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Certain algebraic surfaces of general type with irregularity one and their canonical mappings

by

Tomokuni TAKAHASHI

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Tomokuni TAKAHASHI

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Abstract

In this paper, we show the existence of certain algebraic surfaces of general type with irregularity one, and investigate the canonical mapping of these surfaces. Such a surface has a pencil of non-hyperelliptic curves of genus 3 over an elliptic curve, and is obtained as the minimal resolution of an irreducible relative quartic hypersurface, with at most rational double points as singularities, of the projective plane bundle over an elliptic curve. We use some results on locally free sheaves over elliptic curves by Atiyah and Oda to prove the existence.

Contents

1 Introduction

Let S be a minimal nonsingular complete algebraic surface defined over the complex number field $\mathbf{C}$. S is called a canonical surface if the rational mapping $\Phi_{|K_S|}$ defined by the complete linear system $|K_S|$ of a canonical divisor K_S of S is birational.

In this paper, we show the existence of certain algebraic surfaces of general type with irregularity one, and investigate the canonical mapping of these surfaces. In particular, we check for all values of $p_g(S) \geq 2$ the existence of minimal algebraic surfaces of general type satisfying $K_S^2 = 3p_g(S)$ and $q(S) = 1$, including the cases $p_g(S) \leq 3$. Note that the case $p_g(S) = 1$ was already studied by Catanese and Ciliberto [6].

In general the following inequality holds for the self-intersection number K_S^2 of K_S and the geometric genus $p_g(S)$ of S (cf. [5, Théorème 5.5], [10, Lemma 1.1]):

(I) (Castelnuovo-Horikawa's inequality) If S is a canonical surface, then

$$K_S^2 \geq 3p_g(S) - 7.$$

(II) Castelnuovo classified canonical surfaces which satisfy the equality $K_S^2 = 3p_g(S) - 7$ above. The irregularity of such a surface S satisfies $q(S) = 0$. With a few exceptions such an S is the minimal resolution of an irreducible relative quartic hypersurface of a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ which has at most rational double points as singularities.

In general, the invariants K_S^2, $p_g(S)$, $q(S)$ of an irreducible nonsingular relative quartic hypersurface in a $\mathbf{P}^2$-bundle over a nonsingular curve C of genus b satisfy

$$K_S^2 = 3p_g(S) + 7(b-1), \qquad q(S) = b.$$

We may ask whether a canonical surface S satisfying these equalities is obtained as the minimal resolution of an irreducible relative quartic hypersurface, with at most rational double points, of a $\mathbf{P}^2$-bundle over a nonsingular curve C of genus b. Konno [13, Lemma 3.1, Theorem 3.2] proved that it is the case if $b = 1$. Namely, if S is a canonical surface satisfying $K_S^2 = 3p_g(S)$ and $q(S) = 1$, then S is the minimal resolution of a relative quartic hypersurface in a $\mathbf{P}^2$-bundle over an elliptic curve.

More precisely, S has a pencil $f : S \to C = \mathrm{Alb}(S)$ whose general fiber is a non-hyperelliptic curve of genus 3. Hence, the direct image $f_*\omega_{S/C}$ of the relative dualizing sheaf $\omega_{S/C} := \omega_S \otimes f^*\omega_C^{-1}$ is a locally free sheaf of rank 3 over C. If we let $\pi : W := \mathbf{P}(f_*\omega_{S/C}) \to C$ to be the $\mathbf{P}^2$-bundle associated to $f_*\omega_{S/C}$, $T \in \mathrm{Pic}(W)$ a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong f_*\omega_{S/C}$, and $D \in \mathrm{Pic}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det f_*\omega_{S/C}$, then there exists an irreducible member $S' \in |4T - \pi^*D|$ which has at most rational double points as singularities, and S is the minimal resolution of S' (cf. [13]).

Not all the irreducible relative quartic hypersurfaces in the $\mathbf{P}^2$-bundles over elliptic curves which have at most rational double points as singularities are canonical. For example, since $0 < K_S^2 = 3p_g(S)$ holds, we have the possibilities $p_g(S) = 1,\ 2,\ 3$. Obviously, S is not canonical in these cases.

In this paper, we study whether a complete linear system of $\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$ has members which are irreducible and have at most rational double points as singularities for every locally free sheaf E of rank three over an elliptic curve C, the $\mathbf{P}^2$-bundle $\pi : W := \mathbf{P}(E) \to C$ associated to E and the tautological divisor T with $\pi_*\mathcal{O}_W(T) \cong E$. In particular, we check for all values of $p_g(S) \geq 2$ the existence of minimal algebraic surfaces of general type satisfying $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (Note that the case $p_g(S) = 1$ was already studied by Catanese and Ciliberto [6].) We then investigate their canonical mappings including the cases $p_g(S) \leq 3$. To study the canonical mapping of S, we have to study the rational mapping $\Phi_{|T|}$ of the ambient space W since the canonical sheaf ω_S is isomorphic to the pull back of $\mathcal{O}_W(T)$ to S by the adjunction formula.

We obtain the following results on the existence of minimal algebraic surfaces with $K_S^2 = 3p_g(S)$ and $q(S) = 1$, using the results about vector bundles over elliptic curves by Atiyah [4] and Oda [17].

(1) The case where $f_*\omega_{S/C}$ is isomorphic to the direct sum of three invertible sheaves over C (§4.1): $p_g(S) \geq 3$ is necessary, and conversely, for every integer $N \geq 3$, there

exist minimal algebraic surfaces of general type with $p_g(S) = N$, $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (See Theorem 4.1).

(2) The case where $f_*\omega_{S/C}$ is isomorphic to the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2 over C (§4.2): $p_g(S) \geq 2$ is necessary, and conversely, for every integer $N \geq 2$, there exist minimal algebraic surfaces of general type with $p_g(S) = N$, $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (See Theorem 4.10 and Theorem 4.11).

(3) The case where $f_*\omega_{S/C}$ is indecomposable (§4.3): $p_g(S) \geq 2$ is necessary, and conversely, for every integer $N \geq 2$, there exist minimal algebraic surfaces of general type with $p_g(S) = N$, $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (See Theorem 4.23).

As for the canonical mappings of the above surfaces, we obtain the following results:

(1) In the case where $f_*\omega_{S/C}$ is the direct sum of three invertible sheaves, if $p_g(S) \geq 6$ holds, then the canonical mapping is always birational onto its image with the exception of only one case $f_*\omega_{S/C} \cong L_0^{\oplus 3}$ where L_0 is an invertible sheaf of degree 2 over C.

If $p_g(S) = 5$ and if $f_*\omega_{S/C}$ is not a special locally free sheaf, then the canonical mapping is birational onto its image, too.

If $p_g(S) = 5$ and $f_*\omega_{S/C}$ is a special locally free sheaf, or if $p_g(S) = 4$, then the canonical mapping is birational onto its image in most cases. Although there possibly exists a surface whose canonical mapping is not birational onto its image, we have not obtained any example of such surfaces.

In all the above cases in (1) we obtain some examples of canonical surfaces whose canonical mappings are not holomorphic, and their canonical images are non-normal.

When $p_g(S) = 3$, the canonical mapping is a generically finite mapping onto the projective plane whose degree varies according to the isomorphism class of $f_*\omega_{S/C}$. In most cases, the degree of the canonical mappings are 6, 8 or 9. When the degree of the canonical mapping is 9, then it is holomorphic.

(2) In the case where $f_*\omega_{S/C}$ is the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2, if $p_g(S) \geq 5$ holds, then the canonical mapping is always birational onto its image.

If $p_g(S) = 4$, then the canonical mapping is birational onto its image in most cases. Although there possibly exists a surface whose canonical mapping is not birational onto its image, we have not obtained any example of such surfaces.

In all the above cases in (2) we obtain some examples of canonical surfaces whose canonical mappings are not holomorphic, and their canonical images are non-normal.

If $p_g(S) = 3$, the canonical mapping is a generically finite mapping onto the projective plane whose degree varies according to the isomorphism class of $f_*\omega_{S/C}$. In most cases, the degree of the canonical mapping is 4, 8 or 9. When the degree of the canonical mapping is 9, then it is holomorphic.

If $p_g(S) = 2$, the canonical system is a linear pencil and the genus of a general member of this pencil is 7.

(3) In the case where $f_*\omega_{S/C}$ is indecomposable, if $p_g(S) \geq 5$ holds, then the canonical mapping is always holomorphic and birational onto its image.

If $p_g(S) = 4$, then the canonical mapping is birational onto its image in almost cases. Although there possibly exists a surface whose canonical mapping is not birational onto its image, we have not obtained any example of such surfaces.

If $p_g(S) = 3$, the canonical mapping is a generically finite mapping of degree 8 onto the projective plane in almost cases, but is not holomorphic.

If $p_g(S) = 2$, the canonical system is a linear pencil and the genus of a general member of this pencil is 7.

The case where the canonical mapping is birational but not holomorphic does not appear in the cases treated by Ashikaga [2] and Konno [14].

It is not so easy to study the canonical mapping of a surface S when the rational mapping $\Phi_{|T|}$ of the ambient space is not birational onto its image. Thus Propositions 4.8, 4.9, 4.16, 4.20, 4.22 and Corollary 4.37 require long proofs.

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2 Notation

Let S be a nonsingular complete algebraic surface over the complex number field.

ω_S : the canonical bundle of S.

K_S : the canonical divisor of S.

$p_g(S) := \dim_{\mathbf{C}} H^0(S,\ \omega_S) = \dim_{\mathbf{C}} H^2(S,\ \mathcal{O}_S)$: the geometric genus of S.

$q(S) := \dim_{\mathbf{C}} H^1(S,\ \mathcal{O}_S)$: the irregularity of S.

$\chi(S) := 1 - q(S) + p_g(S)$: the arithmetic genus of S.

For $D \in \mathrm{Div}(S)$, $\Phi_{|D|} : S \cdots \to \mathbf{P}(H^0(S,\ \mathcal{O}_S(D)))$ denotes the rational mapping determined by a complete linear system $|D|$.

3 Preliminaries

Let us mention some results which we need later.

Theorem 3.1 (cf. Konno [13, Corollary 6.4]) *If S is a canonical surface with $q(S) = 1$ and $K_S^2 \le (10/3)\chi(\mathcal{O}_S)$, then a general fiber of the Albanese mapping $f : S \to C := \mathrm{Alb}(S)$ is a nonsingular curve of genus 3.*

Theorem 3.2 (cf. Konno [13, Lemma 3.1, and Theorem 3.2]) *Let $f : S \to C$ be a surjective morphism from a nonsingular surface S to a nonsingular curve C of genus b such that a general fiber is a non-hyperelliptic curve of genus 3. Assume further that f is relatively minimal. Then*

$$(*) \qquad K_S^2 \ge 3\chi(S) + 10(b-1).$$

Furthermore, let $\pi\colon W := \mathbf{P}(f_*\omega_{S/C}) \to C$ be the $\mathbf{P}^2$-bundle over C defined by the locally free sheaf $f_*\omega_{S/C}$ of rank 3, T a tautological divisor such that $\pi_*\mathcal{O}_W(T) \cong f_*\omega_{S/C}$, and $\psi : S \cdots \to W$ the rational mapping over C induced by the natural sheaf homomorphism $f^*f_*\omega_{S/C} \to \omega_{S/C}$. If the equality holds in $(*)$, then ψ is a morphism and the image $S' = \psi(S)$ of ψ has only rational double points as singularities. If we regard S' as a divisor on W, then

$$\mathcal{O}_W(S') \cong \mathcal{O}_W(4T) \otimes \pi^* \det(f_*\omega_{S/C})^\vee$$

holds, where $(f_*\omega_{S/C})^\vee$ is the $\mathcal{O}_C$-module dual to $f_*\omega_{S/C}$.

Remark The inequality stated in the first half of Theorem 3.2 was proved by Horikawa [11], [12, Proposition 2.1] and Reid [18] in a different way. Konno [14, Theorem 2.1] himself also gave another proof.

Proposition 3.3 *Let C be a nonsingular curve of genus b, and E a locally free sheaf of rank 3 over C. Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle over C associated to E, T a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ a divisor on C such that $\mathcal{O}_C(D) \cong \det E$. If the complete linear system $|4T - \pi^*D|$ has an irreducible member S' with at most rational double points as singularities, then the following equalities hold for a minimal resolution S of S'.*

$$\begin{aligned} K_S^2 &= 3\deg E + 16(b-1), \\ p_g(S) &= \deg E + 3(b-1) + \dim H^0(C,\ E^\vee), \\ q(S) &= b + \dim H^0(C,\ E^\vee). \end{aligned}$$

Furthermore, if we let $\nu : S \to S'$ to be the minimal resolution, and if we denote $f := \pi \circ \nu : S \to C$, then we have $f_*\omega_{S/C} \cong E$.

Proof We have $\omega_{S'}^2 = \omega_S^2$, $p_g(S') = p_g(S)$ and $q(S') = q(S)$ by the hypothesis that S' has only rational double points as singularities. From the Euler exact sequence

$$0 \to \Omega^1_{W/C} \to \mathcal{O}_W(-T) \otimes_{\mathcal{O}_W} \pi^* E \to \mathcal{O}_W \to 0,$$

for instance, we have $\omega_{W/C} \cong \mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E$, hence $\omega_W \cong \mathcal{O}_W(-3T) \otimes \pi^*(\omega_C \otimes \det E)$. Thus by the adjunction formula, we get $\omega_{S'} \cong \mathcal{O}_{S'} \otimes_{\mathcal{O}_W} \mathcal{O}_W(T + \pi^* K_C)$. Hence using $T^3 - (\deg E) T^2 F = 0$, we get

$$\omega_{S'}^2 = (T + \pi^* K_C)^2 (4T - \pi^* D) = 4T^3 - T^2 \pi^* D + 8T^2 \pi^* K_C = 3 \deg E + 16(b-1).$$

Next consider the cohomology long exact sequence induced by the exact sequence

$$0 \to \omega_W \to \mathcal{O}_W(T + \pi^* K_C) \to \omega_{S'} \to 0.$$

Since $R^i \pi_* \omega_W = 0$ for $i = 0,\ 1$, we have $H^i(W,\ \omega_W) = 0$ for $i = 0,\ 1$ by the Leray spectral sequence. Hence we get

$$\begin{aligned} p_g(S') &:= \dim H^0(S',\ \omega_{S'}) = \dim H^0(W,\ \mathcal{O}_W(T + \pi^* K_C)) = \dim H^0(C,\ E \otimes \omega_C) \\ &= \deg E + 3(b-1) + \dim H^1(C, E \otimes \omega_C) \quad \text{(by the Riemann-Roch theorem)} \\ &= \deg E + 3(b-1) + \dim H^0(C,\ E^\vee) \quad \text{(by the Serre duality).} \end{aligned}$$

Since $\dim H^2(W,\ \omega_W) = \dim H^1(W,\ \mathcal{O}_W)$ by the Serre duality, and since $\dim H^1(W,\ \mathcal{O}_W) = \dim H^1(C,\ \mathcal{O}_C) = b$ by $\pi_* \mathcal{O}_W \cong \mathcal{O}_C$, $R^1 \pi_* \mathcal{O}_W = 0$ and the Leray spectral sequence, we have $\dim H^2(W,\ \omega_W) = b$. Therefore from $H^2(W,\ \mathcal{O}_W(T + \pi^* K_C)) \cong H^2(C,\ E \otimes \omega_C) = 0$, we get

$$\begin{aligned} q(S') &:= \dim H^1(S',\ \mathcal{O}_{S'}) = \dim H^1(S',\ \omega_{S'}) \qquad \text{(by the Serre duality)} \\ &= \dim H^2(W,\ \omega_W) + \dim H^1(W,\ \mathcal{O}_W(T + \pi^* K_C)) = b + \dim H^0(C,\ E^\vee). \end{aligned}$$

Since S' has at most rational double points as singularities, $\nu^* \omega_{S'/C} \cong \omega_{S/C}$ and $\nu_* \mathcal{O}_S \cong \mathcal{O}_{S'}$ hold. Since $\omega_{S'} \cong (\mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \pi^* \omega_C) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'}$ as we saw above, we have $\omega_{S'/C} \cong \mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'}$. Hence

$$\begin{aligned} f_* \omega_{S/C} &\cong \pi_* \nu_* \omega_{S/C} \cong \pi_* \nu_* \nu^* \omega_{S'/C} \\ &\cong \pi_* \left(\omega_{S'/C} \otimes_{\mathcal{O}_{S'}} \nu_* \mathcal{O}_S \right) \cong \pi_* \omega_{S'/C} \cong \pi_* \left(\mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'} \right). \end{aligned}$$

As the long exact sequence associated to the short exact sequence $0 \to \mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E \to \mathcal{O}_W(T) \to \mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'} \to 0$, we get

$$\begin{aligned} 0 &\to \pi_* \left(\mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E \right) \to \pi_* \mathcal{O}(T) \\ &\to \pi_* \left(\mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'} \right) \to R^1 \pi_* \left(\mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E \right). \end{aligned}$$

Since $R^j \pi_* \left(\mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E \right) \cong \left(R^j \pi_* \mathcal{O}_W(-3T) \right) \otimes_{\mathcal{O}_C} \det E = 0$ for $j = 0,\ 1$, we obtain

$$E = \pi_* \mathcal{O}_W(T) \cong \pi_* \left(\mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'} \right).$$

q.e.d.

Remark By the last assertion of Proposition 3.3, we see that two different $\mathbf{P}^2$-bundles do not contain the same surface.

Theorem 3.4 (cf. Atiyah [4, Theorem 5, Theorem 7 and Corollary, Theorem 9], Oda [17, Theorem 1.2]) *Let C be an elliptic curve and $\mathcal{E}_C(r,\ d)$ $(r,\ d \in \mathbf{Z})$ the set of isomorphism classes of indecomposable locally free sheaves of rank r and degree d over C.*

(1) *If $(r,\ d) = 1$, and if we fix any isogeny $\varphi : \tilde{C} \to C$ of degree r, we have a bijective mapping*

$$\{\ L_0 \in \mathrm{Pic}(\tilde{C}) \mid \deg L_0 = d\ \} \ni L_0 \mapsto \varphi_* L_0 \in \mathcal{E}_C(r,\ d).$$

If we denote $G = \ker \varphi$, then we get

$$\varphi^* \varphi_* L_0 \cong \bigoplus_{\sigma \in G} T_\sigma^* L_0,$$

where $T_\sigma : \tilde{C} \to \tilde{C}$ is the translation by $\sigma \in G$ on $\tilde{C}$.

(2) *For any $r \in \mathbf{N}$, there exists a unique $F_r \in \mathcal{E}_C(r,\ 0)$ such that $H^0(C,\ F_r) \neq 0$. F_r is a successive extension of $\mathcal{O}_C$, and $F_r \cong S^{r-1}(F_2)$ holds. Furthermore, $\dim H^0(C,\ F_r) = \dim H^1(C,\ F_r) = 1$. For $m \in \mathbf{Z}$*

$$\{\ L_0 \in \mathrm{Pic}(C) \mid \deg L_0 = m\ \} \ni L_0 \mapsto F_r \otimes_{\mathcal{O}_C} L_0 \in \mathcal{E}_C(r,\ rm)$$

is a bijective mapping.

Remark Although not necessary in this paper, we have the following in general: If $(r,\ d) = h$, then $\mathcal{E}_C(r/h,\ d/h) \ni F' \mapsto F' \otimes F_h \in \mathcal{E}_C(r,\ d)$ is a bijective mapping.

We use the following lemma in §4.2 and §4.3:

Lemma 3.5 *Let C be an elliptic curve, $\mu : Y = \mathbf{P}(F_2) \to C$ a ruled surface associated to F_2, and $C' \subset Y$ the unique section of μ with $\mu_* \mathcal{O}_Y(C') \cong F_2$. For any point $p \in C$ and for any positive integer i, we have*

$$\mathrm{Bs}\, |iC' + \Gamma_p| = \{y_0\},$$

where $\Gamma_p := \mu^{-1}(p)$ and $y_0 := C' \cap \Gamma_p$. Furthermore, general members of $|iC' + \Gamma_p|$ are nonsingular at y_0, and all the members which are nonsingular at y_0 have the same tangent at y_0. If i and j are positive integers with $i \neq j$, then a nonsingular member of $|iC' + \Gamma_p|$ and a nonsingular member of $|jC' + \Gamma_p|$ have different tangents at y_0.

Proof Since $iC' + \Gamma_p \in |iC' + \Gamma_p|$, the base point of $|iC' + \Gamma_p|$ exists only on $C' \cup \Gamma_p$. Since

$$H^1(Y,\ \mathcal{O}_Y(iC')) \cong H^1(C,\ S^iF_2) \cong H^1(C,\ F_{i+1}) \cong \mathbf{C},$$

and

$$H^1(Y,\ \mathcal{O}_Y(iC' + \Gamma_p)) \cong H^1(C,\ F_{i+1} \otimes \mathcal{O}_C(p)) = 0,$$

the image of the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(iC' + \Gamma_p)) \to H^0(\Gamma_p,\ \mathcal{O}_{\Gamma_p}(iC'))$$

is i-dimensional. Since $H^0(\Gamma_p,\ \mathcal{O}_{\Gamma_p}(iC'))$ $(\cong H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(i)))$ is $(i+1)$-dimensional, there exsits at most one base point on Γ_p. On the other hand, since

$$\dim H^0(Y,\ \mathcal{O}_Y(iC' + \Gamma_p)) = i + 1 \neq i = \dim H^0(Y,\ \mathcal{O}_Y((i-1)C' + \Gamma_p)),$$

C' is not a fixed component of $|iC' + \Gamma_p|$. Furthermore, since $(iC' + \Gamma_p)C' = 1$ and $N_{C'/Y} \cong \mathcal{O}_{C'}$, only $y_0 = C' \cap \Gamma_p$ is the base point of $|iC' + \Gamma_p|$ lying on C'. Hence, we obtain $\mathrm{Bs}\,|iC' + \Gamma_p| = \{y_0\}$.

If all the members of $|iC' + \Gamma_p|$ are singular at y_0, the intersection multiplicity of a member of $|iC' + \Gamma_p|$ and C' at y_0 is at least two. This contradicts $(iC' + \Gamma_p)C' = 1$, and hence, general members of $|iC' + \Gamma_p|$ are nonsingular at y_0.

Let $M \in |iC' + \Gamma_p|$ be a nonsingular member. If we consider the cohomology long exact sequence induced from the exact sequence of sheaves

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(M) \to \mathcal{O}_M(M) \to 0,$$

we obtain

$$\dim H^0(M,\ \mathcal{O}_M(M)) = i + 1,$$

and

$$\dim \mathrm{Im} \left\{ H^0(Y,\ \mathcal{O}_Y(M)) \to H^0(M,\ \mathcal{O}_M(M)) \right\} = i.$$

If we regard y_0 as a point on M, then y_0 can be written as $C'|_M$. Since $\omega_M \cong \omega_Y \otimes \mathcal{O}_Y(M) \otimes \mathcal{O}_M \cong \mathcal{O}_M((i-2)C' + \Gamma_p)$ by the adjunction formula, we have

$$\mathcal{O}_M(M) \cong \omega_M \otimes \mathcal{O}_M(2y_0).$$

The subsystem of the complete linear system of $M|_M$ corresponding to the image of the restriction mapping $H^0(Y,\ \mathcal{O}_Y(M)) \to H^0(M,\ \mathcal{O}_M(M))$ may be regarded as the complete linear system of $M|_M - y_0$, and its dimension is $i - 1$ by what we mentioned above. On the other hand, since

$$\deg \omega_M = (iC' + \Gamma_p)((i-2)C' + \Gamma_p) = 2i - 2,$$

the genus $g(M)$ of M is equal to i. Since $M|_M - 2y_0 \sim K_M$, the complete linear system of $M|_M - 2y_0$ is also $(i-1)$-dimensional. Hence, y_0 is the base point of the complete linear system of $M|_M - y_0$, and the intersection multiplicity of any nonsingular member $M' \in |iC' + \Gamma_p|$ with M at y_0 is at least two, i.e., M and M' have the same tangent.

The last assertion can be proved in the same way as above. q.e.d.

4 Existence and birationality

By Theorem 3.1 and Theorem 3.2, to classify canonical surfaces with $K_S^2 = 3p_g(S)$, and $q(S) = 1$, we need to have a necessary and sufficient condition for the complete linear system $|4T - \pi^*D|$ on the $\mathbf{P}^2$-bundle $W = \mathbf{P}(E)$ associated to a locally free sheaf E of rank 3 over an elliptic curve C to have irreducible members with at most rational double points as singularities, where T is a tautological divisor on W such that $\pi_*\mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ is a divisor such that $\mathcal{O}_C(D) \cong \det E$. We should then choose those members whose nonsingular models have the canonical mappings which are birational onto their images. Locally free sheaves of rank 3 over an elliptic curve C is expressed uniquely up to order as direct sums of indecomposable locally free sheaves (cf. [4]), hence we should consider the following five cases:

(1) E is a direct sum of three invertible sheaves.

(2) E is a direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2.

  (i) The degree of the indecomposable locally free sheaf of rank 2 is odd.

  (ii) The degree of the indecomposable locally free sheaf of rank 2 is even.

(3) E is indecomposable.

  (i) $\deg E$ is not divisible by 3.

  (ii) $\deg E$ is divisible by 3.

We consider each of these cases.

Definition Let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle over the elliptic curve C associated to a locally free sheaf E of rank 3, T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det E$. We say that E satisfies the *condition* (A) if the complete linear system $|4T - \pi^*D|$ has a member S' satisfying the following conditions:

(i) S' is irreducible and has at most rational double points as singularities.
(ii) The minimal resolution S of S' is of general type.
(iii) S satisfies $K_S^2 = 3p_g(S)$ and $q(S) = 1$.

Remark If $H^0(C,\ E^\vee) = 0$, then we have $K_S^2 \neq 3p_g(S)$ and $q(S) \neq 1$ by Proposition 3.3. Hence we only have to consider the locally free sheaves E with $H^0(C,\ E^\vee) = 0$. Furthermore, if E satisfies the condition (A), then S is of general type, and hence, $\chi(\mathcal{O}_S) = \deg E > 0$.

On the other hand, if $f' : S' \to C'$ is a surjective morphism from a nonsingular surface S' to a nonsingular curve C', then $f'_*\omega_{S'/C'}$ is nef, hence, any quotient locally free sheaf of $f'_*\omega_{S'/C'}$ has non-negative degree by Fujita's result [7, (1.2) Proposition]. Hence, we only have to consider nef locally free sheaves.

4.1 The case where E is a direct sum of three invertible sheaves.

Let $L_0,\ L_1,\ L_2$ be invertible sheaves over an elliptic curve C such that $E \cong L_0 \oplus L_1 \oplus L_2$, and denote $d_i := \deg L_i$ $(i = 0,\ 1,\ 2)$. Furthermore, let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to E, and T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$. In §4.1, we prove the existence of a surface S of general type with $K_S^2 = 3p_g(S)$, $q(S) = 1$ and $p_g(S) = N$ for any integer $N \geq 3$ by obtaining necessary and sufficient conditions for the complete linear system of $\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$ to have members with at most rational double points as singularities (Theorem 4.1). We then study the canonical mapping of the surfaces thus obtained. The results about the canonical mappings are stated in Corollaries 4.3 and 4.4, and Propositions 4.6, 4.8 and 4.9.

4.1.1 Existences

We may assume $d_0 \leq d_1 \leq d_2$. We only have to consider the case $d_0 \geq 0,\ d_1 \geq 0$ and $d_2 > 0$ by the remark immedietely before §4.1. By further renumbering of $L_0,\ L_1,\ L_2$ if necessary, we get the following:

Theorem 4.1 *Let $\pi : W = \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle over an elliptic curve C associated to $E \cong L_0 \oplus L_1 \oplus L_2$, T a tautological divisor such that $\pi_*\mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ a divisor on C with $\mathcal{O}_C(D) \cong \det E$. Denote $d_i := \deg L_i (i = 0,\ 1,\ 2)$, and suppose $0 \leq d_0 \leq d_1 \leq d_2$ and $d_2 > 0$. Then the complete linear system $|4T - \pi^*D|$ on W satisfies the condition* (A) *if and only if the following* (1), (2) *and* (3) *hold.*

(1) *One of the following* (i), (ii) *and* (iii) *holds:*

- (i) $d_0 + d_2 < 3d_1$,
- (ii) $L_0 \otimes L_2 \cong L_1^{\otimes 3}$,
- (iii) $2d_0 = 2d_1 = d_2$ *and one of* $L_1^{\otimes 2},\ L_0 \otimes L_1,\ L_0^{\otimes 2},\ L_0^{\otimes 3} \otimes L_1^{-1}$ *is isomorphic to* L_2.

(2) *One of the following* (i), (ii) *and* (iii) *holds:*

- (i) $d_1 < 2d_0$,

(ii) $L_1 \cong L_0^{\otimes 2}$,

(iii) $2d_0 = d_1 = d_2$ *and* $L_2 \cong L_0^{\otimes 2}$.

(3) *If* $d_0 = d_1 = d_2 = 1$ *holds, then one of* $L_0,\ L_1,\ L_2$ *is not isomorphic to the others.*

This proposition can be proved by a method analogous to Ashikaga–Konno [3, Proof of Claim III, pp.523–524], as follows:

Proof $H^0(W,\ \mathcal{O}_W(T)\otimes\pi^*L_i^{-1}) \cong H^0(C,\ L_0\otimes L_i^{-1})\oplus H^0(C,\ L_1\otimes L_i^{-1})\oplus H^0(C,\ L_2\otimes L_i^{-1})$ has a component of the form $H^0(C,\ \mathcal{O}_C)$ for each $i = 0,\ 1,\ 2$. Hence we can choose $X_i \in H^0(W,\ \mathcal{O}_W(T)\otimes\pi^*L_i^{-1})$ $(i = 0,\ 1,\ 2)$ which give homogenous coordinates on each fiber of π. Since

$$H^0(W,\ \mathcal{O}_W(4T)\otimes\pi^*\det E^\vee) = H^0(C,\ S^4E\otimes\det E^\vee)$$

holds, this cohomology group can be written as

$$\bigoplus_{\substack{i,\,j\geq 0\\ i+j\leq 4}} H^0(C,\ L_0^{\otimes(3-i-j)}\otimes L_1^{\otimes(i-1)}\otimes L_2^{\otimes(j-1)})X_0^{4-i-j}X_1^iX_2^j.$$

Hence any $\Psi \in H^0(W,\ \mathcal{O}_W(4T)\otimes\pi^*\det E^\vee)$ can be written as

$$\Psi = \sum_{\substack{i,\,j\geq 0\\ i+j\leq 4}} \psi_{ij}X_0^{4-i-j}X_1^iX_2^j,$$

where $\psi_{ij} \in H^0(C,\ L_0^{\otimes(3-i-j)}\otimes L_1^{\otimes(i-1)}\otimes L_2^{\otimes(j-1)})$.

Suppose that (1) does not hold. When $j = 0$,

$$\deg(L_0^{\otimes(3-i)}\otimes L_1^{\otimes(i-1)}\otimes L_2^{-1}) = (3-i)d_0 + (i-1)d_1 - d_2 \leq -d_0 + 3d_1 - d_2 \leq 0$$

holds, hence if $d_0 + d_2 > 3d_1$, the coefficients of $X_0^4,\ X_0^3X_1,\ X_0^2X_1^2,\ X_0X_1^3,\ X_1^4$ are 0, and hence Ψ is reducible (it is divisible by X_2). If $d_0 + d_2 = 3d_1$, and none of $L_0^{\otimes 3}\otimes L_1^{-1},\ L_0^{\otimes 2},\ L_0\otimes L_1,\ L_1^{\otimes 2},\ L_0^{-1}\otimes L_1^{\otimes 3}$ are isomorphic to L_2, then Ψ is divisible by X_2 again.

Next we suppose that (2) does not hold. Then since we have

$$\begin{aligned}
&\deg(L_0^{\otimes 3}\otimes L_1^{-1}\otimes L_2^{-1}) = 3d_0 - d_1 - d_2 = (2d_0 - d_1) + (d_0 - d_2) \leq 2d_0 - d_1 \leq 0,\\
&\deg(L_0^{\otimes 2}\otimes L_2^{-1}) = 2d_0 - d_2 = (2d_0 - d_1) + (d_1 - d_2) \leq 2d_0 - d_1 \leq 0,\\
&\deg(L_0^{\otimes 2}\otimes L_1^{-1}) = 2d_0 - d_1 \leq 0,
\end{aligned}$$

if $2d_0 < d_1$, then the coefficients of $X_0^4,\ X_0^3X_1,\ X_0^3X_2$ of Ψ are 0. Let $Z_0 \subset W$ be the curve defined by $X_1 = X_2 = 0$. (In the rest of the proof, Z_0 denotes this curve.) The degrees with respect to X_1 and X_2 of each term of Ψ are greater than 1 on Z_0, hence the divisor on W defined by $\Psi = 0$ contains Z_0 as a singular curve. The case where $2d_0 = d_1$ and $L_0^{\otimes 2} \not\cong L_1$ is the same as above, if $d_1 < d_2$. (If $d_1 = d_2$ and $L_0^{\otimes 2} \cong L_2$ hold, interchange L_1 and L_2 and regard this case as the case $L_0^{\otimes 2} \cong L_1$.) When $d_1 = d_2$ and $L_0^{\otimes 2} \ncong L_i$, $(i = 1,\ 2)$ hold, if we assume $3d_0 - d_1 - d_2 = 2d_0 - d_2$, then we have $d_0 = d_1$. However, since $2d_0 = d_1 = d_2$, we have $d_0 = d_1 = d_2$, a contradiction to the assumption. Hence $3d_0 - d_1 - d_2 < 2d_0 - d_2$, and Z_0 is a singular curve in the divisor defined by $\Psi = 0$ on W.

We suppose (3) does not hold, i.e., $d_0 = d_1 = d_2 = 1$ and $L_0 \cong L_1 \cong L_2$ hold. In this case, for all $i,\ j$ satisfying $i,\ j \geq 0$ and $i + j \leq 4$, $L_0^{\otimes(3-i-j)} \otimes L_1^{\otimes(i-1)} \otimes L_2^{\otimes(j-1)}$ are isomorphic to one another, and of degree 1. Hence there exists a point $p \in C$ such that $\psi_{ij} = 0$ holds for all $i,\ j$. Consequently, $F := \pi^{-1}(p)$ is a fixed component of $|4T - \pi^* D|$.

From now on, we assume that (1), (2) and (3) hold.

(I) Let us look at the case where $3d_0 > d_1 + d_2$ or $L_0^{\otimes 3} \cong L_1 \otimes L_2$.

(If, moreover, $d_0 = d_1$ and $L_0^{\otimes 3} \cong L_1 \otimes L_2$ hold, then we have $\deg(L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1}) = \deg(L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1})$. In this case, if $L_1^{\otimes 3} \ncong L_0 \otimes L_2$ holds, we interchange L_0 and L_1 and regard this case as the case $L_0^{\otimes 3} \ncong L_1 \otimes L_2$. So we may assume $L_1^{\otimes 3} \cong L_0 \otimes L_2$ holds when $d_0 = d_1$ and $L_0^{\otimes 3} \cong L_1 \otimes L_2$ hold.)

Since we have $\deg(L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1}) \leq \deg(L_0^{\otimes(3-i-j)} \otimes L_1^{\otimes(i-1)} \otimes L_2^{\otimes(j-1)})$, $|4T - \pi^* D|$ has no base point if and only if $3d_0 - d_1 - d_2 \neq 1$. Indeed, under the assumption of (I), we have $H^0(C,\ L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1}) \neq 0$, $H^0(C,\ L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1}) \neq 0$ and $H^0(C,\ L_0^{-1} \otimes L_1^{-1} \otimes L_2^{\otimes 3}) \neq 0$, and hence we can choose nonzero global sections of $H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ of the form $\Psi_0 := \psi_{00} X_0^4$, $\Psi_1 := \psi_{40} X_1^4$, $\Psi_2 := \psi_{04} X_2^4$. If $3d_0 - d_1 - d_2 \neq 1$ holds, then we have $\deg(L_0^{-1} \otimes L_1^{-1} \otimes L_2^{\otimes 3}) \geq 2$ and $L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1}$, $L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1}$ either have degrees greater than 1 or are isomorphic to $\mathcal{O}_C$. Hence $|4T - \pi^* D|$ has no base point. If $3d_0 - d_1 - d_2 = 1$ holds, there exists a unique point $p_0 \in C$ such that $\psi_{00}(p_0) = 0$ holds for any $\psi_{00} \in H^0(C,\ L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1})$. If we denote $F_0 := \pi^{-1}(p_0)$, then $q_0 := F_0 \cap Z_0$ is an isolated fixed point of $|4T - \pi^* D|$. (In the rest of the proof, p_0 and q_0 denote these points.)

Hence if $3d_0 - d_1 - d_2 \neq 1$ holds, the general member of $|4T - \pi^* D|$ is irreducible and nonsingular by Bertini's theorem.

We claim that even if $3d_0 - d_1 - d_2 = 1$ holds, the general member of $|4T - \pi^* D|$ is also irreducible and nonsingular. Indeed if $-d_0 + 3d_1 - d_2 \geq 2$ and $-d_0 - d_1 + 3d_2 \geq 2$ hold, there exist no base point except q_0. If we let t to be a local coordinate around p_0 on C, and if we denote $x_i := X_i / X_0$ $(i = 1,\ 2)$, then $(t,\ x_1,\ x_2)$ is a local coordinate around q_0 on W. General $\Psi \in H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ can be written as

$$\Psi = t + \psi_{10} x_1 + \psi_{01} x_2 + \cdots$$

around q_0, so the divisor (Ψ) on W is nonsingular at q_0. If $-d_0 + 3d_1 - d_2 = 1$ and $-d_0 - d_1 + 3d_2 \geq 2$ hold, there exists a unique point $p_1 \in C$ with $\psi_{40}(p_1) = 0$ for all $\psi_{40} \in H^0(C,\ L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1})$. If we assume $L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1} \ncong L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1}$, then $p_0 \neq p_1$ holds. In this case, if we let Z_1 to be the curve defined by $X_0 = X_1 = 0$ on W, and if we denote $F_1 := \pi^{-1}(q_1)$ and $q_1 := Z_1 \cap F_1$, then we have $q_0 \neq q_1$ and $\mathrm{Bs}\,|4T - \pi^* D| =$

$\{q_0,\ q_1\}$. We can show that the general member of $|4T - \pi^* D|$ is nonsingular at q_0 and q_1 in the same way as above. (In the rest of the proof, $p_1,\ q_1,\ F_1,\ Z_1$ denote the above points and curves.) If we assume $L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1} \cong L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1}$ and $L_0 \not\cong L_1$, then although p_0 and p_1 coincide, q_0 and q_1 are two distnct points contained in the same fiber of π, and we have $\operatorname{Bs}|4T - \pi^* D| = \{q_0,\ q_1\}$ again. We can show that a general member of $|4T - \pi^* D|$ is nonsingular at q_0 and q_1 in the same way as above. If we assume $L_0 \cong L_1$, then $p_0 = p_1$ holds. In this case, if we let Z' to be the intersection of $F_0 = \pi^{-1}(p_0)$ and the relative hyperplane defined by the equation $X_2 = 0$, then $\operatorname{Bs}|4T - \pi^* D| = Z'$ holds. In this case, we have $H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1}) \cong H^0(C,\ \mathcal{O}_C) \oplus H^0(C,\ \mathcal{O}_C) \oplus H^0(C,\ L_2 \otimes L_0^{-1})$. Let $X_0 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1})$ be the element of the subspace corresponding to the first $H^0(C,\ \mathcal{O}_C)$, and $X_1 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1})$ the element of the subspace corresponding to the second $H^0(C,\ \mathcal{O}_C)$. Any element of the subspace of $H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1})$ corresponding to $H^0(C,\ \mathcal{O}_C) \oplus H^0(C,\ \mathcal{O}_C)$ can be written as $aX_0 + bX_1$ for some $a,\ b \in \mathbf{C}$, and the divisor $(aX_0 + bX_1)$ of W is clearly irreducible. Hence if we denote $X_0' := aX_0 + bX_1$, then $X_0',\ X_1$ and X_2 give homogeneous coordinates of all the fibers of π. There exist constants a and b in $\mathbf{C}$ with $q' = Z' \cap (X_0')$ for any point $q' \in Z'$. For this X_0', if we denote $x_0' := X_0'/X_1$ and $x_2 := X_2/X_1$, and if we let t to be a local coordinate around $p_0 = p_1$, then $(t,\ x_0',\ x_2)$ gives a local coordinate around q' on W. A general $\Psi \in H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ can be written as

$$\Psi = t + \psi_{31} x_2 + \cdots$$

locally, hence the divisor (Ψ) is nonsingular at q'. Thus, a general member of $|4T - \pi^* D|$ is nonsingular at Z'. Next we assume $-d_0 + 3d_1 - d_2 = -d_0 - d_1 + 3d_2 = 1$, i.e., $d_0 = d_1 = d_2 = 1$. Suppose further that $L_0 \cong L_1$. In this case, we have $L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1} \cong L_0^{\otimes 2} \otimes L_2^{-1} \cong L_0 \otimes L_1 \otimes L_2^{-1} \cong L_1^{\otimes 2} \otimes L_2^{-1} \cong L_0^{-1} \otimes L_1^{\otimes 3} \otimes L_2^{-1}$, so there exists a point $p \in C$ such that the coefficients $\psi_{00},\ \psi_{10},\ \psi_{20},\ \psi_{30},\ \psi_{40}$ of $X_0^4,\ X_0^3 X_1,\ X_0^2 X_1^2,\ X_0 X_1^3,\ X_1^4$ vanish on $F := \pi^{-1}(p)$. Hence if we let Z' to be the curve which is the intersection of F and the relative hyperplane defined by the equation $X_2 = 0$, then Z' is contained in $\operatorname{Bs}|4T - \pi^* D|$. Therefore we have then $\operatorname{Bs}|4T - \pi^* D| = \{q_0\} \cup Z'$. We can prove that a general member of $|4T - \pi^* D|$ is also nonsingular along $\{q_0\} \cup Z'$ in this case in the same way as above. In the same way, we obtain the same result when $L_1 \cong L_2$ or $L_2 \cong L_0$. Suppose $L_i \not\cong L_j$ if $i \neq j$ $(i,\ j = 0,\ 1,\ 2)$. There exists a point $p_2 \in C$ such that $\psi_{04}(p_2) = 0$ holds for any global section $\psi_{04} \in H^0(C,\ L_0^{-1} \otimes L_1^{-1} \otimes L_2^{\otimes 3})$. Let Z_2 be the curve defined by $X_0 = X_1 = 0$, and denote $F_2 := \pi^{-1}(p_2)$, $q_2 := Z_2 \cap F_2$. Then we have $q_2 \neq q_0,\ q_1$ and $\operatorname{Bs}|4T - \pi^* D| = \{q_0,\ q_1,\ q_2\}$. We can show that a general member of $|4T - \pi^* D|$ is nonsingular at $q_0,\ q_1,\ q_2$ in the same way as above.

(II) Let us look at the case where $3d_0 < d_1 + d_2$ or ($3d_0 = d_1 + d_2$ and $L_0^{\otimes 3} \not\cong L_1 \otimes L_2$).

Since we have $H^0(C,\ L_0^{\otimes 3} \otimes L_1^{-1} \otimes L_2^{-1}) = 0$, the coefficient ψ_{00} of X_0^4 in Ψ is 0. Hence Z_0 is contained in $\operatorname{Bs}|4T - \pi^* D|$. The other base points are as follows:

(i) There exist no base point except Z_0. $\Longleftrightarrow -d_0 + 3d_1 - d_2 \geq 2$ or $L_1^{\otimes 3} \cong L_0 \otimes L_2$.

(ii) If $-d_0+3d_1-d_2=1$ and $-d_0-d_1+3d_2 \geq 2$, then q_1 is a base point of $|4T-\pi^*D|$.

(iii) If $-d_0+3d_1-d_2=-d_0-d_1+3d_2=1$, then q_1 and q_2 are base points of $|4T-\pi^*D|$.

(iv) If $-d_0+3d_1-d_2=0$ and $L_1^{\otimes 3} \not\cong L_0 \otimes L_2$, then we have $\mathrm{Bs}\,|4T-\pi^*D| = Z_0 \cup Z_1$.

(i), (ii) and (iii) are trivial. So we prove (iv). First, we have $d_0=d_1$. Indeed, if we assume $d_0<d_1$, then we have

$$3d_0-d_1-d_2 < 2d_0-d_2 < d_0+d_1-d_2 < 2d_1-d_2 < -d_0+3d_1-d_2 = 0,$$

hence the coefficients of $X_0^4,\ X_0^3X_1,\ X_0^2X_1^2,\ X_0X_1^3,\ X_1^4$ vanish, and any $\Psi \in H^0(W,\ \mathcal{O}_W(4T)\otimes\pi^*\det E^\vee)$ is reducible. Therefore we have $d_0=d_1$ and hence $2d_0=2d_1=d_2$ holds. Since we have $L_0^{\otimes 3} \not\cong L_1\otimes L_2$ and $L_1^{\otimes 3} \not\cong L_0\otimes L_2$ by assumption, the coefficients of X_0^4 and X_1^4 are 0. On the other hand, since the coefficient of one of $X_0^3X_1,\ X_0^2X_1^2,\ X_0X_1^3$ is not 0 by the assumption of the proposition (the third assumption of (3)), one of $\psi_{10}X_0^3X_1,\ \psi_{20}X_0^2X_1^2$ and $\psi_{30}X_0X_1^3$ gives an effective divisor in W. (It is a union of two relative hyperplanes defined by the equations $X_0=0,\ X_1=0$.)

Since we have $\deg(L_0^{-1}\otimes L_1^{-1}\otimes L_2^{\otimes 3}) = -d_0-d_1+3d_2 = 4d_0 \geq 4 > 2$, we obtain the claim considering the intersection of the divisors defined by the global sections $\psi_{40}X_2^4$ and one of $\psi_{10}X_0^3X_1,\ \psi_{20}X_0^2X_1^2,\ \psi_{30}X_0X_1^3$.

We can show that a general member of $|4T-\pi^*D|$ is nonsingular at q_1 in the case (ii), and at q_1 and q_2 in the case (iii) in the same way as above. Hence in the cases (i), (ii) and (iii), it is sufficient to look at the multiplicity of a general member of $|4T-\pi^*D|$ at Z_0.

Let us look at the case where $2d_0>d_2$ or $L_0^{\otimes 2}\cong L_2$. (When $L_0\cong L_2$, if we assume $d_1=d_2$ and $L_0^{\otimes 2}\not\cong L_1$, further, interchange L_1 and L_2 and regard this case as the case $L_0^{\otimes 2}\not\cong L_2$. Hence, we may assume $L_0^{\otimes 2}\cong L_1$ when $L_0^{\otimes 2}\cong L_2$ and $d_1=d_2$.) Since we have $H^0(C,\ L_0^{\otimes 2}\otimes L_2^{-1})\neq 0$ and $H^0(C,\ L_0^{\otimes 2}\otimes L_1^{-1})\neq 0$, a general member of $|4T-\pi^*D|$ is nonsingular at Z_0 except in the case

$$2d_0-d_1=2d_0-d_2=1, \text{ and } L_1\cong L_2.$$

In this case, since we have $L_0^{\otimes 2}\otimes L_1^{-1}\cong L_0^{\otimes 2}\otimes L_2^{-1}$, there exists a point $p\in C$ such that $\psi(p)=0$ holds for any $\psi\in H^0(C,\ L_0^{\otimes 2}\otimes L_1^{-1})\cong H^0(C,\ L_0^{\otimes 2}\otimes L_2^{-1})$. Denote $F:=\pi^{-1}(p)$, $q:=Z_0\cap F$ and $x_i:=X_i/X_0$ $(i=1,\ 2)$, and let t be a local coordinate around p on C. A general $\Psi\in H^0(W,\ \mathcal{O}_W(4T)\otimes\pi^*\det E^\vee)$ can be written as

$$\begin{aligned}\Psi &= tx_1+ctx_2+\psi_{20}x_1^2+\psi_{11}x_1x_2+\psi_{02}x_2^2+\cdots\\ &= x_1(t+\psi_{20}x_1+\psi_{11}x_2+\cdots)+ctx_2+\psi_{02}x_2^2+\psi_{03}x_2^3+\psi_{04}x_2^4,\end{aligned}$$

where $c\in\mathbf{C}$ is a constant. Hence the divisor defined by $\Psi=0$ on W has a rational double point of type A_1 at q.

Let us look at the case where $2d_0<d_2$ or ($2d_0=d_2$ and $L_0^{\otimes 2}\not\cong L_2$). In this case, the coefficients of $X_0^4,\ X_0^3X_1$ are 0. If $d_1=2d_0$ and $L_0^{\otimes 2}\cong L_1$ hold, then the coefficient of

$X_0^3X_2$ is constant, hence a general member is nonsingular at Z_0. If $d_1 < 2d_0$ holds, and if we let $p \in C$ be a point such that $\psi_{01}(p) = 0$ for $0 \neq \psi_{01} \in H^0(C,\ L_0^{\otimes 2} \otimes L_1^{-1})$, Ψ can be written as

$$\begin{aligned} \Psi &= tx_2 + \psi_{20}x_1^2 + \psi_{11}x_1x_2 + \psi_{02}x_2^2 + \cdots \\ &= x_2(t + \psi_{11}x_1 + \psi_{02}x_2 + \cdots) + \psi_{20}x_1^2 + \psi_{30}x_1^3 + \psi_{40}x_1^4, \end{aligned}$$

around q, where $q,\ t,\ x_1,\ x_2$ are as above. The equation $\Psi = 0$ gives a rational double point of type A_1 at q except in the case

$$L_0 \otimes L_1 \otimes L_2^{-1} \cong L_0^{\otimes 2} \otimes L_1^{-1}, \text{ and } d_0 + d_1 - d_2 = 2d_0 - d_1 = 1.$$

In this case, $\psi_{20} = c't$ holds around q for some constant $c' \in \mathbf{C}$, and the equation $\Psi = 0$ gives a rational double point of type A_2 at q.

In the case (iv), we can show that a general member of $|4T - \pi^*D|$ is irreducible and has at most rational double points of type A_1 on Z_0 and Z_1 in the same way as above. q.e.d.

4.1.2 The canonical mappings

We study the canonical mapping of surfaces classified in Theorem 4.1. The triples $(d_0,\ d_1,\ d_2)$ satisfying $p_g(S) = d_0 + d_1 + d_2 = 4,\ 5,\ 6$ and the conditions in Theorem 4.1 are

$$\begin{aligned} &(d_0,\ d_1,\ d_2) = (1,\ 1,\ 2), && (p_g(S) = 4), \\ &(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2), && (p_g(S) = 5), \\ &(d_0,\ d_1,\ d_2) = (1,\ 2,\ 3),\ (2,\ 2,\ 2), && (p_g(S) = 6). \end{aligned}$$

Lemma 4.2 *Let $L_0,\ L_1,\ L_2$ be invertible sheaves over an elliptic curve C, and denote $d_i := \deg L_i,\ (i = 0,\ 1,\ 2)$. Assume that $L_0,\ L_1,\ L_2$ satisfy the conditions of Theorem 4.1. If $\pi : W := \mathbf{P}(E) \to C$ is the $\mathbf{P}^2$-bundle over C associated to $E := L_0 \oplus L_1 \oplus L_2$, and T is a tautological divisor such that $\pi_*\mathcal{O}_W(T) \cong E$, then $\Phi_{|T|}$ is birational onto its image when one of the following holds.*

(i) $d_0 + d_1 + d_2 \geq 7$.

(ii) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 3)$.

(iii) $(d_0,\ d_1,\ d_2) = (2,\ 2,\ 2)$ *and one of $L_0,\ L_1,\ L_2$ is not isomorphic to the others.*

(iv) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2)$ *and $L_1 \not\cong L_2$.*

Proof First we show that for any general fiber F of π, the restriction of $\Phi_{|T|}$ to F gives an isomorphism of F onto its image. It suffices to show that the restriction mapping $H^0(W,\ \mathcal{O}_W(T)) \to H^0(F,\ \mathcal{O}_F(T))$ is sujective. We only have to show $H^1(W,\ \mathcal{O}_W(T - F)) = 0$ in view of the exact sequence

$$0 \to \mathcal{O}_W(T - F) \to \mathcal{O}_W(T) \to \mathcal{O}_F(T) \to 0.$$

If we denote $q := \pi(F) \in C$, we have

$$H^1(W,\ \mathcal{O}_W(T-F)) = H^1(C,\ E \otimes \mathcal{O}_C(-q))$$
$$\cong H^1(C,\ L_0 \otimes \mathcal{O}_C(-q)) \oplus H^1(C,\ L_1 \otimes \mathcal{O}_C(-q)) \oplus H^1(C,\ L_2 \otimes \mathcal{O}_C(-q)).$$

Since we assume $0 < d_0 \le d_1 \le d_2$, this cohomology group vanishes.

In the rest of the proof, we show that there exists a Zariski open subset of W such that any two points in it contained in different fibers are separated by $|T|$.

Since

$$H^0(W,\ \mathcal{O}_W(T)) \cong H^0(C,\ L_0) \oplus H^0(C,\ L_1) \oplus H^0(C,\ L_2),$$

if we choose $X_i \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_i^{-1})$ $(i = 0,\ 1,\ 2)$ as in the proof of Theorem 4.1, any $\Psi \in H^0(W,\ \mathcal{O}_W(T))$ can be written as

$$\Psi = \psi_0 X_0 + \psi_1 X_1 + \psi_2 X_2, \quad \psi_i \in H^0(C,\ L_i)\ (i = 0,\ 1,\ 2).$$

If $d_2 \ge 3$, then $W \setminus (X_2)$ satisfies the above condition, where (X_2) is the divisor defined by X_2. (Look at all the elements of the form $\psi_2 X_2$.)

If $d_2 = 2$, then we have $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2),\ (2,\ 2,\ 2)$.

If $(d_0,\ d_1,\ d_2) = (2,\ 2,\ 2)$, then at least one of $L_0,\ L_1,\ L_2$ is not isomorphic to the others. We may assume $L_1 \not\cong L_2$ by renumbering $L_0,\ L_1,\ L_2$ if necessary. We see that $W \setminus \{(X_1) \cup (X_2)\}$ satisfies the above condition, where (X_i) is the divisor defined by X_i $(i = 1,\ 2)$. (Look at all the elements of the form $\psi_1 X_1,\ \psi_2 X_2$.)

We obtain the same result when $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2)$, since we assume $L_1 \not\cong L_2$ in this case. q.e.d.

Corollary 4.3 *The canonical mapping of any surface S whose existence is guaranteed by Theorem* 4.1 *and the condition* (A) *is a birational morphism onto its image if one of the following holds.*

(1) $d_0 + d_1 + d_2 \ge 7$ *and* $d_0 \ge 2$,

(2) $(d_0,\ d_1,\ d_2) = (2,\ 2,\ 2)$, *and one of* $L_0,\ L_1,\ L_2$ *is not isomorphic to the others.*

Proof Since $\mathcal{O}_{S'} \otimes_{\mathcal{O}_W} \omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_W(S') \cong \omega_{S'}$ by the adjunction formula, and since $\omega_W \cong \mathcal{O}_W(-3T) \otimes \pi^* \det E$ and $\mathcal{O}_W(S') \cong \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$, we have

$$\omega_{S'} \cong \mathcal{O}_{S'} \otimes_{\mathcal{O}_W} \mathcal{O}_W(T).$$

Furthermore, we have

$$\dim H^i(W,\ \omega_W) = \dim H^{3-i}(W,\ \mathcal{O}_W) = \dim H^{3-i}(C,\ \mathcal{O}_C) = 0,$$

for $i = 0,\ 1$ by the Serre duality and the Leray spectral sequence. Hence in view of the cohomology long exact sequence associated to the short exact sequence

$$0 \to \omega_W \to \mathcal{O}_W(T) \to \omega_{S'} \to 0,$$

we have

$$H^0(W,\ \mathcal{O}_W(T)) \cong H^0(S',\ \omega_{S'}).$$

Since S' has at most rational double points as singularities, we have $\Phi_{|K_S|} = \psi \circ \Phi_{|T|}$, where $\psi : \cdots \to W$ is a morphism by Theorem 3.2. Since ψ is birational onto its image, if $\Phi_{|T|}$ is birational onto its image, then $\Phi_{|K_S|}$ is also birational onto its image. Hence the statement about the birationality follows from Lemma 4.2.

We prove that $\Phi_{|K_S|}$ is holomorphic. Since $d_0 \geq 2$, we see that $\mathrm{Bs}\,|T| = \emptyset$ by considering all the elements in $H^0(W,\ \mathcal{O}_W(T))$ of the form $\psi_0 X_0,\ \psi_1 X_1,\ \psi_2 X_2$. Hence $|K_S|$ also has no base point. q.e.d.

Corollary 4.4 *The complete linear system of the canonical bundle of any surface S whose existence is guaranteed by Theorem* 4.1 *and condition* (A) *has only one isolated base point, and its canonical mapping is birational onto its image, if one of the following holds:*

(1) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 5)$,

(2) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 4)$,

(3) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 3)$,

(4) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2)$ *and* $L_1 \not\cong L_2$.

Furthermore, its canonical image is non-normal.

Proof We use our notation in Theorem 4.1 and Corollary 4.3.

Under the assumption of the corollary, the curve $Z_0 \subset W$ defined by $X_1 = X_2 = 0$ is contained in the set of base points of $|4T - \pi^* D|$ by the proof of Proposition 4.1. Furthermore, we can show that the point $q_0 \in W$ defined by $\psi_0 = X_1 = X_2 = 0$ satisfies $\mathrm{Bs}\,|T| = \{q_0\}$ in the same way as in Corollary 4.3. Since $q_0 \in Z_0$, the complete linear system of the canonical bundle of a general member of $|4T - \pi^* D|$ has only one base point q_0. The birationality of the canonical mapping can be proved in the same way as in the proof of Corollary 4.3.

The restriction of $|T|$ to the fiber F_0 containing q_0 can be regarded as a subsystem of the complete linear system of $\mathcal{O}_{\mathbf{P}^2}(1)$ consisting of all lines going through q_0. Each line of this system intersects the fiber $\mathcal{F}$ of a general member S of $|4T - \pi^* D|$ at four points, one of which is q_0. Hence we have $\deg\left(\Phi_{|K_S|}|_{\mathcal{F}}\right) = 3$, and the canonical image of S is non-normal by Zariski's main theorem. q.e.d.

Lemma 4.5 *If* $(d_0,\ d_1,\ d_2) = (2,\ 2,\ 2)$ *and* $L_0 \cong L_1 \cong L_2$*, then* $\deg \Phi_{|T|} = 2$ *holds.*

Proof If we denote $\nu := \Phi_{|L_0|} : C \to \mathbf{P}^1$, we have $L_0 \cong \nu^*\mathcal{O}_{\mathbf{P}^1}(1)$, and hence $E \cong \nu^*(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 3})$. Therefore, if we denote $\pi_0 : W_0 := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 3}) \to \mathbf{P}^1$, we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\tilde{\nu}} & W_0 \\ \pi \downarrow & & \downarrow \pi_0 \\ C & \xrightarrow[\nu]{} & \mathbf{P}^1 \end{array}$$

If we let T_0 to be a tautological divisor with $\pi_{0*}\mathcal{O}_{W_0}(T_0) \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 3}$, then we have $\tilde{\nu}^*T_0 \sim T$, and

$$\dim H^0(W,\ \mathcal{O}_W(T)) = \dim H^0(C,\ L_0)^{\oplus 3} = 6$$
$$\dim H^0(W_0,\ \mathcal{O}_{W_0}(T_0)) = \dim H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1))^{\oplus 3} = 6,$$

and hence, we get $\Phi_{|T|} = \Phi_{|T_0|} \circ \tilde{\nu}$. Since $\Phi_{|T_0|}$ gives an embedding of W_0 into $\mathbf{P}^5$, we have $\deg \Phi_{|T|} = 2$. q.e.d.

Proposition 4.6 *Let the notation and the assumption be as in Lemma* 4.5*. Then the canonical mapping of a general member of* $|4T - \pi^*D|$ *gives a double covering over a surface of degree* 9 *in* $\mathbf{P}^5$*.*

Proof Since $\mathcal{O}_C(D) \cong \det E \cong \nu^*(\det(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 3}))$, we obtain

$$\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee \cong \tilde{\nu}^*(\mathcal{O}_{W_0}(4T_0) \otimes \pi_0^*\mathcal{O}_{\mathbf{P}^1}(-3)).$$

Since

$$\dim H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee) = \dim H^0(C,\ S^4E \otimes \det E^\vee)$$
$$= \dim H^0(C,\ L_0^{\oplus 15}) = 15 \dim H^0(C,\ L_0) = 30$$

and

$$\dim H^0(W_0,\ \mathcal{O}_{W_0}(4T_0) \otimes \pi_0^*\mathcal{O}_{\mathbf{P}^1}(-3)) = 15 \dim H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1)) = 30,$$

a general member of $|4T - \pi^*D|$ is a pull-back of some member of $|4T_0 - \pi_0^*D_0|$, where $D_0 \in \mathrm{Div}(\mathbf{P}^1)$ is a divisor of degree 3. Hence, the canonical mappings of irreducible and nonsingular members of $|4T - \pi^*D|$ are of degree 2. Since $|T|$ has no base point, we obtain the claim on the degree of the image of S by $\Phi_{|K_S|}$. q.e.d.

Remark The surfaces in Propositon 4.6 can be constructed in another way as follows:

Let $B_1,\ B_2 \subset \mathbf{P}^2$ be nonsingular quartic curves intersecting each other at sixteen points $A_1,\ \cdots,\ A_{16}$ transversally. If B_3 and B_4 are nonsingular members of the pencil generated by B_1 and B_2, then $B_1,\ B_2,\ B_3$ and B_4 intersect one another at $A_1,\ \cdots,\ A_{16}$.

Let $\xi : X \to \mathbf{P}^2$ be the blowing-up at $A_1,\ \cdots,\ A_{16}$, $\tilde{B}_i$ ($i = 1,\ \cdots,\ 16$) the proper transform of B_i, and denote $\mathcal{E}_i := \xi^{-1}(A_i)$. We have $\xi^*B_i = \tilde{B}_i + \sum_{i=1}^{16} \mathcal{E}_i$.

$B := B_1 + B_2 + B_3 + B_4 \sim 16H$ holds, where $H \subset \mathbf{P}^2$ is a hyperplane. Since $\xi^* B = \tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_4 + 4\sum_{i=1}^{16} \mathcal{E}_i$, we have $\tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_4 \sim 16\xi^* H - 4\sum_{i=1}^{16}$. Hence the double covering $S \to X$ branched along $\tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_4$ can be constructed. This S coincides with the surface of Proposition 4.6. Cleariy S is nonsingular by construction.

In the case $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2)$ and $L_1 \cong L_2$, we may assume $L_1 \cong L_0^{\otimes 2}$ by Theorem 4.1.

Lemma 4.7 *Let $L_0,\ L_1,\ L_2$ be invertible sheaves over an elliptic curve C, $\pi : W := \mathbf{P}(E) \to C$ the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E := L_0 \oplus L_1 \oplus L_2$, and T the tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$, and denote $d_i := \deg L_i$ $(i = 0,\ 1,\ 2)$. If $L_0,\ L_1,\ L_2$ satisfy one of the following* (i) *and* (ii)*, then* $\deg \Phi_{|T|} = 2$ *holds:*

(i) $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2)$ *and* $L_0^{\otimes 2} \cong L_1 \cong L_2$.

(ii) $(d_0,\ d_1,\ d_2) = (1,\ 1,\ 2)$.

Proof We use the elementary transformation of W by Maruyama [15, Chapter 1]. First, we consider the case (i). Since we have

$$H^0(W,\ \mathcal{O}_W(T)) \cong H^0(C,\ L_0) \oplus H^0(C,\ L_1) \oplus H^0(C,\ L_2),$$

if we choose $X_0 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1})$ and $X_1,\ X_2 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_1^{-1})$ so that they give homogeneous coordinates for each fiber of π, then any $X \in H^0(W,\ \mathcal{O}_W(T))$ can be written as

$$X = \psi_0 X_0 + \psi_1 X_1 + \psi_2 X_2 \qquad \psi_0 \in H^0(C,\ L_0),\ \psi_1,\ \psi_2 \in H^0(C,\ L_1).$$

Since $\dim H^0(C,\ L_0) = 1$, if we let $q \in W$ to be the point defined by $\psi_0 = X_1 = X_2 = 0$, we have $\mathrm{Bs}\,|T| = \{q\}$. Let $p \in C$ be the point satisfying $L_0 \cong \mathcal{O}_C(p)$, and denote $E' := \mathcal{O}_C \oplus L_1 \oplus L_1$ and $F' := (L_1 \oplus L_1) \otimes \mathcal{O}_p$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & F' & \longrightarrow & E \otimes \mathcal{O}_p & \longrightarrow & \mathcal{O}_p(p) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & \mathcal{O}_p(p) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & E \otimes \mathcal{O}_C(-p) & = & E \otimes \mathcal{O}_C(-p) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the homomorphism $E \to \mathcal{O}_p(p)$ is surjective, we have a subvariety $\mathbf{P}(\mathcal{O}_p(p)) \hookrightarrow W = \mathbf{P}(E)$, which coincides with q. Since the homomorphism $E' \to F'$ is also surjective, if we let $\pi' : W' := \mathbf{P}(E') \to C$ to be the $\mathbf{P}^2$-bundle associated to E', then we have a subvariety $\mathbf{P}(F') \hookrightarrow W'$ contained in a fiber over p with respect to π'. We have the following commutative diagram by Maruyama's result:

$$\begin{array}{ccccc} & & \bar{W} & & \\ & \phi' \swarrow & & \searrow \phi & \\ W' & & & & W \\ & \pi' \searrow & & \swarrow \pi & \\ & & C & & \end{array}$$

where $\phi : \bar{W} \to W$ is the blowing-up at q, and $\phi' : \bar{W} \to W'$ is the blowing-up along $\mathbf{P}(F')$. If we let T' to be the tautological divisor of W' with $\pi_* \mathcal{O}_{W'}(T') \cong E'$, and $\bar{T}$ the proper transform of T by ϕ, then the image of $\bar{T}$ in W' by ϕ' is linearly equivarent to T'. We have the following commutative diagram:

$$\begin{array}{ccccc} \bar{W} & \xrightarrow{\phi'} & W' & \xrightarrow{\Phi} & W_0 \\ \phi \downarrow & & \downarrow \pi' & & \downarrow \pi_0 \\ W & \longrightarrow & C & \longrightarrow & \mathbf{P}^1 \\ & \pi & & \Phi_{|L_1|} & \end{array}$$

where $\pi_0 : W_0 \to \mathbf{P}^1$ is the $\mathbf{P}^2$-bundle associated to a locally free sheaf $E_0 := \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. If T_0 is the tautological divisor of W_0 satisfying $\pi_{0*}\mathcal{O}_{W_0}(T_0) \cong E_0$, then we have $\Phi^* T_0 \sim T'$, and

$$\begin{array}{rclcl} \dim H^0(W',\ \mathcal{O}_{W'}(T')) & = & \dim H^0(C,\ E') & = & 5 \\ \dim H^0(W_0,\ \mathcal{O}_{W_0}(T_0)) & = & \dim H^0(\mathbf{P}^1,\ E_0) & = & 5, \end{array}$$

and hence $\Phi_{|T'|} = \Phi_{|T_0|} \circ \Phi$ holds. We show that $\Phi_{|T_0|}$ is a birational morphism onto the image. If F_0 is any fiber of π_0, we can prove that $\Phi_{|T_0|}|_{F_0}$ is an isomorphism of F_0 onto the image in the same way as in Lemma 4.2. Since $\dim H^0(W_0,\ \mathcal{O}_{W_0}(T_0 - F_0)) = 2$, there exists a section C_0 of π_0 such that $\mathrm{Bs}\,|T_0 - F_0| = C_0$. Let $p,\ q \in W_0 \setminus C_0$ be any two points contained in different fibers. If $T_0' \in |T_0 - F_0|$ is a member which does not contain q, and if F_p is the fiber of π_0 containing p, then $T_0' + F_0 \in |T_0|$ contains p but not q, i.e., p and q are separated by $|T_0|$. Hence $\Phi_{|T_0|}$ is a birational morphism. Therefore we have $\deg \Phi_{|T'|} = \deg \Phi = 2$, and $\deg \Phi_{|T|} = 2$ holds.

Next, we consider the case (ii).

Assume $L_0 \not\cong L_1$. As in the case (i), any $X \in H^0(W,\ \mathcal{O}_W(T))$ can be written as

$$X = \psi_0 X_0 + \psi_1 X_1 + \psi_2 X_2 \qquad \psi_i \in H^0(C,\ L_i)\ (i = 0,\ 1,\ 2).$$

Since $\dim H^0(C,\ L_0) = \dim H^0(C,\ L_1) = 1$ and $L_0 \not\cong L_1$, if we let $q_0,\ q_1 \in W$ to be the points defined by $\psi_0 = X_1 = X_2 = 0$ and $\psi_1 = X_2 = X_0 = 0$, respectively, we have $\mathrm{Bs}\,|T| = \{q_0,\ q_1\}$. If we let $\pi' : W' := \mathbf{P}(E') \to C$ to be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E' := \mathcal{O}_C \oplus \mathcal{O}_C \oplus L_2$ over C, T' the tautological divisor with $\pi'_*\mathcal{O}_{W'}(T) \cong E'$, $\phi : \bar{W} \to W$ the blowing-up at q_0 and q_1, and $\pi_0 : W_0 := \mathbf{P}(E_0) \to \mathbf{P}^1$ the $\mathbf{P}^2$-bundle associated to a locally free sheaf $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ over $\mathbf{P}^1$, then we obtain a commutative diagram similar to that in the case (i). Therefore, there exists a blowing down $\phi' : \bar{W} \to W'$ contracting the proper transform of the fibers of π containing q_0 and q_1 to lines. The image of the proper transform of T by ϕ in W' is linearly equivalent to T'. If Φ is as in (i), and T_0 is the tautological divisor of W_0 satisfying $\pi_{0*}\mathcal{O}_{W_0}(T_0) \cong E_0$, then we have $\Phi^*T_0 \sim T'$ and

$$\begin{array}{rcccl} \dim H^0(W_0,\ \mathcal{O}_{W_0}(T_0)) & = & \dim H^0(\mathbf{P}^1,\ E_0) & = & 4 \\ \dim H^0(W',\ \mathcal{O}_{W'}(T')) & = & \dim H^0(C,\ E') & = & 4, \end{array}$$

and hence $\Phi_{|T'|} = \Phi_{|T_0|} \circ \Phi$ holds. We can prove that $\Phi_{|T_0|}$ is a birational morphism onto the image in the same way as in the case $E_0 := \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$, so we have $\deg \Phi_{|T'|} = \deg \Phi = 2$. Consequently, $\Phi_{|T|} = 2$ holds.

Assume $L_0 \cong L_1$. Since $\pi(q_0) = \pi(q_1)$ in this case, if we let p_0 be this point, we have $L_0 \cong L_1 \cong \mathcal{O}_C(p_0)$. If $Z \subset W$ is the curve defined by $\psi_0 = X_2 = 0$, then $\mathrm{Bs}\,|T| = Z$ holds. We obtain the same commutative diagram as above, and in this case, $\phi : \bar{W} \to W$ is the blowing-up along Z. We can show that $\deg \Phi_{|T|} = 2$ by the same argument as in the case $L_0 \not\cong L_1$ of (ii). q.e.d.

When $(d_0,\ d_1,\ d_2) = (1,\ 2,\ 2)$ and $L_1 \cong L_2$ hold, if we denote $\bar{Y} := \phi^{-1}(q)$, then we have $\phi^*T \sim \bar{T} + \bar{Y}$. Let $\bar{S}$ be the proper transform of S by ϕ. Since S is nonsingular at q, we have $\phi^*S = \bar{S} + \bar{Y}$. On the other hand, since $\phi^*S \sim \phi^*(4T - \pi^*D) \sim 4\bar{T} + 4\bar{Y} - \phi^*\pi^*D$, we obtain $\bar{S} \sim 4\bar{T} + 3\bar{Y} - \phi^*\pi^*D$. The image of $\bar{Y}$ in W' is a fiber Y of π'. Then the image S' of $\bar{S}$ in W' is linearly equivalent to $4T' + 3Y - \pi'^*D$, which is linearly equivalent to $4T' - 2\pi'^*p$, since $L_0^{\otimes 2} \cong L_1$ and $\pi'(Y) = p$.

When $(d_0,\ d_1,\ d_2) = (1,\ 1,\ 2)$ and $L_0 \not\cong L_1$ hold, if we denote $\bar{Y}_i := \phi^{-1}(q_i)$ $(i = 0,\ 1)$, then we have $\phi^*T \sim \bar{T} + Y_0 + Y_1$. Hence we obtain $\bar{S} \sim 4T' - 2\pi^*p_0$ or $\bar{S} \sim 4T' - 2\pi^*p_1$ by an argument similar to that above. (In this case, one of $L_0^{\otimes 2}$ and $L_1^{\otimes 2}$ is isomorphic to L_2 by Theorem 4.1. If $L_0^{\otimes 2} \cong L_2$, then $\bar{S} \sim 4T' - 2\pi'^*p_0$ holds, while if $L_1^{\otimes 2} \cong L_2$, then $\bar{S} \sim 4T' - 2\pi^*p_1$ holds.) We obtain the same result when $(d_0,\ d_1,\ d_2) = (1,\ 1,\ 2)$ and $L_0 \cong L_1$.

Proposition 4.8 *Let the notation and the assumption be as in Lemma 4.7. Then the minimal resolution of a general member of the complete linear system $|4T - \pi^*D|$ is canonical.*

Proof We use the notation of Lemma 4.7.

First, we consider the case $E \cong L_0 \oplus L_1 \oplus L_1$, $(L_0 \in \mathcal{E}_C(1,\ 1),\ L_1 \cong L_0^{\otimes 2})$. Let $S \in |4T - \pi^*D|$ be a general member. We may assume S to be nonsingular. Furthermore, let $\bar{S}$ be a proper transform of S by ϕ, and denote $S' := \phi'(\bar{S})$, $S_0 := \Phi(S') \subset W_0$.

Furthermore, denote $h' := \Phi_{|S'} : S' \to S_0$. Then we have $\deg h' = \deg \Phi_{|K_S|}$. There is nothing to prove if h' is birational onto its image. Thus suppose h' is not birational. Hence h' is an unramified two-to-one covering outside the intersection with S' of the ramification locus $F_0 + F_1 + F_2 + F_3$ of Φ, where $F_i := \pi'^{-1}(p_i)$ $(i = 0,\ 1,\ 2,\ 3)$ with $p_0,\ p_1,\ p_2,\ p_3 \in C$ the ramification points of $\Phi_{|L_1|} : C \to \mathbf{P}^1$. Hence the morphism

$$h'_{|S' \setminus \bigcup_{i=0}^{3} F'_i} : S' \setminus \bigcup_{i=0}^{3} F'_i \to \tilde{S}_0 := h'(S' \setminus \bigcup_{i=0}^{3} F'_i)\ (\subset S_0)$$

is an unramified two-to-one covering. On the elementary transform S of S', we thus have an unramified morphism

$$h : S \setminus \cup_{i=0}^{3} F_i \to \tilde{S}_0$$

where $F_i := \pi^{-1}(p_i) \subset$ $(i = 0,\ 1,\ 2,\ 3)$. Let $C_0 \subset W$ be a curve which is the base locus of $\mathcal{O}_W(T) \otimes \pi^* L_1^{-1}$. S is a canonical surface if h is one-to-one at a point $q \in S \setminus (C_0 \cup_{i=0}^{3} F_i)$. Fix one such point q and let $q' \in W$ be the other point which is mapped to $\Phi_{|T|}(q)$ by the two-to-one map $\Phi_{|T|}$. We show that q' does not belong to a general S.

Since

$$H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1}) \cong H^0(C,\ \mathcal{O}_C) \oplus H^0(C,\ L_0) \oplus H^0(C,\ L_0),$$

we obtain a global section $X_0 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_0^{-1})$ such that X_0 vanishes at q and q' and that the divisor (X_0) is irreducible. Similarly, since

$$H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_1^{-1}) \cong H^0(C,\ \mathcal{O}_C) \oplus H^0(C,\ \mathcal{O}_C),$$

we obtain a global section $X_1 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_1^{-1})$ such that X_1 vanishies at q and q' and that the divisor (X_1) is irreducible. Any $X_2 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L_1^{-1}) \setminus \{0,\ X_1\}$ does not vanish at q and q', and the divisor (X_2) is irreducible. $X_0,\ X_1,\ X_2$ give homogeneous coordinates of each fiber of π. Since

$$H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee) \cong H^0(C,\ S^4 E \otimes \det E^\vee) \cong \bigoplus_{\substack{i,\ j \geq 0 \\ i+j \leq 4}} H^0(C,\ L_0^{\otimes(i+j-1)}),$$

if $\Psi \in H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ is a global section defining S, then Ψ can be written as

$$\Psi = \sum_{\substack{i,\ j \geq 0 \\ i+j \leq 4}} \psi_{ij} X_0^{4-i-j} X_1^i X_2^j, \qquad \psi_{ij} \in H^0(C,\ L_0^{\otimes(i+j-1)}).$$

We have $\Psi(q') = \psi_{04}(q') X_2(q')^4$ by our choice of $X_0,\ X_1,\ X_2$. Hence q' is not contained in S if and only if $\psi_{04}(q') \neq 0$ holds. Since S is general, we are done.

Next, we consider the case $E \cong L_0 \oplus L_1 \oplus L_2$, $(L_0,\ L_1 \in \mathcal{E}_C(1,\ 1),\ L_2 \in \mathcal{E}_C(1,\ 2))$. In this case, at least one of $L_0^{\otimes 3} \otimes L_1^{-1},\ L_0^{\otimes 2},\ L_0 \otimes L_1,\ L_1^{\otimes 2}$ and $L_0^{-1} \otimes L_1^{\otimes 3}$ is isomorphic to L_2.

Let $S \in |4T - \pi^*D|$ be a general member. We may assume S to be nonsingular. Furthermore, let $S' \subset W'$, $S_0 \subset W_0$ and $h' := \Phi_{|S'} : S' \to S_0$ be as above.

If $L_0^{\otimes 3} \otimes L_1^{-1},\ L_0^{-1} \otimes L_1^{\otimes 3} \not\cong L_2$, then $\mathrm{Bs}\,|4T - \pi^*D|$ consists of two points $\{q_0,\ q_1\}$, and these are contained in $\mathrm{Bs}\,|4T - \pi^*D|$. Denote $\mathcal{E}_i := \phi^{-1}(q_i)$ $(i = 0,\ 1)$, and let $\bar{T},\ \bar{S}$ be the proper transform of $T,\ S$ by ϕ, respectively. Then since

$$\bar{S} \sim 4\bar{T} + 3\mathcal{E}_0 + 3\mathcal{E}_1 - \phi^*\pi^*D,$$

we obtain

$$\begin{aligned}\mathcal{O}_{W'}(S') &\cong \mathcal{O}_{W'}(4T') \otimes \pi'^*(L_0^{\otimes 3} \otimes L_1^{\otimes 3} \otimes L_0^{-1} \otimes L_1^{-1} \otimes L_2^{-1}) \\ &\cong \mathcal{O}_{W'}(4T') \otimes \pi'^*(L_0^{\otimes 2} \otimes L_1^{\otimes 2} \otimes L_2^{-1}).\end{aligned}$$

We have $\deg(L_0^{\otimes 2} \otimes L_1^{\otimes 2} \otimes L_2^{-1}) = 2$. If $L_0^{\otimes 2} \otimes L_1^{\otimes 2} \otimes L_2^{-1} \not\cong L_2$ holds, this invertible sheaf cannot be the pull-back by Φ of any invertible sheaf on W_0, and hence S is canonical. We consider the case $L_0^{\otimes 2} \otimes L_1^{\otimes 2} \otimes L_2^{-1} \cong L_2$. Let $p_0,\ p_1,\ p_2,\ p_3 \in C$ be the ramification points of $\Phi_{|L_2|} : C \to \mathbf{P}^1$, and denote $F_i' := \pi'^{-1}(p_i)$ $(i = 0,\ 1,\ 2,\ 3)$. There is nothing to prove if h' is birational onto its image. Thus suppose h' is not birational. Hence h' is an unramified two-to-one covering outside the intersection with S' of the ramification locus $F_0' + F_1' + F_2' + F_3'$ of Φ. Hence, the morphism

$$h'_{S' \setminus \bigcup_{i=0}^{3} F_i'} : S' \setminus \bigcup_{i=0}^{3} F_i' \to \tilde{S}_0 := h'(S' \setminus \bigcup_{i=0}^{3} F_i')\ (\subset S_0)$$

is an unramified two-to-one covering. On the elementary transform S of S', we thus have an unramified morphism

$$h : S \setminus \bigcup_{i=0}^{3} F_i \to \tilde{S}_0,$$

where $F_i := \pi^{-1}(p_i)$ $(i = 0,\ 1,\ 2,\ 3)$. Let $X_2 \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^*L_2^{-1}) \cong \mathbf{C}$ be a non-zero element, and fix any point $q \in S \setminus \left((X_2) \cup \left(\bigcup_{i=0}^{3} F_i\right)\right)$. Let $q' \in W$ be the other point which is mapped to $\Phi_{|T|}(q)$ by the two-to-one map $\Phi_{|T|}$. Since we have $\dim H^0(W,\ \mathcal{O}_W(T) \otimes \pi^*L_i^{-1}) = 2$ for $i = 0,\ 1$, there exist $X_i \in H^0(W,\ \mathcal{O}_W(T) \otimes \pi^*L_i^{-1})$ $(i = 0,\ 1)$ such that the divisors (X_i) are irreducible and that X_i vanish at q and q'. Then $X_0,\ X_1$ and X_2 give homogeneous coordinates of each fiber of π. If $\Psi \in H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ is a global section defining S, then Ψ can be written as

$$\Psi = \sum_{\substack{i,\, j \geq 0 \\ i+j \leq 4}} \psi_{ij} X_0^{4-i-j} X_1^i X_2^j, \qquad \psi_{ij} \in H^0(C,\ L_0^{\otimes(3-i-j)} \otimes L_1^{\otimes(i-1)} \otimes L_2^{\otimes(j-1)}).$$

We have $\Psi(q') = \psi_{04}(q')X_2(q')^4$ by our choice of $X_0,\ X_1,\ X_2$. Hence q' is not contained in S if and only if $\psi_{04}(q') \neq 0$. Since S is general, we are done.

In the case $L_0^{\otimes 3} \otimes L_1^{-1} \cong L_2$ and $L_0^{-1} \otimes L_1^{\otimes 3} \not\cong L_2$, $\mathrm{Bs}\,|T|$ consists of two points $q_0,\ q_1$. One of them is contained in $\mathrm{Bs}\,|4T - \pi^*D|$, while the other is not. We may assume $q_1 \in \mathrm{Bs}\,|4T - \pi^*D|$. In the same notation as above, since

$$\bar{S} \sim 4\bar{T} + 4\mathcal{E}_0 + 3\mathcal{E}_1 - \phi^*\pi^*D,$$

we have

$$\mathcal{O}_{W'}(S') \cong \mathcal{O}_{W'}(4T') \otimes \pi'^*(L_0^{\otimes 3} \otimes L_1^{\otimes 2} \otimes L_2^{-1}).$$

Since $\deg(L_0^{\otimes 3} \otimes L_1^{\otimes 2} \otimes L_2^{-1}) = 3$, this cannot be the pull-back by Φ of any invertible sheaf on W_0. We can obtain the same result in the case $L_0^{\otimes 3} \otimes L_1^{-1} \not\cong L_2$ and $L_0^{-1} \otimes L_1^{\otimes 3} \cong L_2$.

Finally, we consider the case $L_0^{\otimes 3} \otimes L_1^{-1} \cong L_0^{-1} \otimes L_1^{\otimes 3} \cong L_2$. If $L_0 \not\cong L_1$ holds, then $\mathrm{Bs}\,|T|$ consists of two points. Since $\mathrm{Bs}\,|4T - \pi^*D| = \emptyset$, and since S is generic, S does not contain these two points. Hence, in the same notation as above, we have

$$\mathcal{O}_{W'}(S') \cong \mathcal{O}_{W'}(4T') \otimes \pi'^*(L_0^{\otimes 3} \otimes L_1^{\otimes 3} \otimes L_2^{-1}).$$

$\deg(L_0^{\otimes 3} \otimes L_1^{\otimes 3} \otimes L_2^{-1}) = 4$ holds, and if $L_0^{\otimes 3} \otimes L_1^{\otimes 3} \otimes L_2^{-1} \not\cong L_2$ holds, then the above invertible sheaf on W' cannot be the pull-back by Φ of any invertible sheaf on W_0, and hence S is a canonical surface. When $L_0^{\otimes 3} \otimes L_1^{\otimes 3} \otimes L_2^{-1} \cong L_2$, we can prove S to be canonical in the same way as in the case $L_0^{\otimes 3} \otimes L_1^{-1} \not\cong L_2$, $L_0^{-1} \otimes L_1^{\otimes 3} \not\cong L_2$ and $L_0^{\otimes 2} \otimes L_1^{\otimes 2} \otimes L_2^{-1} \cong L_2$. When $L_0 \cong L_1$, we can prove S to be canonical in the same way as above. q.e.d.

Remark In the situation of Proposition 4.8, we have a posibility that there exist special members, with at most rational double points as singularities, of $|4T - \pi^*D|$ whose canonical mapping is of degree 2.

Proposition 4.9 *Let L_0, L_1 and L_2 be invertible sheaves over an elliptic curve C, and denote $d_i := \deg L_i$ ($i = 0,\ 1,\ 2$). Assume that $d_0 = d_1 = d_2 = 1$ holds and one of L_0, L_1 and L_2 is not isomorphic to any of the others. Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E := L_0 \oplus L_1 \oplus L_2$, T a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, $D \in \mathrm{Div}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det E$, and $S \in |4T - \pi^*D|$ a general irreducible nonsingular member. We have the following about $\Phi_{|K_S|}$:*

(1) *If $L_0^{\otimes 2} \not\cong L_1 \otimes L_2$, $L_1^{\otimes 2} \not\cong L_2 \otimes L_0$ and $L_2^{\otimes 2} \not\cong L_0 \otimes L_1$, then $\Phi_{|K_S|}$ gives a covering of degree 9 onto $\mathbf{P}^2$.*

(2) *If only one of $L_0^{\otimes 2} \cong L_1 \otimes L_2$, $L_1^{\otimes 2} \cong L_2 \otimes L_0$ and $L_2^{\otimes 2} \cong L_0 \otimes L_1$ holds, then $|K_S|$ has one isolated base point, and $\Phi_{|K_S|}$ gives a covering of degree 8 over $\mathbf{P}^2$.*

(3) *If all of $L_0^{\otimes 2} \cong L_1 \otimes L_2$, $L_1^{\otimes 2} \cong L_2 \otimes L_0$ and $L_2^{\otimes 2} \cong L_0 \otimes L_1$ hold, then $|K_S|$ has three isolated fixed points, and $\Phi_{|K_S|}$ gives a covering of degree 6 over $\mathbf{P}^2$.*

Proof We investigate the sets of base points of $|T|$ and $|4T - \pi^*D|$.

First we assume that L_0, L_1, L_2 are pairwise non-isomorphic. Since any $X \in H^0(W,\ \mathcal{O}(T))$ can be written as

$$X = \psi_0 X_0 + \psi_1 X_1 + \psi_2 X_2, \qquad \psi_i \in H^0(C,\ L_i)$$

as in the proof of Lemma 4.2, we have $\mathrm{Bs}\,|T| = \{q_0,\ q_1,\ q_2\}$, where $q_0,\ q_1$ and q_2 are the points defined by $\psi_0 = X_1 = X_2 = 0,\ \psi_1 = X_2 = X_0 = 0$ and $\psi_2 = X_0 = X_1 = 0$, respectively. Since any $\Psi \in H^0(W,\ \mathcal{O}(4T) \otimes \pi^* \det E^\vee)$ can be written as

$$\Psi = \sum_{\substack{i,\, j \geq 0 \\ i+j \leq 4}} \psi_{ij} X_0^{4-i-j} X_1^i X_2^j, \qquad \psi_{ij} \in H^0(C,\ L_0^{\otimes(3-i-j)} \otimes L_1^{\otimes(i-1)} \otimes L_2^{\otimes(j-1)})$$

as in the proof of Theorem 4.1, we have $\mathrm{Bs}\,|4T - \pi^* D| = \{q_0',\ q_1',\ q_2'\}$, where $q_0',\ q_1'$ and q_2' are the points defined by $\psi_{00} = X_1 = X_2 = 0,\ \psi_{40} = X_2 = X_0 = 0$ and $\psi_{04} = X_0 = X_1 = 0$, respectively.

Therefore in the case (1), we have $\mathrm{Bs}\,|T| \cap \mathrm{Bs}\,|4T - \pi^* D| = \emptyset$, and hence $\Phi_{|K_S|}$ is a surjective morphism onto $\mathbf{P}^2$. Since $K_S^2 = 9$ and the degree of $\mathbf{P}^2$ is equal to 1, we are done in the case (1).

Next, we consider the case (2). We only have to consider the case $L_0^{\otimes 2} \cong L_1 \otimes L_2$ by renumbering of $L_0,\ L_1$ and L_2 if necessary. In this case, all the members of $|4T - \pi^* D|$ go through q_0. Since $S \in |4T - \pi^* D|$ is general, it does not contain q_1 and q_2.

If we denote $E' := \mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C$, then we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & F' & \longrightarrow & E|_{p_0+p_1+p_2} & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & E \otimes \mathcal{O}_C(-p_0 - p_1 - p_2) & = & E \otimes \mathcal{O}_C(-p_0 - p_1 - p_2) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $F' := \mathcal{O}_{p_1+p_2} \oplus \mathcal{O}_{p_2+p_0} \oplus \mathcal{O}_{p_0+p_1}$ and $F'' := \mathcal{O}_{p_0}(p_0) \oplus \mathcal{O}_{p_1}(p_1) \oplus \mathcal{O}_{p_2}(p_2)$.

Hence we have the following elementary transformation of Maruyama:

$$\begin{array}{ccccc}
 & & \bar{W} & & \\
 & \phi' \swarrow & & \searrow \phi & \\
W' & & & & W \\
 & \pi' \searrow & & \swarrow \pi & \\
 & & C & &
\end{array}$$

where $\pi' : W' := \mathbf{P}(E') \to C$ is the $\mathbf{P}^2$-bundle associated to E', ϕ is the blowing-up at $\mathbf{P}(\mathcal{O}_{p_0}(p_0) \oplus \mathcal{O}_{p_1}(p_1) \oplus \mathcal{O}_{p_2}(p_2)) = \mathbf{P}(\mathcal{O}_{p_0}(p_0)) \cup \mathbf{P}(\mathcal{O}_{p_1}(p_1)) \cup \mathbf{P}(\mathcal{O}_{p_2}(p_2)) = \{\ q_0,\ q_1,\ q_2\ \}$ and ϕ' is the blowing-up along $\mathbf{P}(\mathcal{O}_{p_1+p_2} \oplus \mathcal{O}_{p_2+p_0} \oplus \mathcal{O}_{p_0+p_1}) = \mathbf{P}(\mathcal{O}_{p_1+p_2}) \cup \mathbf{P}(\mathcal{O}_{p_2+p_0}) \cup \mathbf{P}(\mathcal{O}_{p_0+p_1})$. Denote $\mathcal{E}_i := \phi^{-1}(q_i)$ and $F_i' := \phi'(\mathcal{E}_i)$ $(i = 0,\ 1,\ 2)$. If $\bar{S} \subset \bar{W}$ is the proper transform of S by ϕ, then $\phi^* S = \bar{S} + \mathcal{E}_0$ holds. Let $\bar{T}$ be the proper transform of T by ϕ

and $T' \subset W'$ the tautological divisor with $\pi'_* \mathcal{O}_{W'}(T') \cong E'$. We have $T' \sim \phi(\bar{T})$. Since $\phi^* T = \bar{T} + \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2$, we have

$$S' := \phi'(\bar{S}) \sim 4T' + 4(F'_0 + F'_1 + F'_2) - \pi'^* D - F_0 \sim 4T' + 2F'_0 + 3F'_1 + 3F'_2.$$

On the other hand, $W' \cong C \times \mathbf{P}^2$ holds, and $\Phi_{|T'|}$ coincides with the second projection $W' \to \mathbf{P}^2$. Since $\Phi_{|T|}$ factors as a rational mapping into a composite $\Phi_{|T|} : W \cdots \to W' \to \mathbf{P}^2$, we have

$$\deg \Phi_{|K_S|} = \deg \left(\Phi_{|T'|}|_{S'} \right) = (T')^2 (4T' + 2F'_0 + 3F'_1 + 3F'_2) = 8.$$

We obtain the result in the case (3) in the same way as above.

The proof is essentially the same when $L_0 \not\cong L_1 \cong L_2$, $L_1 \not\cong L_2 \cong L_0$, or $L_2 \not\cong L_0 \cong L_1$. q.e.d.

4.2 E is the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2

We denote $E = E_0 \oplus L$, where E_0 is an indecomposable locally free sheaf of rank 2 with $\deg E_0 =: e$, and L is an invertible sheaf over an elliptic curve C with $\deg L =: d$.

We prove the existence of a surface S with $K_S^2 = 3p_g(S)$, $q(S) = 1$ and $p_g(S) = N$ for any integer $N \geq 2$ in §4.2.1 (Theorem 4.10) when e is even, and in §4.2.2 (Theorem 4.11) when e is odd. (When e is even, however, the case $p_g(S) = 2$ does not occur.) In §4.2.3, we study the canonical mapping of the surfaces obtained in §4.2.1 and §4.2.2. The results about the canonical mappings are stated in Corollary 4.15, and Propositions 4.16, 4.20, 4.22 and 4.41.

We only have to consider the case $e \geq 0$, $d \geq 0$ and $(e,\ d) \neq (0,\ 0)$ by the remark immedietely before §4.1.

Let $\pi : W := \mathbf{P}^2(E) \to C$ be the $\mathbf{P}^2$-bundle associated to E, and $T \in \mathrm{Div}(W)$ a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$. We have a section $C_1 := \mathbf{P}(E/E_0) \subset W$ of π. If $\rho : X \to W$ is the blowing-up along C_1, then X is a $\mathbf{P}^1$-bundle $\sigma : X \to Y := \mathbf{P}(E_0)$. Let $\mu : Y \to C$ be the ruling, and denote $Y_1 := \rho^* T$ and $Y_\infty := \rho^{-1}(C_1)$. If $C_0 \in \mathrm{Div}(Y)$ is a tautological divisor with $\mu_* \mathcal{O}_Y(C_0) \cong E_0$, then we have $Y_1 \sim Y_\infty + \sigma^* C_0$, and $\sigma_* \mathcal{O}_X(Y_1) \cong \mathcal{O}_Y(C_0) \oplus \mu^* L$. Let $Y_0 \in \mathrm{Div}(X)$ be a divisor with $\mathcal{O}_X(Y_0) \cong \mathcal{O}_X(Y_1) \otimes \sigma^* \mu^* L^{-1}$, and let $Z_0 \in H^0(X,\ \mathcal{O}_X(Y_0))$, $Z_\infty \in H^0(X,\ \mathcal{O}_X(Y_\infty))$ be global sections with $(Z_0) = Y_0$ and $(Z_\infty) = Y_\infty$. Then Z_0 and Z_∞ give homogeneous coordinates of each fiber of the $\mathbf{P}^1$-bundle σ.

We study the complete linear system of the invertible sheaf $\mathcal{O}_X(4Y_1) \otimes \sigma^* \mu^* \det E^\vee \cong \rho^*(\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ over X. Since we have

$$H^0(X,\ \mathcal{O}_X(4Y_1) \otimes \sigma^* \mu^* \det E^\vee)$$

$$
\begin{aligned}
&\cong H^0(Y,\ S^4(\mathcal{O}_Y(C_0)\oplus\mu^*L)\otimes\mu^*\det E^\vee)\\
&\cong \bigoplus_{j=0}^{4} H^0(Y,\ \mathcal{O}_Y(jC_0)\otimes\mu^*(L^{\otimes(4-j)}\otimes\det E^\vee))\\
&\cong \bigoplus_{j=0}^{4} H^0(C,\ S^j(E_0)\otimes L^{\otimes(4-j)}\otimes\det E^\vee),
\end{aligned}
$$

any $\Psi\in H^0(X,\ \mathcal{O}_X(4Y_1)\otimes\sigma^*\mu^*\det E^\vee)$ can be written as

$$
\Psi=\sum_{j=0}^{4}\psi_j Z_0^{4-j}Z_\infty^j,\qquad \psi_j\in H^0(Y,\ \mathcal{O}_Y(jC_0)\otimes\mu^*(L^{\otimes(4-j)}\otimes\det E^\vee)),\ (j=0,\ \cdots,\ 4).
$$

4.2.1 Existence in the case where e is even

Denote $e=2e_0$. There exist invertible sheaves $L_0\in\mathcal{E}_C(1,\ e_0)$, and $L_1\in\mathcal{E}_C(1,\ d-e_0)$, with $E_0\cong L_0\otimes F_2$ and $L\cong L_0\otimes L_1$, hence we have $E\cong L_0\otimes(F_2\oplus L_1)$.

Theorem 4.10 *Let the conditions and notation be as above. Then the complete linear system $|4T-\pi^*D|$ over W satisfies the condition* (A) *if and only if one of the following* (1), (2) *and* (3) *holds:*

(1) $e=d>0$ *and* $L_0\cong L_1$,

(2) $d<e<4d$,

(3) $e=4d>0$ *and* $L_0\otimes L_1^{\otimes 2}\cong\mathcal{O}_C$.

By the remark after Proposition 3.3, the case $e<0$ and the case $d<0$ may be excluded. Furthermore, the case $E_0\cong F_2$ and the case $L\cong\mathcal{O}_C$ may also be excluded.

If $e=d=0$ and $E_0\not\cong F_2,\ L\not\cong\mathcal{O}_C$ hold, then $H^0(W,\ \mathcal{O}_W(T))\cong H^0(C,\ E)=0$. Since $\omega_{S'}\cong\mathcal{O}_W(T)\otimes_{\mathcal{O}_W}\mathcal{O}_{S'}$ for $S'\in|4T-\pi^*D|$ by the adjunction formula, the minimal resolution of S' cannot be of general type, and this case may be excluded, too.

Therefore, we have $e>0$ and $d>0$.

Since we have $S^j(F_2)\cong F_{j+1}$ (cf. Theorem 3.4), we have

$$
\begin{aligned}
&S^j(E_0)\otimes L^{\otimes(4-j)}\otimes\det E^\vee\\
&\quad\cong\ S^j(F_2)\otimes L_0^{\otimes j}\otimes L_0^{\otimes(4-j)}\otimes L_1^{\otimes(4-j)}\otimes(L_0^{-1})^{\otimes 3}\otimes L_1^{-1}\\
&\quad\cong\ F_{j+1}\otimes L_0\otimes L_1^{\otimes(3-j)},\qquad (j=0,\ \cdots,\ 4).
\end{aligned}
$$

Furthermore, since $\det F_2\cong\mathcal{O}_C$ holds, we have $\det E\cong L_0^{\otimes 3}\otimes L_1$. Hence

$$
H^0(Y,\ \mathcal{O}_Y(jC_0)\otimes\mu^*(L^{\otimes(4-j)}\otimes\det E^\vee))\cong H^0(C,\ F_{j+1}\otimes L_0\otimes L_1^{\otimes(3-j)}).
$$

From now on, we deal with different cases.

(i) The case where ($e = d$ and $L_0 \not\cong L_1$), or ($e < d$). Since we have

$$H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee) \cong H^0(C,\ F_5 \otimes L_0 \otimes L_1^{-1}) = 0,$$

the coefficient ψ_4 of Z_∞^4 in Ψ is always 0, hence the divisor (Ψ) has Y_0 as a component. Therefore, the image of (Ψ) in W by ρ is not irreducible.

(ii) The case where ($e = d > 0$ and $L_0 \cong L_1$), ($d < e < 3d$), or ($e = 3d$ and $L_0 \otimes L_1^{\otimes 3} \cong \mathcal{O}_C$). For general $\Psi \in H^0(X,\ \mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee)$, we may assume $\psi_j \neq 0$, ($j = 0, \cdots, 4$), hence if the complete linear systems of the invertible sheaves $\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee$ on Y and $L^{\otimes 4} \otimes \det E^\vee$ on C do not have base points, then $|4Y_1 - \sigma^*\mu^* D|$ does not have base points either.

Let us look at $L^{\otimes 4} \otimes \det E^\vee$. Since we have $\deg(L^{\otimes 4} \otimes \det E^\vee) = 3d - e \geq 0$, it does not have base points when $3d - e \neq 1$.

If $3d - e = 1$ holds, then there exists a unique point $q \in C$ with $L^{\otimes 4} \otimes \det E^\vee \cong \mathcal{O}_C(q)$. If we denote $\Gamma := Y_\infty \cap (\mu \circ \sigma)^{-1}(q)$, then Γ is contained in the base locus of $|4Y_1 - \sigma^*\mu^* D|$ on X. We claim that Γ is a (-1)-curve on $S'' := (\Psi)$. Indeed, it is clear that Γ is a nonsingular rational curve. Hence it is sufficient to show that the self-intersection number of Γ in S'' is equal to -1. Let $D' \in \mathrm{Div}(C)$ be a divisor on C with $L \cong \mathcal{O}_C(D')$. Then we have

$$\begin{aligned}
(Y_\infty|_{S''})^2 &= Y_\infty^2(4Y_1 - \sigma^*\mu^* D) \ = \ Y_\infty^2(4Y_0 + \sigma^*\mu^*(4D' - D)) \\
&= Y_\infty^2 \sigma^*\mu^*(4D' - D) \\
&= Y_\infty(Y_0 + \sigma^*(\mu^* D' - C_0))\sigma^*\mu^*(4D' - D) \\
&= Y_\infty \sigma^*((\mu^* D' - C_0)(\mu^*(4D' - D)) \\
&= C_0 \mu^*(D - 4D') \ = \ e - 3d \ = \ -1.
\end{aligned}$$

Therefore, the image of Γ in W is a nonsingular point of $S' = \rho(S'')$.

Let us look at $\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee$. Since we have

$$H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^* L_0^{-1}) \cong H^0(C,\ E_0 \otimes L_0^{-1}) \cong H^0(C,\ F_2) \cong \mathbf{C},$$

there exists a section C' of μ with $\mathcal{O}_Y(C') \cong \mathcal{O}_Y(C_0) \otimes \mu^* L_0^{-1}$ on Y. Hence we have

$$H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee) \cong H^0(Y,\ \mathcal{O}_Y(4C') \otimes L_0 \otimes L_1^{-1}).$$

First we consider the case where $e - d \geq 2$. Since $\deg(L_0 \otimes L_1^{-1}) = e - d$ holds, there does not exist a base point in $Y \setminus C'$. We consider the cohomology long exact sequence induced by the exact sequence of sheaves

$$\begin{aligned}
&0 \to \mathcal{O}_Y(3C') \otimes \mu^*(L_0 \otimes L_1^{-1})) \to \mathcal{O}_Y(4C') \otimes \mu^*(L_0 \otimes L_1^{-1}) \\
&\quad \to \mathcal{O}_{C'} \otimes \mathcal{O}_Y(4C') \otimes \mu^*(L_0 \otimes L_1^{-1}) \to 0.
\end{aligned}$$

Since we have

$$\begin{aligned}
&H^1(Y,\ \mathcal{O}_Y(3C') \otimes \mu^*(L_0 \otimes L_1^{-1})) \cong H^1(C,\ S^3(F_2) \otimes L_0 \otimes L_1^{-1}) \\
&\quad \cong H^1(C,\ F_4 \otimes L_0 \otimes L_1^{-1}) \cong 0,
\end{aligned}$$

the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(4C')\otimes\mu^*(L_0\otimes L_1^{-1}))\to H^0(C',\ \mathcal{O}_{C'}\otimes\mathcal{O}_Y(4C')\otimes\mu^*(L_0\otimes L_1^{-1}))$$

is surjective. On the other hand, since we have $(C')^2=0$, the degree of the restriction of $\mathcal{O}_Y(4C')\otimes\mu^*(L_0\otimes L_1^{-1})$ to C' is $e-d\geq 2$, hence there does not exist a base point on C' either.

When $e-d=1$ holds, there exists a unique point $p\in C$ with $L_0\otimes L_1^{-1}\cong\mathcal{O}_C(p)$. If we denote $\Gamma_0:=\mu^{-1}(p)\subset Y$, then $\mathcal{O}_Y(4C_0)\otimes\det E^\vee\cong\mathcal{O}_Y(4C'+\Gamma_0)$. Hence, a general member of $|4C_0-\mu^*D|$ is nonsingular by Lemma 3.5. Thus a general member of the complete linear system $|4Y_1-\sigma^*\mu^*D|$ on X is also irreducible and nonsingular.

If $e-d=0$ holds, since we have $L_0\cong L_1$ by assumption, $\mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee\cong\mathcal{O}_Y(4C')$ holds. Hence $H^0(Y,\ \mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee)\cong H^0(Y,\ \mathcal{O}_Y(4C'))\cong H^0(C,\ F_5)\cong\mathbf{C}$. Thus $C'':=\sigma^{-1}(C')\cap Y_0$ is the base locus of $|4Y_1-\sigma^*\mu^*D|$. We look at the coefficient $\psi_3\in H^0(Y,\ \mathcal{O}_Y(3C_0)\otimes\mu^*(L\otimes\det E^\vee))$ of $Z_0Z_\infty^3$ in Ψ. Since we have $\mathcal{O}_Y(3C_0)\otimes\mu^*(L\otimes\det E^\vee)\cong\mathcal{O}_Y(3C')\otimes\mu^*L_0$, the divisor (ψ_3) on Y defined by general ψ_3 intersects C' at $\deg L_0=e_0$ points transversally. Let p be one of these intersection points, $t,\ u$ local equations for C' and (ψ_3) around p respectively, and denote $z_0:=Z_0/Z_\infty$. Then $(t,\ u,\ z_0)$ gives a local coordinate system of X around $p_0:=\sigma^{-1}(p)\cap Y_0$. Ψ can be written as

$$\Psi=\psi_0z_0^4+\psi_1z_0^3+\psi_2z_0^2+\psi_3z_0+\psi_4=z_0(\psi_0z_0^3+\psi_1z_0^2+\psi_2z_0+u)+t^4$$

around p_0. This is an equation defining a rational double point of type A_3.

We have to consider the case $E\cong L\otimes(F_2\oplus\mathcal{O}_C)$ with $L\in\mathcal{E}_C(1,\ 1)$. (In this case, we have $e=2$ and $d=1$, hence $3d-e=1$ and $e-d=1$ above hold at the same time.) In this case, the coefficient ψ_i of $Z_0^{4-i}Z_\infty^i$ in $\Psi\in H^0(X,\ \mathcal{O}_C(4Y_1)\otimes\sigma^*\mu^*\det E^\vee)$ is the element of $H^0(Y,\ \mathcal{O}_Y(iC_0)\otimes\mu^*(L^{\otimes(4-i)}\otimes\det E^\vee))\cong H^0(Y,\ \mathcal{O}_Y(C')\otimes\mu^*L)$. We have $\mathrm{Bs}\,|iC'+\Gamma_0|=\{y_0\}$ by Lemma 3.5, and hence $\mathrm{Bs}\,|4Y_1-\sigma^*\mu^*D|=\sigma^{-1}(y_0)\cup\{(\mu\circ\sigma)^{-1}(p)\cap Y_\infty\}$. We have already seen that a general member of $|4Y_1-\sigma^*\mu^*D|$ is nonsingular along $(\mu\circ\sigma)^{-1}(p)\cap Y_\infty$. We only have to prove that it is nonsingular along $\sigma^{-1}(y_0)$. Since all the nonsingular members of $|4C'+\Gamma_0|$ have the same tangent at y_0 by Lemma 3.5, we can choose a local coordinate $(t,\ u)$ around y_0 such that $t=0$ is the local equation of Γ_0 and that $u=0$ gives the tangent of nonsingular members of $|4C'+\Gamma_0|$ at y_0. If we denote $z:=Z_0/Z_\infty$, then Ψ can be written as

$$\begin{aligned}\Psi&=a_0tz^4+(a_1t+b_1u+\iota_1(t,u))z^3+(a_2t+b_2u+\iota_2(t,u))z^2\\&\quad+(a_3t+b_3u+\iota_3(t,u))z+(b_4u+\iota_4(t,u))\end{aligned}$$

near $\sigma^{-1}(y_0)\setminus Y_\infty$, where $a_i,\ b_j\in\mathbf{C}$, $(i=0,\ 1,\ 2,\ 3,\ j=1,\ 2,\ 3,\ 4)$, and $\iota_j(t,\ u)$, $(j=1,\ 2,\ 3,\ 4)$ is the sum of all the monomials with respect to t and u with degree at least two. Since Ψ is general, we may assume $a_0\neq 0$ and $b_4\neq 0$. Since the tangent of a nonsingular member of $|iC'+\Gamma_0|$ and the tangent of a nonsingular member of $|jC'+\Gamma_0|$ are distinct when $i \neq j$ by Lemma 3.5, we have $a_1a_2a_3b_1b_2b_3 \neq 0$. We have

$$\frac{\partial \Psi}{\partial t} = z\left\{a_0 z^3 + \left(a_1 + \frac{\partial \iota_1}{\partial t}\right) z^2 + \left(a_2 + \frac{\partial \iota_2}{\partial t}\right) z + \left(a_3 + \frac{\partial \iota_3}{\partial t}\right) + \frac{\partial \iota_4}{\partial t}\right\},$$
$$\frac{\partial \Psi}{\partial u} = \left(b_1 + \frac{\partial \iota_1}{\partial u}\right) z^3 + \left(b_2 + \frac{\partial \iota_2}{\partial u}\right) z^2 + \left(b_3 + \frac{\partial \iota_3}{\partial u}\right) z + b_4 + \frac{\partial \iota_4}{\partial u},$$

and if we fix a_1, a_2 and a_3, then b_1, b_2 and b_3 are uniquely determined. On the other hand, a_0 and b_4 can be chosen independently of them, and hence the two equations $\partial\Psi/\partial t = 0$ and $\partial\Psi/\partial u = 0$ do not have the same solutions, since Ψ is general. Therefore, the divisor (Ψ) is nonsingular along $\sigma^{-1}(y_0)$.

Finally, we investigate the image of $Y_\infty \cap S''$ in W where $S'' := (\Psi)$. If we substitute $Z_\infty = 0$ into $\Psi = 0$, then we get $\psi_0 Z_0^4 = 0$. Since we have $Z_0 \neq 0$ on Y_∞, $\psi_0 = 0$ must hold. From $\psi_0 \in H^0(Y,\ \mu^*(L^{\otimes 4} \otimes \det E^\vee))$, and $\deg(L^{\otimes 4} \otimes \det E^\vee) = 3d - e$, we see that S'' intersects $Y_\infty \cong Y \xrightarrow{\rho} C$ at $3d-e$ fibers. We can obtain $Y_\infty^2 S'' = e - 3d$ by the same caluculation as above. Therefore these are all (-1)-curves. Hence the image of $Y_\infty \cap S''$ in W is a finite set of nonsingular points of $S' := \rho(S'')$.

(iii) The case where ($3d = e > 0$ and $L_0 \otimes L_1^{\otimes 3} \not\cong \mathcal{O}_C$), ($3d < e < 4d$), or ($e = 4d > 0$ and $L_0 \otimes L_1^{\otimes 2} \cong \mathcal{O}_C$). Since we have $H^0(Y,\ \mu^*(L^{\otimes 4} \otimes \det E^\vee)) = H^0(C,\ L_0 \otimes L1^{\otimes 3}) = 0$, the coefficient ψ_0 of Z_0^4 in Ψ is always 0. Hence $S'' := (\Psi)$ has Z_∞ as a component, i.e., the image of S'' in W by ρ contains C_1. In this case, we have to consider the complete linear system of $\mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee \otimes \mathcal{O}_X(-Y_\infty) \cong \mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}_Y(C_0) \otimes \mu^* \det E^\vee)$. Any $\tilde{\Psi} := \Psi/Z_\infty \in H^0(X,\ \mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}_Y(C_0) \otimes \mu^* \det E^\vee))$ can be written as

$$\tilde{\Psi} = \sum_{j=0}^{3} \psi_{j+1} Z_0^{3-j} Z_\infty^j,$$
$$\psi_{j+1} \in H^0(Y,\ \mathcal{O}_Y((j+1)C_0) \otimes \mu^*(L^{\otimes(3-j)} \otimes \det E^\vee)),\ (j = 0,\ \cdots,\ 3).$$

Since we have $\deg(L_0 \otimes L_1^{\otimes 2}) = 2d - e_0 = (1/2)(4d-e) \geq 0$, and $\deg(L_0 \otimes L_1^{-1}) = e - d \geq 0$, we see that $\psi_0 \neq 0$ and $\psi_3 \neq 0$ hold for general $\tilde{\Psi}$ by assumption. Therefore it is sufficient to investigate the base points of the complete linear systems of $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$ and $\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E^\vee)$ on Y to investigate the base points of $\mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}Y(C_0) \otimes \mu^*(\det E^\vee))$.

Let us look at $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$. Since we have $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee) \cong \mathcal{O}_Y(C') \otimes \mu^*(L_0 \otimes L_1^{\otimes 2})$ and $\deg(L_0 \otimes L_1^{\otimes 2}) = (1/2)(4d-e)$, we obtain the following results about the base points. If $4d - e \geq 4$ holds, then base points do not exist. If $4d - e = 2$ holds, then there exists a unique isolated base point on C'. If $4d - e = 0$ holds, then we have $|C'| = \{C'\}$. In each case, we can easily see that a general $\tilde{\Psi}$ is nonsingular over the base points of the complete linear system of $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$ by looking at the above equation for $\tilde{\Psi}$.

Let us look at $\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E^\vee)$. Since $\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E^\vee) \cong \mathcal{O}_Y(4C') \otimes \mu^*(L_0 \otimes L_1^{-1})$ and $\deg(L_0 \otimes L_1^{-1}) = e - d \geq 2$ hold, the complete linear system of $\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E^\vee)$ is base point free.

By what we have seen so far, a general member of the complete linear system of $\mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}_Y(C_0) \otimes \mu^* \det E^\vee)$ is irreducible and nonsingular.

Let S'' be a general member of the complete linear system of $\mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}_Y(C_0) \otimes \mu^*(\det E^\vee))$. We may assume that S'' is irreducible and nonsingular. We study the multiplicity of each point of the image of $S'' \cap Y_\infty$ on $S' := \rho(S'')$ by ρ. If $\tilde{\Psi} \in H^0(X,\ \mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}_Y(C_0) \otimes \mu^*(\det E^\vee))$ is a global section defining S'', and if we substitute $Z_\infty = 0$ into $\tilde{\Psi} = 0$, then we obtain $\psi_1 Z_0^3 = 0$. Since $Z_0 \neq 0$ holds on Y_∞, we have $\psi_1 = 0$. Since ψ_1 is an element of $H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee))$, we investigate the complete linear system of $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$. A general member is irreducible and nonsingular when $4d - e \geq 4$ or $4d - e = 0$ as we saw above. If $4d - e = 2$ holds, since $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee) \cong \mathcal{O}_Y(C') \cong \mu^*(L_0 \otimes L_1^{\otimes 2})$ and $\deg(L_0 \otimes L_1^{\otimes 2}) = 1$ hold, we have $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee) \cong \mathcal{O}_Y(C' + \Gamma)$, where Γ is the fiber of μ such that $\mathcal{O}_C(p) \cong L_0 \otimes L_1^{\otimes 2}$ for $p := \mu(\Gamma)$. If we denote $\Gamma' := \mu^{-1}(p')$ for any $p' \in C$, then we have

$$H^0(Y,\ \mathcal{O}_Y(C' + \Gamma - \Gamma')) \cong H^0(C,\ F_2 \cong \mathcal{O}_C(p - p')) = 0,$$

so no member has any fiber as a component except Γ. On the other hand, we have

$$\dim H^0(Y,\ \mathcal{O}_Y(C' + \Gamma)) = \dim H^0(C,\ F_2 \otimes \mathcal{O}_C(p)) = 2,$$

so a general member of the complete linear system of $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$ is irreducible and nonsingular by what we saw above. Hence $S'' \cap Y_\infty$ is an irreducible section of $Y_\infty \cong Y \xrightarrow{\mu} C$, and does not contain any fiber, so each point of its image by ρ is nonsingular on S'.

(iv) The case where ($e = 4d$ and $L_0 \otimes L_1^{\otimes 2} \not\cong \mathcal{O}_C$) or ($4d < e$). Since we have $H^0(Y,\ \mu^*(L^{\otimes 4} \otimes \det E^\vee)) \cong H^0(C,\ L_0 \otimes L_1^{\otimes 3}) = 0$ and $H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)) \cong H^0(C,\ F_2 \otimes L_0 \otimes L_1^{\otimes 2}) = 0$, the coefficients ψ_0 of Z_0^4 and ψ_1 of $Z_0^3 Z_1$ are always 0. Hence (Ψ) has $2Y_\infty$ as a component. This means that the image of (Ψ) in W contains C_1 as a singular curve. Therefore the complete linear system $|4T - \pi^* D|$ on W does not have irreducible members with at most rational double points as singularities.

4.2.2 Existence in the case where e is odd

Denote $e =: 2e_0 + 1$. If we fix any $F_{2,\,1} \in \mathcal{E}_C(2,\ 1)$, there exist $L_0 \in \mathcal{E}_C(1,\ e_0)$ and $L_1 \in \mathcal{E}_C(1,\ d - e_0)$ with $E_0 \cong L_0 \otimes F_{2,\,1}$ and $L \cong L_0 \otimes L_1$. Hence $E \cong L_0 \otimes (F_{2,\,1} \oplus L_1)$ holds. Let $\mathcal{L}_k$ ($k = 1,\ 2,\ 3$) be the nontrivial line bundles on the elliptic curve C satisfying $\mathcal{L}_k^{\otimes 2} \cong \mathcal{O}_C$.

Theorem 4.11 *Let the conditions and notation be as above. Then the complete linear system* $|4T - \pi^* D|$ *on* W *satisfies the condition* (A) *if and only if one of the following* (1) *and* (2) *holds:*

(1) $e = d > 0$ *and* $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1}$ *is isomorphic to one of* $\mathcal{O}_C$ *and* $\mathcal{L}_k$ $(k = 1,\ 2,\ 3)$.

(2) $d < e < 4d$.

We use the following result by Ashikaga [1] to prove this theorem. Since [1] is unpublished, we give the proof for the readers' convenience.

Lemma 4.12 *If* $\mathcal{L}_k$ $(k = 1,\ 2,\ 3)$ *are the three nontrivial line bundles satisfying* $\mathcal{L}_k^{\otimes 2} \cong \mathcal{O}_C$*, and if* $F_{2,\,1}$ *is an indecomposable locally free sheaf of rank* 2 *and degree* 1 *on an elliptic curve* C*, then the following hold for any nonnegative integer* m*:*

$$
\begin{aligned}
(1) \quad S^{4m}(F_{2,\,1}) \quad &\cong \quad (\mathcal{O}_C^{\oplus(m+1)} \oplus (\bigoplus_{k=1}^{3} \mathcal{L}_k)^{\oplus m}) \otimes (\det F_{2,\,1})^{\otimes 2m} \\
(2) \quad S^{4m+2}(F_{2,\,1}) \quad &\cong \quad (\mathcal{O}_C^{\oplus m} \oplus (\bigoplus_{k=1}^{3} \mathcal{L}_k)^{\oplus(m+1)}) \otimes (\det F_{2,\,1})^{\otimes(2m+1)}.
\end{aligned}
$$

Proof First, we show the statement for $S^2 F_{2,\,1}$. We have

$$
\begin{aligned}
F_{2,\,1} \otimes F_{2,\,1} \quad &\cong \quad S^2 F_{2,\,1} \oplus \det F_{2,\,1} \\
F_{2,\,1} \otimes F_{2,\,1} \quad &\cong \quad (\mathcal{O}_C \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) \otimes M
\end{aligned}
$$

for some $M \in \mathcal{E}_C(1,\ 1)$ by the Clebsch-Gordan formula [4, p.438], and Atiyah's result [4, Theorem 14]. Hence $\det F_{2,\,1}$ is isomorphic to one of M and $M \otimes \mathcal{L}_k$ $(k = 1,\ 2,\ 3)$ by the Krull-Schmidt theorem. If $\det F_{2,\,1} \cong M$ holds, then there is nothing to prove. If $\det F_{2,\,1} \cong \mathcal{L}_1 \otimes M$ holds, then since we have $\mathcal{L}_1^{-1} \cong \mathcal{L}_1$, we obtain $\det F_{2,\,1} \cong \mathcal{L}_1 \otimes M$. Hence we have

$$
\begin{aligned}
S^2 F_{2,\,1} &\cong (\mathcal{O}_C \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) \otimes \mathcal{L}_1 \otimes \det F_{2,\,1} \cong (\mathcal{L}_1 \oplus \mathcal{L}_3 \oplus \mathcal{L}_2) \otimes \det F_{2,\,1} \\
&\cong (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) \otimes \det F_{2,\,1}.
\end{aligned}
$$

If we assume $\det F_{2,\,1} \cong \mathcal{L}_k \otimes M$ $(k = 2,\ 3)$, then we obtain the same result.

To complete the proof, it is sufficient to show the following (i), (ii) and (iii):

(i) $S^4 F_{2,\,1} \cong (\mathcal{O}_C^{\oplus 2} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) \otimes (\det F_{2,\,1})^{\otimes 2}$.

(ii) (1) of the lemma is true under the assumption that the lemma is true for all the even integers less than or equal to $4m - 2$.

(iii) (2) of the lemma is true under the assumption that the lemma is true for all the even integers less than or equal to $4m$.

We show only (ii) here. (i) and (iii) can be shown in the same way.

We have

$$
\begin{aligned}
S^{4m}F_{2,\,1} \otimes S^2F_{2,\,1} &\cong S^{4m+2}F_{2,\,1} \oplus (\det F_{2,\,1}) \otimes S^{4m+1}F_{2,\,1} \otimes F_{2,\,1} \\
&\cong S^{4m+2}F_{2,\,1} \oplus (\det F_{2,\,1}) \otimes \Big((S^{4m}F_{2,\,1} \oplus (\det F_{2,\,1}) \otimes S^{4m-2}F_{2,\,1}\Big) \\
&\cong S^{4m+2}F_{2,\,1} \oplus \Big((\det F_{2,\,1}) \otimes S^{4m}F_{2,\,1}\Big) \oplus \Big((\det F_{2,\,1})^{\otimes 2} \otimes S^{4m-2}F_{2,\,1}\Big)
\end{aligned}
$$

for $m > 0$ by the Clebsch-Gordan formula. On the other hand, we have

$$
\begin{aligned}
&S^{4m}F_{2,\,1} \otimes S^2F_{2,\,1} \\
&\quad\cong \Big(\mathcal{O}_C^{\oplus(m+1)} \oplus (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)^{\oplus m}\Big) \otimes (\det F_{2,\,1})^{\otimes 2m} \\
&\qquad \otimes (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) \otimes (\det F_{2,\,1}) \\
&\quad\cong \Big(\mathcal{O}_C^{\oplus 3m} \oplus (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)^{\oplus(3m+1)}\Big) \otimes (\det F_{2,\,1})^{\otimes(2m+1)}
\end{aligned}
$$

by the induction assumption, and furthermore, we have

$$
\begin{aligned}
&((\det F_{2,\,1}) \otimes S^{4m}F_{2,\,1}) \oplus ((\det F_{2,\,1})^{\otimes 2} \otimes S^{4m-2}F_{2,\,1}) \\
&\quad\cong \Big(\mathcal{O}_C^{\oplus(m+1)} \oplus (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)^{\oplus m}\Big) \otimes (\det F_{2,\,1})^{\otimes(2m+1)} \\
&\qquad \oplus \Big(\mathcal{O}_C^{\oplus(m-1)} \oplus (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)^{\oplus m}\Big) \otimes (\det F_{2,\,1})^{\otimes(2m+1)} \\
&\quad\cong \Big(\mathcal{O}_C^{\oplus 2m} \oplus (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)^{\oplus 2m}\Big) \otimes (\det F_{2,\,1})^{\otimes(2m+1)}.
\end{aligned}
$$

Hence we have

$$
S^{4m+2}F_{2,\,1} \cong \Big(\mathcal{O}_C^{\oplus m} \oplus (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)^{\oplus(m+1)}\Big) \otimes (\det F_{2,\,1})^{\otimes(2m+1)}
$$

by the Krull-Schmidt theorem. q.e.d.

Let us now prove Theorem 4.11.

By the remark after Proposition 3.3, the case $e < 0$, the case $d < 0$ and the case $L \cong \mathcal{O}_C$ may be excluded.

(i) The case where ($e < d$), or ($e = d$ and $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1}$ is isomorphic to none of $\mathcal{O}_C$ and $\mathcal{L}_k$ ($k = 1,\ 2,\ 3$)). We obtain the following isomorphism from Lemma 4.12.

$$
\begin{aligned}
&H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee) \\
&\quad\cong H^0(C,\ S^4(E_0) \otimes \det E^\vee) \\
&\quad\cong H^0(C,\ S^4(F_{2,\,1}) \otimes L_0^{\otimes 4} \otimes (L_0^{-1})^{\otimes 3} \otimes L_1^{-1} \otimes \det F_{2,\,1}^\vee) \\
&\quad\cong H^0(C,\ S^4(F_{2,\,1}) \otimes \det F_{2,\,1}^\vee \otimes L_0 \otimes L_1^{-1}) \\
&\quad\cong H^0(C, (\det F_{2,1}) \otimes L_0 \otimes L_1^{-1})^{\oplus 2} \oplus \left(\bigoplus_{k=1}^{3} H^0(C, (\det F_{2,1}) \otimes L_0 \otimes L_1^{-1} \otimes \mathcal{L}_k)\right) \\
&\quad\cong 0.
\end{aligned}
$$

Therefore, the coefficient ψ_4 for Z_∞^4 of Ψ is always 0, and (Ψ) has Y_0 as a component. This means that the image of (Ψ) in W by ρ is not irreducible.

(ii) The case where ($e = d > 0$ and $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1}$ is isomorphic to one of $\mathcal{O}_C$ and $\mathcal{L}_k$ ($k = 1,\ 2,\ 3$)), ($d < e < 3d$), or ($e = 3d > 0$ and $L_0 \otimes L_1^{\otimes 3} \cong \det F_{2,\,1}$).

For general $\Psi \in H^0(X,\ \mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee)$, we have $\psi_j \neq 0$ ($j = 0,\ \cdots,\ 4$) as in the case of e even. Hence base points of the complete linear system of $\mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee$ exist only over the base points of the complete linear systems of $\mathcal{O}_Y(4C_0) \otimes \det E^\vee$ and $\mu^*(L^{\otimes 4} \otimes \det E^\vee)$ on Y. We investigate the existence of these base points.

Let us look at $L^{\otimes 4} \otimes \det E^\vee$. Since we have $\deg(L^{\otimes 4} \otimes \det E^\vee) = \deg(L_0 \otimes L_1^{\otimes 3} \otimes \det F_{2,\,1}) = 3d - e \geq 0$, and $L_0 \otimes L_1^{\otimes 3} \otimes \det F_{2,\,1} \cong \mathcal{O}_C$ when $3d - e = 0$ holds, there do not exist base points if $3d - e \neq 1$. If $3d - e = 1$ holds, then there exists a point $q \in C$ with $L^{\otimes 4} \otimes \det E^\vee \cong \mathcal{O}_C(q)$. If we denote $\Gamma := Y_\infty \cap \sigma^{-1}\mu^{-1}(q)$, we have $\Gamma \cong \mathbf{P}^1$, and this is contained in the base locus of $|4Y_1 - \sigma^*\mu^* D|$. We can show that Γ is a (-1)-curve of $S'' := (\Psi)$ and $\rho(\Gamma)$ is a nonsingular point of $S' := \rho(S'')$ as before.

Let us now look at $\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee$. We fix any point $q \in C$ and denote $\Gamma := \mu^{-1}(q)$. If the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee) \to H^0(\Gamma,\ \mathcal{O}_\Gamma(4C_0)) \cong H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(4))$$

is surjective, then there do not exist base points on Γ, and since q is an arbitrary point of C, the base locus $\mathrm{Bs}\,|4C_0 - \mu^* D|$ is empty.

Since we have

$$\begin{aligned} &H^1(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^*(\mathcal{O}_C(-q) \otimes \det E^\vee)) \\ &\cong H^1(C,\ \det F_{2,1} \otimes L_0 \otimes L_1^{-1} \otimes \mathcal{O}_C(-q))^{\oplus 2} \\ &\oplus \left(\bigoplus_{k=1}^{3} H^1(C,\ \det F_{2,1} \otimes L_0 \otimes L_1^{-1} \otimes \mathcal{O}_C(-q) \otimes \mathcal{L}_k) \right), \end{aligned}$$

this cohomology group is 0 if $\deg(\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1} \otimes \mathcal{O}_C(-q)) = e - d - 1 \geq 1$, i.e., $e - d \geq 2$, and hence the above restriction mapping is surjective.

We consider the case where $e - d = 0,\ 1$. For that purpose, we need to study the structure of Y more precisely. Let us look at the complete linear system of $\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E_0^\vee)^{\otimes 2}$. Since

$$\begin{aligned} &H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^*(\det E_0^\vee)^{\otimes 2}) \cong H^0(C,\ S^4 E_0 \otimes (\det E_0^\vee)^{\otimes 2}) \\ &\cong H^0(C,\ \mathcal{O}_C)^{\oplus 2} \oplus \left(\bigoplus_{k=1}^{3} H^0(C,\ \mathcal{L}_k) \right), \end{aligned}$$

we have $\dim H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^*(\det E_0^\vee)^{\otimes 2}) = 2$. If we let $D_0 \in \mathrm{Div}(C)$ be a divisor with $\mathcal{O}_C(D_0) \cong \det E_0$, then

$$(4C_0 - 2\mu^* D_0)^2 = 16C_0^2 - 16C_0\mu^* D_0 = 16e - 16e = 0$$

holds. Hence this complete linear system is a linear pencil which has no base point. Let $\zeta : Y \to \mathbf{P}^1$ be the corresponding fibration. The invertible sheaves $\mathcal{M}_k := \mathcal{O}_Y(2C_0) \otimes$

$\mu^*(\mathcal{L}_k \otimes \det F_{2,\,1}^\vee)$ $(k = 1,\ 2,\ 3)$ satisfy $\mathcal{M}_k^{\otimes 2} \cong \mathcal{O}_Y(4C_0) \otimes \mu^*(\det E_0^\vee)^{\otimes 2}$, and

$$H^0(Y,\ \mathcal{M}_k) \cong H^0(C,\ S^2E_0 \otimes \mathcal{L}_k \otimes \det F_{2,\,1}^\vee) \cong H^0(C,\ \mathcal{L}_k \otimes (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)) \cong \mathbf{C}$$

by Lemma 4.12, hence ζ has three multiple fibers $2\mathcal{F}_k$ $(k = 1,\ 2,\ 3)$ with $\mathcal{F}_k$ satisfying $\mathcal{M}_k \cong \mathcal{O}_Y(\mathcal{F}_k)$.

Next, we study the complete linear system of $\mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0^\vee)^{\otimes 2} \otimes \mathcal{L}_k)$. We obtain

$$H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0^\vee)^{\otimes 2} \otimes \mathcal{L}_k)) \cong \mathbf{C}$$

by the same caluculation as above. Therefore, the complete linear system of $\mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0^\vee)^{\otimes 2} \otimes \mathcal{L}_k)$ consists of a unique effective divisor. Since

$$\begin{aligned}
\mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0^\vee)^{\otimes 2} \otimes \mathcal{L}_1) &\cong \mathcal{M}_2 \otimes \mathcal{M}_3 \\
\mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0^\vee)^{\otimes 2} \otimes \mathcal{L}_2) &\cong \mathcal{M}_3 \otimes \mathcal{M}_1 \\
\mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0^\vee)^{\otimes 2} \otimes \mathcal{L}_3) &\cong \mathcal{M}_1 \otimes \mathcal{M}_2,
\end{aligned}$$

the complete linear systems consist of $\mathcal{F}_2 + \mathcal{F}_3,\ \mathcal{F}_3 + \mathcal{F}_1,\ \mathcal{F}_1 + \mathcal{F}_2$, respectively.

If $e - d = 1$, then since $\deg(\det E_0)^{\otimes 2} = 2e = 2d + 2$ and $\deg(\det E) = e + d = 2d + 1$ hold, there exists a unique point $p \in C$ with $(\det E_0)^{\otimes 2} \cong (\det E) \otimes \mathcal{O}_C(p)$. Hence we have

$$\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee \cong (\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E_0)^{\otimes 2}) \otimes \mu^*\mathcal{O}_C(p).$$

Since the complete linear system of $\mathcal{O}_Y(4C_0) \otimes \mu^*(\det E_0^\vee)^{\otimes 2}$ is a pencil without base points, any base point of $|4C_0 - \mu^*D|$ exists only on $\Gamma := \mu^{-1}(p)$. Let $p_k \in C$ be a point with $(\det E_0)^{\otimes 2} \otimes \mathcal{L}_k \cong (\det E) \otimes \mathcal{O}_C(p_k)$, and denote $\Gamma_k := \mu^{-1}(p_k)$ for $k = 1,\ 2,\ 3$. Then we have $\Gamma_1 + \mathcal{F}_2 + \mathcal{F}_3,\ \Gamma_2 + \mathcal{F}_3 + \mathcal{F}_1,\ \Gamma_3 + \mathcal{F}_1 + \mathcal{F}_2 \in |4C_0 - \mu^*D|$. Since $p,\ p_1,\ p_2,\ p_3$ are pairwise different, and since $\mathcal{F}_1,\ \mathcal{F}_2$ and $\mathcal{F}_3$ intersect Γ at different points, we obtain $\mathrm{Bs}\,|4C_0 - \mu^*D| = \emptyset$.

If $e - d = 0$, then $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1}$ is isomorphic to one of $\mathcal{O}_C$ and $\mathcal{L}_k$ $(k = 1,\ 2,\ 3)$ by assumption.

If $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1} \cong \mathcal{O}_C$ holds, then since we have

$$\det E \cong L_0^{\otimes 3} \otimes L_1^{-1} \otimes \det F_{2,\,1} \cong L_0^{\otimes 4} \otimes (\det F_{2,\,1})^{\otimes 2} \cong \det E_0,$$

the complete linear system of $\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee$ is a pencil without base points as above.

If $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1} \cong \mathcal{L}_1$ holds, then since

$$\det E \cong L_0^{\otimes 3} \otimes L_1 \otimes \det F_{2,\,1} \cong L_0^{\otimes 4} \otimes (\det F_{2,\,1})^{\otimes 2} \otimes \mathcal{L}_1 \cong (\det E_0)^{\otimes 2} \otimes \mathcal{L}_1,$$

we have

$$\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee \cong \mathcal{O}_Y(4C_0) \otimes \mu^*((\det E_0)^{\otimes 2} \otimes \mathcal{L}_1).$$

Hence we obtain $|4C_0 - \mu^* D| = \{\mathcal{F}_2 + \mathcal{F}_3\}$. Therefore, $\sigma^{-1}(\mathcal{F}_2 \cup \mathcal{F}_3) \cap Y_0$ is contained in $\mathrm{Bs}\,|4Y_1 - \sigma^*\mu^* D|$. Since $Z_\infty \neq 0$ on Y_0, if we denote $z_0 := Z_0/Z_\infty$, then a general $\Psi \in H^0(X,\ \mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee)$ can be written as

$$\Psi = \psi_0 z_0^4 + \psi_1 z_0^3 + \psi_2 z_0^2 + \psi_3 z_0 + \psi_4, \qquad \psi_i \in H^0(Y,\ \mathcal{O}_Y(iC_0) \otimes \mu^*(L^{\otimes(4-i)} \otimes \det E^\vee),$$

and ψ_4 has zero of order 1 along $\mathcal{F}_2 \cup \mathcal{F}_3$. Hence the divisor (Ψ) on X defined by a general Ψ is nonsingular along $\sigma^{-1}(\mathcal{F}_2 \cup \mathcal{F}_3) \cap Y_0$.

We can obtain the same result in the case $\det F_{2,\,1} \otimes L_0 \otimes L_1^{-1} \cong \mathcal{L}_2,\ \mathcal{L}_3$ by the same calculation.

As in the case of (ii) with e even, we can show that if we denote $S'' := (\Psi)$, then $S'' \cap Y_\infty$ is a disjoint union of $3d - e$ pieces of (-1)-curves and their images by ρ are nonsingular points of $S' := \rho(S'') \subset W$.

(iii) The case where ($e = 3d > 0$ and $L_0 \otimes L_1^{\otimes 3} \not\cong \det F_{2,\,1}$), or ($3d < e < 4d$). Note that since e is assumed to be odd, we have $e \neq 4d$.

Since we have $H^0(Y,\ \mu^*(L^{\otimes 4} \otimes \det E^\vee)) \cong H^0(C,\ \det F_{2,\,1} \otimes L_0 \otimes L_1^{\otimes 3}) = 0$, we have to study the complete linear system of $\mathcal{O}_X(3Y_1) \otimes \sigma^*(\mathcal{O}_Y(C_0) \otimes \mu^* \det E^\vee)$ for the same reason as in the case (iii) with e even. Hence we have to consider the base point of the complete linear systems of $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$ and $\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee$ similarly as before.

Let us look at $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$. We fix any point $q \in C$, and denote $\Gamma := \mu^{-1}(q) \subset Y$. Since we have

$$\begin{aligned} &H^1(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes (\det E^\vee) \otimes \mathcal{O}_C(-q))) \\ &\quad \cong H^1(C,\ F_{2,\,1} \otimes (\det F_{2,\,1}^\vee) \otimes L_0 \otimes L_1^{\otimes 2} \otimes \mathcal{O}_C(-q)) \end{aligned}$$

and

$$\deg(F_{2,\,1} \otimes \det(F_{2,\,1}^\vee) \otimes L_0 \otimes L_1^{\otimes 2} \otimes \mathcal{O}_C(-q)) = 1 + 2(-1 + e_0 + 2d - 2e_0 - 1) = 4d - e - 2,$$

and since e is odd, the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)) \to H^0(\Gamma,\ \mathcal{O}_\Gamma(C_0)) \cong H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1))$$

is surjective when $4d - e \geq 3$ holds. Hence the complete linear system of $\mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee)$ has no base point.

In the case $4d - e = 1$, since we have

$$\begin{aligned} \dim H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^*(L^{\otimes 3} \otimes \det E^\vee) &= \dim H^0(C,\ E_0 \otimes L^{\otimes 3} \otimes \det E^\vee) \\ &= \dim H^0(C,\ F_{2,\,1} \otimes L_0 \otimes L_1^{\otimes 2} \otimes \det F_{2,\,1}^\vee) = 1 + 2(e_0 + 2d - 2e_0 - 1) \\ &= 1 + 4d - 2e_0 - 2 = 1, \end{aligned}$$

the complete linear system of $\mathcal{O}_Y(C_0)\otimes\mu^*(L^{\otimes 3}\otimes\det E^\vee)$ consists of a unique irreducible nonsingular section C' of μ. Therefore, the set of base points of the complete linear system of $\mathcal{O}_X(3Y_1)\otimes\sigma^*(\mathcal{O}_Y(C_0)\otimes\mu^*\det E^\vee)$ contains $\sigma^{-1}(C')\cap Y_\infty$. Since $Z_0\neq 0$ on Y_∞, if we denote $z_\infty := Z_\infty/Z_0$, $\tilde{\Psi}$ can be written as

$$\tilde{\Psi}=\psi_1+\psi_2 z_\infty+\psi_3 z_\infty^2+\psi_4 z_\infty^3$$

on Y_∞, and $\psi_1\in H^0(Y,\ \mathcal{O}_Y(C_0)\otimes\mu^*(L^{\otimes 3}\otimes\det E^\vee))$ has zeros of order 1 along C', so the divisor $(\tilde{\Psi})$ defined by a general $\tilde{\Psi}$ is nonsingular along $\sigma^{-1}(C')\cap Y_\infty$.

Let us now look at $\mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee$. We fix any point $q\in C$, and denote $\Gamma:=\mu^{-1}(q)$. We have

$$\begin{aligned}&H^1(Y,\ \mathcal{O}_Y(4C_0)\otimes\mu^*(\det E^\vee\otimes\mathcal{O}_C(-q)))\\ &\quad\cong H^1(C,\ (\det F_{2,\,1})\otimes L_0\otimes L_1^{-1}\otimes\mathcal{O}_C(-q))^{\oplus 2}\\ &\quad\oplus\left(\bigoplus_{k=1}^{3}H^1(C,\ (\det F_{2,\,1})\otimes L_0\otimes L_1^{-1}\otimes\mathcal{O}_C(-q)\otimes\mathcal{L}_k)\right).\end{aligned}$$

Under our assumption, we have

$$\deg(\det F_{2,\,1}\otimes L_0\otimes L_1^{-1}\otimes\mathcal{O}_C(-q))=\deg(\det F_{2,\,1}\otimes L_0\otimes L_1^{-1}\mathcal{O}_C(-q)\otimes\mathcal{L}_k)=e-d-1>0.$$

Thus $H^1(Y,\ \mathcal{O}_Y(4C_0)\otimes\mu^*(\det E^\vee\otimes\mathcal{O}_C(-q))=0$ holds. Hence the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee)\to H^0(\Gamma,\ \mathcal{O}_\Gamma(4C_o))\cong H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(4))$$

is always surjective. Consequently, the complete linear system of $\mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee$ has no base point.

(iv) The case where $4d<e$. The images of all the members of $|4Y_1-\sigma^*\mu^*D|$ in W have non-isolated singularity for the same reason as in the case (iv) with e even.

Remark (1) We can prove Theorem 4.11 above by using an isogeny $\varphi:\tilde{C}\to C$ with $\deg\varphi=2$ of elliptic curves as in §4.3.4 below where we use an isogeny of degree 3.

(2) The existence of the linear pencil $\zeta: Y\to\mathbf{P}^1$ and the multiple fibers $2\mathcal{F}_k$ ($k=1,\ 2,\ 3$) above was proved by Suwa [19, §4]. What we mentioned in the proof of Theorem 4.11 is a re-interpretation of Suwa's result by means of Lemma 4.12 due to Ashikaga.

4.2.3 The canonical mapping

In this section, we study the canonical mappings of those surfaces whose existences were shown in §§4.10–4.11. Let E_0 be a locally free sheaf of rank 2 and degree e over an elliptic curve C, and L an invertible sheaf of degree d over C. Furthermore, E_0 and L are assumed to satisfy the conditions of Theorem 4.10 when e is even, and Theorem 4.11 when e is odd.

Lemma 4.13 *If $\mu : Y := \mathbf{P}(E_0) \to C$ is the ruled surface associated to $E_0 \in \mathcal{E}_C(2,\ 4)$, and C_0 is a section of μ with $\mu_* \mathcal{O}_Y(C_0) \cong E_0$, then $\Phi_{|C_0|}$ is birational onto its image.*

Proof Let $\delta \in \mathrm{Div}(C)$ be a divisor satisfying $L_0 \cong \mathcal{O}_C(\delta)$. Since

$$H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^* \mathcal{O}_C(-\delta)) \cong H^0(C,\ E_0 \otimes L_0^{-1}) \cong H^0(C,\ F_2) \cong \mathbf{C},$$

there exists a section C' of μ with $|C_0 - \mu^* \delta| = \{C'\}$.

Let $q_1,\ q_2 \in Y \setminus C'$ be any pair of points contained in different fibers of μ, and Γ_1 the fiber of μ containing q_1.

Since $\dim H^0(Y,\ \mathcal{O}_Y(C_0 - \Gamma_1)) = 2$ and $(C_0 - \Gamma_1)^2 = 2 > 0$, $|C_0 - \Gamma_1|$ has base points. If Γ is any fiber of μ, then $H^1(Y,\ \mathcal{O}_Y(C_0 - \Gamma_1 - \Gamma)) \cong H^1(C,\ E_0 \otimes \mathcal{O}_C(-p_1 - p))$, where $p_1 := \mu(q_1)$ and $p := \mu(\Gamma)$. Hence if p satisfies $\mathcal{O}_C(p_1 + p) \not\cong L_0$, then the above cohomology group is 0, and the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(C_0 - \Gamma_1)) \to H^0(\Gamma,\ \mathcal{O}_\Gamma(C_0 - \Gamma_1))$$

is surjective. Let $p_1 \in C$ be the point with $L_0 \cong \mathcal{O}_C(p_1 + p_1')$, and denote $\Gamma_1' := \mu^{-1}(p_1')$. The base points of $|C_0 - \Gamma_1|$ are on Γ_1. Since $(C_0 - \Gamma_1)\Gamma_1' = 1$, the number of the base points is one. On the other hand, since $(C_0 - \Gamma_1')C' = 1$, the base point of $|C_0 - \Gamma_1'|$ is the intersection point of C' and Γ_1'. Since $q_2 \notin C'$, there exists a member C_0' of $|C_0 - \Gamma_1|$ which does not contain q_2. Then $C_0' + \Gamma_1$ contains q_1 but not q_2. Hence $|C_0|$ separates q_1 and q_2, and $\Phi_{|C_0|}$ is birational onto its image. q.e.d.

Lemma 4.14 *Let T be a tautological divisor of the $\mathbf{P}^2$-bundle $\pi : W := \mathbf{P}(E) \to C$ associated to a locally free sheaf $E = E_0 \oplus L$ ($E_0 \in \mathcal{E}_C(2,\ e)$, $L \in \mathcal{E}_C(1,\ d)$) satisfying $\pi_* \mathcal{O}_W(T) \cong E$. Assume that E_0 and L satisfy the conditions of Theorem* 4.10 *(e even) or* 4.11 *(e odd). Then $\Phi_{|T|}$ is a birational mapping onto its image if $e + d \geq 5$ holds.*

Proof We can show that the restriction of $\Phi_{|T|}$ to a general fiber F of π gives an isomorphism of F onto its image as in the proof of Lemma 4.2.

Let $\rho : X \to W$, $\sigma : X \to Y$, $\mu : Y \to C$, C_0, C_1, Y_1, Y_0, Y_∞, Z_0 and Z_∞ be as in the previous section. Since $\Phi_{|Y_1|} = \Phi_{|T|} \circ \rho$, if $\Phi_{|Y_1|}$ is birational onto its image, $\Phi_{|T|}$ is also birational onto its image. Therefore, it suffices to show that there exists a Zariski open subset of X such that any pair of points in it contained in different fibers are separated by $|Y_1|$.

Since we have $H^0(X,\ \mathcal{O}_X(Y_1)) \cong H^0(Y,\ \mathcal{O}_Y(C_0)) \oplus H^0(C,\ L)$, any $\Psi \in H^0(X,\ \mathcal{O}_X(Y_1))$ can be written as

$$\Psi = \psi_0 Z_0 + \psi_\infty Z_\infty, \qquad \psi \in H^0(C,\ L),\ \psi_\infty \in H^0(Y,\ \mathcal{O}_Y(C_0)).$$

Hence if $d \geq 3$, then $X \setminus Y_0$ satisfies the above condition. (Look at all the elements of the form $\psi_0 Z_0$.) If $e \geq 6$, since there exist a section C' of μ and a divisor $\delta \in \mathrm{Div}(C)$

with $\deg \delta \geq 3$ such that $C_0 \sim C' + \mu^*\delta$ holds, $X \setminus \{Y_0 \cup \mu^{-1}(C_0)\}$ satisfies the above condition. (Look at all the elements of the form $\psi_\infty Z_\infty$.) If $(e,\ d) = (5,\ 2)$, then $X \setminus \{Y_0 \cup Y_\infty \cup \mu^{-1}(C_0)\}$ satisfies the above condition. (Look at all the elements of the form $\psi_0 Z_0$ and $\psi_\infty Z_\infty$.)

In the cases $(e,\ d) = (4,\ 2),\ (4,\ 1)$, $X \setminus \{Y_\infty \cup \mu^{-1}(C')\}$ satisfies the above condition by Lemma 4.14. (Look at all the elements of the form $\psi_\infty Z_\infty$.)

If $(e,\ d) = (3,\ 2)$, then we have $\mathrm{Bs}\,|Y_1| = \emptyset$, and hence $\Phi_{|Y_1|}$ is a morphism. Since $Y_1^3 = 5$ and the degree of the image of X cannot be 1, $\Phi_{|Y_1|}$ is a birational morphism. q.e.d.

Corollary 4.15 *The canonical mapping of any surface S whose existence is guaranteed by Theorem 4.10, Theorem 4.11 and condition (A) is birational onto its image if $p_g(S) \geq 5$ holds. If $(e,\ d) \neq (4,\ 1)$, then the canonical mapping is a morphism. If $(e,\ d) = (4,\ 1)$, then $|K_S|$ has a unique isolated base point, and its canonical image is non-normal.*

Proof The birationality can be proved in the same way as in the proof of Corollary 4.3. If $e \geq 3$ and $d \geq 2$, then $\mathrm{Bs}\,|Y_1| = \emptyset$ by the proof of Lemma 4.14, and $\Phi_{|K_S|}$ is birational onto its image in the case $(e,\ d) \neq (4,\ 1)$. In the case $(e,\ d) = (4,\ 1)$, the base locus of $|4T - \pi^*D|$ contains the section $C_1 := \mathbf{P}(E/E_0)$ of π by the proof of Theorem 4.11. On the other hand, the base locus of $|T|$ consists of a unique point contained in C_1. Hence the complete linear system of the canonical bundle of a general member S has only one isolated base point. The non-normality of the canonical image can be shown in the same way as in the proof of Corollary 4.4. q.e.d.

Proposition 4.16 *Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E := E_0 \oplus L$, $(E_0 \in \mathcal{E}_C(2,\ 2),\ L \in \mathcal{E}_C(1,\ 1))$, T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, and $L_0 \in \mathcal{E}_C(1,\ 1)$, $L_1 \in \mathcal{E}_C(1,\ 0)$ the invertible sheaves satisfying $E_0 \cong L_0 \otimes F_2$ and $L \cong L_0 \otimes L_1$. If $S \in |4T - \pi^*D|$ is irreducible and nonsingular, then we have the following:*

(1) $\deg \Phi_{|K_S|} = 9$, *if* $L_1^{\otimes 2} \not\cong \mathcal{O}_C$.

(2) $\deg \Phi_{|K_S|} = 8$, *if* $L_1^{\otimes 2} \cong \mathcal{O}_C$ *and* $L_1 \not\cong \mathcal{O}_C$.

(3) $\deg \Phi_{|K_S|} = 4$, *if* $L_1 \cong \mathcal{O}_C$.

Proof First, we consider the cases (1) and (2). Since $\deg L = d = 1$, there exists a point $p \in C$ with $L \cong \mathcal{O}_C(p)$. Furthermore, there exists a point $q \in Y$ with $\mathrm{Bs}\,|C_0| = \{q\}$ by the proof of Lemma 4.13. Hence we see that the base locus of $|Y_1|$ is the union of the curve $(\mu \circ \sigma)^{-1}(p) \cap Y_\infty$ and the point $\sigma^{-1}(q) \cap Y_0$ in the same way as in the proof of Lemma 4.14.

Recall that any $\Psi \in H^0(X,\ \mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee)$ can be written as

$$\begin{aligned}\Psi = \psi_0 Z_0^4 + \psi_1 Z_0^3 Z_\infty + \psi_2 Z_0^2 Z_\infty^2 + \psi_3 Z_0 Z_\infty^3 + \psi_4 Z_\infty^4, \\ \psi_i \in H^0(Y,\ \mathcal{O}_Y(iC_0) \otimes \mu^*(L^{\otimes(4-i)} \otimes \det E^\vee)).\end{aligned}$$

There exists a section C' of μ with $\mathcal{O}_Y(C') \cong \mathcal{O}_Y(C_0) \otimes \mu^* L_0^{-1}$ by §4.2.1. Since $\mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee \cong \mathcal{O}_Y(4C') \otimes \mu^*(L_0 \otimes L_1^{-1})$ and $\deg(L_0 \otimes L_1^{-1}) = 1$, there exists a point $p_0 \in C$ with $L_0 \otimes L_1^{-1} \cong \mathcal{O}_C(p_0)$, and the intersection point q' of $\mu^{-1}(p_0)$ with C' satisfies $\mathrm{Bs}\,|4C_0 - \mu^* D| = \{q'\}$. Hence we have $\sigma^{-1}(q') \cap Y_0 \subset \mathrm{Bs}\,|4Y_1 - \sigma^*\mu^* D|$.

Since $L^{\otimes 4} \otimes \det E^\vee \cong L_0 \otimes L_1^{\otimes 3}$, $\deg L_0 = 1$ and $\deg L_1 = 0$, we have $\deg(L^{\otimes 4} \otimes \det E^\vee) = 1$, and there exists a point $p' \in C$ with $L^{\otimes 4} \otimes \det E^\vee \cong \mathcal{O}_C(p')$. Hence we have $(\mu \circ \sigma)^{-1}(p') \cap Y_\infty \subset \mathrm{Bs}\,|4Y_1 - \sigma^*\mu^* D|$.

q coincide with q' if and only if $L_0 \cong L_0 \otimes L_1^{-1}$, hence $L_1 \cong \mathcal{O}_C$. p coincide with p' if and only if $L \cong L^{\otimes 4} \otimes \det E^\vee$. Since $L \cong L_0 \otimes L_1$ and $L^{\otimes 4} \otimes \det E^\vee \cong L_0 \otimes L_1^{\otimes 3}$, this is equivalent to $L_1^{\otimes 2} \cong \mathcal{O}_C$.

Hence, if $L_1^{\otimes 2} \not\cong \mathcal{O}_C$ holds, then we have $q \neq q'$ and $p \neq p'$, and the complete linear system of the canonical divisor of a general member of $|4Y_1 - \sigma^*\mu^* D|$ has no base point. Therefore, in this case, we obtain $\deg \Phi_{|K_S|} = 9$.

If $L_1^{\otimes 2} \cong \mathcal{O}_C$ and $L_1 \not\cong \mathcal{O}_C$ hold, then we have $q \neq q'$ and $p = p'$. Hence the complete linear system of the canonical divisor of a general member of $|4T - \pi^* D|$ has one isolated base point. We have the following elementary transformation (cf. [15]):

$$\begin{array}{ccccc} & & \bar{W} & & \\ & \phi \swarrow & & \searrow \phi' & \\ W & & & & W' \\ & \pi \searrow & & \swarrow \pi' & \\ & & C & & \end{array}$$

where $\pi' : W' \to C$ is the $\mathbf{P}^2$-bundle associated to a locally free sheaf $E' := E_0 \oplus \mathcal{O}_C$ of rank 3 over C, ϕ is the blowing-up at the isolated base point of $|4T - \pi^* D|$, and ϕ' is the blowing-up along the line $\mathbf{P}(E_0 \otimes_{\mathcal{O}_C} \mathcal{O}_p) \subset W'$. Let T' be the tautological divisor with $\pi'_* \mathcal{O}_{W'}(T') \cong E'$. The complete linear system $|T|$ on W is mapped to the complete linear system $|T'|$ by this elementary transformation. Furthermore, if $\bar{S}$ is the proper transform of a general member S by ϕ, and if we denote $S' := \phi'(\bar{S})$, then we have $S' \sim 4T'$ by the assumption $L_1^{\otimes 2} \cong \mathcal{O}_C$. Since $|T'|$ has no base point on $\pi'^{-1}(p)$, the complete linear system of $\mathcal{O}_{W'}(T') \otimes_{\mathcal{O}_{W'}} \mathcal{O}_{S'}$ on S' has no base point. Since $\deg \Phi_{|K_S|} = \deg \Phi_{|K_{\bar{S}}|}$, and since $\Phi_{|K_{\bar{S}}|}$ factors as

$$\Phi_{|K_{\bar{S}}|} : \bar{S} \to S' \to \Phi_{|T'|}(S') \hookrightarrow \mathbf{P}^n, \quad (n := p_g(S) - 1),$$

we have $\deg \Phi_{|K_S|} = T'^2 S' = 4T'^3 = 4 \deg E' = 8$.

Finally, we consider the case (3), i.e., the case $E \cong L \otimes (F_2 \oplus \mathcal{O}_C)$. If p, p', q and q' are as above, we have $p = p'$, $q = q'$ and $\mu(q) = p$. We can prove

$$\mathrm{Bs}\,|Y_1| = \mathrm{Bs}\,|4Y_1 - \sigma^*\mu^*D| = \sigma^{-1}(q) \cup \{(\mu\circ\sigma)^{-1} \cap Y_\infty\}$$

in the same way as above, and hence, $\mathrm{Bs}\,|T| = \mathrm{Bs}\,|4T - \pi^*D|$ is a line contained in a fiber $\pi^{-1}(p) \subset W$.

We have the same elementary transformation as in the case (2). (We use the same notation as above.) In this case, $\mathrm{Bs}\,|T'|$ consists of one point contained in $\pi'^{-1}(p)$, and the image S' of the proper transform of a general member $S \in |4T - \pi^*D|$ in W' goes through this point. If T_0' is the image of $\mathbf{P}(E/L) = \mathbf{P}(L \otimes F_2) \hookrightarrow W$ in W', then we have $T_0' \sim T'$. Let us regard $C_0,\ C'$ and $\mu^{-1}(p)$ as divisors on $\mathbf{P}(E/L)$ or T_0' in view of $Y \cong \mathbf{P}(E/L) \cong T_0'$. In this notation, we have $\mathcal{O}_{W'}(T_0') \otimes_{\mathcal{O}_W} \mathcal{O}_{T_0'} \cong \mathcal{O}_{T_0'}(C_0)$ $\left(\cong \mathcal{O}_{T_0'}(C' + \mu^{-1}(p))\right)$. Since the restriction of S to $\mathbf{P}(E/L)$ is linearly equivalent to $4C' + \mu^{-1}(p)$ and since $S' \sim 4T'$, the restriction of S' to T_0' is the sum of a divisor G which is linearly equivalent to $4C' + \mu^{-1}(p)$ and $3\mu^{-1}(p)$. G goes through $q = \mu^{-1}(p) \cap C'$, and since S is generic, G is nonsingular at q. C_0 also goes through q and nonsingular at q, and C_0 and G have different tangents.

Let $\nu : \tilde{W} \to W'$ be the blowing-up at q, $\tilde{T}$ and $\tilde{S}$ be the proper transforms of T' and S' respectively, and denote $\tilde{\mathcal{E}} := \nu^{-1}(q)$. Since $\nu^*T' \sim \tilde{T} + \tilde{\mathcal{E}}$, we can prove $\nu^*S' \sim \tilde{S} + 4\tilde{\mathcal{E}}$ by the above result. Although $|\tilde{T}|$ has one isolated base point, $\tilde{S}$ does not go through the point. Hence we have

$$\begin{aligned}\deg \Phi_{|K_S|} &= \deg\left(\Phi_{|\tilde{T}|}|_{\tilde{S}}\right) = \tilde{T}^2\tilde{S} = (\nu^*T' - \tilde{\mathcal{E}})^2(\nu^*S' - 4\tilde{\mathcal{E}}) \\ &= \left((\nu^*T')^2 - 2(\nu^*T')\tilde{\mathcal{E}} + \tilde{\mathcal{E}}\right)(\nu^*S' - 4\tilde{\mathcal{E}}) = T'^2S' - 4\tilde{\mathcal{E}}^3 = 4T'^3 - 4 = 4.\end{aligned}$$

q.e.d.

We treat the case $(e,\ d) = (1,\ 1)$ in Propsition 4.41 in the next section.

In the case $(e,\ d) = (3,\ 1)$, we have the following:

Lemma 4.17 *Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E := E_0 \oplus L,\ (E_0 \in \mathcal{E}_C(2,\ 3),\ L \in \mathcal{E}_C(1,\ 1))$, and T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$. Then $\Phi_{|T|}$ is a triple covering of W over $\mathbf{P}^3$.*

We prove the following lemma to show Lemma 4.17:

Lemma 4.18 *If $\mu : Y := \mathbf{P}(E_0) \to C$ is the ruled surface associated to the locally free sheaf $E_0 \in \mathcal{E}_C(2,\ 3)$, and if C_0 is a section of μ with $\mu_*\mathcal{O}_Y(C_0) \cong E_0$, then we have $\mathrm{Bs}\,|C_0| = \emptyset$.*

Proof Let Γ be any fiber of μ, and denote $p := \mu(\Gamma) \in C$. Since we have

$$H^1(Y,\ \mathcal{O}_Y(C_0 - \Gamma)) \cong H^1(C,\ E_0 \otimes \mathcal{O}_C(-p)) = 0,$$

the restriction mapping

$$H^0(Y,\ \mathcal{O}_Y(C_0)) \to H^0(\Gamma,\ \mathcal{O}_\Gamma(C_0)) \cong H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1))$$

is surjective, and $|C_0|$ has no base point. q.e.d.

Proof of Lemma 4.17 First we consider the pull-back of $|T|$ to X. Since

$$H^0(X,\ \mathcal{O}_X(Y_1)) \cong H^0(Y,\ \mathcal{O}_Y(C_0)) \oplus H^0(C,\ L),$$

any $Z \in H^0(X,\ \mathcal{O}_X(Y_1))$ can be written as

$$Z = \psi_0 Z_0 + \psi_\infty Z_\infty, \qquad \psi_0 \in H^0(C,\ L),\ \psi_\infty \in H^0(Y,\ \mathcal{O}_Y(C_0)).$$

Since $\mathrm{Bs}\,|C_0| = \emptyset$ by Lemma 4.18, and since $\deg L = 1$, the base locus of $|Y_1|$ is the curve $Y_\infty \cap (\mu \circ \sigma)^{-1}(p)$, where $p \in C$ is the point with $L \cong \mathcal{O}_C(p)$. This curve is contracted to a point q by $\rho : X \to W$, and we have $\mathrm{Bs}\,|T| = \{q\}$.

Let $\pi' : W' := \mathbf{P}(E') \to C$ be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E' := E_0 \oplus \mathcal{O}_C$, and T' the tautological divisor with $\pi'_*\mathcal{O}_{W'}(T') \cong E'$. We obtain an elementary transformation

$$\begin{array}{ccccc} & & \bar{W} & & \\ & \phi' \swarrow & & \searrow \phi & \\ W' & & & & W \end{array}$$

as before where ϕ is the blowing-up at q. The image by ϕ' of the proper transform of T by ϕ is linearly equivalent to T'.

If F' is any fiber of π', we have

$$\begin{aligned} H^1(W',\ \mathcal{O}_{W'}(T' - F')) &\cong H^1(C,\ E' \otimes \mathcal{O}_C(-p)) \\ &\cong H^1(C,\ E_0 \otimes \mathcal{O}_C(-p)) \oplus H^1(C,\ \mathcal{O}_C(-p)) \end{aligned}$$

and

$$H^1(W',\ \mathcal{O}_{W'}(T')) \cong H^1(C,\ E') \cong H^1(C,\ E_0) \oplus H^1(C,\ \mathcal{O}_C).$$

These cohomology groups are one-dimensional and hence the canonical homomorphism $H^1(W',\ \mathcal{O}_{W'}(T'-F')) \to H^1(W',\ \mathcal{O}_{W'}(T'))$ is an isomorphism. Therefore the restriction mapping

$$H^0(W',\ \mathcal{O}_{W'}(T')) \to H^0(F',\ \mathcal{O}_{F'}(T'))$$

is surjective, and we obtain $\mathrm{Bs}\,|T'| = \emptyset$. Hence $\deg \Phi_{|T|} = \deg \Phi_{|T'|} = (T')^3 = 3$ holds. q.e.d.

In this case, we may assume $E \cong L \otimes (F_{2,\,1} \oplus \mathcal{O}_C)$ by denoting $E_{2,\,1} := E_0 \otimes L^{-1}$.

Lemma 4.19 *Let $\mu : Y \to C$ be the ruled surface associated to a locally free sheaf $E_0 \in \mathcal{E}_C(2, 3)$ of rank 2 over C, and C_0 a section of μ with $\mu_*\mathcal{O}_Y(C_0) \cong E_0$. Then $\Phi_{|4C_0 - \mu^*D|}$ is a birational morphism onto its image for any divisor $D \in \mathrm{Div}(C)$ of degree 4.*

Proof It is known that Y is isomorphic to the symmetric product of C of degree 2 (cf. [6]). Let $\eta : C \times C \to Y$ be the quotient morphism. A point q of Y may be written as $q = (p, p')$ for two points $p, p' \in C$ (possibly $p = p'$). The ruling $\mu : Y \to C$ is expressed as $\mu(q) = p + p'$ where $+$ is the group addition of the elliptic curve C once an appropriate point of C is chosen as 0. $C \times \{p\}$ and $\{p\} \times C$ are mapped to the same curve C_p on Y by η for any point $p \in C$. Since $(C \times \{p\} + \{p\} \times C)^2 = 2$ and $\deg \eta = 2$, this curve C_p is a section of μ with self-intersection number 1. There exists a fiber Γ of μ with $C_0 \sim C_p + \Gamma$.

Hence the complete linear system $|4C_0 - \mu^*D|$ contains a member of the form $\sum_{i=1}^4 C_{p_i}$, $(p_i \in C,\ i = 1,\ 2,\ 3,\ 4)$. Since $\eta^{-1}(\cup_{i=1}^4 C_{p_i}) = \bigcup_{i=1}^4 \{(C \times \{p_i\}) \cup (\{p_i\} \times C)\}$, for any point $q \in Y$, there exist points p_i $(i = 1,\ 2,\ 3,\ 4)$ such that $\cup_{i=1}^4 C_{p_i}$ does not contain q. Hence we have $\mathrm{Bs}\,|4C_0 - \mu^*D| = \emptyset$.

Let $q,\ q^* \in Y$ be distinct points which are not contained in the image under η of the diagonal of $C \times C$, and denote $q = (p,\ p')$. Then C_p and $C_{p'}$ are two distinct sections of μ with self-intersection numbers 1. Since $C_pC_{p'} = 1$, at least one of C_p and $C_{p'}$ does not go through q^*. We may assume that C_p does not go through q^*. We can show that the complete linear system $|4C_0 - \pi^*D - C_p|$ contains a member of the form $\sum_{i=1}^3 C_{p_i}$, $(p_i \in C,\ i = 1,\ 2,\ 3)$, and there exist points $p_i \in C$ $(i = 1,\ 2,\ 3)$ such that $\sum_{i=1}^3 C_{p_i}$ does not go through q'. Hence the complete linear system $|4C_0 - \mu^*D|$ separates q and q'. q.e.d.

Proposition 4.20 *Let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E \cong L \otimes (F_{2,1} \oplus \mathcal{O}_C)$, $(L \in \mathcal{E}_C(1,\ 1),\ F_{2,1} \in \mathcal{E}_C(2,\ 1))$, T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det E$. Then a general member $S \in |4T - \pi^*D|$ is a canonical surface.*

Proof By the proof of Theorem 4.11, we know that $\mathrm{Bs}\,|4T - \pi^*D| = \emptyset$ holds when $L \cong \det F_{2,1}$, and that $C_1 := \mathbf{P}(E/E_0) \subset W$ is the base locus of $|4T - \pi^*D|$ when $L \not\cong \det F_{2,1}$.

First, we consider the case $L \cong \det F_{2,1}$.

Since $\deg \Phi_{|T|} = 3$ by Lemma 4.17, we have $\deg \Phi_{|K_S|} = 1,\ 2$ or 3. There is nothing to prove if $\deg \Phi_{|K_S|} = 1$.

Assume $\deg \Phi_{|K_S|} = 2$. Since $\mathrm{Bs}\,|T|$ consists of one point $q \in W$, a general member $S \in |4T - \pi^*D|$ does not contain q. Let $\phi : \bar{W} \to W$ be the blowing-up at q and $\bar{T} \subset \bar{W}$ the proper transform of T by ϕ, and denote $\mathcal{E} := \phi^{-1}(q)$. In this case, the proper transform $\bar{S}$ of S is linearly equivalent to $4\bar{T} + 4\mathcal{E} - \phi^*\pi^*D$. If $\phi' : \bar{W} \to W'$ is the elementary transformation appearing in the proof of Lemma 4.17, then $S' := \phi'(\bar{S})$ is linearly equivalent to $4T' + \pi'^*(4p - D) \sim 4T'$, where $p \in C$ is the point with $\mathcal{O}_C(p) \cong L$. (In the rest of the proof, p denots this point.) Since S does not contain q, we may identify as $S = \bar{S}$, and $\Phi_{|K_S|}$ is factored as

$$\Phi_{|K_S|} : S \to S' \to \Phi_{|T'|}(S') \ \left(\subset \mathbf{P}^3\right).$$

Since $(4T')(T')^2 = 12$ and $\deg \Phi_{|K_S|} = 2$, if $H \subset \mathbf{P}^3$ is a hyperplane, we have $\Phi_{|T'|}(S') \sim 6H$. Since $\Phi_{|T'|}^* H \sim T'$, we have $\Phi_{|T'|}^*\left(\Phi_{|T'|}(S')\right) \sim 6T'$, and there exists a relative hyperquadric surface $Q \in |2T'|$ with

$$\Phi_{|T'|}^*\left(\Phi_{|T'|}(S')\right) = S' + Q.$$

Since $\deg \Phi_{|T'|} = 3$ and $\deg \Phi_{|K_S|} = 2$, we see that Q is birationaly equivalent to $\Phi_{|K_S|}(S)$. On the other hand, Q is birationaly equivalent to a ruled surface over C. Thus S' is birationaly equivalent to a double covering of a ruled surface over C. Hence S' has an irrational pencil whose general fiber is a rational curve, an elliptic curve or a hyperelliptic curve, which is absurd.

Assume $\deg \Phi_{|K_S|} = 3$. Denote $C_1 := \mathbf{P}(E/E_0)$, and let $q_0 \in S' \setminus C_1$ be a point such that $\Phi_{|T'|}^{-1}\left(\Phi_{|T'|}(q_0)\right)$ consists of three distinct points $q_0,\ q_1,\ q_2$. Since we can prove that the restriction of $\Phi_{|T'|}$ to any fiber of π is an isomorphism onto its image by the same proof as in Lemma 4.2, $q_1,\ q_2,\ q_3$ are contained in distinct fibers of π'. Let $C_1 \subset W,\ \rho : X \to W,\ \sigma : X \to Y, \mu : Y \to C,\ T',\ C_0,\ Y_1,\ Y_0,\ Y_\infty,\ Z_0$ and Z_∞ be as in Theorem 4.11. Since $Z_0 \in H^0(X,\ \mathcal{O}_X(Y_1) \otimes \sigma^*\mu^*L^{-1})$ and

$$\begin{aligned} &H^0(X,\ \mathcal{O}_X(Y_1) \otimes \sigma^*\mu^*L^{-1}) \cong H^0(Y,\ (\mathcal{O}_Y(C_0) \oplus \mu^*L) \otimes \mu^*L^{-1}) \\ &\quad \cong H^0(Y,\ \mathcal{O}_Y(C_0 - \mu^*L) \oplus \mathcal{O}_Y) \cong H^0(C,\ (E_0 \otimes L^{-1}) \oplus \mathcal{O}_Y), \end{aligned}$$

this cohomology group is of dimension two. Since $q_0 \notin C_1$, we may choose Z_0 in such a way that $Z_0(q_0) = 0$ and that the divisor (Z_0) is irreducible. The global section $\Psi \in H^0(X,\ \mathcal{O}_X(4Y_1) \otimes \sigma^*\mu^* \det E^\vee)$ defining the proper transform $\tilde{S}$ of S by ρ can be written as

$$\begin{aligned} &\Psi = \sum_{i=0}^{4} \psi_i Z_0^{4-i} Z_\infty^i, \\ &\quad \psi_i \in H^0(Y,\ \mathcal{O}_Y(iC_0) \otimes \mu^*L^{\otimes(4-i)} \otimes \det E^\vee),\ (i = 0,\ 1,\ \cdots,\ 4). \end{aligned}$$

Since the complete linear system $|T|$ does not separate q_0, q_1 and q_2, and since $Y_1 (= \rho^*T) \sim (Z_0) + \phi^{-1}(\pi^{-1}(p))$, we have $Z_0(q_1) = Z_0(q_2) = 0$. Hence we have $\Psi(q_i) = \psi_4(q_i) Z_\infty(q_i)^4$, and q_i $(i = 1,\ 2)$ is contained in S' if and only if $\psi_4(q_i) = 0$. On the other hand, since $\psi_4 \in H^0(Y,\ \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee)$, we have $\psi_4(q_i) \neq 0$ for a general S by Lemma 4.19, and we obtain a contradiction.

Next, we consider the case $L \ncong \det F_{2,\,1}$.

Since $\operatorname{Bs}|T|$ consists of the intersection point of C_1 with $\pi^{-1}(p)$, if we denote $\phi : \bar{W} \to W,\ \mathcal{E},\ \bar{T}$ and $\bar{S}$ as above, we have

$$\bar{S} \sim 4\bar{T} + 3\mathcal{E} - \phi^*\pi^*D,$$

and

$$S' \sim 4T' - \pi^*(p'),$$

where $S' = \phi'(\bar{S}) \subset W'$, and $p' \in C$ is the point with $\mathcal{O}_C(p') \cong \det F_{2,\,1}$. Hence, the invertible sheaf $\mathcal{O}_{W'}(4T') \otimes \pi'^*\mathcal{O}_C(-p')$ on W' cannot be the pull-back of any invertible sheaf over $\mathbf{P}^3$, and we have $\deg \Phi_{|K_S|} \neq 3$.

We can prove that $\Phi_{|K_S|}$ does not give rise to a double covering onto its image as in the case $L \cong \det F_{2,\,1}$. Therefore, S is canonical in this case, too. q.e.d.

In the case $(e,\ d) = (2,\ 2)$, we have the following:

Lemma 4.21 *Let $E \cong E_0 \oplus L$ be a locally free sheaf of rank 3 over an elliptic curve C with $E_0 \in \mathcal{E}_C(2,\ 2)$ and $L \in \mathcal{E}_C(1,\ 2)$. Furthermore, let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to E, and T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$. Then the rational mapping $\Phi_{|T|}$ defined by the complete linear system $|T|$ has degree two.*

In the notation of Lemma 4.21, there exists an invertible sheaf $L \in \mathcal{E}_C(1,\ 1)$ with $E_0 \cong L \otimes F_2$. Let $p_0 \in C$ be the point with $L \cong \mathcal{O}_C(p_0)$, $\mu : Y \to C$ the ruled surface assciated to E_0 and C_0 the section of μ with $\mu_*\mathcal{O}_Y(C_0) \cong E_0$. Since

$$H^0(Y,\ \mathcal{O}_Y(C_0) \otimes \mu^*L^{-1}) \cong H^0(C,\ E_0 \otimes L^{-1}) \cong H^0(C,\ F_2) \cong \mathbf{C},$$

there exists a unique section C' with $C' \sim C_0 - \Gamma_0$, where $\Gamma_0 := \mu^{-1}(p_0)$.

Proof Let $\rho : X \to W$, $\sigma : X \to Y$, $\mu : Y \to C$, Y_1, Y_0, Y_∞, Z_0 and Z_∞ be as above. Since $H^0(X,\ \mathcal{O}_X(Y_1)) \cong H^0(C,\ L) \oplus H^0(Y,\ \mathcal{O}_Y(C_0))$, any $Z \in H^0(X,\ \mathcal{O}_X(Y_1))$ can be written as

$$Z = \tilde{\psi}_0 Z_0 + \tilde{\psi}_\infty Z_\infty, \quad \tilde{\psi}_0 \in H^0(C,\ L),\ \tilde{\psi}_\infty \in H^0(Y,\ \mathcal{O}_Y(C_0)).$$

Let $y_0 \in Y$ be as in Lemma 3.5. We have $\mathrm{Bs}\,|Y_1| = \{q_0\}$ by Lemma 3.5 and by $\deg L = 2$, where $q_0 := \sigma^{-1}(y_0) \cap Y_0$. If we identify q_0 with $\rho(q_0)$ and so that $q_0 \in W$, then we have $\mathrm{Bs}\,|T| = \{q_0\}$. Again by Lemma 3.5, two different general members of $|C_0|$ intersect at y_0 with multiplicity 2. Hence if we let $\zeta_1 : W_1 \to W$ be the blowing-up at q_0, and T' the proper transform of T, then the complete linear system $|T'|$ has one base point q_0'. If we let $\zeta_2 : W_2 \to W_1$ be the blowing-up at q_0', and T'' the proper transform of T', then we have $\mathrm{Bs}\,|T''| = \emptyset$, $\dim |T''| = \dim |T| = 3$ and $(T'')^3 = T^3 - 2 = 2$, and $\Phi_{|T''|} : Y_1'' \to \mathbf{P}^3$ is the double covering. q.e.d.

Proposition 4.22 *Let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to the locally free sheaf $E \cong E_0 \oplus L$ with $E_0 \cong L_0 \otimes F_2$ and $L \cong L_0^{\otimes 2}$ for some $L_0 \in \mathcal{E}_C(1,\ 1)$, T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ the divisor with $\mathcal{O}_C(D) \cong \det E$. Then a general member $S \in |4T - \pi^*D|$ is a canonical surface.*

Proof We use the notation in the proof of Lemma 4.21. Furthermore, we regard Y_0 to be a relative hyperplane of W by identifying $\rho(Y_0)$ with Y_0.

The canonical mapping of the minimal resolution of singularities of a general member $S' \in |4T - \pi^* D|$ has degree one or two by Lemma 4.21.

Since $\mathrm{Bs}\,|4T - \pi^* D| = C'$ and since S' has a rational double point of type A_3 at $q_0 \in C'$ by the proof of Theorem 4.10, we have

$$S'_1 \sim 4T' + 2\mathcal{E}_1 - \zeta_1^* \pi^* D,$$

where $\mathcal{E}_1 := \zeta_1^{-1}(q_0)$ and S'_1 is the proper transform of S' by ζ_1. S_1 has a rational double point of type A_1. On the other hand, since the support of the intersection of S' with Y_0 is C', this rational double point does not coincide with q'_0. Hence the proper transform S'_2 of S'_1 by ζ_2 satisfies

$$S'_2 \sim 4T'' + 6\mathcal{E}_2 + 2\mathcal{E}'_1 - \zeta_2^* \zeta_1^* \pi^* D,$$

where $\mathcal{E}_2$ is the exceptional divisor of ζ_2, and $\mathcal{E}'_1$ is the proper transform of $\mathcal{E}_1$ by ζ_2. Since $6\mathcal{E}_2 + 2\mathcal{E}'_1 \not\sim \zeta_2^* \zeta_1^* \pi^* D$, S'_2 cannot be the pull-back of any effective divisor of $\mathbf{P}^3$ by $\Phi_{|T''|}$. Therefore, S is a canonical surface. q.e.d.

4.3 E is indecomposable

Let E be an indecomposable locally free sheaf of rank 3 over an elliptic curve C. Denote $d := \deg E$. We prove the following theorem in §§4.3.1–4.3.4. We consider the case $d \not\equiv 0 \pmod 3$ and $d \neq 1,\ 2$ in §4.3.1, the case $d \equiv 0 \pmod 3$ and $d \neq 3$ in §4.3.2, the case $d = 3$ in §4.3.3, and the case $d = 2$ in §4.3.4. We omit the case $d = 1$ because it was investigated by Catanese and Ciliberto [6]. In §4.3.5, we study the canonical mapping of the surfaces obtained in §§4.3.1–4.3.4. The results about the canonical mappings are stated in Corollaries 4.37 and 4.39, and Propositions 4.40 and 4.41.

We only have to consider the case $d > 0$ by the remark immedietely before §4.1.

Theorem 4.23 *Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to E, T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$ and $D \in \mathrm{Div}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det E$. The complete linear system $|4T - \pi^* D|$ on W satisfies the condition* (A) *if and only if $d \geq 1$.*

Remark In this case, the complete linear system $|4T - \pi^* D|$ turns out not to have base points except the case $d = 3$, and hence its general member is irreducible and nonsingular by Bertini's theorem.

The restriction of $\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$ to a fiber F of π is isomorphic to $\mathcal{O}_{\mathbf{P}^2}(4)$. The complete linear system of $\mathcal{O}_{\mathbf{P}^2}(4)$ is base point free. Therefore to prove that the complete linear system of $\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$ is base point free, it suffices to show the following:

Lemma 4.24 *In the notation of Theorem* 4.23*, if we assume* $d \geq 4$*, then the restriction mapping*

$$H^0(W,\ \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee) \to H^0(F,\ \mathcal{O}_F(4T)) \cong H^0(\mathbf{P}^2,\ \mathcal{O}_{\mathbf{P}^2}(4))$$

is surjective for any fiber F *of* π.

4.3.1 The proof when $\deg E \neq 1,\ 2$ is not divisible by 3

Denote $d := \deg E$. By Theorem 3.4, if we choose and fix an arbitrary isogeny $\varphi : \tilde{C} \to C$ of degree 3, there exists an invertible sheaf L_0 of degree d over $\tilde{C}$ such that $\varphi_* L_0 \cong E$. Furthermore, if we denote $G := \ker \varphi = \{0,\ \sigma,\ 2\sigma\}$ ($\sigma \in G$, $\sigma \neq 0$, $3\sigma = 0$) and $L_1 := T_\sigma^* L_0$, $L_2 := T_{2\sigma}^* L_0$ where $T_{i\sigma}$ ($i = 1,\ 2$) is the translation by $i\sigma \in G$, then we have $\varphi^* E \cong L_0 \oplus L_1 \oplus L_2$. Denote $\tilde{E} := \varphi^* E$.

Let $\pi : W := \mathbf{P}(E) \to C$ and $\tilde{\pi} : \tilde{W} := \mathbf{P}(\tilde{E}) \to \tilde{C}$ be the $\mathbf{P}^2$-bundles associated to E and $\tilde{E}$, respectively. Let T and $\tilde{T}$ be tautological divisors on W and $\tilde{W}$, respectively, such that $\pi_* \mathcal{O}_W(T) \cong E$ and $\tilde{\pi}_* \mathcal{O}_{\tilde{W}}(\tilde{T}) \cong \tilde{E}$. Consider the following diagram:

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\Phi} & W \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \tilde{C} & \xrightarrow[\varphi]{} & C \end{array}$$

If we denote by Φ the morphism from $\tilde{W}$ to W in the above diagram, and choose a divisor $\tilde{D} \in \mathrm{Div}(\tilde{C})$ such that $\mathcal{O}_{\tilde{C}}(\tilde{D}) \cong \det \tilde{E}$, then $\tilde{T} \sim \Phi^* T$ and hence $\Phi^* \mathcal{O}_W(4T - \pi^* D) \cong \mathcal{O}_{\tilde{W}}(4\tilde{T} - \tilde{\pi}^* \tilde{D})$.

We prove Theorem 4.23 in the case $d \geq 4$, i.e., Lemma 4.24. If F is any fiber of π, and if we denote $\mathcal{L} = \mathcal{O}_W(4T - F) \otimes \pi^* \det E^\vee$, then it suffices to show that $H^1(W,\ \mathcal{L}) = 0$ holds.

The kernel $\ker \varphi^*$ of the isogeny $\varphi^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\tilde{C})$ corresponding to $\varphi : \tilde{C} \to C$ is of the form $\{\mathcal{O}_C,\ \mathcal{M},\ \mathcal{M}^{\otimes 2}\}$ with $\mathcal{M} \not\cong \mathcal{O}_C$, $\mathcal{M}^{\otimes 3} \cong \mathcal{O}_C$.

Lemma 4.25 *In the above notation, we have*

$$\varphi_* \mathcal{O}_{\tilde{C}} \cong \mathcal{O}_C \oplus \mathcal{M} \oplus \mathcal{M}^{\otimes 2}.$$

Proof The exact sequence $0 \to \mathcal{O}_C \to \phi_* \mathcal{O}_{\tilde{C}}$ of sheaves splits by the homomorphism $(1/3)\,\mathrm{tr} : \phi_* \mathcal{O}_{\tilde{C}} \to \mathcal{O}_C$, where tr is the trace mapping. Hence there exists a locally free sheaf $\mathcal{E}$ of rank 2 over C with $\phi_* \mathcal{O}_{\tilde{C}} \cong \mathcal{O}_C \oplus \mathcal{E}$. On the other hand, since $\mathcal{M} \otimes \phi_* \mathcal{O}_{\tilde{C}} \cong \phi_*(\mathcal{O}_{\tilde{C}} \otimes \phi^* \mathcal{M}) \cong \phi_* \mathcal{O}_{\tilde{C}}$ holds by the projection formula, we have $\mathcal{M} \oplus (\mathcal{M} \otimes \mathcal{E}) \cong \mathcal{O}_C \oplus \mathcal{E}$. Similarly, since $\mathcal{M}^{\otimes 2} \otimes \phi_* \mathcal{O}_{\tilde{C}} \cong \phi_*(\mathcal{O}_{\tilde{C}} \otimes \phi^* \mathcal{M}^{\otimes 2}) \cong \phi_* \mathcal{O}_{\tilde{C}}$ holds, we have $\mathcal{M}^{\otimes 2} \oplus (\mathcal{M}^{\otimes 2} \otimes \mathcal{E}) \cong \mathcal{O}_C \oplus \mathcal{E}$. Hence we have $\mathcal{E} \cong \mathcal{M} \oplus \mathcal{M}^{\otimes 2}$ by Krull-Schmidt's theorem (cf. [4]). q.e.d.

Remark In the proof of Lemma 4.25, we do not use the condition $d \geq 4$. Namely, this lemma also holds in the case $d \leq 2$ and $d \not\equiv 0 \pmod 3$.

By Lemma 4.25, we get

$$H^1(\tilde{W},\ \Phi^*\mathcal{L}) \cong H^1(W,\ \mathcal{L}) \oplus H^1(W,\ \mathcal{L} \otimes \mathcal{M}) \oplus H^1(W,\ \mathcal{L} \otimes \mathcal{M}^{\otimes 2}).$$

Since the action of G on $\tilde{W}$ is fixed point free, we have $H^1(W,\ \mathcal{L}) = H^1(\tilde{W},\ \Phi^*\mathcal{L})^G$ (cf., e.g., [9, p. 202, Corollaire]). On the other hand, if $\tilde{F_0},\ \tilde{F_1},\ \tilde{F_2}$ are the fibers of $\tilde{\pi}$ which are in the inverse image of F by Φ, then $\Phi^*\mathcal{L} \cong \mathcal{O}_{\tilde{W}}(4\tilde{T} - \tilde{F_0} - \tilde{F_1} - \tilde{F_2}) \otimes \tilde{\pi}^* \det \tilde{E}^\vee$ holds. Hence if we denote $q_i := \tilde{\pi}(\tilde{F_i})$ $(i = 0,\ 1,\ 2)$, then we get

$$\begin{aligned} H^1(\tilde{W},\ \Phi^*\mathcal{L}) &\cong H^1(\tilde{C},\ S^4\tilde{E} \otimes \tilde{\pi}^*(\det \tilde{E}^\vee \otimes \mathcal{O}_{\tilde{C}}(-q_0 - q_1 - q_2))) \\ &\cong \bigoplus_{\substack{\alpha,\ \beta,\ \gamma \geq 0 \\ \alpha+\beta+\gamma=4}} H^1(\tilde{C},\ L_0^{\otimes(\alpha-1)} \otimes L_1^{\otimes(\beta-1)} \otimes L_2^{\otimes(\gamma-1)} \otimes \mathcal{O}_{\tilde{C}}(-q_0 - q_1 - q_2)). \end{aligned}$$

Since $d \geq 4$ by our assumption, this cohomology group is equal to 0, and Theorem 4.23 in the case $d \geq 4$ is proved. q.e.d.

4.3.2 The proof when $\deg E \neq 3$ is divisible by 3

If we denote $d_0 = d/3$, there exists an invertible sheaf L of degree d_0 such that $E \cong L \otimes F_3$.

First we prove Theorem 4.23 in the case $d \geq 6$, i.e., Lemma 4.24. If we let $p := \pi(F)$, then we have $H^1(W,\ \mathcal{O}_W(4T - F) \otimes \pi^* \det E^\vee) \cong H^1(C,\ S^4E \otimes \det E^\vee \otimes \mathcal{O}_C(-p)) \cong H^1(C,\ S^4F_3 \otimes L \otimes \mathcal{O}_C(-p))$. On the other hand, since $F_3 \cong S^2(F_2)$ by Atiyah [4, Theorem 9], we have an isomorphism,

$$S^4(F_3) \cong S^4(S^2(F_2)) \cong F_9 \oplus F_5 \oplus \mathcal{O}_C$$

by [8, p. 156]. Therefore we have an isomorphism

$$\begin{aligned} &H^1(W,\ \mathcal{O}_W(4T - F) \otimes \pi^* \det E^\vee) \\ &\cong H^1(C,\ F_9 \otimes L \otimes \mathcal{O}_C(-p)) \oplus H^1(C,\ F_5 \otimes L \otimes \mathcal{O}_C(-p)) \oplus H^1(C,\ L \otimes \mathcal{O}_C(-p)). \end{aligned}$$

Since $\deg(F_i \otimes L \otimes \mathcal{O}_C(-p)) = i(d_0 - 1) > 0$ for $i = 1,\ 5,\ 9$ (with $F_1 = \mathcal{O}_C$), these summands are all 0 if $d_0 \geq 2$. Hence we have

$$H^1(W,\ \mathcal{O}_W(4T - F) \otimes \pi^* \det E^\vee) = 0.$$

if $d_0 \geq 2$. q.e.d.

4.3.3 The proof when $\deg E = 3$ holds

Let E be a locally free sheaf of rank 3 and degree 3 over an elliptic curve C. There exists an invertible sheaf $L \in \mathcal{E}_C(1,\ 1)$ with $E \cong L \otimes F_3$. Let $p_0 \in C$ be a point satisfying $L \cong \mathcal{O}_C(p_0)$.

Let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to E and T the tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$.

Lemma 4.26 $\mathrm{Bs}\,|T|$ *consists of one point.*

Proof Let F be a fiber of π over a point $p \in C \setminus \{p_0\}$. Since

$$H^1(W,\ \mathcal{O}_W(T-F)) \cong H^1(C,\ E \otimes \mathcal{O}_C(-p)) \cong H^1(C,\ F_3 \otimes \mathcal{O}_C(p_0 - p)) = 0,$$

the following restriction mapping is surjective:

$$H^0(W,\ \mathcal{O}_W(T)) \longrightarrow H^0(F,\ \mathcal{O}_F(T)) \ \left(\cong H^0(\mathbf{P}^2,\ \mathcal{O}_{\mathbf{P}^2}(1))\right).$$

Hence there is no base point on F.

Denote $F_0 := \pi^{-1}(p_0)$. Since

$$H^1(W,\ \mathcal{O}_W(T - F_0)) \cong H^1(C,\ E \otimes \mathcal{O}_C(-p_0)) \cong H^1(C,\ F_3) \cong \mathbf{C}$$

and

$$H^1(W,\ \mathcal{O}_W(T)) \cong H^1(C,\ E) = 0,$$

the image of the following restriction mapping is two-dimensional:

$$H^0(W,\ \mathcal{O}_W(T)) \longrightarrow H^0(F_0,\ \mathcal{O}_{F_0}(T)) \ \left(\cong H^0(\mathbf{P}^2,\ \mathcal{O}_{\mathbf{P}^2}(1))\right).$$

q.e.d.

Let F_0 be as in the proof of Lemma 4.26. Since we have $\det E \cong \mathcal{O}_W(3p_0)$, we have to consider the complete linear system $|4T - 3F_0|$.

Since

$$H^0(W,\ \mathcal{O}_W(T) \otimes \pi^* L^{-1}) \cong H^0(C,\ E \otimes L^{-1}) \cong H^0(C,\ F_3) \cong \mathbf{C},$$

there exists a unique relative hyperplane T_0 with $T_0 \sim T - F_0$.

Lemma 4.27 T_0 *is isomorphic to the ruled surface* $\mathbf{P}(F_2)$. *Furthermore, if* $C_0 \subset T_0$ *is a section of* $\mu := \pi_{|T_0} : T_0 \to C$ *with* $\mu_* \mathcal{O}_{T_0}(C_0) \cong F_2$, *then we have* $N_{T_0/W} \cong \mathcal{O}_{T_0}(C_0)$, *where* $N_{T_0/W}$ *is a normal bundle of* T_0 *in* W.

Proof Since $T_0^3 = 0$, $N_{T_0/W}$ is isomorphic to $\mathcal{O}_{T_0}$ or an invertible sheaf induced from a nonzero divisor on T_0 with self-intersection number zero.

If $N_{T_0/W} \cong \mathcal{O}_W$ holds, since we have $\mathrm{Pic}(W) \cong \mathbf{Z} \cdot T_0 \oplus \pi^* \mathrm{Pic}(C)$, the restriction of any divisor Z on W to T_0 consists of fibers of μ. Hence we must have $Z^2 T_0 = 0$. On the other hand, we have $T^2 T_0 = T^2(T - F_0) = T^3 - T^2 F_0 = 3 - 1 = 2 \neq 0$, which contradicts $Z^2 T_0 = 0$.

Since we have $T_0^2 F = 1$ for any fiber F of π, if $C' \subset T_0$ is some section of μ, then there exists a divisor δ on C with $N_{T_0/W} \cong \mathcal{O}_{T_0}(C' + \mu^* \delta)$. On the other hand, we have

$$\dim H^0(T_0,\ N_{T_0/W}) = \dim H^1(T_0,\ N_{T_0/W}) = 0,\ 1,$$

by the cohomology long exact sequence induced from the exact sequence of sheaves $0 \to \mathcal{O}_W \to \mathcal{O}_W(T_0) \to N_{T_0/W} \to 0$. The pairs of a ruled suface and a divisor on it satisfying the above conditions are as follows:

(1) $T_0 = \mathbf{P}(\mathcal{O}_C \oplus L)$ with $L \in \mathrm{Pic}^0(C) \setminus \{\mathcal{O}_C\}$, and $C_0 + \mu^*\delta$ with $\mu_*\mathcal{O}_{T_0}(C_0) \cong \mathcal{O}_C \oplus L$ and $\deg \delta = 0$,

(2) $T_0 = \mathbf{P}(F_2)$ and $C_0 + \mu^*\delta$ with $\mu_*\mathcal{O}_{T_0}(C_0) \cong F_2$ and $\deg \delta = 0$,

where C_0 is a section of μ.

Assume (1) holds. Let $\delta_0 \in \mathrm{Div}(C)$ be a divisor with $L \cong \mathcal{O}_C(\delta_0)$.

Since $N_{T_0/W} \cong \mathcal{O}_{T_0}(T_0) \cong \mathcal{O}_{T_0}(C_0 + \mu^*\delta)$, if we let $p_0' \in C$ be a point with $p_0' \sim p_0 + \delta$, then we have $\mathcal{O}_{T_0}(T) \cong \mathcal{O}_{T_0}(C_0 + \mu^* p_0')$. Since

$$\dim H^0(T_0,\ \mathcal{O}_{T_0}(C_0 + \mu^* p_0')) = \dim H^0(C,\ (\mathcal{O}_C \oplus L) \otimes \mathcal{O}_C(p_0)) = 2,$$

and since $(C_0 + \mu^* p_0')^2 = 2$, the complete linear system $|C_0 + \mu^* p_0'|$ has one or two isolated base points. If we let $p_0'' \in C$ be a point with $p_0'' \sim p_0' + \delta_0$, then we have

$$\begin{aligned} H^0(T_0,\ \mathcal{O}_{T_0}(C_0 + \mu^*(p_0' - p_0''))) &\cong H^0(C,\ (\mathcal{O}_C \oplus L) \otimes \mathcal{O}_C(p_0' - p_0'')) \\ &\cong H^0(C,\ \mathcal{O}_C(p_0' - p_0'')) \oplus H^0(C,\ L \otimes \mathcal{O}_C(p_0' - p_0'')), \end{aligned}$$

and this cohomology group is one-dimensional since $L \ncong \mathcal{O}_C$. Hence there exists a section $C_0' \subset T_0$ of μ with

$$C_0' \sim C_0 + \mu^*(p_0' - p_0'').$$

Since $\delta_0 \neq 0$, we have $p_0' \neq p_0''$ and $C_0 \neq C_0'$. Hence, $C_0 + \mu^* p_0'$ and $C_0' + \mu^* p_0''$ are mutually distinct members of $|C_0 + \mu^* p_0'|$. Since $C_0 C_0' = 0$, the intersection points of these two divisor are two distinct points $C_0 \cap \mu^* p_0''$ and $C_0 \cap \mu^* p_0'$, which are the base points of $|C_0 + \mu^* p_0'|$, a contradiction to Lemma 4.26. Therefore we have $T_0 \cong \mathbf{P}(F_2)$.

We only have to show that $\delta = 0$ in the notation of (2). Asumme $\delta \neq 0$. There exists a point $p_1 \in C$ with

$$T_{|T_0} \sim (T_0 + F_0)_{|T_0} \sim C_0 + \mu^*(\delta + p_0) \sim C_0 + \mu^* p_1.$$

We can prove that the complete linar system of $T_{|T_0}$ on T_0 has one base point on $\mu^{-1}(p_1)$ and no other base points in the same way as in the proof of Lemma 3.5, a contradiction to the fact that the base point of $|T|$ is in the fiber of π over p_0 since we have $p_0 \neq p_1$. Hence we have $\delta = 0$ and $N_{T_0/W} \cong \mathcal{O}_{T_0}(C_0)$. q.e.d.

In the notation of the proof of Lemma 4.27, since $T_{|T_0} \sim C_0 + \Gamma_0$, we have $\mathrm{Bs}\,|T| = \{q_0\}$, where $q_0 := C_0 \cap \Gamma_0$ by Lemma 3.5.

Lemma 4.28 $\mathrm{Bs}\,|4T - 3F_0| = \{q_0\}$ *holds.*

To prove this lemma, we need the following lemma:

Lemma 4.29 *The restriction mapping*

$$H^0(W,\ \mathcal{O}_W(4T_0+F_0)) \longrightarrow H^0(T_0,\ \mathcal{O}_{T_0}(4C_0+\Gamma_0))$$

is surjective.

Proof We only have to prove $H^1(W,\ \mathcal{O}_W(3T_0+F_0)) = 0$ in the view of the cohomology long exact sequence induced from the exact sequence of sheaves

$$0 \to \mathcal{O}_W(3T_0+F_0) \to \mathcal{O}_W(4T_0+F_0) \to \mathcal{O}_{T_0}(4C_0+\Gamma_0) \to 0.$$

Since $S^3F_3 \cong S^3(S^2F_2) \cong F_3 \oplus F_7$ (cf. [4, Theorem 9], [8, p.156]), we have

$$\begin{aligned} H^1(W,\ \mathcal{O}_W(3T_0+F_0)) &\cong H^1(C,\ (S^3F_3)\otimes L) \\ &\cong H^1(C,\ F_3\otimes L) \oplus H^1(C,\ F_7\otimes L) = 0. \end{aligned}$$

q.e.d.

Proof of Lemma 4.28 We can show that there is no base point of $|4T-3F_0|$ on any fiber except F_0 in the same way as in the proof of Lemma 4.26. Furthermore, the base points of $|4T-3F_0|$ exist only on the line $T_0 \cap F_0 \cong \mathbf{P}^1$, since $3T_0+T \in |4T-3F_0|$.

Since $4T-3F_0 \sim 4T_0+F_0$, we have $\mathcal{O}_W(4T-3F_0)\otimes_{\mathcal{O}_W}\mathcal{O}_{T_0} \cong \mathcal{O}_{T_0}(4C_0+\Gamma_0)$. On the other hand, since the restriction mapping

$$H^0(W,\ \mathcal{O}_W(4T-3F_0)) \longrightarrow H^0(T_0,\ \mathcal{O}_{T_0}(4C_0+\Gamma_0))$$

is surjective by Lemma 4.29, q_0 is the only base point of $|4T-3F_0|$ by Lemma 3.5. q.e.d.

The restriction of a general member S of $|4T-3F_0|$ to T_0 is nonsingular by Lemmas 3.5 and 4.29. Hence S is nonsingular. q.e.d.

4.3.4 The proof when $\deg E = 2$ holds

We fix an isogeny $\varphi : \tilde{C} \to C$ of degree 3 as in §4.3.1. We let L_i $(i = 0,\ 1,\ 2)$, $\tilde{E}$, $\tilde{\pi} : \tilde{W} \to \tilde{C}$, $\tilde{T}$, $\Phi : \tilde{W} \to W$, $D \in \mathrm{Div}(C)$ and $\tilde{D} \in \mathrm{Div}(\tilde{C})$ be as in §4.3.1.

Denote $\mathcal{U} := \{\ \Phi^*S \in |4\tilde{T} - \tilde{\pi}^*\tilde{D}| \mid S \in |4T - \pi^*D|\ \}$.

$G := \ker\varphi = \{0,\ \sigma,\ 2\sigma\}$ acts on $H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)$ by $\{\mathrm{id},\ T_\sigma^*,\ T_{2\sigma}^*\}$. Let $H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)^G$ be the subspace which consists of all the members which are invariant under this action.

Lemma 4.30 *Let* (Ψ) *be the divisor defined by* $\Psi \in H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)$. *Then we have*

$$\mathcal{U} = \{\ (\Psi) \mid \Psi \in H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)^G\ \}.$$

Proof Since we have $\Phi_*\mathcal{O}_{\tilde{W}} \cong \pi^*\varphi_*\mathcal{O}_{\tilde{C}}$ by the base change theorem (cf., e.g., Mumford [16]), if we denote $\mathcal{L} := \mathcal{O}_W(4T)\otimes\pi^*\det E^\vee$, then we obtain isomorphisms

$$\begin{aligned} H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee) &= H^0(\tilde{W},\ \Phi^*\mathcal{L}) \cong H^0(W,\ \mathcal{L}\otimes\Phi_*\mathcal{O}_{\tilde{W}}) \\ &\cong H^0(W,\ \mathcal{L}\otimes\pi^*\varphi_*\mathcal{O}_{\tilde{C}}) \cong H^0(W,\ \mathcal{L}\otimes\pi^*(\mathcal{O}_C\oplus\mathcal{M}\oplus\mathcal{M}^{\otimes 2})) \\ &\cong H^0(W,\ \mathcal{L})\oplus H^0(W,\ \mathcal{L}\otimes\pi^*\mathcal{M})\oplus H^0(W,\ \mathcal{L}\otimes\pi^*\mathcal{M}^{\otimes 2}). \end{aligned}$$

All the elements of the subspaces corresponding to $H^0(W,\ \mathcal{L})$, $H^0(W,\ \mathcal{L}\otimes\pi^*\mathcal{M})$ and $H^0(W,\ \mathcal{L}\otimes\pi^*\mathcal{M}^{\otimes 2})$ are G-semi-invariant, and hence these subspaces correspond to the eigenspaces of the isomorphism T^*_σ on $H^0(\tilde{W},\ \Phi^*\mathcal{L})$. The eigenspace $H^0(\tilde{W},\ \Phi^*\mathcal{L})^G$ for the eigenvalue 1 corresponds to $H^0(W,\ \mathcal{L})$, and is the image of the injection $H^0(W,\ \mathcal{L})\hookrightarrow H^0(\tilde{W},\ \Phi^*\mathcal{L})$. q.e.d.

We investigate the base locus $\mathrm{Bs}\,\mathcal{U}$ of $\mathcal{U}$. To do so, we describe the action of G on $H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)$.

We choose and fix $0\neq X_i\in H^0(\tilde{W},\mathcal{O}_{\tilde{W}}(\tilde{T})\otimes\tilde{\pi}^*L_i^{-1}) = H^0(\tilde{C},\ \tilde{E}\otimes L_i^{-1}) \cong H^0(\tilde{C},\ L_0\otimes L_i^{-1})\oplus H^0(\tilde{C},\ L_1\otimes L_i^{-1})\oplus H^0(\tilde{C},\ L_2\otimes L_i^{-1}) \cong \mathbf{C}$ $(i=0,\ 1,\ 2)$ such that $X_1 = T^*_\sigma X_0$ and $X_2 = T^*_{2\sigma}X_0$ hold. Then any $\Psi\in H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)$ can be written as

$$\Psi = \sum_{\substack{\alpha,\,\beta,\,\gamma\geq 0\\ \alpha+\beta+\gamma=4}} \psi_{\alpha\beta\gamma}X_0^\alpha X_1^\beta X_2^\gamma, \qquad \psi_{\alpha\beta\gamma}\in H^0(\tilde{C},\ L_0^{\otimes(\alpha-1)}\otimes L_1^{\otimes(\beta-1)}\otimes L_2^{\otimes(\gamma-1)}).$$

Since we have

$$T^*_\sigma\Psi = \sum_{\substack{\alpha,\,\beta,\,\gamma\geq 0\\ \alpha+\beta+\gamma=4}} (T^*_\sigma\psi_{\alpha\beta\gamma})X_1^\alpha X_2^\beta X_0^\gamma,$$

we get $\Psi\in H^0(\tilde{W},\ (4\tilde{T})\otimes\tilde{\pi}^*\det\tilde{E}^\vee)^G$ if and only if $T^*_\sigma\psi_{\alpha\beta\gamma} = \psi_{\gamma\alpha\beta}$ $(\alpha,\ \beta,\ \gamma\geq 0,\ \alpha+\beta+\gamma=4)$.

Let $\Lambda : \tilde{C}\to\mathrm{Pic}^0(\tilde{C})$ be defined by $\Lambda(y) := T^*_y L_0\otimes L_0^{-1}$ for $y\in C$ where T_y is the translation by y on $\tilde{C}$. Then it is a group homomorphism by the theorem of square [16]. Since we have $L_1 = Ł_0\otimes\Lambda(\sigma)$, $L_2 = L_0\otimes\Lambda(2\sigma)$ and $\Lambda(3\sigma) = \Lambda(0) = \mathcal{O}_{\tilde{C}}$, we have isomorphisms

$$\begin{array}{ccccccccc} L_0^{\otimes 3}\otimes L_1^{-1}\otimes L_2^{-1} & \cong & L_0 & \cong & L_1^{\otimes 2}\otimes L_2^{-1} & \cong & L_1^{-1}\otimes L_2^{\otimes 2} & \cong & L_0^{-1}\otimes L_1\otimes L_2 \\ L_0^{-1}\otimes L_1^{\otimes 3}\otimes L_2^{-1} & \cong & L_1 & \cong & L_0^{-1}\otimes L_2^{\otimes 2} & \cong & L_0^{\otimes 2}\otimes L_2^{-1} & \cong & L_0\otimes L_1^{-1}\otimes L_2 \\ L_0^{-1}\otimes L_1^{-1}\otimes L_2^{\otimes 3} & \cong & L_2 & \cong & L_0^{\otimes 2}\otimes L_1^{-1} & \cong & L_0^{-1}\otimes L_1^{\otimes 2} & \cong & L_0\otimes L_1\otimes L_2^{-1}. \end{array}$$

Hence we have

$$\begin{aligned} &\psi_{400},\ \psi_{211},\ \psi_{130},\ \psi_{103},\ \psi_{022}\in H^0(\tilde{C},\ L_0) \\ &\psi_{040},\ \psi_{121},\ \psi_{013},\ \psi_{310},\ \psi_{202}\in H^0(\tilde{C},\ L_1) \\ &\psi_{004},\ \psi_{112},\ \psi_{301},\ \psi_{031},\ \psi_{220}\in H^0(\tilde{C},\ L_2). \end{aligned}$$

Since we assume $d=2$, we have $\dim H^0(\tilde{C},\ L_i) = 2$ $(i=0,\ 1,\ 2)$ by the Riemann-Roch theorem, the Serre duality and $\mathcal{O}_{\tilde{C}}(K_{\tilde{C}})\cong\mathcal{O}_{\tilde{C}}$.

Let $\{s_1,\ s_2\} \subset H^0(\tilde{C},\ L_0)$ be a basis as a $\mathbf{C}$-vector space, and denote $t_j := T_\sigma^* s_j \in H^0(\tilde{C},\ L_1)$, $u_j := T_{2\sigma}^* s_j \in H^0(\tilde{C},\ L_2)$ $(j = 1,\ 2)$. Then we can choose a basis of $H^0(\tilde{W},\ \mathcal{O}_{\tilde{W}}(4\tilde{T}) \otimes \tilde{\pi}^* \det \tilde{E})^G$ consisting of the following ten elements

$$\begin{aligned}
\Psi_{1j} &:= s_j X_0^4 + t_j X_1^4 + u_j X_2^4, \\
\Psi_{2j} &:= s_j X_0^2 X_1 X_2 + t_j X_0 X_1^2 X_2 + u_j X_0 X_1 X_2^2, \\
\Psi_{3j} &:= s_j X_0 X_1^3 + t_j X_1 X_2^3 + u_j X_0^3 X_2, \\
\Psi_{4j} &:= s_j X_0 X_2^3 + t_j X_0^3 X_1 + u_j X_1^3 X_2, \\
\Psi_{5j} &:= s_j X_1^2 X_2^2 + t_j X_0^1 X_2^2 + u_j X_0^2 X_1^2
\end{aligned}$$

for $j = 1,\ 2$.

Lemma 4.31 *We can choose the basis $\{s_1,\ s_2\}$ of $H^0(\tilde{C},\ L_0)$ so that $s_j(p)t_j(p)u_j(p) \neq 0$ holds for any $p \in \tilde{C}$ and for at least one of $j = 1,\ 2$. Furthermore, we have $s_j(p)s_j(p')s_j(p'') \neq 0$, where $p' := T_\sigma(p)$ and $p'' := T_{2\sigma}(p)$.*

Proof To avoid confusion in this proof, we denote by (q) the divisor on $\tilde{C}$ determined by $q \in \tilde{C}$. Let $(q_1') + (q_2')$ be the divisor defined by a global section $s \in H^0(\tilde{C},\ L_0)$, and $p_1 \in \tilde{C}$ a point satisfying $2p_1 = q_1' + q_2'$ with respect to the group addition. Then we have $L_0 \cong \mathcal{O}_{\tilde{C}}(2(p_1))$ by Abel's theorem. Since we assume $\deg L_0 = 2$, there exists a point $p_2 \in \tilde{C} \setminus \{p_1\}$ with $L_0 \cong \mathcal{O}_{\tilde{C}}(2(p_1)) \cong \mathcal{O}_{\tilde{C}}(2(p_2))$. If we denote $p_i' := T_{-\sigma}(p_i)$ and $p_i'' := T_{-2\sigma}(p_i)$ $(i = 1,\ 2)$, we have $\{p_1,\ p_1',\ p_1''\} \cap \{p_2,\ p_2',\ p_2''\} = \emptyset$. Indeed, since $p_1' = p_1 - \sigma$ holds (where $-$ is the group subtraction on $\tilde{C}$), if $p_1' = p_2$ holds, then we obtain $2p_2 = 2p_1 - 2\sigma$ by doubling both sides of the equality. On the other hand, since $2p_1 = 2p_2$ holds by Abel's theorem, we obtain $2\sigma = 0$ and this contradicts the definition of σ. We can obtain the same results in the other cases.

Let $s_1,\ s_2 \in H^0(\tilde{C},\ L_0)$ be the global sections defining the divisors $2(p_1),\ 2(p_2)$ respectively, and denote $t_j := T_\sigma^* s_j$, and $u_j := T_{2\sigma}^* s_j$ $(j = 1,\ 2)$. Then since we have $\mathrm{supp}(s_j) = \{p_j\}$, $\mathrm{supp}(t_j) = \{p_j'\}$ and $\mathrm{supp}(u_j) = \{p_j''\}$ $(j = 1,\ 2)$, if we choose $\{s_1,\ s_2\}$ as a basis of $H^0(\tilde{C},\ L_0)$, then one of s_1 and s_2 satisfies the condition of the lemma for any point except $p_1,\ p_2$. Moreover, s_2 satisfies the condition of the lemma for p_1 while s_1 satisfies the condition of the lemma for p_2. q.e.d.

We choose a basis $\{s_1,\ s_2\} \subset H^0(\tilde{C},\ L_0)$ satisfying the condition of Lemme 4.31, fix an arbitrary point $p \in \tilde{C}$, and denote $p' := T_\sigma(p)$ and $T_{2\sigma}(p)$. We assume that $j \in \{1,\ 2\}$ satisfies $s_j(p)s_j(p')s_j(p'') \neq 0$. Let us restrict Ψ_{ij} $(i = 1, \cdots,\ 5,\ j = 1,\ 2)$ to a fiber of $\tilde{\pi}$ over p, and investigate whether they have common solutions on the fiber. Note that Ψ_{2j} can be decomposed into the product of four linear forms as $\Psi_{2j} = X_0 X_1 X_2 (s_j X_0 + t_j X_1 + u_j X_2)$.We consider the condition so that each of these linear forms and other Ψ_{ij} have common solutions.

Lemma 4.32 *If we fix $j \in \{1,\ 2\}$ satisfying $s_j(p)s_j(p')s_j(p'') \neq 0$, then $X_i = 0,\ \Psi_{1j} = 0$ and $\Psi_{3j} = 0$ do not have common solutions for any $i = 0,\ 1,\ 2$.*

Proof If we substitute $X_0 = 0$ into $\Psi_{1j} = 0$, we have $t_j X_1^4 + u_j X_2^4 = 0$. Therefore, if we let $t^{1/4}$ and $u^{1/4}$ be one of the fourth roots of $t_j(p)$ and $u_j(p)$, respectively, and ζ_8 a primitive eighth root of 1, then $(p,\ (0 : u^{1/4} : \zeta_8^k t^{1/4})$ $(k = 1,\ 3,\ 5,\ 7)$ is a common solution of $X_0 = 0$ and $\Psi_{1j} = 0$. On the other hand, if we substitute $X_0 = 0$ into $\Psi_{3j} = 0$, we obtain $t_j X_1 X_2^3 = 0$, and since $t_j(p) \neq 0$ holds, $(p,\ (0 : 1 : 0))$ and $(p,\ (0 : 0 : 1))$ are the common solutions of $X_0 = 0$ and $\Psi_{3j} = 0$. Since we have $tu \neq 0$, we see that $X_0 = 0$, $\Psi_{1j} = 0$ and $\Psi_{3j} = 0$ have no common solution. We can obtain the same results for $X_1 = 0$ and $X_2 = 0$. q.e.d.

In view of Lemma 4.32, we consider only the solutions satisfying $X_0 X_1 X_2 \neq 0$ in the rest of our argument. Denote $\Psi_{0j} := s_j X_0 t_j X_1 u_j X_2$.

Lemma 4.33 *If we fix $j \in \{1,\ 2\}$ with $s_j(p)s_j(p')s_j(p'') \neq 0$, then $(p,\ (1 : a : b))$ is a common solution of $\Psi_{ij} = 0$ $(i = 0,\ 1,\ 3,\ 4,\ 5)$ if and only if a, b are cube roots of 1, and $s_j(p) + at_j(p) + bu_j(p) = 0$.*

Proof Since we have $\Psi_{1j} + \Psi_{3j} + \Psi_{4j} = \Psi_{0j}(X_0^3 + X_1^3 + X_2^3)$, we may exclude Ψ_{1j} from our consideration. Since we have

$$X_0^5 \Psi_{0j} - X_0^2(\Psi_{3j} + \Psi_{4j}) + X_1 X_2 \Psi_{5j} = s_j(X_0^3 - X_1^3)(X_0^3 - X_1^3),$$

if there exist common solutions, one of $X_1^3 = X_0^3$ and $X_2^3 = X_0^3$ must hold. If $X_1^3 = X_0^3$ holds, however, since we have $X_0^3 \Psi_{0j} - \Psi_{4j} = s_j X_0(X_0^3 - X_2^3)$ and since we assume $s_j(p) \neq 0$ and $X_0 \neq 0$, we obtain $X_2^3 = X_0^3$. Similarly, if we assume $X_2^3 = X_0^3$, we have $X_0^3 \Psi_{0j} - \Psi_{3j} = s_j X_0(X_0^3 - X_1^3)$ and we obtain $X_1^3 = X_0^3$. Hence if there exist common solutions, they must satisfy $X_0^3 = X_1^3 = X_2^3$. If we denote by ω a primitive cube root of 1, then the common solutions satisfying this condition are

$$\begin{array}{lll} (p,\ (1:1:1)) & (p,\ (1:1:\omega)) & (p,\ (1:1:\omega^2)) \\ (p,\ (1:\omega:1)) & (p,\ (1:\omega:\omega)) & (p,\ (1:\omega:\omega^2)) \\ (p,\ (1:\omega^2:1)) & (p,\ (1:\omega^2:\omega)) & (p,\ (1:\omega^2:\omega^2)). \end{array}$$

If $(p,\ (1 : 1 : 1))$ is a common solution, we obtain $s_j(p) + t_j(p) + u_j(p) = 0$ by substituting $(p,\ (1 : 1 : 1))$ into $\Psi_{ij} = 0$ $(i = 0,\ 1,\ 3,\ 4,\ 5)$. We can obtain the same result in the other cases. q.e.d.

Proposition 4.34 *$\mathcal{U}$ has no base point. Hence a general member of $\mathcal{U}$ is irreducible and nonsingular by Bertini's theorem.*

Proof Assume that $(p,\ (1 : a : b))$ is a base point of $\mathcal{U}$. If $g : \tilde{C} \to \mathbf{P}^1$ is the holomorphic mapping defined by the complete linear system of L_0, then we have $g^*\mathcal{O}_{\mathbf{P}^1}(1) \cong L_0$. Denote $p' := T_\sigma(p)$ and $p'' := T_{2\sigma}(p)$.

First assume that $g(p) = g(p')$ holds. Since $(p,\ (1 : a : b))$ is a base point, $s(p) + at(p) + bu(p) = 0$ holds for any $s \in H^0(\tilde{C},\ L_0)$ by Lemma 4.33, where $t := T_\sigma^* s$ and

$u := T_{2\sigma}^* s$. Since we have $L_0 \cong \mathcal{O}_{\tilde{C}}(-p-p')$, if we let $s' \in H^0(\tilde{C},\ L_0)$ be a global section defining the divisor $p+p'$, then $s'(p) = 0,\ s'(p') = 0$ and $s'(p'') \neq 0$ hold, and hence $s'(p) + as'(p') + bs'(p'') \neq 0$. This contradicts the assumption. We can obtain the same results when $g(p') = g(p'')$ or $g(p'') = g(p)$ holds.

Next, we assume that $g(p),\ g(p')$ and $g(p'')$ are pairwise different. We may assume that the homogeneous coordinates of these points are $(1:0),\ (1:1),\ (0:1)$, respectively. In terms of the homogeneous coordinate $(z_0 : z_1)$ with $z_0,\ z_1 \in H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1))$, a global section $\xi \in H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1))$ can be written as $\xi = Az_0 + Bz_1$ for some $A,\ B \in \mathbf{C}$. Since we have $\xi(1:0) = A,\ \xi(1:1) = A+B$ and $\xi(0:1) = B$, if we choose $A,\ B \in \mathbf{C}$ with $(a+1)A + (b+1)B \neq 0$ and if we let $s \in H^0(\tilde{C},\ L_0)$ be the image of ξ under the natural isomorphism $H^0(\mathbf{P}^1,\ \mathcal{O}_{\mathbf{P}^1}(1)) \xrightarrow{\sim} H^0(\tilde{C},\ L_0)$, then s satisfies $s(p) + as(p') + bx(p'') \neq 0$. This contradicts the assumption. q.e.d.

Let $\tilde{S} \in \mathcal{U}$ be a general member. We may assume that $\tilde{S}$ is irreducible and nonsingular. $S := \Phi(\tilde{S})$ is contained in the complete linear system $|4T - \pi^* D|$ on W, and $\Phi_{|\tilde{S}} : \tilde{S} \to S$ is the quotient with respect to the restriction of the action of G on $\tilde{W}$. On the other hand, this action is compatible with the action of G on $\tilde{C}$, and hence has no fixed point. Therefore S is irreducible and nonsingular as well. q.e.d.

Remark Instead of our argument in §4.3.1, we can use the above argument also in the case $d \geq 4$ and $d \not\equiv 0 \pmod 3$.

4.3.5 The canonical mapping

In this section, we study the canonical mappings of those surfaces whose existences were shown in §§4.3.1–4.3.4.

Let E be an indecomposable locally free sheaf of rank 3, and degree d over an elliptic curve C, i.e., $E \in \mathcal{E}(3,\ d)$. If $d = 1$, we have $p_g(S) = 1$ and the canonical mapping $\Phi_{|K_S|}$ is trivial. Moreover, we showed the non-existence in the case $d < 0$ in §§4.3.1–4.3.2. Hence we may assume $d \geq 2$.

Since we have $\Phi_{|T|}|_{S'} \circ \psi = \Phi_{|K_S|}$, we investigate $\Phi_{|T|}$.

Lemma 4.35 *Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to $E \in \mathcal{E}_C(3,\ d)$, and T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$. If we assume $d \geq 4$, then we have $\operatorname{Bs}|T| = \emptyset$.*

Proof Let F be any fiber of π, and denote $q := \pi(F)$. Since we have $\dim H^1(W,\ \mathcal{O}_W(T-F)) = \dim H^1(C,\ E \otimes \mathcal{O}_C(-q)) = 0$, the restriction mapping

$$H^0(W,\ \mathcal{O}_W(T)) \to H^0(F,\ \mathcal{O}_F(T)) \cong H^0(\mathbf{P}^2,\ \mathcal{O}_{\mathbf{P}^2}(1))$$

is surjective. Since the complete linear system of $\mathcal{O}_{\mathbf{P}^2}(1)$ has no base point, $|T|$ has no base point on F. Since F is any fiber, we are done. q.e.d.

Corollary 4.36 *Let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to a locally free sheaf $E \in \mathcal{E}_C(3,\ 4)$, and T a tautological divisor of W satisfying $\pi_*\mathcal{O}_W(T) \cong E$. Then we have* $\deg \Phi_{|T|} = 4$.

Proof $|T|$ has no base point by Lemma 4.35. Since $T^3 = 4 > 0$, the image of W under $\Phi_{|T|}$ is 3-dimensional, and hence $\Phi_{|T|}$ gives a covering of W onto $\mathbf{P}^3$ of degree 4. q.e.d.

Corollary 4.37 *In the notation of Corollary* 4.36*, any irreducible and nonsingular members of $|4T - \pi^*D|$ are canonical surfaces.*

Proof Let $S \in |4T - \pi^*D|$ be a general nonsingular member. Since $\mathrm{Bs}\,|T| = \emptyset$ by Lemma 4.35, $\Phi_{|K_S|}$ is a morphism. Since $\deg \Phi_{|T|} = 4$ by Corollary 4.36, the degree of $\Phi_{|K_S|}$ is 1, 2, 3 or 4.

Since $K_S^2 = T^2S = 12$ holds, if $\deg \Phi_{|K_S|} = 4$, then $S'' := \Phi_{|K_S|}(S) \subset \mathbf{P}^3$ is a cubic surface. Hence, if we let $H \subset \mathbf{P}^3$ be a hyperplane, then we have $S'' \sim 3H$. Since $\Phi_{|T|}^*H \sim T$ holds, we have $\Phi_{|T|}^*S'' \sim T$, which is absurd since $S \sim 4T - \pi^*D$.

If $\deg \Phi_{|K_S|} = 3$, then S'' is a quartic surface. Hence $S'' \sim 4H$ holds, and we have $\Phi_{|T|}^*S'' \sim 4T$. Therefore, there exist fibers $F_1,\ F_2,\ F_3,\ F_4$ of π satisfying $\Phi_{|T|}^*S'' = S + F_1 + F_2 + F_3 + F_4$. Since we have $\deg \Phi_{|T|} = 4$ and since we assume $\deg \Phi_{|K_S|} = \deg \Phi_{|K_S|}|_S = 3$, we see that $\Phi_{|T|}$ is a birational morphism of $F_1 \cup F_2 \cup F_3 \cup F_4$ onto its image. This means that the image is not irreducible, and we obtain a contradiction.

Finally, we show that the case $\deg \Phi_{|K_S|} = 2$ does not occur. Let $p,\ p' \in C$ be two distinct general points. Furthermore, denote $F_p := \pi^{-1}(p)$ and $F_{p'} := \pi^{-1}(p')$, and let T_p and $T_{p'}$ be the relative hyperplanes of W satisfying $T \sim T_p + F_p \sim T_{p'} + F_{p'}$. Since $p,\ p' \in C$ and $S \in |4T - \pi^*D|$ are generic, $S \cap T_p \cap F_{p'}$, $S \cap T_{p'} \cap F_p$ and $S \cap T_p \cap T_{p'}$ all consist of four distinct points set-theoretically. Since any fiber of π is mapped onto its image in $\mathbf{P}^3$ by $\Phi_{|T|}$, if $\deg \Phi_{|K_S|} = 2$, then some point of $S \cap T_p \cap F_{p'}$ and some point of $S \cap T_{p'} \cap F_p$ are mapped to the same point by $\Phi_{|T|}$. Hence if we fix any point $q \in S \cap T_p \cap F_{p'}$ and any point $q' \in S \cap T_{p'} \cap F_p$, we only have to find a member of $|T|$ containing q but not q'.

It is well-known that W is isomorphic to the symmetric product of C of degree 3. (cf., e.g., [6, pp.310–311] Let $\zeta : C \times C \times C \to W$ be the quotient morphism. Since the self-intersection number of the divisor $C \times C \times \{p\} + C \times \{p\} \times C + \{p\} \times C \times C$ of $C \times C \times C$ is 6 for any point $p \in C$, and since $\deg \zeta = 6$, the image of $(C \times C \times \{p\}) \cup (C \times \{p\} \times C) \cup (\{p\} \times C \times C)$ in W is a relative hyperplane with self-intersection number 1. Therefore, for a general point of W, there exist three distinct relative hyperplanes with self-intersection number one containing the point.

Since $p,\ p' \in C$ and $S \in |4T - \pi^*D|$ are general, there exist two distinct reltive hyperplanes T_p' and T_p'' diffrent from T_p and containing q. If F_p' and F_p'' are fibers of π

satisfying $T \sim T'_p + F'_p \sim T''_p + F''_p$, respectively, then one of $T'_p + F'_p$ and $T''_p + F''_p$ does not contain q'.

Hence $\Phi_{|K_S|}$ is a birational morphism onto its image. q.e.d.

Corollary 4.38 *Let $\pi : W \to C$ be the $\mathbf{P}^2$-bundle associated to $E \in \mathcal{E}_C(3,\ d)$, and T a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$. If we assume $d \geq 5$, then $\Phi_{|T|}$ is birational onto its image.*

Proof First, we consider the case $d \geq 7$. It suffices to show the existence of a member of $|T|$ which contains p and does not contain q for any pair of distinct points $p,\ q \in W$. If p and q are contained in the same fiber of π, we easily see that such a member exists by the proof of Lemma 4.35. Suppose p and q are contained in different fibers. $|T - F|$ has no base point in the fiber F containing p by Lemma 4.35. If we let $T_0 \in |T - F|$ be a member which does not contain q, then $T_0 + F \in |T|$ contains p and does not contain q. Hence $\Phi_{|T|}$ is injective.

When $d = 6$, we can show that $\Phi_{|T|}$ is birational onto the image using the same argument as above for points $p,\ q \in W$ contained in general fibers by Lemma 4.28

If $d = 5$, then since $5 = T^3 = \deg \Phi_{|T|} \deg \Phi_{|T|}(W)$ and $\deg \Phi_{|T|}(W) \geq 2$, we see that $\Phi_{|T|}$ is birational onto the image. q.e.d.

Corollary 4.39 *Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to $E \in \mathcal{E}_C(3,\ d)$, T a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$ and $D \in \mathrm{Div}(C)$ a divisor with $\det E \cong \mathcal{O}_C(D)$. If $d \geq 5$ holds, then the canonical mapping of the minimal resolution of a member of $|4T - \pi^*D|$, which is irreducible and has at most rational double points as singularities, is birational onto the image.*

Next, we investigate the canonical mapping in the case $p_g(S) = d = 3$. We use the notation of §4.3.3.

Recall that $\mathrm{Bs}\,|C_0 + \Gamma_0| = \mathrm{Bs}\,|4C_0 + \Gamma_0| = \{q_0\}$, where $q_0 := C_0 \cap \Gamma_0$, and that all the nonsingular members of $|C_0 + \Gamma_0|$ have the same tangent.

Proposition 4.40 *Let $\pi : W \to C$ be a $\mathbf{P}^2$-bundle associated to a locally free sheaf $E \in \mathcal{E}_C(3,\ 3)$ over an elliptic curve C, T the tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$, L the invertible sheaf with $E \cong L \otimes F_3$ and $p_0 \in C$ the point with $L \cong \mathcal{O}_C(p_0)$, and denote $F_0 := \pi^{-1}(p_0)$. Then the canonical mapping of a nonsingular member $S \in |4T - 3F_0|$ has degree* 8.

Proof Since $\mathrm{Bs}\,|T| = \mathrm{Bs}\,|4T - 3F_0| = \{q_0\}$, the canonical system $|K_S|$ of a general nonsingular member $S \in |4T - 3F_0|$ has one base point. If $\nu : \bar{W} \to W$ is the blowing-up at q_0, the complete linear system of the proper transform $\bar{T}$ of T by ν has one base point by Lemma 3.5. On the other hand, the proper transform $\bar{S}$ of S by ν does not go through the base point of $|\bar{T}|$ by Lemma 3.5. Hence, if we denote $\mathcal{E} := \nu^{-1}(q_0)$, we have

$$\begin{aligned}\deg \Phi_{|K_S|} &= \deg \Phi_{|K_{\bar{S}}|} = \bar{T}^2(4\bar{T} + 3\mathcal{E} - 3F_0) \\ &= 4\bar{T}^3 + 3\bar{T}^2\mathcal{E} - 3\bar{T}^2F_0 = 8 + 3 - 3 = 8.\end{aligned}$$

q.e.d.

Finally, we study the canonical mapping in the case $p_g(S) = 2$.

In §4.2.2, we proved the existence of a surface S with $K_S^2 = 3p_g(S)$, $q(S) = 1$ and $p_g(S) = 2$, but did not study the canonical mapping $\Phi_{|K_S|}$ in the case $E \cong E_0 \oplus L$, $(E_0 \in \mathcal{E}_C(2,\ 1),\ L \in \mathcal{E}_C(1,\ 1))$. On the other hand, we showed the existence of a surface S with the same invariants in the case $E \in \mathcal{E}_C(3,\ 2)$. We obtain the following result in these two cases:

Proposition 4.41 *Let E be one of the following:*

(1) $E := E_0 \oplus L$ *with* $E_0 \in \mathcal{E}_C(2,\ 1),\ L \in \mathcal{E}_C(1,\ 1)$.

(2) $E \in \mathcal{E}_C(3,\ 2)$.

Let $\pi : W := \mathbf{P}(E) \to C$ be the $\mathbf{P}^2$-bundle associated to E, T a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$ and $D \in \mathrm{Div}(C)$ a divisor with $\det E \cong \mathcal{O}_C(D)$. The canonical mapping of the minimal resolution of a member of $|4T - \pi^*D|$, which is irreducible and has at most rational double points as singularities, gives a linear pencil whose general members are irreducible nonsingular curves of genus 7.

Proof Let $S \in |4T - \pi^*D|$ be a general member. We may assume that S is irreducible and has at most rational double points as singularities. Since $H^0(S,\ \omega_S) \cong H^0(W,\ \mathcal{O}_W(T))$ and $\dim H^0(W,\ \mathcal{O}_W(T)) = 2$, the canonical mapping of S clearly gives a linear pencil. Therefore it suffices to show that the intersection $T \cap S$ is a nonsingular curve of genus 7 for a general member T of $|T|$.

Since $\omega_W \cong \mathcal{O}_W(-3T) \otimes \det E$, we have $\omega_T \cong (\mathcal{O}_W(-2T) \otimes \det E)|_T$. Since we may assume that T is irreducible and nonsingular, we have

$$\omega_Z \cong (\mathcal{O}_W(S) \otimes \mathcal{O}_W(-2T) \otimes \det E)|_Z \cong (\mathcal{O}_W(2T))|_Z,$$

where $Z := T \cap S$. Hence we have

$$g(Z) = \frac{1}{2}T(2T)(4T - \pi^*D) + 1 = 4T^3 - T^2\pi^*D + 1 = 7.$$

q.e.d.

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Faculty of General Education
Ichinoseki National College of Technology
Ichinoseki 021
Japan

e-mail: tomokuni@ichinoseki.ac.jp