\nu(a)$-periodic (a=small parameter) &

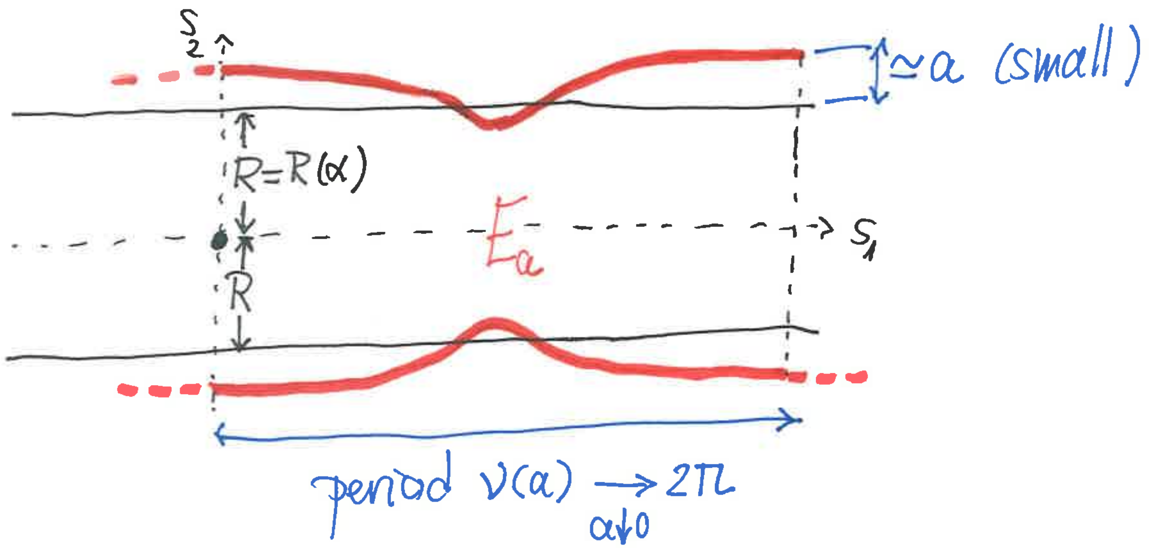
  $E_a = \{(s_1, s_2) \in \mathbb{R}^2 : -u_a(s_1) < s_2 < u_a(s_1)\}$

  have all same NMC $\equiv h_R > 0$ (ctt), $E_a \xrightarrow[a \downarrow 0]{} \{-R < s_2 < R\}$ = a band &

  $a \neq a' \Rightarrow E_a \neq E_{a'}$ & $\nu(a) \xrightarrow[a \downarrow 0]{} 2\pi$

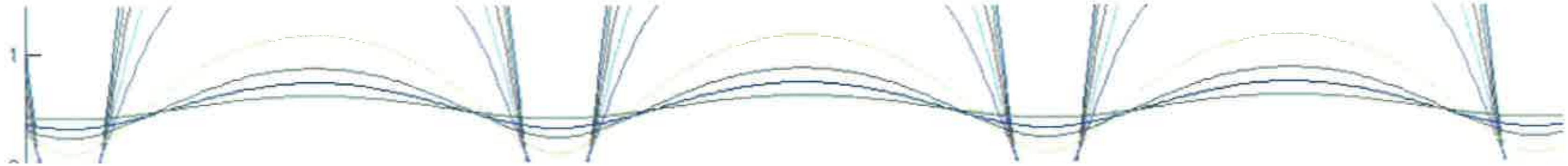
- Hence:

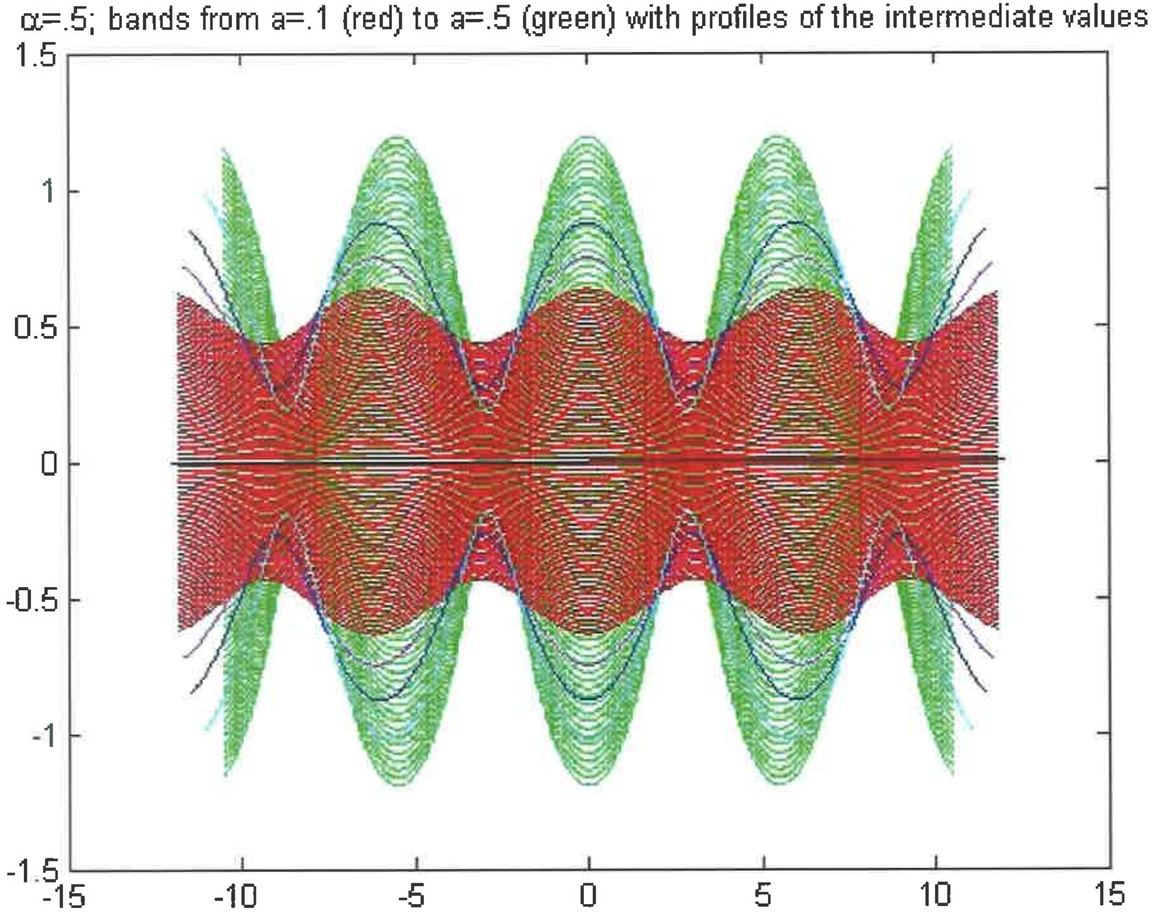
From a straight band in $\mathbb{R}^2$, a family of periodic bands $\{-u_a(s_1) < s_2 < u_a(s_1)\}$ bifurcate. They all have the same NMC (but their periods are different)



- Our next paper:

the same holds in $\mathbb{R}^n$, $n \geq 3$ ← Periodic CNMC cylinders

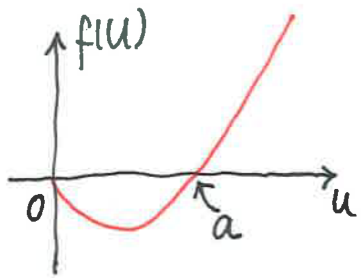


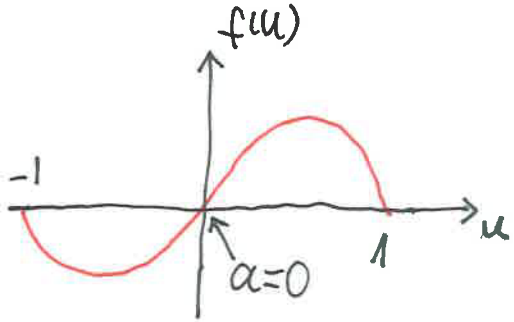

α=.5; bands from a=.1 (red) to a=.5 (green) with profiles of the intermediate values
1.5
1
0.5
0
-0.5
-1
-1.5
-15
-10
-5
0
5
10
15

[Cabré & Solà-Morales '15]: same method (Lyapunov-Schmidt reduction)
to establish existence of periodic (small) solutions for

$$(-\Delta)^{\alpha} u = f(u) \quad \text{in } \mathbb{R}$$

close to $u \equiv a$ (= ctt) if $f(a)=0$, $f'(a)>0$.





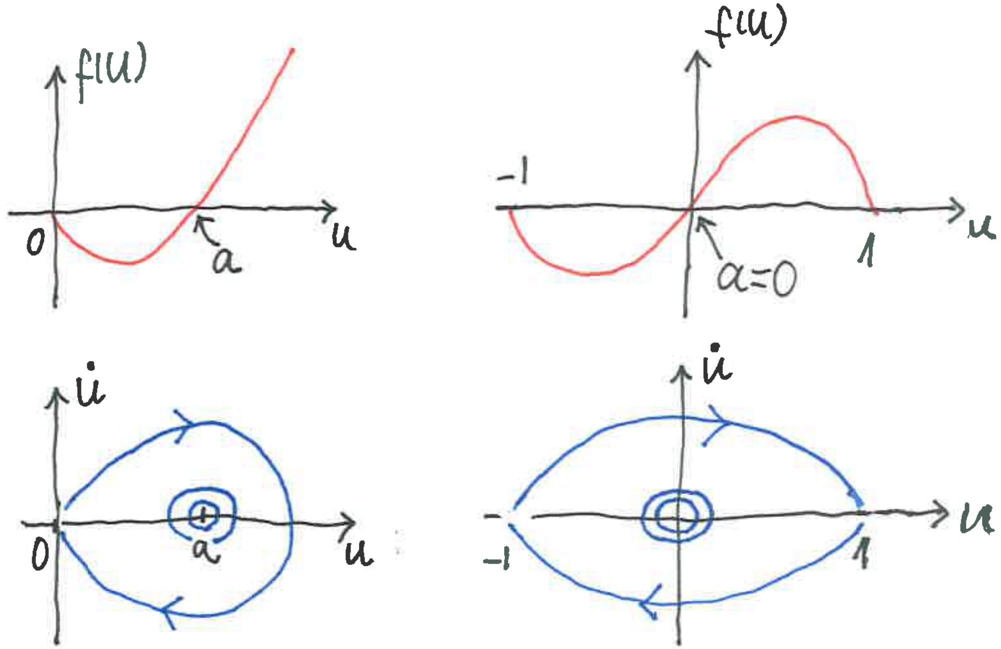
[Cabré & Solà-Morales '15]: same method (Lyapunov-Schmidt reduction) to establish existence of periodic (small) solutions for

$$(-\Delta)^\alpha u = f(u) \quad \text{in } \mathbb{R}$$

close to $u \equiv a$ ($=$ ctt) if $f(a)=0$, $f'(a)>0$.

• Periodic orbits when $\alpha = 1/2$ & $f(u) = -u + u^2$ or $f(u) = \sin(\pi u)$ found by [Amick-Toland, Acta Math 1991] & [Toland, JFA '97]

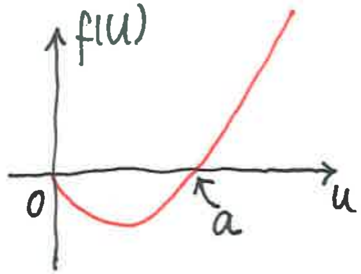
$(-\Delta)$: $\alpha = 1$ →

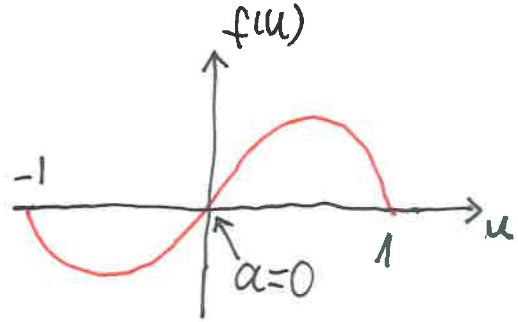


[Cabré & Solà-Morales '15]: same method (Lyapunov-Schmidt reduction) to establish existence of periodic (small) solutions for

$$(-\Delta)^{\alpha} u = f(u) \quad \text{in } \mathbb{R}$$

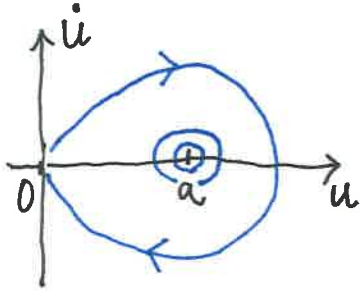
close to $u \equiv a$ (= ctt) if $f(a)=0,\ f'(a)>0$.

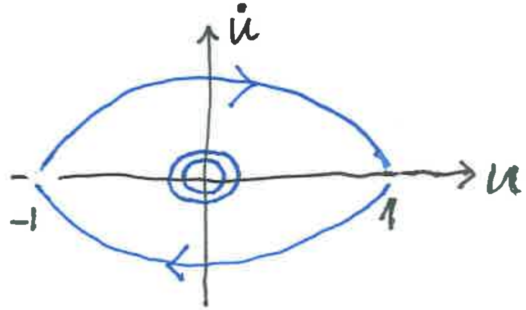




• Periodic orbits when $\alpha = 1/2$ & $f(u) = -u + u^2$ or $f(u) = \sin(\pi u)$ found by [Amick-Toland, Acta Math 1991] & [Toland, JFA '97]

$(-\Delta)$: $\alpha = 1$ →





↓

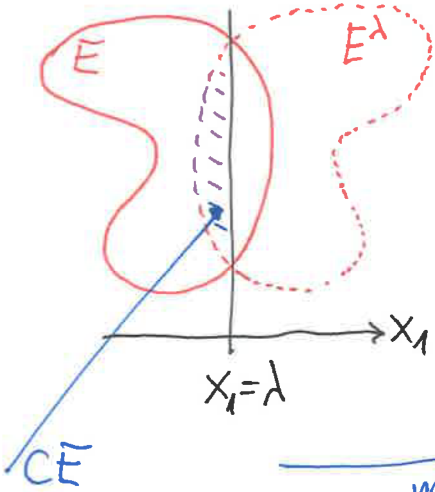
These 2 f's give a completely integrable Hamiltonian system →

That $(-\Delta)^{\alpha} u = f(u)$ in $\mathbb{R}$ has ($\forall f$) Hamiltonian structure found by [Cabré & Solà-Morales, CPAM '05] ($\alpha = 1/2$) [Cabré & Sire '14] ($\forall \alpha \in (0,1)$)

→ Hamiltonian used by [Frank-Lenzmann-Silvestre, CPAM '15]: $\exists!$ GROUND STATES

- **Proof of Thm 1 (Aleksandrov)**: use moving planes method for

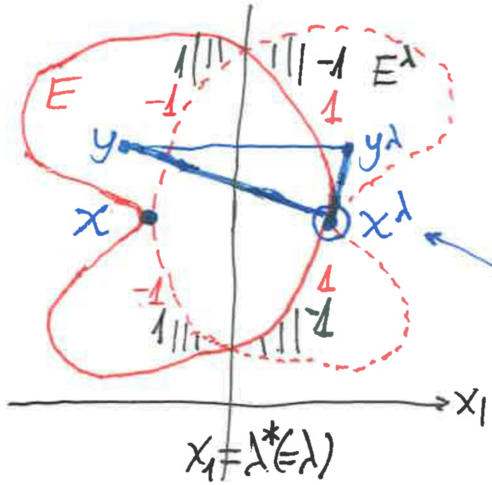
$$H_E(x) = \int_{\mathbb{R}^n} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|x-y|^{n+\alpha}}\, dy \equiv ctt \quad \forall x \in \partial E$$



continue:

make $\lambda \downarrow$

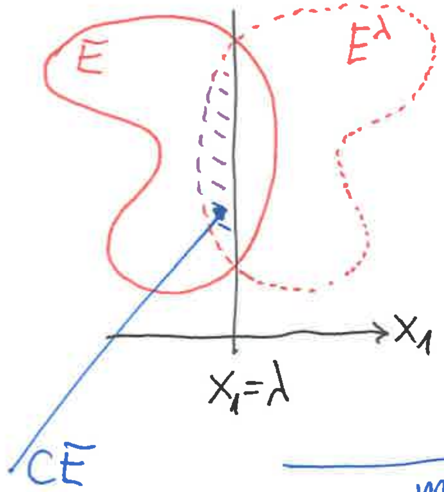
Two possible obstructions → $\boxed{1^{st}}$



$$H_E(x) = H_{E^\lambda}(x^\lambda)$$
$$\parallel$$
$$H_E(x^\lambda)$$

• Proof of Thm 1 (Aleksandrov): use moving planes method for

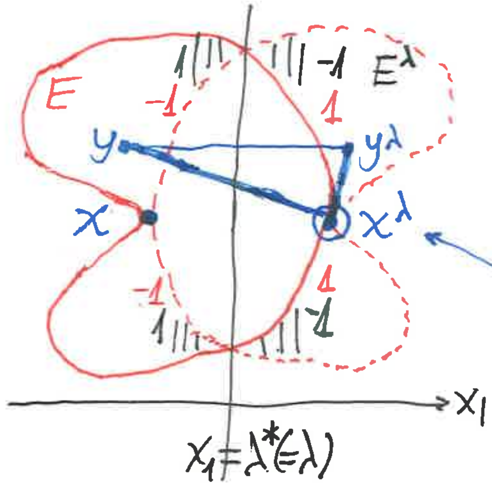
$$H_E(x) = \int_{\mathbb{R}^n} \frac{1\!\!1_{E^c}(y) - 1\!\!1_E(y)}{|x-y|^{n+\alpha}}\, dy \equiv ctt \quad \forall x \in \partial E$$



continue: make $\lambda\downarrow$

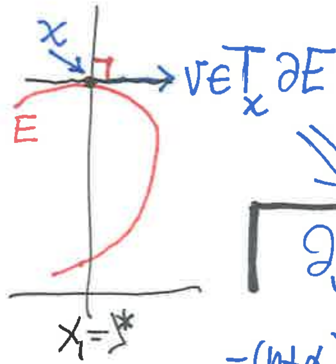
Two possible obstructions

1st



$$H_E(x) = H_{E^\lambda}(x^\lambda)$$
$$\| $$
$$H_E(x^\lambda)$$

2nd



Lemma

$$\partial_v H_E(x) = (n+\alpha)\int_{\mathbb{R}^n} (1\!\!1_E - 1\!\!1_{E^c})(y)\, |x-y|^{-n-\alpha-2}(x-y)\cdot v \, dy$$

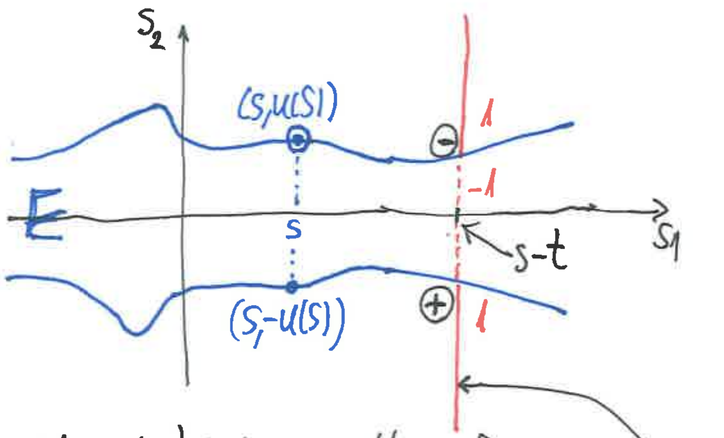
- Proof of Thm 2 : $\exists$ of CNMC periodic bands

We use a Lyapunov-Schmidt reduction $\oplus$ Implicit Function Theorem applied to a quasilinear-type fractional elliptic equation.

- STEP 1 : The setting, equation, and functional spaces:

$u : \mathbb{R} \to \mathbb{R}_+$, $0 < m_1 \le u \le m_2$

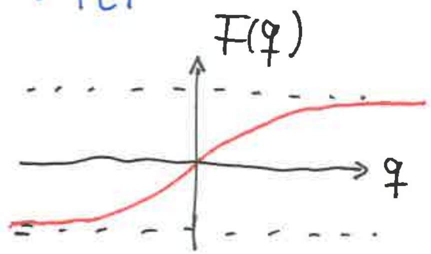
$E = \{ -u(s_1) < s_2 < u(s_1) \} \subset \mathbb{R}^2 \Rightarrow$



$$\boxed{\begin{aligned} \frac{1}{2} H_E(s,u(s)) &=: \frac{1}{2} H(u)(s) = \\ &= \int_{\mathbb{R}} F\left(\frac{u(s)-u(s-t)}{|t|}\right) \frac{dt}{|t|^{1+\alpha}} \\ &\quad - \int_{\mathbb{R}} \left\{ F\left(\frac{u(s)+u(s-t)}{|t|}\right) - F(+\infty) \right\} \frac{dt}{|t|^{1+\alpha}} \end{aligned}}$$

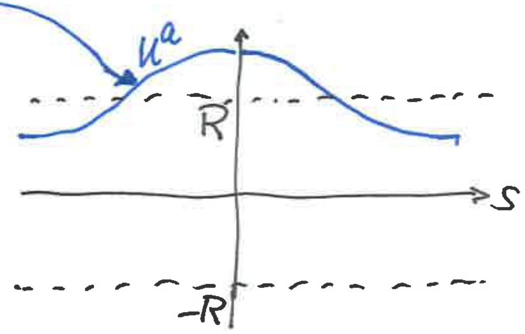
Use Fubini for the $\int_{\mathbb{R}^2}$: integrate first here ds_2

where $F(q) = \int_0^q \frac{d\tau}{(1+\tau^2)^{\frac{2+\alpha}{2}}}$



Want $H(u_a)(s) \equiv$ ctt indep. of a

where $u_a(s) := R + \frac{a}{\lambda}\{\cos(\lambda s) + v_a(\lambda s)\}$



want $\lambda = \lambda(a)$ & $v_a = v(a)$

Period $= \frac{2\pi}{\lambda(a)}$

v_a even f'cn

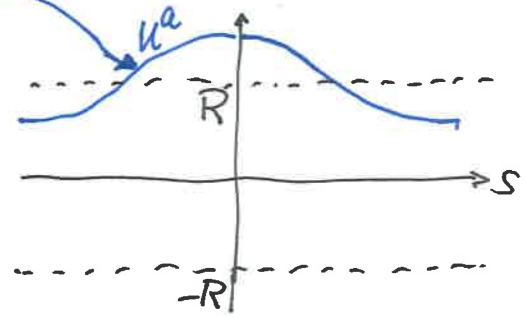
Rescale $\overset{s_1}{\overset{\shortparallel}{s}}$-variable & the $\overset{s_2}{\overset{\shortparallel}{u}}$-variable to make all fcns 2π-periodic

NMC rescales like λ^α

Want $\underline{H(u_a)(s) \equiv \text{ctt indep. of } a}$

where $u_a(s) := R + \frac{a}{\lambda}\{\cos(\lambda s) + v_a(\lambda s)\}$

want $\lambda = \lambda(a)$ & $v_a = v(a)$



Period $= \dfrac{2\pi}{\lambda(a)}$

v_a even fcn

Rescale $\overset{s_1}{\overset{\shortparallel}{s}}$-variable & the $\overset{s_2}{\overset{\shortparallel}{u}}$-variable to make all fcns 2π-periodic

NMC rescales like λ^α

NEW EQUATION after dividing by a; as in [CRANDALL-RABINOWITZ]

$$u(s) = \lambda R + a\{\cos(s) + v_a(s)\}$$
$$= \lambda R + a\,\varphi(s)$$

$$\Phi(a,\lambda,\varphi)(s) := \int_{\mathbb{R}} \frac{1}{a} F\left(a\,\frac{\varphi(s)-\varphi(s-t)}{|t|}\right)\frac{dt}{|t|^{1+\alpha}}$$
$$- \int_{\mathbb{R}} \frac{1}{a}\left\{F\left(\frac{2\lambda R + a(\varphi(s)+\varphi(s-t))}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right)\right\}\frac{dt}{|t|^{1+\alpha}} = 0$$

WANT

Solve $\underline{0 = \Phi(a,\lambda,\varphi)(s)} := \int_{\mathbb{R}} \frac{1}{a} F\left(a\,\frac{\varphi(s)-\varphi(s-t)}{|t|}\right) \frac{dt}{|t|^{1+\alpha}}$

$$- \int_{\mathbb{R}} \frac{1}{a} \left\{ F\left(\frac{2\lambda R + a(\varphi(s)+\varphi(s-t))}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right) \right\} \frac{dt}{|t|^{1+\alpha}}$$

Get $\left\{ \begin{array}{l} \lambda = \lambda(a) \\ v = v(a) \end{array} \right.$ for $|a|$ small.

$\boxed{\text{Spaces}}$:

$X = C^{1,\beta}_{p,e} = \{\varphi : \mathbb{R} \to \mathbb{R},\ C^{1,\beta}(\mathbb{R}),\ 2\pi\text{-periodic, even}\}$

$Y = C^{0,\beta-\alpha}_{p,e} = \{\tilde{\varphi} : \mathbb{R} \to \mathbb{R},\ C^{0,\beta-\alpha}(\mathbb{R}),\ 2\pi\text{-periodic, even}\}$

$$\underline{\Phi(0,\lambda,\varphi)(s)} = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\,dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}} F'\left(\frac{2\lambda R}{|t|}\right) dt.$$

Solve $\underline{0 = \Phi(a,\lambda,\varphi)(s)} := \int_{\mathbb{R}} \frac{1}{a} F\left(a \frac{\varphi(s)-\varphi(s-t)}{|t|}\right) \frac{dt}{|t|^{1+\alpha}}$

$$- \int_{\mathbb{R}} \frac{1}{a} \left\{ F\left(\frac{2\lambda R + a(\varphi(s)+\varphi(s-t))}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right) \right\} \frac{dt}{|t|^{1+\alpha}}$$

Get $\begin{cases} \lambda = \lambda(a) \\ v = v(a) \end{cases}$ for $|a|$ small.

$\boxed{\text{Spaces}}$:

$X = C^{1,\beta}_{p,e} = \{\varphi : \mathbb{R} \to \mathbb{R},\ C^{1,\beta}(\mathbb{R}),\ 2\pi\text{-periodic, even}\}$

$Y = C^{0,\beta-\alpha}_{p,e} = \{\tilde{\varphi} : \mathbb{R} \to \mathbb{R},\ C^{0,\beta-\alpha}(\mathbb{R}),\ 2\pi\text{-periodic, even}\}$

$$\underline{\Phi(0,\lambda,\varphi)(s)} = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\, dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}} F'\left(\frac{2\lambda R}{|t|}\right) dt.$$

- **Lemma**. $\exists! R$ such that $\Phi(a=0, \lambda=1, \varphi=\cos(\cdot)) = 0$

before scaling, bands have period $2\pi/\lambda$

■ STEP 2 : Linearization at $(a=0, \lambda=1, \varphi=\cos(\cdot))$

$2+\alpha = 1 + 2\,\frac{1+\alpha}{2}$

This term is $\boxed{c_\alpha (-\Delta)^{\frac{1+\alpha}{2}} \varphi(s)}$ $\Big\}$ loose $1+\alpha$ derivatives

$$\Phi(0,\lambda,\varphi)(s) = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\,dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}} F'\left(\frac{2\lambda R}{|t|}\right) dt.$$

$$\underline{D_\lambda \Phi(0,1,\cos(\cdot))}(s) = -\int_{\mathbb{R}} \{\cos(s)+\cos(s-t)\} \frac{2R\,F''(2R/|t|)}{|t|^{3+\alpha}}\,dt$$

$$= \underline{c_\alpha \cdot \cos(s)} \quad !!!$$

$$\boxed{Lw(s) := D_\varphi \Phi(0,1,\cos(\cdot))\cdot w(s)} = \Phi(0,1,w)(s)$$

$$= \left\{ \underline{c_\alpha(-\Delta)^{\frac{1+\alpha}{2}} w} - \left(\int_{\mathbb{R}} P_R\right) \underline{w} - \underline{P_R * w} \right\}(s)$$

$P_R \in (L^1 \cap L^\infty)(\mathbb{R})$ even fcn

$$\Phi(0,\lambda,\varphi)(s) = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\,dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}}\, F'\!\left(\frac{2\lambda R}{|t|}\right) dt.$$

$$D_\lambda \Phi(0,1,\cos(\cdot))(s) = -\int_{\mathbb{R}} \{\cos(s)+\cos(s-t)\}\, \frac{2R\, F''(2R/|t|)}{|t|^{3+\alpha}}\, dt$$

$$= \underline{c_\alpha \cdot \cos(s)} \quad !!!$$

$$Lw(s) := D_\varphi \Phi(0,1,\cos(\cdot))\cdot w(s) = \Phi(0,1,w)(s)$$

$$= \left\{ c_\alpha (-\Delta)^{\frac{1+\alpha}{2}} w - \left(\int_{\mathbb{R}} P_R\right) w - P_R * w \right\}(s)$$

$P_R \in (L^1 \cap L^\infty)(\mathbb{R})$ even fcn

- **Lemma** $w_k(s) = \cos(ks)$, $k = 0,1,2,3,\dots$ are eigenfunctions of L in X with eigenvalues

$$\lambda_0 < 0 = \lambda_1 < \lambda_2 < \lambda_3 < \dots \nearrow +\infty \quad \& \quad \frac{\lambda_k}{k^{1+\alpha}} \xrightarrow[k\to\infty]{} ct > 0$$

$\Updownarrow$

choice of R to have $\Phi(0,1,\cos(\cdot)) = 0$.

$\Rightarrow \quad X \overset{L}{\rightleftarrows} Y$

$w_1 = \cos(\cdot)$ is in the kernel of $D_\varphi \Phi(0,1,\cos(\cdot))$ & does not belong to its image,

BUT is in the image of $D_\lambda \Phi(0,1,\cos(\cdot))$!!

$\Big\downarrow$ Lyapunov-Schmidt, I.F.Thm

$\begin{cases} \lambda = \lambda(a) \\ \varphi = \varphi(a) \end{cases}$ for $|a|$ small. $\blacksquare$

$w_1 = \cos(\cdot)$ is in the kernel of $D_\varphi \Phi(0,1,\cos(\cdot))$ & does not belong to its image,

BUT is in the image of $D_\lambda \Phi(0,1,\cos(\cdot))$!!

$\Big\downarrow$ Lyapunov-Schmidt, I.F. Thm

$\begin{cases} \lambda = \lambda(a) \\ \varphi = \varphi(a) \end{cases}$ for $|a|$ small. $\square$

■ Step 3 : $\Phi = \Phi(a,\lambda,\varphi)$ is C^1 from $A \subset \mathbb{R} \times \mathbb{R} \times X$ into Y

• Proposition

Hölder spaces $C^{1,\beta}$ & $C^{0,\beta-\alpha}$ of even & 2π-periodic fcns

Main part of Φ is :

$$\Phi_1(a,\varphi)(s) := \int_{\mathbb{R}} F_1\left(a, \frac{\varphi(s)-\varphi(s-t)}{|t|}\right) \frac{dt}{|t|^{1+\alpha}}$$

where $F_1(a,q) = \int_0^q \frac{d\tau}{(1+a^2\tau^2)^{\frac{2+\alpha}{2}}}$

$$= \frac{1}{2} \int_{\mathbb{R}} \frac{2\varphi(s)-\varphi(s-t)-\varphi(s+t)}{|t|^{2+\alpha}} F_3\left(a, \frac{\varphi(s)-\varphi(s-t)}{|t|}, \frac{\varphi(s)+\varphi(s+t)}{|t|}\right)$$

with F_3 similar to F_1.

$(1+\alpha)$-deriv.

1^{st}-deriv

elliptic **quasi-linear** operator.