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COSET CONSTRUCTIONS OF CONFORMAL BLOCKS

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COSET CONSTRUCTIONS OF CONFORMAL BLOCKS

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Contents

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Chapter I

Conformal Field Theory
— its origin and developments

Typeset by $\mathcal{AMS}$-TEX

The scaling hypothesis in the theory of second-order phase transition says that at the critical point, a field $\phi_i(x)$ having an anomalous dimension h_i transforms like

$$\phi_i(x) \to \lambda^{h_i} \phi_i(\lambda x), \tag{0.1}$$

under the scaling transformation

$$x \to \lambda x \tag{0.2}$$

of the space. It means that the N-point function $\langle \phi_1(x_1) \cdots \phi_{N}(x_{N}) \rangle$ does not change under the transformation (0.1). The scaling invariance, for example, fixes the form of the 2-point functions:

$$\langle \phi_i(x_1)\, \phi_i(x_2) \rangle \propto |x_1 - x_2|^{-2h_i}. \tag{0.3}$$

A. M. Polyakov pointed out that the N-point function at the critical point has not only the scaling invariance but also the conformal invariance. The conformal symmetry leads to stronger constraints on the N-point functions. For example, the 3-point functions must have the following simple form

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle \propto |x_1 - x_2|^{-h_1-h_2+h_3} |x_2 - x_3|^{-h_2-h_3+h_1} |x_1 - x_3|^{-h_3-h_1+h_2}. \tag{0.4}$$

In 1984, A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov [BPZ] introduced the Virasoro algebra into the study of two-dimensional quantum field theory with conformal symmetry. They made the best use of the fact that the group of conformal transformations in two-dimensional space is infinite dimensional. Symmetry of the theory with respect to the conformal group was suitably expressed in terms of representation theory of the Virasoro algebra, the central extension of the Lie algebra of the algebraic vector fields on the complex plane punctured at the origin. They showed that the basic fields in the theory are classified according to the irreducible representations of the Virasoro algebra and the N-point correlation functions are completely determined by the conformal invariance.

They also discovered remarkable "exactly solvable" theories associated with some classes of the degenerate representations of the Virasoro algebra. In such a theory, a finite number of basic fields are involved and their anomalous dimensions are known exactly. All the N-point functions satisfy hypergeometric-type linear differential equations. Such a theory is called the "minimal" conformal quantum field theory. It was argued in [BPZ] that such minimal theories should describe critical fluctuations in two-dimensional statistical models at the second order phase transition point.

The two-dimensional conformal field theories have been generalized to the ones having larger conformal symmetries than the Virasoro algebra. In particular, V. G. Knizhnik and A. B. Zamolodchikov [KZ] developed the conformal field theory with the symmetry of affine Lie algebra, called the Wess-Zumino-Witten model. They introduced the notion of primary fields with gauge symmetry and gave the differential equations for the N-point functions known as the Knizhnik-Zamolodchikov equations.

Mathematically rigorous foundations for the Wess-Zumino-Witten model on the Riemann sphere were given by A. Tsuchiya and Y. Kanie [TK], where the notion of primary fields were defined in terms of representation theory, the N-point functions were characterized by a system of equations including the Knizhnik-Zamolodchikov equations, and the monodromies of the equations were determined.

This thesis is devoted to the study of the minimal conformal field theories in connection with the Wess-Zumino-Witten models. Before going into the details of the results, we shall present an introduction to some notions of conformal field theories.

Throughout of the article, $\mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{N}$ denote the ring of integers, the field of rational numbers, the field of complex numbers (the complex plane), the set of positive integers respectively, and $\mathbb{P}^1$ denote the Riemann sphere (complex projective line). We also denote by $\mathbb{C}^\times := \mathbb{C} - \{0\}$ the complex plane punctured at the origin.

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1. Review of Conformal Field Theory

Let us sketch, according to [BPZ], some basic ideas of conformal field theory in the physical language. Although some statements in this subsection seem to be mathematically obscure, we believe that they will illuminate the mathematical formulation that follows them.

Note that we shall only deal with the *chiral* (or holomorphic) theory, the half of the physical theory, although the real theory must be built upon both the chiral and the anti-chiral theories.

1.1. Primary fields. In the complex plane $\mathbb{C}$ with the coordinate z and the metric $|\,dz\,|^2$, consider an infinitesimal conformal transformation

$$z \mapsto z + \varepsilon(z), \tag{1.1}$$

where $\varepsilon(z)$ is an infinitesimal analytic function. The variation of an arbitrary field $A(z)$ under the transformation (1.1) is written in the form

$$\delta_\varepsilon A(z) = \frac{1}{2\pi i} \oint_{C_z} d\zeta\, \varepsilon(\zeta)\, T(\zeta) A(z), \tag{1.2}$$

where $T(\zeta)$ is the stress-energy tensor and C_z is a small contour surrounding the point z.

If a field $\phi(z)$ transforms in the following simple form

$$\delta_\varepsilon \phi(z) = \varepsilon(z) \frac{\partial}{\partial z} \phi(z) + h\varepsilon'(z)\, \phi(z), \quad h, \in \mathbb{C} \tag{1.3}$$

we shall call the field $\phi(z)$ a *primary field* of a conformal dimension h. The primary fields are the basic ingredients of the theory.

It is useful to introduce the operators $L_n, n \in \mathbb{Z}$ as coefficients of the Laurent expansion

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}. \tag{1.4}$$

The operators $L_n, n \in \mathbb{Z}$ are shown to satisfy the commutation relations:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \delta_{m+n,0}\, \frac{m^3-m}{12}\, c, \tag{1.5}$$

where $c \in \mathbb{C}$ is a parameter of the theory called the central charge. This is known as the commutation relations of the Virasoro algebra. The transformation law (1.3) is equivalent to the following commutation relations:

$$[L_n, \phi_h(z)] = z^n (z\frac{\partial}{\partial z} + h(n+1))\, \phi_h(z), \quad n \in \mathbb{Z}. \tag{1.6}$$

The vacuum $|\,\mathrm{vac}\,\rangle$ is the ground state of the operator L_0. The vacuum satisfies the equation

$$L_n|\,\mathrm{vac}\,\rangle = 0, \quad \text{if} \quad n \geq -1. \tag{1.7}$$

The dual vacuum $\langle\,\mathrm{vac}\,|$ satisfies

$$\langle\,\mathrm{vac}\,|L_n = 0, \quad \text{if} \quad n \leq 1. \tag{1.8}$$

Let $\phi_h(z)$ be a primary field of a conformal dimension h. Let us introduce the vector (the primary state)

$$|\,h\rangle = \lim_{z\to 0} \phi_h(z)|\,\mathrm{vac}\,\rangle. \tag{1.9}$$

Using (1.6), one can get

$$L_n|\,h\rangle = 0 \quad \text{if} \quad n > 0, \quad L_0|\,h\rangle = h|\,h\rangle. \tag{1.10}$$

Let us define the "out" primary states by the formula

$$\langle\,h\,| = \lim_{z\to\infty} \langle\,\mathrm{vac}\,|\,\phi_h(z)\, z^{2L_0}. \tag{1.11}$$

These vectors satisfy the equations

$$\langle\,h\,|L_n = 0 \quad \text{if} \quad n < 0, \quad \langle\,h\,|L_0 = h\langle\,h\,|. \tag{1.12}$$

The orthogonality condition

$$\langle\,h|\,h'\rangle = \delta_{h,h'} \tag{1.13}$$

holds.

1.2. Descendants. From the primary field $\phi_h(z)$, infinitely many other fields called secondary fields or descendants

$$\phi_h^{(-k_1,-k_2,\dots,-k_n)}(z) \quad \text{for} \quad k_j \geq 1 \;\; \text{and} \;\; n = 1, 2, \dots, \tag{1.14}$$

come into the theory. The fields (1.14) are defined as

$$\phi_h^{(-k_1,-k_2,\dots,-k_n)}(z) = L_{-k_1}(z) \cdots L_{-k_n}(z)\, \phi_h(z), \tag{1.15}$$

where the operators $L_{-k}(z)$ are given by the contour integrals

$$L_{-k}(z) = \frac{1}{2\pi i} \oint \frac{d\zeta\, T(\zeta)}{(\zeta - z)^{k+1}}. \tag{1.16}$$

The contours associated with each of the operators $L_{-k_j}(z)$ in (1.17) enclose the point z as well as the points $\zeta_{j+1}, \zeta_{j+2}, \dots, \zeta_n$.

An infinite set of the fields (1.14) constitutes the conformal family $[\phi_h]$, which form a representation of the Virasoro algebra.

Note that

$$\phi_h^{(-1,-k_1,-k_2,\dots,-k_n)}(z) = \frac{\partial}{\partial z}\, \phi_h^{(-k_1,-k_2,\dots,-k_n)}(z). \tag{1.17}$$

1.3. Conformal Ward identities. The fundamental problem in the theory is to calculate the N-point correlation function

$$\langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle := \langle \, \mathrm{vac} \,|\phi_1(z_1)\, \phi_2(z_2) \cdots \phi_N(z_N)|\, \mathrm{vac} \,\rangle \tag{1.18}$$

of the primary fields $\phi_1, \dots, \phi_N$. The conformal Ward identities enable us to calculate correlation functions of arbitrary descendants from the correlators of the from (1.18).

The conformal Ward identity is of the following form:

$$\begin{aligned}
&\langle T(\zeta)\, T(\zeta_1) \cdots T(\zeta_M)\, \phi_1(z_1)\, \phi_2(z_2) \cdots \phi_N(z_N) \rangle \\
&= \left\{ \sum_{i=1}^{N} \left\{ \frac{h_i}{(\zeta - z_i)^2} + \frac{1}{\zeta - z_i} \frac{\partial}{\partial z_i} \right\} + \sum_{j=1}^{M} \left\{ \frac{2}{(\zeta - \zeta_j)^2} + \frac{1}{\zeta - \zeta_j} \frac{\partial}{\partial \zeta_j} \right\} \right\} \\
&\quad \times \langle T(\zeta_1) \cdots T(\zeta_M)\, \phi_1(z_1)\, \phi_2(z_2) \cdots \phi_N(z_N) \rangle \\
&\quad + \sum_{j=1}^{M} \frac{c}{(\zeta - \zeta_j)^4} \langle T(\zeta_1) \cdots T(\zeta_{j-1}) T(\zeta_{j+1}) \cdots T(\zeta_M) \phi_1(z_1) \cdots \phi_N(z_N) \rangle,
\end{aligned} \tag{1.19}$$

where $\phi_1, \dots, \phi_N$ are the primary fields of conformal dimensions $h_1, \dots, h_N$ respectively. By the formula and (1.16), vacuum expectation values of the composition of arbitrary secondary fields can be expressed in terms of the correlators of the primary fields (1.18). Hence, all the information about the conformal field theory is contained in the correlators of the primary fields (1.18).

1.4. Degenerate theories.

We saw that a primary field $\phi_h(z)$ of a conformal dimension h generates a conformal family $[\phi_h]$, which is a representation of the Virasoro algebra. This representation is known as the Verma module. It is irreducible unless the conformal dimension h takes some special values. For these values, the representation contain a singular vector $|\,\chi\,\rangle$ i.e.

$$L_n|\,\chi\,\rangle = 0, \quad \text{if} \quad n > 0, \quad L_0|\,\chi\,\rangle = (h+d)|\,\chi\,\rangle \tag{1.20}$$

for some $d > 0$. In the conformal family $[\phi_h]$, there is the corresponding secondary field $\chi(z)$, which possesses the conformal property of a primary field of the conformal dimension $h+d$. Such a field is called the *null field*. Since $|\,\chi\,\rangle$ is orthogonal to any state, it follows that any correlation function of the form

$$\langle \chi(z)\,\phi_1(z_1)\cdots\phi_{N}(z_{N})\rangle \tag{1.21}$$

vanishes. Therefore the null field can be regarded as zero in the theory:

$$\chi(z) = 0. \tag{1.22}$$

This condition implies that any secondary field generated from $\chi(z)$ is also zero:

$$[\chi] = 0, \tag{1.23}$$

where $[\chi]$ is the conformal family generated by the primary field $\chi(z)$. The conformal field theory with the constraint (1.23) for all the null field is called a degenerate conformal field theory.

The Ward identity implies that for any $\{k\} = (-k_1, \ldots, -k_n)$, there exists a linear partial differential operator $\mathcal{P}^{\{k\}}(z, z_1, \ldots, z_{N})$ such that

$$\langle \phi_h^{\{k\}}(z)\,\phi_1(z_1)\cdots\phi_{N}(z_{N})\rangle = \mathcal{P}^{\{k\}}(z, z_1, \ldots, z_{N})\,\langle \phi_h(z)\,\phi_1(z_1)\cdots\phi_{N}(z_{N})\rangle. \tag{1.24}$$

Since the null field $\chi(z)$ is a linear combination of the secondary fields $\phi_h^{\{k\}}(z)$, the equation (1.22) leads to a differential equation for the function $\langle \phi_h(z)\,\phi_1(z_1)\cdots\phi_{N}(z_{N})\rangle$.

For example, the vector

$$|\,\chi\,\rangle = (L_{-1}^2 - \frac{2(2\,h+1)}{3}L_{-2})|\,h\,\rangle \tag{1.25}$$

satisfies (1.20) with $d = 2$ provided that h takes any of the two values

$$h = \frac{5 - c \pm \sqrt{(c-1)(c-25)}}{16}. \tag{1.26}$$

The corresponding null field is

$$\chi(z) = \frac{\partial^2}{\partial z^2}\phi_h(z) - \frac{2(2h+1)}{3}\phi_h^{(-2)}(z). \tag{1.27}$$

Therefore we have the differential equation

$$\left\{\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N}\frac{h_i}{(z-z_i)^2} - \sum_{i=1}^{N}\frac{1}{z-z_i}\frac{\partial}{\partial z}\right\}\langle \phi_h(z)\,\phi_1(z_1)\cdots\phi_{N}(z_{N})\rangle = 0, \tag{1.28}$$

where $h_1, \ldots, h_{N}$ are the conformal dimensions of the primary fields $\phi_1, \ldots, \phi_{N}$.

1.5. The minimal models. For a pair of coprime positive integers p, q greater than 2, a highly degenerate class of the representations of the Virasoro algebra exists. Define the central charge

$$c(p,q) = 1 - \frac{6(p-q)^2}{pq}. \tag{1.29}$$

In the theory with the central charge $c(p,q)$, only the finite number of conformal families are involved. We shall call such a theory a *minimal* conformal field theory or simply a minimal model.

Conformal dimensions of primary fields in the theory are given:

$$h_{r,s} = h_{r,s}(p,q) = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \quad (r, s \in \mathbb{Z},\ 0 < r < q\,, 0 < s < p). \tag{1.30}$$

Let $\phi_{r,s}(z)$ be the primary field of the conformal dimension $h_{r,s}$. The operators in the conformal families $[\phi_{r,s}]$ $r, s \in \mathbb{Z}, 0 < r < q\,, 0 < s < p$ form "the closed operator algebra". As $h_{r,s} = h_{q-r,p-s}$, we have $\phi_{r,s}(z) = \phi_{q-r,p-s}(z)$ and the number of distinct primary fields in the theory is $(p-1)(q-1)/2$.

The *fusion rules* of the operator algebra have the form

$$\psi_{r_1,s_1}\, \psi_{r_2,s_2} = \sum_{r=|r_1-r_2|+1}^{r_{\max}} \sum_{s=|s_1-s_2|+1}^{s_{\max}} [\psi_{r,s}\,], \tag{1.31}$$

where r (resp. s) runs over the even integers, provided that $r_1 + r_2$ (resp. $s_1 + s_2$) is odd and vice versa, and

$$r_{\max} := \min\{r_1 + r_2 - 1, 2q - r_1 - r_2 - 1\}, \tag{1.32}$$

$$s_{\max} := \min\{s_1 + s_2 - 1, 2p - s_1 - s_2 - 1\}. \tag{1.33}$$

Since the Verma module of the highest weight $h_{r,s}$ has two independent singular vectors [FFu], any correlation function involving the field $\phi_{r,s}(z)$ satisfies two linear partial differential equations, one of the order rs, and the other of the order $(q-r)(p-s)$.

2. Formulation of the Virasoro CFT

Let us translate the contents of the preceding section into the mathematical language. This can be carried out by means of the ideas in [TK2].

In physical context, *operators* are given at the beginning. After that, physical arguments show that such *operators* obey some laws of representation theory. For example, the energy momentum tensor $T(z)$ is considered to be a more fundamental object than the representations it generates. The primary field $\phi_h(z)$ is introduced before the primary states $|\,h\,\rangle$ is *defined*. For our purpose, however, it is convenient to reverse the story so that the representations are given at the outset and study the operators on them.

2.1. Highest weight representations. The condition (1.10) and (1.12) are known as the highest weight condition in representation theory. We summarize some basic facts on highest weight representations of the Virasoro algebra Vir.

Definition 2.1. Let c and h be complex numbers.

(1) A left representation V of the Virasoro algebra is called a left highest weight representation of highest weight h and central charge c if V is generated by a nonzero vector $|\,h\,\rangle \in V$ and

$$L_n|\,h\,\rangle = 0 \quad \text{if} \quad n > 0, \quad L_0|\,h\,\rangle = h|\,h\,\rangle, \quad C|\,h\,\rangle = c\,|\,h\,\rangle. \tag{2.1}$$

(2) A right representation $V^\dagger$ of the Virasoro algebra is called a right highest weight representation of highest weight h and central charge c if $V^\dagger$ is generated by a nonzero vector $\langle\,h\,| \in V^\dagger$ and

$$\langle\,h\,|L_n = 0 \quad \text{if} \quad n < 0, \quad \langle\,h\,|L_0 = h\langle\,h\,|, \quad \langle\,h\,|C = c\langle\,h\,|. \tag{2.2}$$

The vectors $|\,h\,\rangle$ and $\langle\,h\,|$ are called *highest weight vectors*.

Proposition 2.2.

(1) *For any $c, h \in \mathbb{C}$, there is a left (resp. right) highest weight representation $M_{c,h}$ (resp. $M^\dagger_{c,h}$) unique up to isomorphisms such that any left (resp. right) highest weight representation is isomorphic to a quotient of $M_{c,h}$ (resp. $M^\dagger_{c,h}$).*

(2) *The vectors*

$$L_{-k_n}\cdots L_{-k_2}L_{-k_1}|\,h\,\rangle \quad (n \geq 0,\ 1 \leq k_1 \leq k_2 \leq \cdots \leq k_n)$$

constitute a basis of $M_{c,h}$, and the vectors

$$\langle\,h\,|\,L_{k_1}L_{k_2}\cdots L_{k_n} \quad (n \geq 0,\ 1 \leq k_1 \leq k_2 \leq \cdots \leq k_n)$$

constitute a basis of $M^\dagger_{c,h}$.

(3) *There is a unique bilinear form*

$$M^\dagger_{c,h} \times M_{c,h} \to \mathbb{C}, \quad (\langle\,w\,|, |\,v\,\rangle) \mapsto \langle w\,|\,v\rangle, \tag{2.3}$$

such that $\langle w \,|\{ L_n| \, v \,\rangle\} = \{\langle \, w \,|L_n\} \,| \, v \,\rangle$ *for all* $| \, v \,\rangle \in M_{c,h}, \langle \, w \,| \in M^{\dagger}_{c,h}, n \in \mathbb{Z}$, *and*

$$\langle h \,|\, h \rangle = 1. \tag{2.4}$$

(4) $M_{c,h}$ (*resp.* $M^{\dagger}_{c,h}$) *contains a unique proper maximal submodule* $N_{c,h}$ *(resp.* $N^{\dagger}_{c,h}$) *and we have*

$$N_{c,h} = \{| \, v \,\rangle \in M_{c,h} \,|\, \langle w \,|\, v \rangle = 0 \text{ for all } \langle \, w \,| \in M^{\dagger}_{c,h}\}, \tag{2.5}$$

$$N^{\dagger}_{c,h} = \{\langle \, w \,| \in M^{\dagger}_{c,h} \,|\, \langle w \,|\, v \rangle = 0 \text{ for all } | \, v \,\rangle \in M_{c,h}\}. \tag{2.6}$$

It follows from the above proposition that among the left (resp. right) modules with highest weight h and central charge c there is a unique irreducible one, namely the module

$$V_{c,h} := M_{c,h}/N_{c,h} \quad (\text{resp. } V^{\dagger}_{c,h} := M^{\dagger}_{c,h}/N^{\dagger}_{c,h}). \tag{2.7}$$

Hence the bilinear form (2.3) induces a non-degenerate bilinear form on $V^{\dagger}_{c,h} \times V_{c,h}$. We denote it also by the same notation.

2.2. Operators. In the following, we fix a central charge $c \in \mathbb{C}$, and denote $V_{c,h}$ (resp. $V^{\dagger}_{c,h}$) by V_h (resp. $V^{\dagger}_h$).

Proposition 2.3. *The modules* V_h *and* $V^{\dagger}_h$ *have the eigenspace decomposition with respect to* L_0 :

$$V_h = \oplus_{d \geq 0} V_h[d] \quad \text{and} \quad V^{\dagger}_h = \oplus_{d \geq 0} V^{\dagger}_h[d], \tag{2.8}$$

where $V_h[d]$ *and* $V^{\dagger}_h[d]$ *are the eigenspaces of the eigenvalue* $h+d$. *Moreover* $\dim_{\mathbb{C}} V_h[d] = \dim_{\mathbb{C}} V^{\dagger}_h[d] < +\infty$, *and* $\langle V^{\dagger}_h[m] \,|\, V_h[n] \rangle = 0$ *if* $m \neq n$.

Introduce the products $\widehat{V}_h := \Pi_{d \geq 0} V_h[d]$ and $\widehat{V}^{\dagger}_h := \Pi_{d \geq 0} V^{\dagger}_h[d]$. The bilinear form can be naturally extended to $\widehat{V}^{\dagger}_h \times V_h$ and $V^{\dagger}_h \times \widehat{V}_h$.

Definition 2.4. By an *operator* Φ of the type $(\lambda_0, \lambda_\infty)$, we mean an element of the space $\mathrm{Hom}_{\mathbb{C}}(V_{h_0}, \widehat{V}_{h_\infty})$.

To give an operator $\Phi : V_{h_0} \to \widehat{V}_{h_\infty}$ is equivalent to give a bilinear map $\widehat{\Phi} : V^{\dagger}_{h_\infty} \times V_{h_0} \to \mathbb{C}$ and also a linear map $\Phi^{\dagger} : V^{\dagger}_{h_\infty} \to \widehat{V}^{\dagger}_{h_0}$ by the condition that for any $| \, v \,\rangle \in V_{h_0}$ and $\langle \, w \,| \in V^{\dagger}_{h_\infty}$,

$$\widehat{\Phi}(\langle \, w \,|, | \, v \,\rangle) = \{\langle \, w \,|\Phi^{\dagger}\} \,|\, v \rangle = \langle w \,|\, \{\Phi \,|\, v \,\rangle\}. \tag{2.9}$$

We shall denote the value (2.9) simply by $\langle w \,|\Phi|\, v \rangle$.

2.3. Definition of primary fields. The condition for the primary fields (1.6) can be translated into the language of representation.

Definition 2.5. Let U be an open set in a complex manifold X. A family of operators

$$\phi(z) : V_{h_0} \to \widehat{V}_{h_\infty}, \tag{2.10}$$

depending on $z \in U$ is called an operator-valued holomorphic function on U, if the matrix element $\langle w \,|\, \phi(z) \,|\, v \rangle$ is a holomorphic function of z for all $w \in V^\dagger_{h_\infty}$ and $v \in V_{h_0}$.

The space of the operator-valued functions on U is isomorphic to

$$\mathrm{Hom}_{\mathbb{C}}(V^\dagger_{h_\infty} \otimes V_{h_0}, \mathcal{O}_X(U)), \tag{2.11}$$

where $\mathcal{O}_X(U)$ is the space of holomorphic functions on U.

Definition 2.6. An operator-valued, multi-valued, holomorphic function $\phi_h(z)$: $V_{h_0} \to \widehat{V}_{h_\infty}$ on $\mathbb{C}^\times$ is called a primary field of the type $(h_0, h; h_\infty)$ if

$$[L_n, \phi_h(z)] = z^n (z \frac{\partial}{\partial z} + h\,(n+1))\,\phi_h(z) \quad \text{for all } n \in \mathbb{Z}. \tag{2.12}$$

We shall prepare the following general definition.

Definition 2.7. Let X be a complex manifold and $\widetilde{X}$ the universal covering manifold. We call a function $\Phi(z)$ on $\widetilde{X}$ with values in the space

$$\mathrm{Hom}_{\mathbb{C}}(V^\dagger_{h'_M} \otimes \cdots \otimes V^\dagger_{h'_1} \otimes V_{h_N} \otimes \cdots \otimes V_{h_1}, \mathbb{C})$$

a *field* of the type $(h_1, \dots h_N; h'_1, \dots h'_M)$ on X, if it is *holomorphic* in the following sense: for all vectors $v_i \in V^\dagger_{h'_i}$ $(i = 1, \dots, M)$ and $u_j \in V_{h_j}$ $(j = 1, \dots, N)$, the function

$$\langle\, v_M \,| \cdots \langle\, v_1 \,|\Phi(z)|\, u_N \,\rangle \cdots |\, u_1 \,\rangle := \Phi(z)(v_M \otimes \cdots \otimes v_1 \otimes u_N \otimes \cdots \otimes u_1) \tag{2.13}$$

on $\widetilde{X}$, obtained by the evaluation at the vector $v_M \otimes \cdots \otimes v_1 \otimes u_N \otimes \cdots \otimes u_1$, is a holomorphic function on $\widetilde{X}$.

For example, a primary field $\phi_h(z) : V_{h_0} \to \widehat{V}_{h_\infty}$ is a field of the type $(h_0; h_\infty)$, and $T(z)$ can be considered as a field of the type $(h; h)$ for any $h \in \mathbb{C}$.

2.4. Compositions of the operators.

Definition 2.8. Let $A : V_{h_0} \to \widehat{V}_h$, $B : V_h \to \widehat{V}_{h_\infty}$ be two operators. Fix dual bases $\{v_{d,i}\}_{i=1}^{n_d}$ of $V_h[d]$ and $\{v_d^i\}_{i=1}^{n_d}$ of $V_h^\dagger[d]$. An ordered pair (B, A) of operators is *composable*, if

$$\sum_{d=0}^{\infty} \left| \sum_{i=1}^{n_d} \langle\, w \,|\, B \,|\, v_{d,i} \,\rangle \, \langle\, v_d^i \,|\, A \,|\, v \,\rangle \right| < +\infty \tag{2.14}$$

for all $w \in V^\dagger_{h_3}$ and $v \in V_{h_1}$. Then the *composed operator* $BA : V_{h_0} \to \widehat{V}_{h_\infty}$ is defined by the values

$$\langle w \,|\, BA \,|\, v \rangle = \sum_{d=0}^{\infty} \left| \sum_{i=1}^{n_d} \langle\, w \,|\, B \,|\, v_{d,i} \,\rangle \, \langle\, v_d^i \,|\, A \,|\, v \,\rangle \right| \tag{2.15}$$

for $w \in V^\dagger_{h_\infty}$ and $v \in V_{h_0}$.

Example 2.9. *Let $\phi_h(z)$ be a primary field, then $(T(\zeta), \phi_h(z))$ is composable for $|\zeta| > |z| > 0$, and $(\phi_h(z), T(\zeta))$ is composable for $|z| > |\zeta| > 0$, and $T(\zeta)\,\phi_h(z)$ and $\phi_h(z)\,T(\zeta)$ are analytically continued to a field on $\{(\zeta, z) \in (\mathbb{C}^\times)^2 \,|\, \zeta \neq z\}$, simply denoted by $T(\zeta)\,\phi_h(z)$ or $\phi_h(z)\,T(\zeta)$. Moreover we have*

$$T(\zeta)\,\phi_h(z) = \frac{h}{(\zeta - z)^2}\phi_h(z) + \frac{1}{\zeta - z}\frac{\partial}{\partial z}\phi_h(z) + \sum_{k=0}^{\infty}(\zeta - z)^k \phi_h^{(-k-2)}(z), \tag{2.16}$$

where $\phi_h^{(k)}(z)\,(k \leq -2)$ are the fields on $\mathbb{C}^\times$ defined by this equation (see the next subsection).

The situation in the above example is very common in conformal field theory. We prepare the following terminology.

Definition 2.10. Let $\phi(z) : V_{h_0} \to \widehat{V}_{h_\infty}$ be a field on $\mathbb{C}^\times$.

(1) Assume $(T(\zeta), \phi(z))$ is composable in the region $|\zeta| > |z| > 0$, and $(\phi(z), T(\zeta))$ is composable in the region $|z| > |\zeta| > 0$, and composed fields $T(\zeta)\phi(z)$ and $\phi(z)T(\zeta)$ are analytically continued to a field on $\{(\zeta, z) \in (\mathbb{C}^\times)^2 \,|\, \zeta \neq z\}$. Then we say that a radial ordered product $T(\zeta)\phi(z)\,(= \phi(z)T(\zeta)\,)$ exists.

(2) If a radial ordered product $T(\zeta)\phi(z)$ has an expansion of the following form

$$T(\zeta)\phi(z) = \sum_{k\in\mathbb{Z}} \frac{A_k(z)}{(\zeta - z)^k}, \tag{2.17}$$

where $A_k(z)$ are the fields on $\mathbb{C}^\times$ such that $A_k(z) = 0$ for $k \gg 1$, then we say that $T(\zeta)\phi(z)$ has the operator product expansion (OPE for abbreviation) (2.17) .

Example 2.11. There exists the following OPE;

$$T(\zeta)T(z) = \frac{c/2}{(\zeta - z)^4} + \frac{2\,T(z)}{(\zeta - z)^2} + \frac{1}{\zeta - z}\frac{\partial}{\partial z}T(z) + (\text{regular at } \ \zeta = z). \tag{2.18}$$

2.5. Chiral Vertex Operators. Let $\phi_h(z) : V_{h_0} \to \widehat{V}_{h_\infty}$ be a nonzero primary field. In [BPZ], they say that the descendants (secondary fields)

$$\phi^{(-k_n,\dots,-k_1)}(z) : V_{h_0} \to \widehat{V}_{h_\infty}, \quad (1 \leq k_1 \leq \cdots \leq k_n) \tag{2.19}$$

form the conformal family $[\phi_h] \cong V_h$. We shall discuss the existence of such operators.

The OPE

$$T(\zeta)\phi_h(z) = \sum_{k\leq 0}(\zeta - z)^{-k-2}\phi_h^{(k)}(z) \tag{2.20}$$

can be considered as the definition of $\phi_h^{(k)}(z)$ $(k \in \mathbb{Z})$ namely

$$\phi_h^{(k)}(z) = \oint_{C_z} (\zeta - z)^{k+1} T(\zeta)\phi_h(z) d\zeta \quad \text{for } k \in \mathbb{Z} \tag{2.21}$$

where C_z is a small contour encircling z. We have

$$\phi_h^{(0)}(z) = h\phi_h(z), \quad \phi_h^{(k)}(z) = 0 \text{ for } k > 0. \tag{2.22}$$

and

$$\phi_h^{(-1)}(z) = \frac{d}{dz}\phi_h(z). \tag{2.23}$$

Next we consider the following OPE

$$T(\zeta)\phi_h^{(-k)}(z) = \sum_{l \geq -k} (\zeta - z)^{l-2}\phi_h^{(-l,-k)}(z), \tag{2.24}$$

with some operators $\phi_h^{(-l,-k)}(z)$, defined by the OPE. It is easy to show the existence of the above OPE. In this way we can manage to define the operators

$$\phi_h^{(-k_n,\dots,-k_1)}(z) \quad \text{for} \quad k_1, \dots, k_n \in \mathbb{Z} \tag{2.25}$$

using the OPE's like (2.24) as a recurrence formula.

The question is whether the operators $\phi_h^{(-k_n,\dots,-k_1)}(z)$ from the representation V_h by the correspondence

$$\phi_h^{(-k_n,\dots,-k_1)}(z) \leftrightarrow L_{-k_n} \cdots L_{-k_1} v_h \in V_h. \tag{2.26}$$

First we note that the OPE (2.18) guarantees that on the space of the *operators*

$$[\phi_h] = \{\phi_h^{(-k_n,\dots,-k_1)}(z) \mid k_1, \dots, k_n \in \mathbb{Z}\}, \tag{2.27}$$

the endomorphisms on $[\phi_h]$,

$$\widehat{L}_k := \oint_{C_z} d\zeta(\zeta - z)^{k+1}T(\zeta) \; (k \in \mathbb{Z}) \tag{2.28}$$

give a representation of the Virasoro algebra.

The equation (2.21) says that the representation $[\phi_h]$ is a highest weight representation of highest weight h. What is nontrivial here is the following.

Theorem 2.12. *The representation $[\phi_h]$ is irreducible, and hence is isomorphic to V_h.*

The above fact is implicit in the work of [BPZ]. We presented, for the convenience of reader, the proof of this theorem in the next chapter [Proposition 5.2.5]. This proof is due to G. Kuroki [Ku, Appendix B].

Let us denote $\phi_h(v, z) \in [\phi_h]$ for $v \in V_h$ by the correspondence (2.26). We define also

$$\phi_h(z) : V_h \otimes V_{h_0} \to \widehat{V}_{h_\infty}, \tag{2.29}$$

by $\phi_h(v\otimes u)(z)=\phi_h(v;z)(u),\ v\in V_h, u\in V_h$. We call such $\phi_h(z):V_h\otimes V_{h_0}\to \widehat{V}_{h_\infty}$, a chiral vertex operator (CVO for abbreviation).

Let $\phi_h(z):V_h\otimes V_{h_0}\to \widehat{V}_{h_\infty}$ be a chiral vertex operator (CVO for abbreviation). For $v\in V_h$, define the field $\phi_h(v;z):V_{h_0}\to \widehat{V}_{h_\infty}$ by $\phi_h(v;z)v_0=\phi_h(z)(v\otimes v_0)$ $(v_0\in V_{h_0})$. If $v\in V_h[m]$, then the field $\phi_h(v;z)$ has an expansion

$$\phi_h(v;z)=\sum_{n\in\mathbb{Z}}\phi_h(v)_n\, z^{-n-m-h_0-h+h_\infty}, \tag{2.30}$$

where $\phi_h(v)_n$ $(n\in\mathbb{Z})$ satisfies

$$[L_0,\phi_h(v)_n]=(h_\infty-h_0-n)\phi_h(v)_n \quad (n\in\mathbb{Z}), \tag{2.31}$$

namely $\phi_h(v)_n:V_{h_0}[d]\to V_{h_\infty}[d-n]$ $(n\in\mathbb{Z})$.

2.6. Formulation of the Ward identities, Conformal blocks.

Fix an $(N+2)$-ple $\boldsymbol{h}=(h_0,h_1,\dots,h_N;h_\infty)\in\mathbb{C}^N$ of highest weights. Consider for any $h'_0,h'_1,\dots,h'_N\in\mathbb{C}$ with $h'_0=h_0, h'_N=h_\infty$ and N chiral vertex operators

$$\phi_i(z_i):V_{h_i}\otimes V_{h'_{i-1}}\to \widehat{V}_{h'_i},\quad i=1,\dots,N. \tag{2.32}$$

For $v_i\in V_{h_i}[m_i]$ $(i=1,\dots,N)$, define the formal Laurent series of $z_1,\dots,z_N$ with coefficients in the space $\mathrm{Hom}_{\mathbb{C}}(V_{h_0},V_{h_\infty})$ by

$$\begin{aligned}&\phi_N(v_N;z_N)\cdots\phi_1(v_1;z_1)\\ &=\sum_{n_1,\dots,n_N\in\mathbb{Z}}\phi_N(v_N)_{n_N}\cdots\phi_1(v_1)_{n_1}\, z_N^{-n_N-m_N-\Delta_N}\cdots z_1^{-n_1-m_1-\Delta_1},\end{aligned} \tag{2.33}$$

where $\Delta_i= h'_{i-1}+h_i-h'_i$ $(i=1,\dots,N)$.

We shall discuss the validity of the following statement in the subsection 2.8. In this subsection, however, we shall only discuss the consequences of the statement to motivate our definition of the Ward identities, and the conformal blocks.

Statement. *For all $v_0\in V_{h_0}$ and $v_\infty\in V^\dagger_{h_\infty}$, the matrix coefficient*

$$\langle\, v_\infty\,|\,\phi_N(v_N;z_N)\cdots\phi_1(v_1;z_1)\,|\,v_0\,\rangle \tag{2.34}$$

is absolutely convergent in the region $\{(z_1,\dots,z_N)\in\mathbb{C}^N\mid |z_N|>\cdots>|z_1|>0\}$, and is analytically continued to a multi-valued holomorphic function on

$$X_N:=\{(z_1,\dots,z_N)\in(\mathbb{C}^\times)^N\mid z_i\neq z_j \text{ if } i\neq j\}.$$

Granted that the statement holds, we have a $\mathrm{Hom}_{\mathbb{C}}(V_{h_N}\otimes\cdots\otimes V_{h_0},\widehat{V}_{h_\infty})$-valued, multi-valued, holomorphic function $\Phi(z_1,\dots,z_N)=\Phi(z)$ defined by

$$\langle\, v_\infty\,|\Phi(z)|\, v_N\,\rangle\cdots|\,v_1\,\rangle|\,v_0\,\rangle:=\langle\, v_\infty\,|\,\phi_N(v_N;z_N)\cdots\phi_1(v_1;z_1)\,|\,v_0\,\rangle, \tag{2.35}$$

that is to say, a field of the type $(h_0,\dots,h_N;h_\infty)$ on X_N. Define

$$\pi_i(L_n)\langle\, v_\infty\,|\Phi(z)|\, v_N\,\rangle\cdots|\,v_1\,\rangle|\,v_0\,\rangle:=\langle\, v_\infty\,|\Phi(z)|\, v_N\,\rangle\cdots|\,L_nv_i\,\rangle\cdots|\,v_0\,\rangle$$

for $1=0,1,\dots,N$, and

$$\pi_\infty(L_n)\langle\, v_\infty\,|\Phi(z)|\, v_N\,\rangle\cdots|\,v_1\,\rangle|\,v_0\,\rangle:=\langle\, v_\infty L_n\,|\Phi(z)|\, v_N\,\rangle\cdots|\,v_1\,\rangle|\,v_0\,\rangle.$$

The function $\Phi(z)$ satisfies the following conditions, which is our interpretation of the Ward identities.

The Ward identities. (W.1) *For any fixed $z \in X_{N}$, the following $(N+2)$ Laurent series*

$$\sum_{k\in\mathbb{Z}} \pi_i(L_k)\langle\, v_\infty\, |\Phi(z)|\, v_{N}\,\rangle \cdots |\, v_1\,\rangle |\, v_0\,\rangle\ (\zeta - z_i)^{-k-2} \in \mathbb{C}((\zeta - z_i)) \text{ for } i = 1, \ldots, N. \tag{2.36}$$

$$\sum_{k\in\mathbb{Z}} \pi_0(L_k)\langle\, v_\infty\, |\Phi(z)|\, v_{N}\,\rangle \cdots |\, v_1\,\rangle |\, v_0\,\rangle\ \zeta^{-k-2} \in \mathbb{C}((\zeta)), \tag{2.37}$$

$$\sum_{k\in\mathbb{Z}} \pi_\infty(L_k)\langle\, v_\infty\, |\Phi(z)|\, v_{N}\,\rangle \cdots |\, v_1\,\rangle |\, v_0\,\rangle\ \zeta^{-k-2} \in \mathbb{C}((\zeta^{-1})), \tag{2.38}$$

are convergent, and they are analytically continued to a unique rational function of ζ regular except at the points $0, z_1, \ldots, z_{N}, \infty \in \mathbb{P}^1$, which we denote by

$$\langle\, v_\infty\, |T(\zeta)\Phi(z)|\, v_{N}\,\rangle \cdots |\, v_1\,\rangle |\, v_0\,\rangle. \tag{2.39}$$

(W.2) *For each $i = 1, \ldots, N$, the function $\Phi(z)$ satisfies the differential equation*

$$\frac{\partial}{\partial z_i}\langle\, v_\infty\, |\Phi(z)|\, v_{N}\,\rangle \cdots |\, v_1\,\rangle |\, v_0\,\rangle = \pi_i(L_{-1})\langle\, v_\infty\, |\Phi(z)|\, v_{N}\,\rangle \cdots |\, v_1\,\rangle |\, v_0\,\rangle. \tag{2.40}$$

For each $z \in X_{N}$, the condition in (W.1) defines linear equations on the space $\mathrm{Hom}_{\mathbb{C}}(V^\dagger_{h_\infty} \otimes V_{h_{N}} \otimes \cdots \otimes V_{h_0}, \mathbb{C})$. In the trivial bundle with the fiber $\mathrm{Hom}_{\mathbb{C}}(V^\dagger_{h_\infty} \otimes V_{h_{N}} \otimes \cdots \otimes V_{h_0}, \mathbb{C})$, the condition (W.1) defines the subbundle $\mathcal{V}_{c,\boldsymbol{h}}$. The differential equations (2.24) can be read as the equations for the horizontal sections of the bundle $\mathcal{V}_{c,\boldsymbol{h}}$ with respect to an integrable connection ∇ (see Chapter II, Lemma 3.3.4). We give here a definition of the notion of conformal blocks.

Definition 2.13. $\mathrm{Hom}_{\mathbb{C}}(V^\dagger_{h_\infty} \otimes V_{h_{N}} \otimes \cdots \otimes V_{h_1} \otimes V_{h_0}, \mathbb{C})$-valued holomorphic function $\Phi(z)$ on $\widetilde{X}_{N}$ is called a conformal block (of the Virasoro CFT) of the type $\boldsymbol{h}$, if it satisfies the system of equations (W.1) and (W.2). Let $\mathcal{V}^\nabla_{c,\boldsymbol{h}}$ denote the space of conformal blocks of the type $\boldsymbol{h}$.

Let us illustrate the way the Ward identity in the form of Section 1 follows from the above equations. Let $h_0 = h_\infty = 0$ and v_{h_i} be the highest weight vectors of V_{h_i} $(i = 1, \ldots, N)$. Define the N-point function

$$\langle\, \phi_{N}(z_{N}) \cdots \phi_1(z_1)\,\rangle := \langle\, \mathrm{vac}\, |\Phi(z)|\, h_{N}\,\rangle \cdots |\, h_1\,\rangle |\, \mathrm{vac}\,\rangle, \tag{2.41}$$

where $\langle\, \mathrm{vac}\,| \in V_0^\dagger$ and $|\, \mathrm{vac}\,\rangle \in V_0$, denote the vacua, the highest weight vectors. The function

$$\langle\, T(\zeta)\, \phi_{N}(z_{N}) \cdots \phi_1(z_1)\,\rangle := \langle\, \mathrm{vac}\, |T(\zeta)\Phi(z)|\, h_{N}\,\rangle \cdots |\, h_1\,\rangle |\, \mathrm{vac}\,\rangle \tag{2.42}$$

is regular at $0, \infty$ and has the Laurent expansions

$$\sum_{k\in\mathbb{Z}} (\zeta - z_i)^{-k-2}\pi_i(L_k)\langle\, \mathrm{vac}\, |\Phi(z)|\, h_{N}\,\rangle \cdots |\, h_1\,\rangle |\, \mathrm{vac}\,\rangle \ (i = 1, \ldots, N) \tag{2.43}$$

at $\zeta = z_i$ $(i = 1, \ldots, N)$, with the singular parts

$$\frac{h_i}{(\zeta - z_i)^2} \langle \phi_{_N}(z_{_N}) \cdots \phi_1(z_1) \rangle + \frac{1}{\zeta - z_i} \frac{\partial}{\partial z_i} \langle \phi_{_N}(z_{_N}) \cdots \phi_1(z_1) \rangle. \tag{2.44}$$

Therefore the function

$$\langle T(\zeta) \phi_{_N}(z_{_N}) \cdots \phi_1(z_1) \rangle - \sum_{i=1,\ldots,N} \left\{ \frac{h_i}{(\zeta - z_i)^2} + \frac{1}{\zeta - z_i} \frac{\partial}{\partial z_i} \right\} \langle \phi_{_N}(z_{_N}) \cdots \phi_1(z_1) \rangle \tag{2.45}$$

is regular at every point of $\mathbb{P}^1$, and hence a constant. The constant is proved to be zero (see the expansion at ∞).

2.7. Degenerate representations. We assumed the representations involved in the definition of conformal blocks are irreducible. This corresponds to consider the degenerate conformal field theory. The image of the maximal proper submodule $N_{c,h}$ gives nontrivial linear relations among the monomials

$$L_{-k_n} \cdots L_{-k_1} v_h \in V_h.$$

These relations and the Ward identities lead to the nontrivial linear differential equations such as (1.28) satisfied by the conformal blocks.

V. G. Kac [Ka1] determined the explicit formula of the determinant for the bilinear form (2.3) restricted to the subspaces $M^\dagger_{c,h}[d] \times M_{c,h}[d]$ $(d \geq 0)$, and obtained the following.

Theorem 2.14 [Ka1]**.** *The Verma module $M_{c,h}$ is reducible if and only if there exist $r, s \in \mathbb{N}$ and $t \in \mathbb{C}^\times$ such that*

$$c = c(t) := 6t + 13 + 6t^{-1}, \quad h = h_{r,s}(t) := \frac{1 - r^2}{4t} + \frac{1 - rs}{2} + \frac{1 - s^2}{4} t. \tag{2.46}$$

2.8. The minimal models, revisited. Fix the central charge $c = c(p, q)$. Minimal representations of the central charge $c = c(p, q)$ are parametrized, up to isomorphisms, by the set of highest weights

$$H = \{h_{r,s} \in \mathbb{Q} \,|\, r, s \in \mathbb{Z}, 0 < r < q, 0 < s < p\}. \tag{2.47}$$

We call a pair (r, s) a representative for $h = h_{r,s} \in H$. Each $h = h_{r,s} \in H$ has exactly two representatives, namely (r, s) and $(q - r, p - s)$.

Theorem 2.15 [Ku,Wa]**.** *Let $h_\infty, h, h_0 \in H$. There exists a nonzero chiral vertex operator*

$$\phi_h(v; z) : V_{h_0} \to \widehat{V}_{h_\infty} \ (v \in V_h),$$

if and only if: h_0, h, h_∞ have representatives $(r_1, s_1), (r_2, s_2), (r_3, s_3)$ respectively, and

$$|r_1 - r_2| + 1 \leq r_3 \leq \min\{r_1 + r_2 - 1, 2q - r_1 - r_2 - 1\}, \quad r_3 \equiv r_1 + r_2 - 1 \bmod 2, \tag{2.48}$$

$$|s_1 - s_2| + 1 \leq s_3 \leq \min\{s_1 + s_2 - 1, 2p - s_1 - s_2 - 1\}, \quad s_3 \equiv s_1 + s_2 - 1 \bmod 2. \tag{2.49}$$

Moreover, nonzero chiral vertex operators of a fixed type $(h_0, h; h_\infty) \in H^3$ are unique up to constant multiples.

We still lack a proof of the following fundamental statement, called the factorization property, which is one of the main goal of our study. In Section 5, we shall explain our approach.

The factorization property. *Let* $\boldsymbol{h} = (h_0, h_1, \ldots, h_{_N}; h_\infty) \in H^{N+2}$. *For any* $\boldsymbol{p} = (h'_0, \ldots, h'_{_N}) \in H^{N+1}$ *in the set*

$$\begin{aligned}\mathcal{P}(\boldsymbol{h}) :=\Big\{ & \boldsymbol{p} = (h'_0, \ldots, h'_{_N}) \in H^{N+1} \mid h'_0 = h_0, h'_{_N} = h_\infty, \\ & \textit{There is a nonzero CVO;} \quad \phi_i(z_i) : V_{h_i} \otimes V_{h'_{i-1}} \to \widehat{V}_{h'_i} \Big\},\end{aligned} \tag{2.50}$$

the composition $\Phi_{\boldsymbol{p}}(z)$ *of the CVO's* $\phi_1(z_1), \ldots, \phi_{_N}(z_{_N})$ *is convergent in the region* $\{ |z_{_N}| > \cdots > |z_1| > 0\}$ *and is analytically continued to a multi-valued holomorphic function on* $X_{_N}$. *Moreover the functions* $\Phi_{\boldsymbol{p}}(z)$ $(\boldsymbol{p} \in \mathcal{P}(\boldsymbol{h}))$ *form locally a basis of the local system* $\mathcal{V}^{\nabla}_{c,\boldsymbol{h}}$ *of the conformal blocks.*

Let us remark on the convergence of the series. Introduce the coordinates

$$\tau_i = z_i/z_{i+1} \ (i = 1, \ldots, N-1), \tau_{_N} = z_{_N}, \tag{2.51}$$

and put $\varepsilon_i = h'_i - h'_0 - \sum_{j=1}^{i} h_j$ $(i = 1, \ldots, N)$. Then $\Phi_{\boldsymbol{p}}(z)$ can be viewed as an element of the space

$$\mathrm{Hom}_{\mathbb{C}}(V^{\dagger}_{h_\infty} \otimes V_{h_{_N}} \otimes \cdots \otimes V_{h_0}, \tau_{_N}^{\varepsilon_N} \cdots \tau_1^{\varepsilon_1} \mathbb{C}((\tau)) \ ),$$

where $\mathbb{C}((\tau))$ is the field of fractions of the ring of formal power series $\mathbb{C}[[\tau_1, \ldots, \tau_{_N}]]$. To prove the convergence of this series, it is sufficient to prove that our connection has regular singularities along the divisor $\tau_1 \cdots \tau_{_N} = 0$. At present, however, we do not have a solution to the problem of the regularity at the boundary. We note that our construction (Theorem C in Section 5) must give a way toward this problem.

3. Conformal field theory with gauge symmetry

V. G. Knizhnik and A. B. Zamolodchikov [KZ] developed the conformal field theory with gauge symmetry, called the Wess-Zumino-Witten (WZW for abbreviation) model. A. Tsuchiya and Y. Kanie [TK2] clarified the mathematical meaning of the work [KZ] on the foundation of representation theory of affine Lie algebra and proved the factorization property. They also determined the monodromy representation of the braid group obtained from the local system of conformal blocks.

We attempt to give an introductory article on the factorization property of the WZW model putting an emphasis on the Ward identities.

3.1. Affine Lie algebra. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be the Borel subalgebra such that $\mathfrak{b} \supset \mathfrak{h}$. By $\triangle$ ($\triangle_+$) we denote the (positive) root system. Fix an invariant bilinear form $(\cdot|\cdot)$ such that $(\theta|\theta) = 2$ for the highest root θ.

Let P_+ be the set of dominant integral weights. There is a one to one correspondence between the set P_+ and the set of isomorphism classes of finite dimensional irreducible representations of $\mathfrak{g}$. For $\lambda \in P_+$, we denote by L_λ the irreducible representation of $\mathfrak{g}$ with the highest weight λ.

The affine Lie algebra $\widehat{\mathfrak{g}}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K \tag{3.1}$$

where K is an element of the center of $\widehat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$[X \otimes f(t), Y(t) \otimes g(t)] = [X, Y] \otimes f(t)g(t) + (X|Y) \operatorname*{Res}_{t=0} (f'(t)g(t)\, dt)\, K \tag{3.2}$$

for

$$X, Y \in \mathfrak{g}, \quad f(t), g(t) \in \mathbb{C}((t)).$$

Define the subalgebras

$$\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[t]t, \quad \widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[t]t^{-1}. \tag{3.3}$$

We have the decomposition

$$\widehat{\mathfrak{g}} = \mathfrak{p}_+ \oplus \widehat{\mathfrak{g}}_- \tag{3.4}$$

where $\mathfrak{p}_+$ is the subalgebra $\widehat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C}K$. For a representation M of $\mathfrak{g}$ and a complex number $k \in \mathbb{C}$, an action of $\mathfrak{p}_+$ can be given by letting $\widehat{\mathfrak{g}}_+$ act by zero and K by multiplication by k. We denote this $\mathfrak{p}_+$-module by M_k. Consider the induced representation

$$\operatorname{Ind}_{\mathfrak{p}_+}^{\widehat{\mathfrak{g}}} M_k = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{p}_+)} M_k, \tag{3.5}$$

where $U(\mathfrak{a})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{a}$.

The induced module $\operatorname{Ind}_{\mathfrak{p}_+}^{\widehat{\mathfrak{g}}} (L_\lambda)_k$ has a unique irreducible quotient denoted by $L_{k,\lambda}$. If k is a non-negative integer and λ is contained in the set

$$P_k = \{\lambda \in P_+ \,|\, 0 \le (\lambda|\theta) \le k\}, \tag{3.6}$$

then the representation $L_{k,\lambda}$ is integrable [Ka2] in the sense that for all root vector $X_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta$) of $\mathfrak{g}$ and $m \in \mathbb{Z}$, $X_\alpha[m]$ is locally nilpotent, namely for each $v \in L_{k,\lambda}$, there exists $n \in \mathbb{N}$ such that

$$X_\alpha[m]^l v = 0 \quad \text{for all} \quad l \geq n. \tag{3.7}$$

We denote by v_λ a fixed highest weight vector of $L_{k,\lambda}$. In the following, we assume that k is a positive integer and $\lambda \in P_k$.

Remark. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. We shall identify P_+ with the set of non-negative integers; for a non-negative integer l, L_l denotes the unique irreducible representation of dimension $l+1$. The subset P_k is identified with the set $\{0, 1, \ldots, k\}$.

3.2. Sugawara construction. Let $\{J_a\}_{a=1}^{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, and $\{J^a\}_{a=1}^{\dim \mathfrak{g}}$ be the dual basis with respect to $(\cdot|\cdot)$, namely $(J_a|J^b) = \delta_{a,b}$. Let $X[n]$ denote the element $X \otimes t^n$ of $\widehat{\mathfrak{g}}$, for $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Let us define the normal product $\substack{\circ \\ \circ} \cdot \substack{\circ \\ \circ}$ by

$$\substack{\circ \\ \circ} X[m]\, Y[n] \substack{\circ \\ \circ} = \begin{cases} X[m]\, Y[n] & (m < n) \\ \frac{1}{2}(X[m]\, Y[n] + Y[n]\, X[m]) & (m = n) \\ Y[n]\, X[m] & (m > n) \end{cases}. \tag{3.8}$$

Define the endomorphisms on $L_{k,\lambda}$ by

$$T[n] = \frac{1}{2(k+g)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim_{\mathbb{C}} \mathfrak{g}} \substack{\circ \\ \circ} J_a[m]\, J^a[n-m] \substack{\circ \\ \circ} \quad (n \in \mathbb{Z}), \tag{3.9}$$

where $g \in \mathbb{N}$ is the dual Coxeter number of $\mathfrak{g}$.

Proposition 3.1. *We have the relations*

$$[\, T[m], X[n]\,] = -n\, X[m+n], \quad \textit{for} \quad X \in \mathfrak{g}, \tag{3.10}$$

and

$$[\, T[m], T[n]\,] = (m-n)\, T[m+n] + \delta_{m+n,0} \frac{m^3 - m}{12}\, c, \tag{3.11}$$

where

$$c = \frac{k \dim_{\mathbb{C}} \mathfrak{g}}{k+g}. \tag{3.12}$$

$T[n]$ ($n \in \mathbb{Z}$) are called the Sugawara operators.

Let

$$\Delta_{k,\lambda} = \frac{(\lambda|\lambda + 2\rho)}{2(k+g)}, \tag{3.13}$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{3.14}$$

Then we have the eigenspace decomposition with respect to $T[0]$,

$$L_{k,\lambda} = \bigoplus_{d=0}^{\infty} L_{k,\lambda}\,[d], \tag{3.15}$$

where $L_{k,\lambda}\,[d]$ is the eigenspace of the eigenvalue $(\Delta_{k,\lambda} + d)$. Note that the space $L_{k,\lambda}\,[0]$ is a irreducible $\mathfrak{g}$-submodule isomorphic to L_λ.

Proposition 3.2. *There exists a unique right $\widehat{\mathfrak{g}}$-module $L^{\dagger}_{k,\lambda}$ generated by the highest weight vector $\langle v_\lambda \,|$ such that there is a unique non-degenerate bilinear form $\langle \cdot \mid \cdot \rangle : L^{\dagger}_{k,\lambda} \times L_{k,\lambda} \to \mathbb{C}$ satisfying $\langle v_\lambda \,|\, v_\lambda \rangle = 1$, $\langle u\, A \,|\, v \rangle = \langle u \,|\, A\, v \rangle$ for all $u \in L^{\dagger}_{k,\lambda}, v \in L_{k,\lambda}$ and $A \in \widehat{\mathfrak{g}}$. We have the eigenspace decomposition with respect to $T[0]$:*

$$L^{\dagger}_{k,\lambda} = \bigoplus_{d=0}^{\infty} L^{\dagger}_{k,\lambda}\,[d], \tag{3.16}$$

where $L^{\dagger}_{k,\lambda}\,[d]$ is the eigenspace of the eigenvalue $\triangle_{k,\lambda} + d$, and we have

$$\langle\, L^{\dagger}_{k,\lambda}[m] \,|\, L_{k,\lambda}[n] \,\rangle = 0 \text{ for } m \neq n.$$

The space $L^{\dagger}_{k,\lambda}[0]$ is a irreducible (right) $\mathfrak{g}$-submodule of the highest weight λ. Introduce the product $\widehat{L}^{\dagger}_{k,\lambda} = \prod_{d=0}^{\infty} L^{\dagger}_{k,\lambda}[d]$ and $\widehat{L}_{k,\lambda} = \prod_{d=0}^{\infty} L_{k,\lambda}[d]$. The bilinear form $\langle \cdot|\cdot \rangle$ can be extended to $\widehat{L}^{\dagger}_{k,\lambda} \times L_{k,\lambda}$ and $L^{\dagger}_{k,\lambda} \times \widehat{L}_{k,\lambda}$. The form $\langle \cdot|\cdot \rangle$, induces the linear isomorphisms $\widehat{L}_{k,\lambda} \simeq \mathrm{Hom}_{\mathbb{C}}(L^{\dagger}_{k,\lambda}, \mathbb{C})$ and $\widehat{L}^{\dagger}_{k,\lambda} \simeq \mathrm{Hom}_{\mathbb{C}}(L_{k,\lambda}, \mathbb{C})$. We shall freely use this type of identification in the following.

3.3. Primary fields with gauge symmetry. We leave the definitions of the *operators* and the *fields* to the reader, which is given in the the same way as in the Viraroso CFT.

Definition 3.3. A family of fields on $\mathbb{C}^{\times}$, depending linearly on $v \in L_\lambda$,

$$\phi_\lambda(v; z) : L_{k,\lambda_0} \to \widehat{L}_{k,\lambda_\infty} \; (v \in L_\lambda)$$

is called a primary field of the type $(\lambda_0, \lambda; \lambda_\infty)$ if:

$$[X[n], \phi_\lambda(v; z)] = z^n\, \phi_\lambda(Xv; z), X \in \mathfrak{g},\; v \in L_\lambda,\; n \in \mathbb{Z}, \tag{3.17}$$

$$[T[n], \phi_\lambda(v; z)] = z^n (z \frac{\partial}{\partial z} + \triangle_{k,\lambda}\, (n+1))\, \phi_\lambda(v; z), n \in \mathbb{Z}. \tag{3.18}$$

3.4. Chiral Vertex Operators of the WZW models. Let $\phi_\lambda(v; z) : L_{k,\lambda_0} \to \widehat{L}_{\lambda_\infty}$ $(v \in L_\lambda)$ be a nonzero primary field. We shall leave the question of the existence of such a $\phi_\lambda(\cdot\,; z)$ until the next subsection, and discuss the secondary fields of it.

As in the case of the Virasoro CFT, we can define the fields

$$(\widehat{X}_n[-l_n] \cdots \widehat{X}_1[-l_1]\phi_\lambda)(v; z) \text{ for } X_1, \ldots, X_n \in \mathfrak{g},\; l_1, \ldots, l_n \in \mathbb{Z}, \tag{3.19}$$

using the following OPE

$$X(\zeta)\phi_\lambda(v; z) = \frac{\phi_\lambda(Xv; z)}{\zeta - z} + (\text{regular at } \; \zeta = z) \tag{3.20}$$

as the starting point, where $X \in \mathbb{Z}$ and $X(\zeta) = \sum_{n\in\mathbb{Z}} X[n]z^{-n-1}$.

By the OPE

$$X(\zeta)Y(z) = \frac{k\,(X|Y)}{(\zeta - z)^2} + \frac{[X, Y](z)}{\zeta - z} + (\text{regular at } \ \zeta = z) \quad \text{for } X, Y \in \mathfrak{g}, \tag{3.21}$$

we can show that the fields (3.19) form the highest weight representation $[\phi_\lambda]$ of $\widehat{\mathfrak{g}}$ with the highest weight (k, λ). The action of $X[n]$ ($X \in \mathfrak{g}$, $n \in \mathbb{Z}$) on the field $\phi(z) \in [\phi_\lambda]$ is given

$$(\widehat{X}[n]\phi)(z) := \frac{1}{2\pi i}\oint_{C_z} (\zeta - z)^n X(\zeta)\phi(z)d\zeta. \tag{3.22}$$

As in the case of the Virasoro CFT, we have the following.

Theorem 3.4 [TK2]. *The representation $[\phi_\lambda]$ is irreducible, and hence is isomorphic to $L_{k,\lambda}$.*

We shall denote by $\phi_\lambda(v; z) \in [\phi_\lambda]$ the field corresponding to $v \in L_{k,\lambda}$. We also define the field

$$\phi(z) : L_{k,\lambda} \otimes L_{k,\lambda_0} \to \widehat{L}_{k,\lambda_\infty} \tag{3.23}$$

by $\phi(z)(v \otimes v_0) = \phi_\lambda(v; z)v_0,\ v \in L_{k,\lambda}, v_0 \in L_{k,\lambda_0}$. We shall call the family of operators $\phi_\lambda(v; z)$ ($v \in L_{k,\lambda}$) or the field $\phi(z)$ defined by (3.23) a chiral vertex operator (CVO for short). In the following, we shall identify the primary field $\phi_\lambda(v; z)$ ($v \in L_\lambda$), given at the beginning, with the corresponding CVO $\phi(z)$.

3.5. Initial terms. We are going to discuss the existence of a nonzero chiral vertex operator (or primary field). Let $\phi(z) : L_{k,\lambda} \otimes L_{k,\lambda_0} \to \widehat{L}_{k,\lambda_\infty}$ be a CVO. Then there exists an element ϕ° of $\mathrm{Hom}_{\mathfrak{g}}(L_\lambda \otimes L_{\lambda_0}, L_{\lambda_\infty})$ such that

$$\phi(z)|_{L^\dagger_{\lambda_\infty}\otimes L_\lambda \otimes L_{\lambda_0}} = z^{-\triangle}\phi^\circ, \tag{3.24}$$

where $\triangle{=}\triangle_{\lambda_0} + \triangle_\lambda - \triangle_{\lambda_\infty}$, and we used the identification $\mathrm{Hom}_{\mathbb{C}}(L_\lambda \otimes L_{\lambda_0}, L_{\lambda_\infty}) = \mathrm{Hom}_{\mathbb{C}}(L^\dagger_{\lambda_\infty} \otimes L_\lambda \otimes L_{\lambda_0}, \mathbb{C})$. We shall call $z^{-\triangle}\phi^\circ$ the initial term of $\phi(z)$.

Let $\mathcal{L}^{\nabla}_{k,\boldsymbol{\lambda}}$ denote the space of CVO's of the type $\boldsymbol{\lambda} = (\lambda_0, \lambda; \lambda_\infty)$. Consider the linear map

$$\mathrm{ini} : \mathcal{L}^{\nabla}_{k,\boldsymbol{\lambda}} \to z^{-\triangle}\mathrm{Hom}_{\mathfrak{g}}(L_\lambda \otimes L_{\lambda_0}, L_{\lambda_\infty}) \tag{3.25}$$

defined by $\mathrm{ini}(\phi(z)) = z^{-\triangle}\phi^\circ$. Then we have the following.

Lemma 3.5. *The linear map* ini *is injective.*

In the next subsection we shall characterize the image of ini, and hence obtain the condition for the existence of a nonzero CVO of a fixed type.

3.6. Existence of primary fields, the fusion rule. Let $E_\theta \in \mathfrak{g}_\theta$, $F_\theta \in \mathfrak{g}_{-\theta}$, $H_\theta \in \mathfrak{h}$ be such that $[H_\theta, E_\theta] = 2E_\theta, [H_\theta, F_\theta] = -2F_\theta, [E_\theta, F_\theta] = H_\theta$. The subalgebra $\mathfrak{k}_\theta := \mathbb{C}E_\theta \oplus \mathbb{C}H_\theta \oplus \mathbb{C}F_\theta$ is the principle 3-dimensional subalgebra of $\mathfrak{g}$. Let $W_{\lambda,l}$ (resp. $W^\dagger_{\lambda,l}$) be the sum of $\mathfrak{k}_\theta$-submodules in L_λ (resp. $L^\dagger_\lambda$) isomorphic to the $(l + 1)$-dimensional irreducible $\mathfrak{k}_\theta$-module. We have $L_\lambda = \oplus_{0\le l\le(\theta|\lambda)} W_{\lambda,l}$ and $L^\dagger_\lambda = \oplus_{0\le l\le(\theta|\lambda)} W^\dagger_{\lambda,l}$.

Theorem 3.6 [GW, TK2]**.** *The image of the map* ini : $\mathcal{L}_{k,\boldsymbol{\lambda}} \to \mathrm{Hom}_{\mathfrak{g}}(L_\lambda \otimes L_{\lambda_0}, L_{\lambda_\infty})$ *is characterized by the following condition for* $\phi^\circ \in \mathrm{Hom}_{\mathfrak{g}}(L_\lambda \otimes L_{\lambda_0}, L_{\lambda_\infty})$:

$$\phi^\circ|_{W^\dagger_{\lambda_\infty,l}\otimes W_{\lambda,m}\otimes W_{\lambda,n}} = 0 \quad \text{if} \quad l+m+n > 2k. \tag{3.26}$$

Corollary 3.7. *Let* $\mathfrak{g} = \mathfrak{sl}_2$. *Then a nonzero CVO* $\phi_m(z) : L_{k,m} \otimes L_{k,l} \to \widehat{L}_{k,n}$ *exists if and only if;*

$$|l-m| \le n \le \min\{l+m, 2k-l-m\}, \quad n \equiv l+m \bmod 2. \tag{3.27}$$

Moreover, nonzero CVO's of a fixed type are unique up to constant multiples.

3.7. Compositions of CVO's. Fix an $(N+2)$-ple $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_N; \lambda_\infty) \in P_k^{N+2}$ of weights. For each $(N+1)$-ple $\boldsymbol{\mu} = (\mu_0, \dots, \mu_N) \in P_k^{N+1}$ such that $\mu_0 = \lambda_0$ and $\mu_N = \lambda_\infty$, let

$$\phi_i(z_i) : L_{k,\lambda_i} \otimes L_{k,\mu_{i-1}} \to \widehat{L}_{k,\mu_i}, \quad i = 1, \dots, N \tag{3.28}$$

be CVO's. The composed field $\Phi(z)$ is defined by

$$\langle v_\infty | \Phi(z) | v_N \rangle \cdots | v_1 \rangle | v_0 \rangle = \langle v_\infty | \phi_N(v_N; z_N) \cdots \phi_1(v_1; z_1) | v_0 \rangle, \tag{3.29}$$

where $v_\infty \in L_{k,\lambda_\infty}$ and $v_i \in L_{k,\lambda_i}$ $(i = 0, 1, \dots, N)$.

$\Phi(z)$ is a formal Laurent series, with the coefficients in the space

$$\mathrm{Hom}_{\mathbb{C}}(L_{k,\lambda_N} \otimes \cdots L_{k,\lambda_0}, L_{k,\lambda_\infty}).$$

Recall the coordinates τ (2.52). As in the case of the Virasoro CFT, $\Phi(z)$ can be viewed as an element of the space

$$\mathrm{Hom}_{\mathbb{C}}(L^\dagger_{k,\lambda_\infty} \otimes L_{k,\lambda_N} \otimes \cdots L_{k,\lambda_0}, \tau_N^{\varepsilon_N} \cdots \tau_1^{\varepsilon_1} \mathbb{C}((\tau))),$$

where

$$\varepsilon_i := \Delta_{k,\mu_i} - \Delta_{k,\mu_0} - \sum_{j=1}^{i} \Delta_{k,\lambda_j} \quad (i = 1, \dots, N). \tag{3.30}$$

Theorem 3.8 [TK2]**.** *The series* $\Phi(z)$ *is absolutely convergent in the region* $\{|z_N| > \cdots > |z_1| > 0\}$, *and is analytically continued to a multi-valued holomorphic function on* X_N.

In [TK2], the above Theorem is proved by the study of the equations for the N-point functions, which we shall discuss in the next subsection.

3.8. N**-point functions.** Assume, in the same notation of the preceding subsection, $\lambda_0 = \lambda_\infty = 0$. We shall take the CVO's $\phi_i(\cdot\,; z_i)$ as the primary fields, $\phi_i(v; z_i)$ $(v \in L_{\lambda_i})$. Consider the following formal Laurent series called the N-point function

$$\langle \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) \rangle := \langle \, \mathrm{vac}\, | \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) | \, \mathrm{vac} \, \rangle \tag{3.31}$$

with the coefficients in the space

$$\mathrm{Hom}_{\mathbb{C}}(L_{\lambda_{N}} \otimes \cdots L_{\lambda_1}, \mathbb{C}).$$

Theorem 3.9 [KZ,GW,TK2]. *The N-point function $\langle \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) \rangle$ is absolutely convergent in the region $\{|z_{N}| > \cdots |z_1| > 0\}$, and is analytically continued to a multi-valued, $\mathrm{Hom}_{\mathbb{C}}(L_{\lambda_{N}} \otimes \cdots L_{\lambda_1}, \mathbb{C})$-valued holomorphic function on X_{N}, and satisfies the following equations:*

(1) *For $m = -1, 0$ and 1,*

$$\sum_{i=1}^{N} z_i^m \left(z_i \frac{\partial}{\partial z_i} + (m+1)\, \triangle_{k,\lambda_i} \right) \langle \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) \rangle = 0. \tag{3.32}$$

(2) *For $X \in \mathfrak{g}$,*

$$\sum_{i=1}^{N} \pi_i(X) \langle \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) \rangle = 0. \tag{3.33}$$

(3) *For each $i = 1, \ldots, N$,*

$$(k+g) \frac{\partial}{\partial z_i} \langle \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) \rangle = \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{\Omega_{i,j}}{z_i - z_j} \langle \phi_{N}(\cdot\,; z_{N}) \cdots \phi_1(\cdot\,; z_1) \rangle, \tag{3.34}$$

where

$$\Omega_{i,j} := \sum_{a=1}^{\dim_{\mathbb{C}} \mathfrak{g}} \pi_i(J_a) \pi_j(J^a). \tag{3.35}$$

(4) *For each $i = 1, \ldots, N$ and $v_j \in L_{\lambda_j}$ $(j = 1, \ldots, N,\ j \neq i)$,*

$$\sum_{\substack{n_1, \ldots, n_{N} \geq 0 \\ n_1 + \cdots + n_{N} \\ = k - (\theta|\lambda_i) + 1}} \frac{k - (\theta|\lambda_i) + 1}{n_1! \cdots n_{N}!} \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{\pi_j(E_\theta)}{z_j - z_i} \right)^{n_j} \langle \phi_{N}(v_{N}; z_{N}) \cdots \phi_i(|\, \lambda_i \,\rangle; z_i) \cdots \phi_1(v_1; z_1) \rangle = 0, \tag{3.36}$$

where $|\, \lambda_i \,\rangle$ is the highest weight vector of L_{λ_i}.

3.9. The Ward identities, the gauge conditions.

Definition 3.10. $\mathrm{Hom}_C(L^\dagger_{k,\lambda_\infty} \otimes L_{k,\lambda_N} \otimes \cdots \otimes L_{k,\lambda_0}, \mathbb{C})$-valued, multi-valued holomorphic function $\Phi(z)$ on X_N is called a conformal block of the WZW model if it satisfies the conditions:

(W.3) For any fixed $z \in X_N$ and $X \in \mathfrak{g}$, the following $(N+2)$ Laurent series

$$\sum_{n\in\mathbb{Z}} \pi_i(X[n])\langle\, v_\infty \,|\Phi(z)|\, v_N\,\rangle \cdots |\, v_1\,\rangle|\, v_0\,\rangle\; (\zeta - z_i)^{-n-1} \in \mathbb{C}((\zeta - z_i)) \text{ for } i = 1, \ldots, N, \tag{3.37}$$

$$\sum_{n\in\mathbb{Z}} \pi_0(X[n])\langle\, v_\infty \,|\Phi(z)|\, v_N\,\rangle \cdots |\, v_1\,\rangle|\, v_0\,\rangle\; \zeta^{-n-1} \in \mathbb{C}((\zeta)), \tag{3.38}$$

$$\sum_{n\in\mathbb{Z}} \pi_\infty(X[n])\langle\, v_\infty \,|\Phi(z)|\, v_N\,\rangle \cdots |\, v_1\,\rangle|\, v_0\,\rangle\; \zeta^{-n-1} \in \mathbb{C}((\zeta^{-1})), \tag{3.39}$$

are convergent, and they are analytically continued to a unique rational function of ζ regular except at the points $0, z_1, \ldots, z_N, \infty \in \mathbb{P}^1$, which we denote by

$$\langle\, v_\infty \,|X(\zeta)\Phi(z)|\, v_N\,\rangle \cdots |\, v_1\,\rangle|\, v_0\,\rangle \tag{3.40}$$

(W.2') For each $i = 1, \ldots, N$, the function $\Phi(z)$ satisfies the differential equation

$$\frac{\partial}{\partial z_i}\langle\, v_\infty \,|\Phi(z)|\, v_N\,\rangle \cdots |\, v_1\,\rangle|\, v_0\,\rangle = \pi_i(T[-1])\langle\, v_\infty \,|\Phi(z)|\, v_N\,\rangle \cdots |\, v_1\,\rangle|\, v_0\,\rangle. \tag{3.41}$$

Let $\mathcal{L}^\nabla_{k,\boldsymbol{\lambda}}$ denote the space of conformal blocks of the type $\boldsymbol{\lambda}$. When $N = 1$ the notion of CVO and conformal block are proved to coincide.

3.10. The factorization property. For $\lambda, \mu, \nu \in P_k$, define the space

$$V^\nu_{\mu,\lambda} := \{\phi^\circ \in \mathrm{Hom}_{\mathfrak{g}}(L_\mu \otimes L_\lambda, L_\nu) \mid \phi^\circ \text{ satisfies the condition (3.26)}\}. \tag{3.42}$$

$V^\nu_{\mu,\lambda}$ is the space of the initial terms of the primary fields of the type $(\lambda, \mu; \nu)$. For $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_N; \lambda_\infty) \in P_k^{N+2}$,

$$\boldsymbol{V}_{k,\boldsymbol{\lambda}} := \bigoplus_{\substack{\boldsymbol{\mu}\in P_k^{N+1} \\ \mu_0=\lambda_0,\, \mu_N=\lambda_\infty}} V^{\mu_N}_{\lambda_N,\mu_{N-1}} \otimes V^{\mu_{N-1}}_{\lambda_{N-1},\mu_{N-2}} \otimes \cdots \otimes V^{\mu_2}_{\lambda_2,\mu_1} \otimes V^{\mu_1}_{\lambda_1,\mu_0}. \tag{3.43}$$

Theorem 3.11 [TK2]. *For each $i = 1, \ldots, N$ and $\phi^\circ_i \in V^{\mu_i}_{\lambda_i,\mu_{i-1}}$, let*

$$\phi_i(z_i) : L_{k,\lambda_i} \otimes L_{k,\mu_{i-1}} \to \widehat{L}_{k,\mu_i}$$

be the CVO having the initial term $z_i^{-\Delta_i}\phi^\circ_i$. Let $\Phi(z)$ be the field of the composition $\phi_N(z_N)\cdots\phi_1(z_1)$. Then we have the following:

1. *$\Phi(z)$ is a conformal block of the type $\boldsymbol{\lambda}$.*
2. *The linear map*

$$z_1{}^{-\Delta_1} \cdots z_N^{-\Delta_N} \boldsymbol{V}_{k,\boldsymbol{\lambda}} \to \mathcal{L}^\nabla_{k,\boldsymbol{\lambda}} \tag{3.44}$$

defined by

$$z_1{}^{-\Delta_1} \cdots z_N^{-\Delta_N} \phi^\circ_N \otimes \cdots \otimes \phi^\circ_1 \mapsto \Phi(z) \tag{3.45}$$

is a linear isomorphism.

4. Coset construction of the unitarizable Virasoro modules

Definition 4.1. A representation M of the Virasoro algebra is called a unitarizable if it carries a positive definite Hermitian form $(\cdot|\cdot)$ such that $(L_n v|u) = (v|L_{-n}u)$ for all $v, u \in M, n \in \mathbb{Z}$.

We present here the following beautiful result on the unitarizability of representations of the Virasoro algebra.

Theorem 4.2 [GKO,KW,TK,FQS,L]**.** *The following is a complete list of pairs* (c, h) *for which* $V_{c,h}$ *is unitarizable:*

(1) $c \geq 1$ *and* $h \geq 0$:

(2) $(c, h) = (c^{(k)}, h^{(k)}_{r,s}), \quad k, r, s \in \mathbb{Z}, k \geq 0, 1 \leq r \leq s \leq k+1,$

$$\text{where } c^{(k)} = 1 - \frac{6}{(k+2)(k+3)} \quad \text{and} \quad h^{(k)}_{r,s} = \frac{((k+3)r-(k+2)s)^2-1}{4(k+3)(k+2)}.$$

Remark. We note $c^{(k)} = c(k+3, k+2)$ and $h^{(k)}_{r,s} = h_{r,s}(k+3, k+2)$.

The series of the representations (2) of the above Theorem is often called the *discrete series* of the Virasoro algebra. P. Goddard, A. Kent, D. Olive [GKO] invented a method of constructing the discrete series representations out of representations of affine Lie algebras. Although, the construction is valid for an arbitrary pair $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{p}})$ of affine Lie algebras such that $\widehat{\mathfrak{g}} \supset \widehat{\mathfrak{p}}$, we only use a simple case $(\widehat{\mathfrak{sl}}_2 \times \widehat{\mathfrak{sl}}_2, \widehat{\mathfrak{sl}}_2)$.

We shall now consider the tensor product of two integrable representations of $\widehat{\mathfrak{sl}}_2$, namely $L_{k,l}$ and $L_{1,m}$, where $l \in P_k$ and $m \in P_1$. Define the space

$$B_{l,n} := \{v \in L_{k,l} \otimes L_{1,m} \mid \widehat{\mathfrak{g}}_+ v = 0, Hv = nv\}.$$

Then by the complete irreducibility, we have the decomposition

$$L_{k,l} \otimes L_{1,m} = \bigoplus_{0 \leq n \leq k+1} L_{k+1,n} \otimes B_{l,n}.$$

The coset construction by [GKO] gives a representation of the Virasoro algebra on $L_{k,l} \otimes L_{1,m}$, which commutes with $\widehat{\mathfrak{sl}}_2$, and hence the Virasoro algebra maps $B_{l,n}$ into itself (see Chapter II, section2). The central charge of the representation $B_{l,n}$ is $c^{(k)}$. It is easy to see $B_{l,n} = 0$ unless $n \equiv l + m \bmod 2$. Moreover the following result is well-known.

Theorem 4.3 [GKO,KW,TK1]**.** *We have the decomposition as* $\widehat{sl}_2 \times Vir$*-modules:*

$$L_{k,l} \otimes L_{1,m} = \bigoplus_{\substack{0 \leq n \leq k+1 \\ n \equiv l+m \bmod 2}} L_{k+1,n} \otimes V_{c^{(k)}, h^{(k)}_{l+1,n+1}},$$

$$\text{where } c^{(k)} = 1 - \frac{6}{(k+2)(k+3)} \quad \text{and} \quad h^{(k)}_{r,s} = \frac{((k+3)r-(k+2)s)^2-1}{4(k+3)(k+2)}.$$

The unitarizability of the discrete series follows from this Theorem.

5. Our main results

The following should be basic problems concerning the minimal theories.

Problem 1. *Determine the rank of* $\mathcal{V}_{c,\boldsymbol{h}}$.

Problem 2. *Determine the monodromy representation of* $\mathcal{V}_{c,\boldsymbol{h}}^{\nabla}$.

Problem 3. *Give a realization of the functions in* $\mathcal{V}_{c,\boldsymbol{h}}^{\nabla}$.

The aim of this thesis is to attack these Problems by mainly using the coset construction. As can be seen in the preceding sections, the WZW model is easy to handle compared with the minimal models. In this thesis, we studied the relation between the WZW models and the minimal models induced by the coset construction. We hope that the relationship will give us a better understanding of the minimal models as well as the WZW models.

As the first step to Problem 1, we obtained the following.

Theorem A. *Let* $\boldsymbol{h}$ *be a sequence of highest weights corresponding to degenerate representations, then* $\mathcal{V}_{c,\boldsymbol{h}}$ *is a coherent* $\mathcal{O}_X$*-module.*

Since representations are infinite dimensional, the above fact is nontrivial. Theorem A implies that the $\mathcal{D}_X$-module $\mathcal{V}_{c,\boldsymbol{h}}$ is an integrable connection, namely it is a locally free $\mathcal{O}_X$-module of finite rank, or a vector bundle. It is desirable to obtain the corresponding result for an arbitrary Riemann surface (see [Appendix]), in particular for the case of the minimal models.

The following is our main tool to solve the problems. Let $\mathcal{E}^{\vee}$ denote the $\mathcal{O}_X$ dual of an $\mathcal{O}_X$-module $\mathcal{E}$.

Theorem B. *Let* $\boldsymbol{l} \in P_k^N, \boldsymbol{n} \in P_{k+1}^N$. *Then there exists a canonical map of integrable connections*

$$\varphi : \mathcal{L}_{k,\boldsymbol{l}} \otimes_{\mathcal{O}_X} \mathcal{L}_{k+1,\boldsymbol{n}}^{\vee} \to \mathcal{L}_{1,\boldsymbol{m}}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{V}_{c,\boldsymbol{h}},$$

where we defined $\boldsymbol{m} \in P_1^N$ *and* $\boldsymbol{h} \in H^N$ *by* $m_i \equiv l_i + n_i \bmod 2$ *and* $h_i = h_{l_i+1,n_i+1}$.

In this thesis we call the above result *coset constructions of conformal blocks*. We remark that an extension of the construction of Theorem B to arbitrary Riemann surface C is straightforward, with N-points and C fixed. However, the development of the construction to a family of N-pointed Riemann surfaces is nontrivial, in particular, the description of a natural connection, or a twisted $\mathcal{D}_X$-module structure, is interesting. We shall discuss this elsewhere.

We believe that the map φ obtained in Theorem B are isomorphisms. Of course, if we prove that the map are isomorphisms, Problems 1, 2 can be solved in view of the corresponding results for the WZW models. We partially confirmed this conjecture:

Theorem C. *For* $k = 1$ *and* $N = 3$ *the each map* φ *in the Theorem B is an isomorphism.*

Details are presented in the sections 6, 7 of Chapter II. In the proof of Theorem C, we utilized a realization of representations given by Clifford algebra. Such a description also enable us to give a realization of chiral vertex operators in terms of the Clifford algebra, and hence we answer Problem 3 in this case. We note that the condition $N = 3$ in Theorem C is not thought to be so restrictive, because of the principle of factorization. However, the condition $k = 1$ seems to be hard to extend to an arbitrary positive integer.

APPENDIX

The Ward identities on an arbitrary Riemann surface

We shall summarize here, after E. Witten [Wi, Appendix], how to formulate the Ward identities for the Virasoro CFT on an arbitrary Riemann surface C. The author thanks G. Kuroki for his explanation of the contents in the following form.

Global generalization of the Virasoro algebra. The Virasoro algebra Vir can be described as

$$Vir = \mathbb{C}((z))\frac{d}{dz} \oplus \mathbb{C}C \tag{A.1}$$

with the Lie algebra structure over $\mathbb{C}$;

$$[f(z)\frac{d}{dz}, g(z)\frac{d}{dz}] = (f\frac{dg}{dz} - \frac{df}{dz}g)\frac{d}{dz} + \operatorname*{Res}_{z=0} \omega(f,g)\, dz\ \frac{C}{24}, \tag{A.2}$$

where

$$\omega(f,g) := \frac{d^3 f}{dz^3} g - f\frac{d^3 g}{dz^3}. \tag{A.3}$$

The common generators L_n are given as $L_n = z^{n+1} d/dz\, (n \in \mathbb{Z})$.

Let us consider the transformation property of $\omega(f,g)$ under a change of local parameter z to $w(z)$. The functions f and g transform as

$$\widetilde{f}(w) = \frac{dw}{dz} f(z), \quad \widetilde{g}(w) = \frac{dw}{dz} g(z) \tag{A.4}$$

Put

$$\widetilde{\omega}(w) = \frac{d^3 \widetilde{f}}{dw^3}\widetilde{g} - \widetilde{f}\frac{d^3 \widetilde{g}}{dw^3},$$

and

$$l(z) = f(z)g'(z) - f'(z)g(z),\ \widetilde{l}(w) = \widetilde{f}(w)\widetilde{g}'(w) - \widetilde{f}'(w)\widetilde{g}(w).$$

The straightforward calculation shows that

$$\widetilde{\omega}(w) = \left(\frac{dw}{dz}\right)^{-1} \omega(z) + 2\left(\frac{dw}{dz}\right)^{-1} \{w,z\}\, l(z), \tag{A.5}$$

where $\{w,z\}$ is the Schwarzian derivative

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2.$$

Since the second term in (A.5) does not vanish in general, ω can not be thought as a one form.

Let us recall the composition law for the Schwarzian derivative. If u, w, z are three local coordinates, we have

$$\{u, z\} = \{u, w\} \left(\frac{dw}{dz}\right)^2 + \{w, z\}. \tag{A.6}$$

If we define

$$U_{w,z} = \begin{pmatrix} (dw/dz)^{-1} & \frac{c}{12}\{w, z\}(dw/dz)^{-1} \\ 0 & (dw/dz) \end{pmatrix}, \tag{A.7}$$

then a cocycle condition

$$U_{u,w}\, U_{w,z} = U_{u,z} \tag{A.8}$$

holds in view of (A.6), and (A.4) and (A.5) can be written in the form

$$\begin{pmatrix} c\,\widetilde{\omega}/24 \\ \widetilde{l} \end{pmatrix} = U_{w,z} \begin{pmatrix} c\,\omega/24 \\ l \end{pmatrix}. \tag{A.9}$$

Let us interpreted U's as transition functions a rank 2 vector bundle $\mathcal{U}_c$. There is an exact sequence of $\mathcal{O}_C$-modules

$$0 \to \Omega_C \to \mathcal{U}_c \to \Theta_C \to 0, \tag{A.10}$$

where $\Omega_C \to \mathcal{U}_c$ is a natural map defined by

$$\omega(z)\, dz \mapsto \begin{pmatrix} \omega(z) \\ 0 \end{pmatrix}, \tag{A.11}$$

and $\mathcal{U}_c \to \Theta_C$ is a natural map defined by

$$\begin{pmatrix} \omega(z) \\ f(z) \end{pmatrix} \mapsto f(z)\frac{d}{dz}. \tag{A.12}$$

Let us explain that the sequence (A.10) splits. An open covering of C by open sets O_i ($i \in I$) with local parameter z_i is called a projective structure on C if the coordinate transformation on every $O_i \cap O_j$ is given by a fractional linear transformation $z_j = (az_i + b)/(cz_i + d)$. It is known that there exists a projective structure (see [Gu]) on C. Fix a projective structure $P = \{\, (O_i, z_i)\,\}_{i \in I}$ Then we have $\{z_i, z_j\} = 0$ for all $i, j \in I$, so the transition matrices U_{z_i,z_j} are of diagonal form. This means that the exact sequence (A.1) splits as $\mathcal{U}_c \cong \Omega_C \oplus \Theta_C$, namely, for a projective structure $P = \{\, (O_i, z_i)\,\}_{i \in I}$,

$$f_i(z_i)\frac{d}{dz_i} \mapsto \begin{pmatrix} 0 \\ f_i(z_i) \end{pmatrix} \text{ on } O_i, \tag{A.13}$$

gives a splitting $\sigma : \Theta_C \to \mathcal{U}_c$. We note that every other splitting differs by an element of

$$\mathrm{Hom}_{\mathcal{O}_C}(\Theta_C, \Omega_C) \cong H^0(C, \Omega_C^{\otimes 2}),$$

where $\Omega_C^{\otimes 2}$ is the sheaf of quadratic differentials (see the proof of Lemma).

Define an $\mathcal{O}_C$-bilinear map $[\cdot,\cdot] : \mathcal{U}_c \times \mathcal{U}_c \to \mathcal{U}_c$ by

$$\left[\begin{pmatrix} \omega \\ f \end{pmatrix}, \begin{pmatrix} \eta \\ g \end{pmatrix}\right] = \begin{pmatrix} \frac{c}{24}(f'''g - fg''') \\ fg' - f'g \end{pmatrix}, \tag{A.14}$$

this is well-defined by (A.9), and satisfies

$$\left[\left[\begin{pmatrix} \omega \\ f \end{pmatrix}, \begin{pmatrix} \eta \\ g \end{pmatrix}\right], \begin{pmatrix} \theta \\ h \end{pmatrix}\right] + \left[\left[\begin{pmatrix} \eta \\ g \end{pmatrix}, \begin{pmatrix} \theta \\ h \end{pmatrix}\right], \begin{pmatrix} \omega \\ f \end{pmatrix}\right] + \left[\left[\begin{pmatrix} \theta \\ h \end{pmatrix}, \begin{pmatrix} \omega \\ f \end{pmatrix}\right], \begin{pmatrix} \eta \\ g \end{pmatrix}\right]$$

$$= \begin{pmatrix} dW(f,g,h) \\ 0 \end{pmatrix}, \tag{A.15}$$

where

$$W(f,g,h) = \frac{c}{12}\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}. \tag{A.16}$$

For meromorphic vector fields f, g, h, it is easy to see that W is independent of the choice of coordinate, and gives a meromorphic function on C.

Adeles. For a vector bundle $\mathcal{E}$ on C, consider the restricted product

$$\mathbb{A}(\mathcal{E}) = \prod_{p\in C}{}' \mathcal{E}_p \otimes_{\mathcal{O}_p} \widehat{K}_p \tag{A.17}$$

consisting of products $\prod_{p\in C} e_p$ with $e_p \in \mathcal{E}_p \otimes_{\mathcal{O}_p} \widehat{O}_p$ for all but finitely many $p \in C$, where $\mathcal{E}_p$ is the stalk of $\mathcal{E}$ at $p \in C$, $\widehat{O}_p$ is the completion of the local ring $\mathcal{O}_p$ of C at $p \in C$, $\widehat{K}_p$ is the field of fractions of $\widehat{O}_p$. We call the vector space $\mathbb{A}(\mathcal{E})$ the adele of $\mathcal{E}$, and we shall simply denote the ring $\mathbb{A}(\mathcal{O}_C)$ by $\mathbb{A}$.

By (A.10), we have the following exact sequence

$$0 \to \mathbb{A}(\Omega_C) \to \mathbb{A}(\mathcal{U}_c) \to \mathbb{A}(\Theta_C) \to 0. \tag{A.18}$$

As (A.13), a projective structure gives a splitting $\sigma : \mathbb{A}(\Theta_C) \to \mathbb{A}(\mathcal{U}_c)$. Consider the following commutative diagram with all rows and columns are exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathbb{A} \arrow[r, equal] \arrow[d, "d"'] & \mathbb{A} \arrow[d] & & \\
0 \arrow[r] & \mathbb{A}(\Omega_C) \arrow[r] \arrow[d, "\mathrm{Res}"'] & \mathbb{A}(\mathcal{U}_c) \arrow[r] \arrow[d] & \mathbb{A}(\Theta_C) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C}^{\oplus C} \arrow[r] \arrow[d] & \mathbb{A}(\mathcal{U}_c)/d\,\mathbb{A} \arrow[r] \arrow[d] & \mathbb{A}(\Theta_C) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}, \tag{A.19}$$

where $\mathrm{Res} : \mathbb{A}(\Omega_C) \to \mathbb{C}^{\oplus C}$ is defined by $(\omega_p)_p \mapsto (\mathrm{Res}_p\, \omega_p)_p$.

We define $\mathcal{V}ir_C$ by the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{Ker}\,\Sigma \arrow[r, equal] \arrow[d] & \mathrm{Ker}\,\Sigma \arrow[d] & & \\
0 \arrow[r] & \mathbb{C}^{\oplus C} \arrow[r] \arrow[d, "\Sigma"'] & \mathbb{A}(\mathcal{U}_c)/d\,\mathbb{A} \arrow[r] \arrow[d] & \mathbb{A}(\Theta_C) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathcal{V}ir_C \arrow[r] \arrow[d] & \mathbb{A}(\Theta_C) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}, \tag{A.20}$$

where $\Sigma : \mathbb{C}^{\oplus C} \to \mathbb{C}$ is defined by $(r_p)_p \mapsto \sum_{p\in C} r_p$, and all rows and columns are exact. The bilinear map $[\cdot,\cdot] : \mathbb{A}(\mathcal{U}_c) \times \mathbb{A}(\mathcal{U}_c) \to \mathbb{A}(\mathcal{U}_c)$ defined by the same formula (A.14) induces a Lie algebra structure on $\mathcal{V}ir_C$.

Let Θ_η denote the Lie algebra of global meromorphic vector fields on C, and $\Theta_\eta \hookrightarrow \mathbb{A}(\Theta_C)$ be a natural injection.

Lemma. *Let $\sigma : \mathbb{A}(\Theta_C) \to \mathbb{A}(\Omega_C)$ be a splitting, and consider the following linear map*

$$j_\sigma : \Theta_\eta \hookrightarrow \mathbb{A}(\Theta_C) \xrightarrow{\sigma} \mathbb{A}(\Omega_C) \twoheadrightarrow \mathcal{V}ir_C. \tag{A.21}$$

Then

(1) *j_σ does not depend on the choice of σ,*
(2) *j_σ is a Lie algebra homomorphism.*

Proof. Let $P = \{\ (O_i, z_i)\ \}_{i \in I}$ be a projective structure on C corresponding to σ, namely σ is given by

$$f_i(z_i)\frac{d}{dz_i} \mapsto \begin{pmatrix} 0 \\ f_i(z_i) \end{pmatrix} \tag{A.22}$$

on O_i. Every other splitting $\sigma' : \mathbb{A}(\Theta_C) \to \mathbb{A}(\mathcal{U}_c)$ is obtained, by a unique holomorphic quadratic form $\alpha_i(z_i)\,dz_i^2$ on C, in the way

$$f_i(z_i)\frac{d}{dz_i} \mapsto \begin{pmatrix} f_i(z_i)\,\alpha_i(z_i) \\ f_i(z_i) \end{pmatrix}. \tag{A.23}$$

If $f_i(z_i)$ gives a global meromorphic vector field on C, then $\alpha_i(z_i)\,f(z_i)$ gives a meromorphic one form ω on C, and hence $\sum_{p \in C} \mathrm{Res}_p\, \omega_p = 0$. This proves the first assertion.

Let the splitting

$$\sigma : \mathbb{A}(\Theta_C) \to \mathbb{A}(\Omega_C) \tag{A.24}$$

is given by (A.13). For $f, g \in \mathbb{A}(\Theta_C)$,

$$[\sigma(f), \sigma(g)] - \sigma([f,g]) = \begin{pmatrix} \frac{c}{24}(f_i'''(z_i)g_i(z_i) - f_i(z_i)g_i'''(z_i)) \\ 0 \end{pmatrix} \tag{A.25}$$

on (O_i, z_i). Since P is a projective structure, if f, g are global meromorphic sections, the collection of functions

$$\frac{c}{24}\,(f_i'''(z_i)g_i(z_i) - f_i(z_i)g_i'''(z_i))$$

defines a meromorphic one form on C, and hence the image of $[\sigma(f), \sigma(g)] - \sigma([f,g])$ is 0 in $\mathcal{V}ir_C$. ■

Let $p_1, \dots, p_N$ be distinct points in C, and D be a divisor $\sum_{i=1}^N p_i$. For a vector bundle $\mathcal{E}$ on C, define

$$\mathbb{A}_D(\mathcal{E}) = \prod_{i=1}^N \mathcal{E}_{p_i} \otimes_{\mathcal{O}_{p_i}} \widehat{K}_{p_i}. \tag{A.26}$$

There exist the natural inclusions

$$H^0(C, \mathcal{E}(*D)) \subset \mathbb{A}_D(\mathcal{E}) \subset \mathbb{A}(\mathcal{E}), \tag{A.27}$$

where $H^0(C, \mathcal{E}(*D))$ is the space of global meromorphic sections of $\mathcal{E}$ with possible poles at D.

Let us define a Lie algebra $\mathcal{V}ir_{C,D}$ by the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{Ker}\Sigma \arrow[r, equal] \arrow[d] & \operatorname{Ker}\Sigma \arrow[d] & & \\
0 \arrow[r] & \mathbb{C}^{\oplus D} \arrow[r] \arrow[d, "\Sigma"'] & \mathbb{A}_D(\mathcal{U}_c)/d\,\mathbb{A}_D \arrow[r] \arrow[d] & \mathbb{A}_D(\Theta_C) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathcal{V}ir_{C,D} \arrow[r] \arrow[d] & \mathbb{A}_D(\Theta_C) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \quad , \tag{A.28}$$

where all rows and columns are exact.

Formulation of the Ward identities. A splitting $\sigma : \Theta_C \to \mathcal{U}_c$ induces the injective Lie algebra homomorphism

$$j : H^0(C, \Theta_C(*D)) \hookrightarrow \mathcal{V}ir_{C,D}, \tag{A.29}$$

which does not depend on the choice of σ. The splitting σ also induces the splitting

$$\mathcal{V}ir_{C,D} \cong \mathbb{A}_D(\Theta_C) \oplus \mathbb{C}. \tag{A.30}$$

If we fix an identification of the $\mathbb{C}$-algebras $\mathbb{C}[[t_i]] = \widehat{\mathcal{O}}_{p_i}$ $(i = 1, \ldots, N)$ then this induces the identification

$$\mathbb{A}_D(\Theta_C) = \bigoplus_{i=1}^{N} \mathbb{C}((t_i)) \frac{d}{dt_i}. \tag{A.31}$$

The Lie algebra structure of

$$\mathcal{V}ir_{C,D} \cong \bigoplus_{i=1}^{N} \mathbb{C}((t_i)) \frac{d}{dt_i} \oplus \mathbb{C} \tag{A.32}$$

can be described as

$$[(f_i \frac{d}{dt_i})_i, (g_i \frac{d}{dt_i})_i] = (\ (f_i\, g_i' - f_i'\, g_i) \frac{d}{dt_i}\ )_i + \sum_{i=1}^{N} \operatorname*{Res}_{t_i=0} \omega(f_i, g_i)\, dt_i \frac{c}{24}. \tag{A.33}$$

Let $V_1, \ldots, V_N$ be representations of the Virasoro algebra of the same central charge c. The Lie algebra $\mathcal{V}ir_{C,D}$ acts on the space $\boldsymbol{V} = V_1 \otimes \cdots \otimes V_N$ naturally. Consider the space of coinvariants for $j(H^0(C, \Theta_C(*D)))$, namely

$$\boldsymbol{V}/j(H^0(C, \Theta_C(*D)))\boldsymbol{V} \tag{A.34}$$

and call its dual space the space of conformal blocks, associated with the data

$$(C, D; t_1, \ldots, t_N; V_1, \ldots, V_N).$$

Example. On the Riemann sphere $C = \mathbb{P}^1$, a projective structure is given by $(\mathbb{P}^1 - \infty, z), (\mathbb{P}^1 - 0, w)$ with $w = z^{-1}$. Let $z_1, \ldots, z_N \in \mathbb{C}^\times$ be pairwise distinct points. Put $p_0 \leftrightarrow 0$, $p_i \leftrightarrow z_i$ $(i = 1, \ldots, N)$, $p_\infty \leftrightarrow \infty$. Take the local parameters $t_0 = z, t_i = z - z_i$ $(i = 1, \ldots, N), t_\infty = w$. Consider the modules $V_i = V_{c,h_i}$ $(i = 0, 1, \ldots, N)$, $V_\infty = (V^\dagger_{c,h_\infty})^\tau$, where $\tau : Vir \to Vir$ is an anti-involution defined by $\tau(L_n) = L_{-n}$ $(n \in \mathbb{Z})$, and $(V^\dagger_{c,h_\infty})^\tau$ is the space $V^\dagger_{c,h_\infty}$ with the left Vir action given by $Xv := v\tau(X)$ $(v \in V^\dagger_{c,h_\infty}, X \in Vir)$.

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Chapter II

Coset constructions of conformal blocks

1. Introduction

The principal objectives of conformal field theory (CFT) are the sheaves of conformal blocks. The aim of this work is to study a relationship of conformal blocks (on $\mathbb{P}^1$) of two types of CFT's, the Wess-Zumino-Witten (WZW) model associated with the integrable representations of the affine Lie algebra $\hat{\mathfrak{sl}}_2$ and the minimal model introduced in [BPZ].

The minimal models are related to a class of the most degenerate representations of the Virasoro algebra. Among these representations, the unitarizable representations are classified by the sequence of the central charges

$$c^{(k)} = 1 - \frac{6}{(k+2)(k+3)} \quad \text{for} \quad k = 1, 2, 3, \ldots .$$

The coset construction gives a method for constructing unitarizable representations of the Virasoro algebra out of representations of affine Lie algebras.

Let us recall the coset construction. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra $\hat{\mathfrak{sl}}_2 \times \hat{\mathfrak{sl}}_2$ and $\hat{\mathfrak{p}}$ its diagonal subalgebra. Consider an integrable highest weight $\hat{\mathfrak{g}}$-module of the level $(k, 1)$. Then we have the operators $T^{\cos}[n]\,(n \in \mathbb{Z})$ that commute with $\hat{\mathfrak{p}}$ and satisfy the relations of the Virasoro algebra of the central charge $c^{(k)}$. The obtained representations of the Virasoro algebra are known to be unitarizable and irreducible ([GKO],[KW],[TK1]).

The embedding $\hat{\mathfrak{p}} \subset \hat{\mathfrak{g}}$ induces the decomposition of the conformal blocks for $\hat{\mathfrak{g}}$ into the sum of the conformal blocks for $\hat{\mathfrak{p}}$. By the decomposition, we have canonical maps that relate the two types of CFT's, the unitarizable minimal model and the Wess-Zumino-Witten (WZW) model (Theorem 4.1). We conjectured that the maps are isomorphisms. For $k = 1$, we confirmed the conjecture in the case of three point conformal blocks (Theorem 4.4).

We remark that the map has a counterpart for every other pair of affine Lie algebras. In [NT], the pair $\hat{\mathfrak{sl}}_l \times \hat{\mathfrak{sl}}_r \times \hat{\mathfrak{a}} \subset \hat{\mathfrak{gl}}_{rl}$, where $\hat{\mathfrak{a}}$ is an affine extension of the abelian subalgebra of the central elements of $\mathfrak{gl}_{rl}$, was studied in detail. As an application, they proved "level-rank duality" in the WZW model on $\mathbb{P}^1$.

The paper is organized as follows. In Section 2 we recall the coset construction of representations of the Virasoro algebra. In Section 3 definitions of conformal blocks are presented for both WZW models and the minimal models. We also discuss the coherency of the conformal blocks for the Virasoro algebra (Theorem 3.3.8). In Section 4 we formulate coset constructions of conformal blocks (Theorem 4.1) and make a conjecture that obtained maps from the WZW models to the minimal models are isomorphisms. In Section 5 we review on some results on three point conformal blocks. In Section 6 we prepare some facts on spinor realizations of the integrable $\hat{\mathfrak{g}}$-modules. In Section 7 we confirm the conjecture in the case of $k = 1, N = 3$ using the spinor realizations.

2. The coset constructions of unitarizable representations of the Virasoro algebra

The Virasoro algebra Vir is the Lie algebra defined by $Vir := \oplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} C$ with the following Lie algebra structure:

$$[C, L_n] = 0, \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C.$$

Let $\mathfrak{n}_-$ be the subalgebra generated by $L_n\,(n<0)$. For arbitrary complex numbers c, h, the Verma module $M_{c,h}$ of the central charge c and the conformal dimension h is the module whose restriction to $\mathfrak{n}_-$ is a free module of rank one and the generator v_h, assumed fixed, satisfies $Cv_h = c\,v_h,\ L_0 v_h = h\,v_h$. The module $M_{c,h}$ has a unique irreducible quotient denoted by $V_{c,h}$.

Let $\mathfrak{g}$ be the complex Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 matrices with complex entries. $\mathfrak{g}$ is spanned by the matrices $E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $(\cdot\,|\,\cdot)$ be the invariant bilinear form on $\mathfrak{g}$ defined by $(X|\,Y) := \operatorname{tr} XY$ for $X, Y \in \mathfrak{g}$, where tr means the trace as 2×2 matrices. The subspace $\mathfrak{h} := \mathbb{C}H$ is a Cartan subalgebra of $\mathfrak{g}$. Its dual $\mathfrak{h}^*$ is spanned by the element α, defined by $\alpha(H) = 2$. For $\lambda \in \mathfrak{h}^*$, we denote by L_λ the irreducible highest weight representation of $\mathfrak{g}$ of highest weight $\lambda \in \mathfrak{h}^*$.

The non-twisted affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined by $\hat{\mathfrak{g}} := \mathfrak{g}\otimes\mathbb{C}((\,t\,))\oplus \mathbb{C}K$ with the Lie algebra structure

$$[K, \hat{\mathfrak{g}}\,] = 0,$$

$$[X\otimes f(t), Y\otimes g(t)\,] = [X,Y]\otimes f(t)\,g(t) + (X|\,Y)\operatorname*{Res}_{t=0}(g(t)\,df(t))\,K$$

$$\text{for } X, Y \in \mathfrak{g},\ f(t), g(t) \in \mathbb{C}((\,t\,)).$$

Fix a complex number k. The action of $\mathfrak{g}$ on L_λ can be extended to the Lie subalgebra $\mathfrak{b}_+ := \mathfrak{g}\otimes\mathbb{C}[[t]]\oplus\mathbb{C}K$ of $\hat{\mathfrak{g}}$, by letting $\mathfrak{g}\otimes t\mathbb{C}[[t]]$ act by zero and the central element K by multiplication by $k \in \mathbb{C}$. The induced module $U(\hat{\mathfrak{g}})\otimes_{U(\mathfrak{b}_+)} L_\lambda$ has a unique irreducible quotient denoted by $L_{k,\lambda}$. If the number k, called the level of the module $L_{k,\lambda}$, is a positive integer, we set $P_k := \{\lambda\,|\,\lambda(H)\in\mathbb{Z}_{\geq 0}, (\lambda|\alpha)\leq k\}$. We have $P_k = \{m\alpha/2\,|\,m = 0, 1, \ldots, k\}$. If $\lambda \in P_k$, then $L_{k,\lambda}$ is an integrable $\hat{\mathfrak{g}}$ -module (see [K]). Let v_λ be a fixed highest weight vector of $L_{k,\lambda}$. In the following, we shall often denote the weight $l\alpha/2$ simply by l. For example, we denote $L_{k,l\alpha/2}$ simply by $L_{k,l}$.

Let $U_c(\hat{\mathfrak{g}})$ be the restricted completion of the universal enveloping algebra of $\hat{\mathfrak{g}}$ (see [K, §12.8]), and put $U_c(\hat{\mathfrak{g}})_k = U_c(\hat{\mathfrak{g}})/U_c(\hat{\mathfrak{g}})(K - k\cdot 1)$. Set $X[n] := X\otimes t^n$ for $X \in \mathfrak{g}, n \in \mathbb{Z}$. Let us define the normal product ${}^\circ_\circ \cdot {}^\circ_\circ$ by

$${}^\circ_\circ X[m]\,Y[n]\,{}^\circ_\circ = \begin{cases} X[m]\,Y[n] & (m<n) \\ \frac{1}{2}(X[m]\,Y[n] + Y[n]\,X[m]) & (m=n) \\ Y[n]\,X[m] & (m>n) \end{cases}.$$

Consider the formal Laurent series $X(z) = \sum_{n\in\mathbb{Z}} X[n]z^{-n-1}$ for $X \in \mathfrak{g}$. Let us define the series $T^k(z)$ called the energy momentum tensor by the Segal-Sugawara construction:

$$T^k(z) := \frac{1}{2(k+2)}\,{}^\circ_\circ\,\frac{1}{2}H(z)^2 + E(z)F(z) + F(z)E(z)\,{}^\circ_\circ.$$

Expand $T^k(z)$ in the form $\sum_{n\in\mathbb{Z}} T^k[n]z^{-n-2}$ to get $T^k[n] \in U_c(\hat{\mathfrak{g}})_k$. Since $L_{k,\lambda}$ is a restricted $\mathfrak{g}$-module (see [K, §12.8]), $T^k[n]$ gives an element of $\operatorname{End} L_{k,\lambda}$ which we denote by the same letter.

$T^k[0]$ is diagonalizable on $L_{k,\lambda}$ and the eigenspace decomposition has the following form $L_{k,\lambda} = \oplus_{d\in\mathbb{N}} L_{k,\lambda}[d]$, where we set

$$\triangle_\lambda := \frac{(\lambda|\lambda+\alpha)}{2(k+2)},\ L_{k,\lambda}[d] := \{v \in L_{k,\lambda} \,|\, T^k[0]v = (\triangle_\lambda + d)v\}.$$

Note that the image of $1 \otimes_{U_{\mathfrak{b}_+}} L_\lambda$ coincides with the space $L_{k,\lambda}[0]$ and it is isomorphic as a $\mathfrak{g}$-module to L_λ.

Definition 2.1. We shall call $L_{k,\lambda}[0]$ the underlying $\mathfrak{g}$-module for $L_{k,\lambda}$.

The algebra homomorphism $\triangle : U(\hat{\mathfrak{g}}) \to U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{g}})$ is defined by $\triangle(a) = a \otimes 1 + 1 \otimes a$ ($a \in \hat{\mathfrak{g}}$), that has a natural extension $\triangle : U_c(\hat{\mathfrak{g}}) \to U_c(\hat{\mathfrak{g}} \times \hat{\mathfrak{g}})$. Let J be the ideal of $U_c(\hat{\mathfrak{g}} \times \hat{\mathfrak{g}})$ generated by $(K - k \cdot 1, 0)$ and $(0, K - 1)$. Let us introduce the following elements of $U_c(\hat{\mathfrak{g}} \times \hat{\mathfrak{g}})/J$ by

$$T^{\cos}[n] := T^{\mathrm{tot}}[n] - T^{\triangle}[n],$$

where we set $T^{\mathrm{tot}}[n] := T^k[n] \otimes 1 + 1 \otimes T^1[n]$ and $T^{\triangle}[n] := \triangle(T^{k+1}[n])$. The following is well-known.

Proposition 2.2. *The elements $T^{\cos}[n]$ satisfy the relations of the Virasoro algebra with $C = c^{(k)}$. Moreover $T^{\cos}[n]$ commute with the diagonal subalgebra of $\hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$.*

Let us chose weights $\lambda = l\alpha/2 \in P_k$, $\mu = m\alpha/2 \in P_1$. Consider the tensor product representation $L_{k,\lambda} \otimes L_{1,\mu}$. The elements $T^{\cos}[n]$ give an action of the Virasoro algebra on the module. Thus we have the action of $\triangle(\hat{\mathfrak{g}}) \times Vir$ on $L_{k,\lambda} \otimes L_{1,\mu}$. The structure of the modules are well-known.

Lemma 2.3 ([GKO], [KW], [TK1]). *We have the following decomposition as $\triangle(\hat{\mathfrak{g}}) \times Vir$-modules*

$$L_{k,\lambda} \otimes L_{1,\mu} \cong \bigoplus_{\substack{n\alpha/2 \in P_{k+1} \\ n \equiv l+m \bmod 2}} L_{k+1,n\alpha/2} \otimes V_{c^{(k)},h^{(k)}_{l+1,n+1}},$$

$$\textit{where } h^{(k)}_{r,s} := \frac{((k+3)r - (k+2)s)^2 - 1}{4(k+3)(k+2)}.$$

In Sections 6,7, we treat the following.

Example 2.4. We have the following decompositions as $\triangle(\hat{\mathfrak{g}}) \times Vir$-modules

$$L_{1,0} \otimes L_{1,0} \cong L_{2,0} \otimes V_{1/2,0} \oplus L_{2,2} \otimes V_{1/2,1/2},$$
$$L_{1,1} \otimes L_{1,1} \cong L_{2,2} \otimes V_{1/2,0} \oplus L_{2,0} \otimes V_{1/2,1/2},$$
$$L_{1,0} \otimes L_{1,1} \cong L_{2,1} \otimes V_{1/2,1/16},$$
$$L_{1,1} \otimes L_{1,0} \cong L_{2,1} \otimes V_{1/2,1/16}.$$

3. Conformal blocks

3.1. Consider the affine space $\mathbb{C}^{N-1}$ with a coordinate $z = (z_1, \ldots, z_{N-1})$ and its open set $X := \mathbb{C}^{N-1} - \bigcup_{i \neq j}\{z_i = z_j\}$. Put $Y := \mathbb{P}^1 \times X$. Let $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) be the sheaf of holomorphic functions on X (resp. Y). Let D_i ($i = 1, 2, \ldots , N$) be the divisor of Y defined by the equation $w = z_i$ where w is a inhomogeneous coordinate of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and we set $z_N = \infty$. Put $D := \sum_i D_i$. Let $\pi : Y \to X$ be the second projection. Let us define t_i by $t_i = w - z_i$ if $i \neq N$ and $t_N := w^{-1}$. On each fiber $\pi^{-1}(z)$, t_i gives the local parameter around the point $D_i \cap \pi^{-1}(z)$. By the Laurent expansion with respect to the local parameters, we have an embedding

$$\pi_* \mathcal{O}_Y(*D) := \operatorname*{indlim}_k \pi_* \mathcal{O}_Y(kD) \hookrightarrow \bigoplus_{i=1,\ldots,N} \mathcal{O}_X((t_i)).$$

3.2. Conformal blocks of the WZW model.
The sheaf $\widetilde{\mathfrak{g}}_N$ of affine Lie algebra attached to $\pi : Y \to X$ is a sheaf of $\mathcal{O}_X$-module $\widetilde{\mathfrak{g}}_N := \bigoplus_{i=1,\ldots,N} \mathfrak{g} \otimes \mathcal{O}_X((t_i)) \oplus \mathcal{O}_X K$ with the following commutation relations:

$$[K, \widetilde{\mathfrak{g}}_N] = 0,$$

$$[\oplus_i X_i \otimes f_i \;, \oplus_i Y_i \otimes g_i] = \oplus_i [X_i, Y_i] \otimes f_i\, g_i + \sum_i (X_i | \, Y_i) \operatorname*{Res}_{t_i=0} (g_i \, df_i) \; K,$$

$$\text{where } X_i, Y_i \in \mathfrak{g}, \;\; f_i, g_i \in \mathcal{O}_X((t_i)).$$

By the embedding above, we regard $\mathfrak{g} \otimes \pi_* \mathcal{O}_Y(*D)$ as a subsheaf of $\widetilde{\mathfrak{g}}_N$. Moreover, $\mathfrak{g} \otimes \pi_* \mathcal{O}_Y(*D)$ is a Lie subalgebra in virtue of the residue theorem. Suppose that N weights $\lambda_1, \ldots, \lambda_N \in P_k$ are given. Set $\boldsymbol{\lambda} := (\lambda_1, \ldots , \lambda_N)$. Let us introduce the space $\boldsymbol{L}_{k,\boldsymbol{\lambda}} = \bigotimes_{i=1,\ldots,N} L_{k,\lambda_i}$ on which $\widetilde{\mathfrak{g}}_N$ acts naturally. On the free $\mathcal{O}_X$-module $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{L}_{k,\boldsymbol{\lambda}}, \mathcal{O}_X) \cong \mathcal{O}_X \otimes \boldsymbol{L}^*_{k,\boldsymbol{\lambda}}$, $\widetilde{\mathfrak{g}}_N$ acts by the right. We shall use the notation $A^{(i)}$ for $A \in \operatorname{End} L_{k,\lambda_i}$ to denote $\operatorname{Id} \otimes \cdots \otimes \operatorname{Id} \otimes A \otimes \operatorname{Id} \otimes \cdots \otimes \operatorname{Id} \in \operatorname{End} \boldsymbol{L}_{k,\boldsymbol{\lambda}}$, where A is in the i-th factor. Here we define an $\mathcal{O}_X$-submodule $\mathcal{L}_{k,\boldsymbol{\lambda}}$ of $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{L}_{k,\boldsymbol{\lambda}}, \mathcal{O}_X)$ as follows.

Definition 3.2.1. A local section $\Phi(z)$ of $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{L}_{k,\boldsymbol{\lambda}}, \mathcal{O}_X)$ is in $\mathcal{L}_{k,\boldsymbol{\lambda}}$ if and only if:

$$\sum_{i=1,\ldots,N} \Phi(z) \{ \operatorname*{Res}_{w_i=0} X(w_i) f_i(w_i)\, dw_i \}^{(i)} = 0$$

$$\text{for all } f = (f_1(t_i), \ldots , f_N(t_N)) \in \pi_* \mathcal{O}_Y(*D) \text{ and } X \in \mathfrak{g}.$$

The $\mathcal{O}_X$-module $\mathcal{L}_{k,\boldsymbol{\lambda}}$ is equipped with the following integrable connection called the Knizhnik-Zamolodchikov connection ([KZ]).

Lemma 3.2.2. *The operators*

$$\nabla_i = \frac{\partial}{\partial z_i} - (T^k[-1])^{(i)}, \; i = 1, \ldots , N-1,$$

acting on $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{L}_{k,\boldsymbol{\lambda}}, \mathcal{O}_X)$, commute with each other and preserve $\mathcal{L}_{k,\boldsymbol{\lambda}}$. Therefore they induce an integrable connection on $\mathcal{L}_{k,\boldsymbol{\lambda}}$.

Proof. Commutativity is trivial. For the second part, the reader can consult [FFR, Lemma 4] (see also the proof of Lemma 3.3.4). ■

Lemma 3.2.3. *$\mathcal{L}_{k,\boldsymbol{\lambda}}$ is a coherent $\mathcal{O}_X$-module.*

Proof. The reader is referred to [TUY, Example 2.2.8]. ■

By Lemmas 3.2.2, 3.2.3, a standard argument implies that $\mathcal{L}_{k,\boldsymbol{\lambda}}$ is a locally free $\mathcal{O}_X$-module (of finite rank). Let $\mathcal{L}_{k,\boldsymbol{\lambda}}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_{k,\boldsymbol{\lambda}}, \mathcal{O}_X)$ denote its $\mathcal{O}_X$-dual with the dual connection. The sheaf of horizontal sections $\mathcal{L}_{k,\boldsymbol{\lambda}}^{\nabla}$ is called the sheaf of conformal blocks of the WZW model (on $\mathbb{P}^1$).

Lemma 3.2.4 ([GW], [TK2]). *Let $\boldsymbol{\lambda} = (l_1, l_2, l_3) \in P_k^3$ be given. The rank of $\mathcal{L}_{k,\boldsymbol{\lambda}}$ is at most one. More precisely, it is one if and only if:*

$$l_1 + l_2 + l_3 \text{ is even}, \quad l_1 + l_2 + l_3 \leq 2k,$$

$$l_i + l_j - l_k \geq 0 \text{ for all the permutations } (i, j, k) \text{ of } (1, 2, 3).$$

Definition 3.2.5. We call $\boldsymbol{\lambda} \in P_k^3$ a fusion triple if $\operatorname{rank} \mathcal{L}_{k,\boldsymbol{\lambda}} \neq 0$.

3.3. Conformal blocks of the minimal models.

Let $c, h_1, \ldots, h_N \in \mathbb{C}$ be given. Set $\boldsymbol{h} := (h_1, \ldots, h_N)$. Consider N-fold tensor product $\boldsymbol{V}_{c,\boldsymbol{h}} = \bigotimes_i V_{c,h_i}$. We have an embedding

$$\pi_* \Theta_{Y/X}(*D) := \operatorname*{indlim}_k \pi_* \Theta_{Y/X}(kD) \hookrightarrow \oplus_{i=1,\ldots,N} \mathcal{O}_X((t_i)) \frac{d}{dt_i},$$

where $\Theta_{Y/X}$ is the sheaf of relative vector fields of π. We now define an $\mathcal{O}_X$-submodule $\mathcal{V}_{c,\boldsymbol{h}}$ of $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{V}_{c,\boldsymbol{h}}, \mathcal{O}_X) \cong \mathcal{O}_X \otimes \boldsymbol{V}_{c,\boldsymbol{h}}^*$. Since $L_n v = 0 \, (v \in V_{c,h})$ for $n \gg 0$, the expression $\operatorname{Res}_{w=0} T(w) \, l(w) \, dw$ for $l(w) \in \mathbb{C}((w))$ is well-defined as an element of $\operatorname{End} V_{c,h}$.

Definition 3.3.1. A local section $\Phi(z)$ of $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{V}_{c,\boldsymbol{h}}, \mathcal{O}_X)$ is in $\mathcal{V}_{c,\boldsymbol{h}}$ if and only if:

$$\Phi(z) \sum_{i=1,\ldots,N} \{ \operatorname*{Res}_{w_i=0} T(w_i) \, l_i(w_i) \, dw_i \}^{(i)} = 0$$

$$\text{for all} \quad (l_1(t_1) \frac{d}{dt_1}, \ldots, l_N(t_N) \frac{d}{dt_N}) \in \pi_* \Theta_{Y/X}(*D).$$

Remark 3.3.2. Conformal blocks has the invariance under the projective linear transformations (see for example [BPZ], [TK2]).

Remark 3.3.3. Conformal blocks for the WZW models also satisfy the similar condition to the preceding definition, if we take $T(z)$ as given by the Segal-Sugawara construction (see for example [TUY, §2.4]).

Lemma 3.3.4. *The operators*

$$\nabla_i = \frac{\partial}{\partial z_i} - L_{-1}^{(i)}, \quad i = 1, \ldots, N-1,$$

acting on $\mathcal{H}om_{\mathbb{C}_X}(\boldsymbol{V}_{c,\boldsymbol{h}}, \mathcal{O}_X)$, *commute with each other and preserve* $\mathcal{V}_{c,\boldsymbol{h}}$. *Therefore they induce an integrable connection on* $\mathcal{V}_{c,\boldsymbol{h}}$.

Proof. Commutativity is clear by the definition.

Let $T\,[\,l(w)d/dw\,]$ denote $\sum_{i=1,\ldots,N}\{\mathrm{Res}_{w_i=0}\,T(w_i)\,l_i(w_i)\,dw_i\}^{(i)}$. For the second part, we shall show that $[\ T\,[\,l(w)d/dw\,]\,,\nabla_i\,]$ is again expressed as $T\,[\,p(w)d/dw\,]$ for some $p(w)d/dw \in \pi_*\Theta_{Y/X}(*D)$. It suffices to consider the vector fields $(w - z_i)^{m+1}d/dw$ $(i \neq N)$, $w^{m+1}d/dw$ $(m \in \mathbb{Z})$. By the binomial theorem, we have the following expressions.

$$T\,[\,(w-z_i)^{m+1}\frac{d}{dw}\,] \tag{3.1}$$

$$= L_m^{(i)} + \sum_{\substack{j\neq i,N\\ n\geq -1}} \binom{m+1}{n+1}(z_j - z_i)^{m-n}L_n^{(j)} - \sum_{n\geq -m}\binom{m+1}{m+n}(-z_i)^{m+n}L_n^{(N)},$$

$$T\,[w^{m+1}\frac{d}{dw}\,] = \sum_{j\neq N}\sum_{n\geq -1}\binom{m+1}{n+1} z_j^{m-n}L_n^{(j)} - L_{-m}^{(N)}, \tag{3.2}$$

where $\displaystyle\binom{a}{n} := a(a-1)\cdots(a-n+1)/n!$.

Straightforward calculation using the relation $[L_n, L_{-1}] = (n+1)\,L_{n-1}$ yields

$$[\,T\,[\,(w-z_i)^{m+1}\frac{d}{dw}\,]\,,\nabla_j\,] = -(m+1)\,\delta_{i,j}\;T\,[\,(w-z_i)^m\frac{d}{dw}\,],\ [\,T\,[w^{m+1}\frac{d}{dw}\,]\,,\nabla_j\,] = 0,$$

and hence the lemma follows. ■

In the following, we shall consider some degeneracy condition on the representations.

Definition 3.3.5. *Nonzero element* v *of the Verma module* $M_{c,h}$ *is called a singular vector of degree* $d \in \mathbb{Z}_{\geq 0}$, *if* $L_n v = 0$ *for* $n > 0$ *and* $L_0 v = (h+d)v$.

Remark 3.3.6. A singular vector of degree $d > 0$ belongs to the maximal proper submodule of $M_{c,h}$. Therefore its image in $L_{c,h}$ is null.

Lemma 3.3.7 ([FFu]). (1) *The Verma module* $M_{c,h}$ *has a singular vector of degree* $d > 0$ *if and only if there are* $r, s \in \mathbb{Z}_{>0}$ *and* $t \in \mathbb{C}^\times$ *such that* $rs = d$ *and*

$$c = c(t) := 6t + 13 + 6t^{-1}, \quad h = h_{r,s}(t) := \frac{1-r^2}{4t} + \frac{1-rs}{2} + \frac{1-s^2}{4}t.$$

(2) *For* $r, s \in \mathbb{Z}_{>0}$, *there exists*

$$S_{r,s}(t) := \sum_{\substack{k_1,\ldots,k_d\geq 0\\ i_1+2i_2+\cdots+di_d=d}} p_{r,s}^{k_1,\ldots,k_d}(t)\,L_{-d}^{k_d}\cdots L_{-2}^{k_2}L_{-1}^{k_1}$$

$$(d := rs,\ p_{r,s}^{k_1,\ldots,k_d}(t) \in \mathbb{C}[t,t^{-1}],\ p_{r,s}^{d,0,\ldots,0}(t) \equiv 1)$$

such that $S_{r,s}(t)v_{h_{r,s}(t)}$ *is a singular vector of* $M_{c(t),h_{r,s}(t)}$ *of degree* d.

We have the following sufficient condition for the coherency of the sheaves $\mathcal{V}_{c,\boldsymbol{h}}$.

Theorem 3.3.8. *Let $t \in \mathbb{C}^\times$ and $r_i, s_i \in \mathbb{Z}_{>0}$ $(i = 1, \dots, N)$ be given. Put $c = c(t)$ and $h_i = h_{r_i,s_i}(t)$. Then $\mathcal{V}_{c,\boldsymbol{h}}$ is a coherent $\mathcal{O}_X$-module.*

Proof. We call the elements of the form $M = L_{-n}^{k_n} \cdots L_{-1}^{k_1} \in U(\mathfrak{n}_-)$ the monomials. We set its standard degree $d(M) = \sum_{i=1}^n k_i$. Put also $d'(M) = \sum_{i=2}^n k_i$. We define the standard filtration $\{F_n\}_{n\in\mathbb{Z}}$ of $U(\mathfrak{n}_-)$ by, $F_n U(\mathfrak{n}_-) = 0$ for $n < 0$ and $F_n U(\mathfrak{n}_-) = \sum_{d(M)\le n} \mathbb{C}M$ for $n \ge 0$ where the summation runs through all the monomials of the degree less than or equal to n. For $a \in F_n U(\mathfrak{n}_-) - F_{n-1}U(\mathfrak{n}_-)$, we define its degree $d(a)$ as n. The induced filtration of $V_{c,h}$ is defined by $F_n V_{c,h} = F_n U(\mathfrak{n}_-) v_h$. For $n \ge 0$, we define the subspace $W_n V_{c,h} := \sum_{0\le m\le n} \mathbb{C}L_{-1}^m v_h$ of $F_n V_{c,h}$. Put $F_n \boldsymbol{V}_{c,\boldsymbol{h}} := \sum_{n_1+\cdots+n_N=n} \bigotimes_{i=1,\dots,N} F_{n_i} V_{c,h_i}$, $W_n \boldsymbol{V}_{c,\boldsymbol{h}} := \sum_{n_1+\cdots+n_N=n} \bigotimes_{i=1,\dots,N} W_{n_i} V_{c,h_i}$.

Claim. *Put $d_i = r_i s_i$. The map $\mathcal{V}_{c,h} \to \mathcal{H}om_{\mathbb{C}_X}(W_{\sum_{i=1}^N (d_i-1)} \boldsymbol{V}_{c,\boldsymbol{h}}, \mathcal{O}_X)$ given by the restriction is injective.*

The theorem follows from this claim immediately.

Proof of Claim.

Step 1. Let $v \in F_n \boldsymbol{V}_{c,\boldsymbol{h}}$. We shall now prove the following statement $\mathrm{S}(v)$:
There is a set of functions $f_{k_1,\dots,k_N}(z) \in \mathcal{O}_X(X)$ $(k_i \in \mathbb{Z}_{\ge 0}, \sum_{i=1}^N k_i \le n)$ such that

$$\Phi(z)(v) = \sum_{\substack{k_1,\dots,k_N\in\mathbb{Z}_{\ge0} \\ \sum_{i=1}^N k_i \le n}} f_{k_1,\dots,k_N}(z) \Phi(z)(L_{-1}^{k_1} v_{h_1} \otimes \cdots \otimes L_{-1}^{k_N} v_{h_N}) \text{ for all } \Phi(z) \in \mathcal{V}_{c,\boldsymbol{h}}.$$

It is enough to show $\mathrm{S}(v)$ for the vectors of the form $v := M_1 v_1 \otimes \cdots \otimes M_N v_N \in F_n \boldsymbol{V}_{c,\boldsymbol{h}}$ with M_i monomials, where we put $v_i := v_{h_i}$ for brevity. Put $p := \sum_i d'(M_i)$, the total number of $L_{-2}, L_{-3}, \dots$ that appear in the monomials $M_1, \dots, M_N$. Let $\mathrm{S}(n,p)$ $(0 \le p \le n)$ be the following statement: For all $v \in F_n \boldsymbol{V}_{c,\boldsymbol{h}}$ of the above form with $p = \sum_i d'(M_i)$, $\mathrm{S}(v)$ holds. We shall prove $\mathrm{S}(n,p)$ for all n and p by the induction. $\mathrm{S}(n,0)$ $(n \ge 0)$ are trivial since $p = 0$ means $v \in W_n \boldsymbol{V}_{c,\boldsymbol{h}}$. Thus we consider the cases $1 \le p \le n$. Assume $\mathrm{S}(m,q)$ for $m \le n-1$, $0 \le q \le m$ and $\mathrm{S}(n,q)$ for $q \le p-1$ hold. We shall prove $\mathrm{S}(n,p)$ in the following.

Since $p \ge 1$, some M_i is of the form $M_i = L_{-l}^{k_l} L_{-l+1}^{k_{l-1}} \cdots L_{-1}^{k_1}$ $(l \ge 2, k_l > 0)$. Put $N_i = L_{-l}^{k_l - 1} L_{-l+1}^{k_{l-1}} \cdots L_{-1}^{k_1}$. Let $w_{m,j}$ $(j \ne i, m \ge -1)$ denote the vectors

$$M_1 v_1 \otimes \cdots \otimes L_m M_j v_j \otimes \cdots \otimes N_i v_i \otimes \cdots \otimes M_N v_N \in F_n \boldsymbol{V}_{c,\boldsymbol{h}}.$$

Here we put $d := \sum_{i=1}^n i k_i$. Then it is easy to see that $w_{m,j} = 0$ for $m > d$.
By means of (3.1) and (3.2), we obtain

$$\begin{aligned} &\Phi(z)(\otimes_j M_j v_j) \\ &= -\sum_{\substack{j\ne i,N \\ -1\le m\le d}} \binom{-l+1}{m+1} z_{ji}^{-l-m} \Phi(z)(w_{m,j}) + \sum_{l\le m\le d} \binom{-l+1}{-l+m} (-z_i)^{-l+m} \Phi(z)(w_{m,N}) \end{aligned}$$

when $i \neq N$, where $z_{ji} := z_j - z_i$, and

$$\Phi(z)(\otimes_j M_j v_j) = \sum_{\substack{j \neq N \\ -1 \leq m \leq d}} \binom{l+1}{m+1} z_j^{l-m} \Phi(z)(w_{m,j})$$

when $i = N$.

By the above formulas, it is clear that $\mathrm{S}(n,p)$ follows from $\mathrm{S}(w_{m,j})$ $(j \neq i, m \geq -1)$. It is easy to verify that $L_m\,(m \geq 0)$ preserves $F_l V_{c,h_j}$. Hence the vectors $w_{m,j}$ $(j \neq i, m \geq 0)$ belong to $F_{n-1}\boldsymbol{V}_{c,\boldsymbol{h}}$. By the induction hypothesis, we have $\mathrm{S}(w_{m,j})$ for $j \neq i, m \geq 0$.

It remains to prove $\mathrm{S}(w_{-1,j})$. Write the element $L_{-1}M_j \in U(\mathfrak{n}_-)$ as a linear combination of the monomials, i.e. $L_{-1}M_j = \sum c_K K$. Then we have $d(K) \leq d(M_j) + 1$ and $d'(K) \leq d'(M_j)$. Let v_K be the vector $M_1 v_1 \otimes \cdots \otimes K v_j \otimes \cdots \otimes N_i v_i \otimes \cdots \otimes M_N v_N \in F_n \boldsymbol{V}_{c,\boldsymbol{h}}$. Since $d'(N_i) = d'(M_i) - 1$ we have the inequality

$$\sum_{k \neq i,j} d'(M_k) + d'(K) + d'(N_i) < p.$$

Thus by the induction hypothesis, we have $\mathrm{S}(v_K)$. Hence $\mathrm{S}(w_{-1,j})$ follows.

Step 2. We shall prove the following: For every $v \in \boldsymbol{V}_{c,\boldsymbol{h}}$ there is a set of functions $g_{k_1,\dots,k_N}(z) \in \mathcal{O}_X(X)$ $(0 \leq k_i < d_i)$ such that

$$\Phi(z)(v) = \sum_{\substack{0 \leq k_i < d_i \\ i=1,2,\dots,N}} g_{k_1,\dots,k_N}(z)\Phi(z)(L_{-1}^{k_1} v_{h_1} \otimes \cdots \otimes L_{-1}^{k_N} v_{h_N}) \text{ for all } \Phi(z) \in \mathcal{V}_{c,\boldsymbol{h}}.$$

Claim follows from this immediately.

To show this, we shall utilize the relations (cf. Lemma 3.3.7)

$$L_{-1}^{d_i} v_{h_i} = - \sum_{\substack{k_1 + 2k_2 + \cdots + d_i k_{d_i} = d_i \\ (k_1, \cdots, k_{d_i}) \neq (d_i, 0, \cdots, 0)}} p_{r_i,s_i}^{k_1,\cdots,k_{d_i}}(t)\, L_{-d_i}^{k_{d_i}} \cdots L_{-2}^{k_2} L_{-1}^{k_1} v_{h_i}. \tag{3.3}$$

First of all we observe that each monomial appears in the right hand side has the degree less than d_i.

Let $v \in F_n \boldsymbol{V}_{c,\boldsymbol{h}}$. We use the induction on n. We may assume v in $W_n \boldsymbol{V}_{c,\boldsymbol{h}}$ by virtue of Step 1. Let $v = \otimes_i L_{-1}^{l_i} v_i \in W_n \boldsymbol{V}_{c,\boldsymbol{h}}$ $(\sum_i l_i = n)$. If n is so small that $l_i < d_i$ for all i, then the statement is trivial. Suppose $l_i \geq d_i$ for some i. By means of (3.3) we obtain

$$\Phi(z)(v) = - \sum_{\substack{k_1 + 2k_2 + \cdots + d_i k_{d_i} = d_i \\ (k_1, \cdots, k_{d_i}) \neq (d_i, 0, \cdots, 0)}} p_{r_i,s_i}^{k_1,\cdots,k_{d_i}}(t)\Phi(z)(L_{-1}^{l_i} v_1 \otimes \cdots \otimes M_{k_1,\dots,k_{d_j}} v_j \otimes \cdots \otimes L_{-1}^{l_N} v_N)$$

where we put

$$M_{k_1,\dots,k_{d_j}} := L_{-1}^{l_j - d_j}(L_{-d_j}^{k_{d_j}} \cdots L_{-2}^{k_2} L_{-1}^{k_1})$$

and $M_{k_1,\dots,k_{d_j}} v_j$ are in the j-th factor. By the observation below (3.3), we have the inequality $d(L_{-d_j}^{k_{d_j}} \cdots L_{-2}^{k_2} L_{-1}^{k_1}) < d_i$, and hence $d(M_{k_1,\dots,k_{d_j}}) < l_i$. It follows that the vectors

$$L_{-1}^{l_i} v_1 \otimes \cdots \otimes M_{k_1,\dots,k_{d_j}} v_j \otimes \cdots \otimes L_{-1}^{l_N} v_N$$

belong to $F_{n-1}\boldsymbol{V}_{c,\boldsymbol{h}}$. Thus the induction completes. ■

Corollary 3.3.9. *Let $c, h_1, \dots, h_N$ be as in Theorem 3.3.8. Then the sheaf $\mathcal{V}_{c,\boldsymbol{h}}$ is a locally free $\mathcal{O}_X$-module (of finite rank).*

The minimal models are related to the following special values of c and h's. Fix a pair of coprime positive integers p, q greater than 2. In the following, we are concerned with the following values

$$c = 1 - \frac{6(p-q)^2}{pq},$$

$$h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \quad (r, s \in \mathbb{Z},\ 0 < r < q\,, 0 < s < p).$$

Note that $c = c(t)$ and $h_{r,s} = h_{r,s}(t)$ for $t = -q/p$. Define the finite set $R_c := \{\, h_{r,s} \,|\, r, s \in \mathbb{Z},\ 0 < r < q, 0 < s < p\}$. Since $h_{r,s} = h_{q-r,p-s}$, we have $\#R_c = (p-1)(q-1)/2$.

Definition 3.3.10. *When c and h's are selected as above, we call the sheaf $\mathcal{V}_{c,\boldsymbol{h}}^{\nabla}$ of horizontal sections the sheaf of conformal blocks for the minimal model (on $\mathbb{P}^1$).*

Remark 3.3.11. If $p = k+3$, $q = k+2$ for a positive integer k, then $c = c^{(k)}$ and $h_{r,s} = h_{r,s}^{(k)}$.

Definition 3.3.12. If rank $\mathcal{V}_{c,\boldsymbol{h}} \neq 0$, we call $\boldsymbol{h} \in R_c^3$ a fusion triple.

Lemma 3.3.13 ([W, Ku]). *Suppose $N = 3$. Then the rank of $\mathcal{V}_{c,\boldsymbol{h}}$ is at most one, and $\boldsymbol{h} \in R_c^3$ is a fusion triple if and only if: There exist integers r_i, s_i $(1 \le i \le 3,\ 0 < r_i < q, 0 < s_i < p)$ such that $h_i = h_{r_i,s_i}$ and*

$$r_1 + r_2 + r_3,\ s_1 + s_2 + s_3 \quad \text{are odd}\,,$$

$$r_1 + r_2 + r_3 \le 2q - 1,\ s_1 + s_2 + s_3 \le 2p - 1,$$

$$r_i + r_j - r_k \ge 1,\ s_i + s_j - s_k \ge 1 \text{ for all the permutations } (i, j, k) \text{ of } (1, 2, 3)\,.$$

4 Coset constructions of conformal blocks

We have the one to one correspondence $P_k \times P_{k+1} \cong P_1 \times R_{c^{(k)}}$ given by the coset construction, i.e. $P_k \times P_{k+1} \ni (l, n) \leftrightarrow (m, h_{l+1,n+1}) \in P_1 \times R_{c^{(k)}}$ where $n \equiv l + m \bmod 2$. Let $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in P_k^3 \times P_{k+1}^3$ correspond to $(\boldsymbol{\mu}, \boldsymbol{h}) \in P_1^3 \times R_{c^{(k)}}^3$. Then we observe the following: $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are fusion triples if and only if $\boldsymbol{\mu}$ and $\boldsymbol{h}$ are fusion triples.

Now we can formulate the coset constructions of the conformal blocks.

Theorem 4.1. *Let $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in P_k^N \times P_{k+1}^N$ correspond to $(\boldsymbol{\mu}, \boldsymbol{h}) \in P_1^N \times R_{c^{(k)}}^N$. Then there exists a natural map of integrable connections*

$$\varphi : \mathcal{L}_{k,\boldsymbol{\lambda}} \otimes_{\mathcal{O}_X} \mathcal{L}_{k+1,\boldsymbol{\nu}}^{\vee} \to \mathcal{L}_{1,\boldsymbol{\mu}}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{V}_{c^{(k)},\boldsymbol{h}}.$$

Proof. Since $\mathcal{L}_{k,\boldsymbol{\lambda}}$ etc. are locally free of finite ranks, it suffices to construct $P : \mathcal{L}_{k,\boldsymbol{\lambda}} \otimes_{\mathcal{O}_X} \mathcal{L}_{1,\boldsymbol{\mu}} \to \mathcal{L}_{k+1,\boldsymbol{\nu}} \otimes_{\mathcal{O}_X} \mathcal{V}_{c^{(k)},\boldsymbol{h}}$.

Let us put $\boldsymbol{\lambda} = (l_1, \dots, l_N)$, $\boldsymbol{\mu} = (m_1, \dots, m_N)$ and $\boldsymbol{\nu} = (n_1, \dots, n_N)$. Then $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{\lambda}, \boldsymbol{\nu}) := (h_{l_1+1,n_1+1}, \dots, h_{l_N+1,n_N+1})$ and $n_i \equiv l_i + m_i \bmod 2$. Suppose $\Phi(z) \in \mathcal{L}_{k,\boldsymbol{\lambda}}$ and $\Psi(z) \in \mathcal{L}_{1,\boldsymbol{\mu}}$ are given. By Lemma 2.3, we have

$$\boldsymbol{L}_{k,\boldsymbol{\lambda}} \otimes \boldsymbol{L}_{1,\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\nu}} \boldsymbol{L}_{k+1,\boldsymbol{\nu}} \otimes \boldsymbol{V}_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\boldsymbol{\nu})},$$

where $\boldsymbol{\nu}$ runs through the set $\{\boldsymbol{\nu} = (n_1, \dots, n_N) \in P_{k+1}^N \,|\, n_i \equiv l_i + m_i \bmod 2,\ i = 1, \dots, N\}$. Let P be the $\mathcal{O}_X$-linear map obtained by the composition of the inclusion $\mathcal{L}_{k,\boldsymbol{\lambda}} \otimes_{\mathcal{O}_X} \mathcal{L}_{k,\boldsymbol{\mu}} \hookrightarrow \mathcal{O}_X \otimes (\boldsymbol{L}_{k,\boldsymbol{\lambda}} \otimes \boldsymbol{L}_{1,\boldsymbol{\mu}})^*$ and the projection $\mathcal{O}_X \otimes (\boldsymbol{L}_{k,\boldsymbol{\lambda}} \otimes \boldsymbol{L}_{1,\boldsymbol{\mu}})^* \twoheadrightarrow \mathcal{O}_X \otimes (\boldsymbol{L}_{k+1,\boldsymbol{\nu}} \otimes \boldsymbol{V}_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\,\boldsymbol{\nu})})^*$ induced by the decomposition above. We shall now prove that the image of P is in $\mathcal{L}_{k+1,\boldsymbol{\nu}} \otimes_{\mathcal{O}_X} \mathcal{V}_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\,\boldsymbol{\nu})}$.

Let $\{u_l^{\boldsymbol{\nu}}\}_{l=0}^{\infty}$ be a basis of the space $\boldsymbol{V}_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\,\boldsymbol{\nu})}$. Put $v_l^{\boldsymbol{\nu}} \in \boldsymbol{V}^*_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\,\boldsymbol{\nu})}$ be the elements defined by $v_l^{\boldsymbol{\nu}}(u_m^{\boldsymbol{\nu}}) = \delta_{l,m}$. We can express $P(\Phi(z) \otimes \Psi(z))$ as $\sum_{l=0}^{\infty} f_l^{\boldsymbol{\nu}}(z) \otimes v_l^{\boldsymbol{\nu}}$, $(f_l^{\boldsymbol{\nu}}(z) \in \mathcal{O}_X \otimes \boldsymbol{L}^*_{k+1,\boldsymbol{\nu}})$.

Since $\Phi(z)$ and $\Psi(z)$ are conformal blocks, we have

$$\begin{aligned}
0 &= \left\{ \Phi(z) \sum_{i=1,\dots,N} \{\operatorname*{Res}_{w_i=0} X(w_i) f_i(w_i)\, dw_i\}^{(i)} \right\} \otimes \Psi(z) \\
&\quad + \Phi(z) \otimes \left\{ \Psi(z) \sum_{i=1,\dots,N} \{\operatorname*{Res}_{w_i=0} X(w_i) f_i(w_i)\, dw_i\}^{(i)} \right\} \\
&= \Phi(z) \otimes \Psi(z) \sum_{i=1,\dots,N} \{\operatorname*{Res}_{w_i=0} \triangle(X)(w_i) f_i(w_i)\, dw_i\}^{(i)} \\
&= \sum_{\boldsymbol{\nu}} \sum_{l=0}^{\infty} \left\{ f_l^{\boldsymbol{\nu}}(z) \sum_{i=1,\dots,N} \{\operatorname*{Res}_{w_i=0} X(w_i) f_i(w_i)\, dw_i\}^{(i)} \right\} \otimes v_l^{\boldsymbol{\nu}}.
\end{aligned}$$

Linear independence of $\{v_l^{\boldsymbol{\nu}}\}_{l=0}^{\infty}$ implies

$$f_l^{\boldsymbol{\nu}}(z) \sum_{i=1,\dots,N} \{\operatorname*{Res}_{w_i=0} X(w_i) f_i(w_i)\, dw_i\}^{(i)} = 0$$

for all $\boldsymbol{\nu}$ and l. This means $f_l^{\boldsymbol{\nu}}(z)$ are sections of $\mathcal{L}_{k+1,\,\boldsymbol{\nu}}$.

Locally, we chose a free basis $\{\Xi_l^{\boldsymbol{\nu}}(z)\}_{l=1}^{r_{\boldsymbol{\nu}}}$ of $\mathcal{L}_{k+1,\boldsymbol{\nu}}$, where $r_{\boldsymbol{\nu}} := \operatorname{rank}\mathcal{L}_{k+1,\boldsymbol{\nu}}$. We can rewrite $\Phi(z)\otimes\Psi(z)$ as $\sum_{\boldsymbol{\nu},l}\Xi_l^{\boldsymbol{\nu}}(z)\otimes\Pi_l^{\boldsymbol{\mu}}(z)$, where $\Pi_l^{\boldsymbol{\nu}}(z)$ are in $\mathcal{O}_X\otimes\boldsymbol{V}^*_{c,\boldsymbol{h}(\boldsymbol{\lambda},\boldsymbol{\nu})}$. Next we prove that $\Pi_l^{\boldsymbol{\mu}}(z)$ are conformal blocks for the minimal model for the operators $T^{\cos}[n]$. Let $l(w)d/dw = (l_1(t_1)d/dt_1,\dots,l_N(t_N)d/dt_N)$ be an arbitrary vector field in $\pi_*\Theta_{Y/X}(*D)$.

Let $T^{\mathrm{tot}}[\,l(w)d/dw\,]$ denote $\sum_{i=1,\dots,N}\{\operatorname{Res}_{w_i=0}T^{\mathrm{tot}}(w_i)\,l_i(w_i)\,dw_i\}^{(i)}$ etc. By the identity

$$T^{\cos}[l(w)\frac{d}{dw}] = T^{\mathrm{tot}}[l(w)\frac{d}{dw}] - T^{\triangle}[l(w)\frac{d}{dw}],$$

we have (cf. Remark 3.3.3)

$$0 = \left\{\sum_{\boldsymbol{\nu},l}\Xi_l^{\boldsymbol{\nu}}(z)\otimes\Pi_l^{\boldsymbol{\nu}}(z)\right\}\sum_{i=1,\dots,N}\{\operatorname*{Res}_{w_i=0}\ T^{\mathrm{tot}}(w_i)\ l_i(w_i)\ dw_i\}^{(i)}$$

$$= \sum_{\boldsymbol{\nu},l}\left\{\Xi_l^{\boldsymbol{\nu}}(z)\sum_{i=1,\dots,N}\{\operatorname*{Res}_{w_i=0}T^{\triangle}(w_i)\ l_i(w_i)\ dw_i\}^{(i)}\right\}\otimes\Pi_l^{\boldsymbol{\nu}}(z)$$

$$+\sum_{\boldsymbol{\nu},l}\Xi_l^{\boldsymbol{\nu}}(z)\otimes\left\{\Pi_l^{\boldsymbol{\nu}}(z)\sum_{i=1,\dots,N}\{\operatorname*{Res}_{w_i=0}T^{\cos}(w_i)\ l_i(w_i)\ dw_i\}^{(i)}\right\},$$

where the first term vanish since $\Xi_l^{\boldsymbol{\nu}}(z)\in\mathcal{L}_{k+1,\boldsymbol{\nu}}$.

Thus we obtain

$$0 = \sum_{\boldsymbol{\nu},l}\Xi_l^{\boldsymbol{\nu}}(z)\otimes\left\{\Pi_l^{\boldsymbol{\nu}}(z)\sum_{i=1,\dots,N}\{\operatorname*{Res}_{w_i=0}T^{\cos}(w_i)\ l_i(w_i)\ dw_i\}^{(i)}\right\}.$$

Linear independence of $\Xi_l^{\boldsymbol{\nu}}(z)$ imply $\Pi_l^{\boldsymbol{\nu}}(z)\in\mathcal{V}_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\boldsymbol{\nu})}$. Hence P is an element of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_{k,\boldsymbol{\lambda}}\otimes_{\mathcal{O}_X}\mathcal{L}_{1,\boldsymbol{\mu}},\mathcal{L}_{k+1,\boldsymbol{\nu}}\otimes_{\mathcal{O}_X}\mathcal{V}_{c^{(k)},\boldsymbol{h}(\boldsymbol{\lambda},\,\boldsymbol{\nu})})$.

The commutativity with the connections is easily verified by the fact that $\nabla_i^{\mathrm{tot}} = \nabla_i^{\triangle}+\nabla_i^{\cos}$ on $\mathcal{O}_X\otimes(\boldsymbol{L}_{k,\boldsymbol{\lambda}}\otimes\boldsymbol{L}_{1,\boldsymbol{\mu}})^*$ and Lemmas 3.2.2 and 3.3.4. ∎

Conjecture 4.2. (*G. Kuroki*) *The maps φ are isomorphisms.*

Remark 4.3. The factorization property in CFT (see for example [TUY, Theorem 6.2.6]) for the minimal model and the observation given at the beginning of this section imply $\operatorname{rank}(\mathcal{L}_{k,\boldsymbol{\lambda}}\otimes_{\mathcal{O}_X}\mathcal{L}^\vee_{k+1,\boldsymbol{\nu}}) = \operatorname{rank}(\mathcal{L}^\vee_{1,\boldsymbol{\mu}}\otimes_{\mathcal{O}_X}\mathcal{V}_{c^{(k)},\boldsymbol{h}})$. Moreover, by the factorization property, the proof of the conjecture for arbitrary N can be reduced to the case of $N=3$.

The following of this paper shall be devoted to the proof of the next Theorem.

Theorem 4.4. *The conjecture is true for $N=3, k=1$.*

5. Three point conformal block

We summarize some facts on conformal blocks for $N = 3$. In the following, we shall work on $\mathbb{C}^{\times} = \mathbb{P}^1 - \{0, \infty\}$ rather than on $X = \{(z_1, z_2) \in \mathbb{C}^2 | \, z_1 \neq z_2\}$ i.e. we always put z_1 as 0 and consider the dependence on z_2 only (see Remark 3.3.2).

5.1. Dual modules.

It is sometimes convenient for us to consider the representation at ∞ as a dual right module. The followings are standard.

Proposition 5.1.1. (1) *There exists a unique right $\hat{\mathfrak{g}}$-module $L^{\dagger}_{k,\lambda}$ generated by the highest weight vector $\langle \, v_\lambda \, |$ such that there is a unique non-degenerate bilinear form $\langle \cdot \mid \cdot \rangle : L^{\dagger}_{k,\lambda} \times L_{k,\lambda} \to \mathbb{C}$ satisfying $\langle \, v_\lambda \, | \, v_\lambda \rangle = 1$, $\langle \, u \, a \, | \, v \rangle = \langle \, u \, | \, a \, v \rangle$ for all $u \in L^{\dagger}_{k,\lambda}, v \in L_{k,\lambda}$ and $a \in \hat{\mathfrak{g}}$. Moreover we have $\langle \, L^{\dagger}_{k,\lambda}[m] \, | \, L_{k,\lambda}[n] \, \rangle = 0$ for $m \neq n$.*

(2) *There exists a unique right Vir-module $V^{\dagger}_{c,h}$ generated by the highest weight vector $\langle \, v_h \, |$ such that there is a unique non-degenerate bilinear form $\langle \cdot \mid \cdot \rangle : V^{\dagger}_{c,h} \times V_{c,h} \to \mathbb{C}$ satisfying $\langle \, v_h \, | \, v_h \rangle = 1$, $\langle \, u \, a \, | \, v \rangle = \langle \, u \, | \, a \, v \rangle$ for all $u \in V^{\dagger}_{c,h}, v \in V_{c,h}$ and $a \in Vir$. Moreover we have $\langle \, V^{\dagger}_{c,h}[m] \, | \, V_{c,h}[n] \, \rangle = 0$ for $m \neq n$.*

We have linear the isomorphisms $\hat{L}_{k,\lambda} \cong \mathrm{Hom}_{\mathbb{C}} \, (L^{\dagger}_{k,\lambda}, \mathbb{C})$, $\hat{V}^{\dagger}_{c,h} \cong \mathrm{Hom}_{\mathbb{C}} \, (V^{\dagger}_{c,h}, \mathbb{C})$, where we put $\hat{L}_{k,\lambda} := \prod_{d=0}^{\infty} L_{k,\lambda}[d]$, $\hat{V}_{c,h} := \prod_{d=0}^{\infty} V_{c,h}[d]$. Moreover, we use the following.

Proposition 5.1.2. (1) *There is a unique linear isomorphism $\tau : L_{k,\lambda} \to L^{\dagger}_{k,\lambda}$ such that $\tau \, (| \, v_\lambda \, \rangle) = \langle \, v_\lambda \, |$, $\tau(X[n] | \, v \, \rangle) = -\langle \, v \, | X[-n]$ for any $v \in L_{k,\lambda}$ and $X[n] \in \hat{\mathfrak{g}}$.*

(2) *There is a unique linear isomorphism $\tau : V_{c,h} \to V^{\dagger}_{c,h}$ such that $\tau \, (| \, v_h \, \rangle) = \langle \, v_h \, |$, $\tau(L_n | \, v \, \rangle) = \langle \, v \, | L_{-n}$ for any $v \in V_{c,h}$ and $n \in \mathbb{Z}$.*

Remark 5.1.3. The τ's for both $\hat{\mathfrak{g}}$ and Vir are consistent with the Segal-Sugawara construction.

By the preceding two propositions, we have the following.

Lemma 5.1.4. *Let M_i $(i = 1, 2, 3)$ be the modules L_{k,λ_i} or V_{c,h_i}. There is a canonical linear isomorphism:*

$$(M_3 \otimes M_2 \otimes M_1)^* \cong \mathrm{Hom}_{\mathbb{C}}(M_2 \otimes M_1, \hat{M}_3).$$

More precisely, for Φ in the left hand side, the corresponding operator in the right hand side, denoted by the same letter, is given by

$$\langle \, \tau(u) \, | \Phi(v \otimes w) \rangle = \Phi(u \otimes v \otimes w), \text{ for any } u \in M_3, v \in M_2 \text{ and } w \in M_1.$$

In the following, we shall use the above identification.

5.2. Primary fields. In this subsection, the notion of a primary field is introduced. We start with the case for the WZW model.

Definition 5.2.1 ([KZ], [TK2, §2]). *A linear operator $\Phi(z) : L_{\lambda_2} \otimes L_{k,\lambda_1} \to \hat{L}_{k,\lambda_3}$ is called a primary field of type $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ if the following conditions are satisfied:*

(1) *$\Phi(v, z)$ depends (multi-valued) holomorphically on $z \in \mathbb{C}^\times$,*

(2) *$[X[m], \Phi(v, z)\,] = z^m \Phi(Xv, z)$ for $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$,*

(3) *$[T^k[m], \Phi(v, z)] = z^m (z\dfrac{d}{dz} + \triangle_{\lambda_2}(m+1))\,\Phi(v, z)$ for $m \in \mathbb{Z}$,*

where $\Phi(v, z) : L_{k,\lambda_1} \to \hat{L}_{k,\lambda_3}$ is the operator defined by $\Phi(v, z)\,u = \Phi(z)(v \otimes u)$, $(v \in L_{\lambda_2}, u \in L_{k,\lambda_1})$.

The sheaf of the primary fields of type $\boldsymbol{\lambda}$ is denoted by $\mathcal{L}^{\mathrm{pr}}_{k,\boldsymbol{\lambda}}$. Recall the isomorphism $L_\lambda = L_{k,\lambda}[0]$. We have the following correspondence of primary fields and conformal blocks.

Proposition 5.2.2 ([TK2, Theorem 2.9]). *Let $\Phi(z) : L_{\lambda_2} \otimes L_{k,\lambda_1} \to \hat{L}_{k,\lambda_3}$ be a primary field of type $\boldsymbol{\lambda}$. Then there is a unique extension $\widetilde{\Phi}(z) : L_{k,\lambda_2} \otimes L_{k,\lambda_1} \to \hat{L}_{k,\lambda_3}$ such that it is a conformal block of type $\boldsymbol{\lambda}$. Moreover, this map $\mathcal{L}^{\mathrm{pr}}_{k,\boldsymbol{\lambda}} \ni \Phi(z) \mapsto \widetilde{\Phi}(z) \in \mathcal{L}^{\nabla}_{k,\boldsymbol{\lambda}}$ is an isomorphism of local systems.*

Suppose $\boldsymbol{h} \in R_c^3$ be given. We have the parallel results for the minimal models.

Definition 5.2.3. *A linear operator $\Phi(z) : V_{c,h_1} \to \hat{V}_{c,h_3}$ is called a primary field of type $\boldsymbol{h}$ if the following conditions are satisfied:*

(1) *$\Phi(z)$ depends (multi-valued) holomorphically on $z \in \mathbb{C}^\times$,*

(2) *$[\,L_m, \Phi(z)\,] = z^m (z\dfrac{d}{dz} + h_2(m+1))\,\Phi(z) \quad (m \in \mathbb{Z})$.*

The sheaf of the primary fields of type $\boldsymbol{h}$ is denoted by $\mathcal{V}^{\mathrm{pr}}_{c,\boldsymbol{h}}$.

For $h \in \mathbb{C}$ and $m \in \mathbb{Z}$, let $\mathcal{L}^{(h)}_m$ denote the differential operator

$$-z^m (z\frac{d}{dz} + h(m+1))$$

on $\mathbb{C}^\times$. They satisfy the relations of the Virasoro algebra with $C = 0$. For $r, s \in \mathbb{Z}_{>0}$, let $\mathcal{D}^{(h)}_{r,s}(t)$ be the differential operators obtained by substituting $\mathcal{L}^{(h)}_{-m}$'s into $S_{r,s}(t)$ in place of L_{-m} 's:

$$D^{(h)}_{r,s}(t) := \sum_{\substack{k_1,k_2,\dots,k_d \geq 0 \\ k_1+2k_2+\cdots+dk_d=d}} p^{k_1,\dots,k_d}_{r,s}(t)\,\mathcal{L}^{k_d}_{-d} \cdots \mathcal{L}^{k_2}_{-2}\mathcal{L}^{k_1}_{-1}.$$

Lemma 5.2.4. *Define the function $f_{r,s}(h_1, h_2, h_3; t) \in \mathbb{C}[h_1, h_2, h_3, t, t^{-1}]$ by the formula*

$$\mathcal{D}^{(h_2)}_{r,s}(t)\,z^{-h_1-h_2+h_3} = f_{r,s}(h_1, h_2, h_3, t)\,z^{-h_1-h_2+h_3-rs}.$$

Then

$$f_{r_1,s_1}(h_{r_1,s_1}(t), h_2,\, h_{r_3,s_3}(t); t)^{\,2} = \prod_{i=1}^{r_1}\prod_{j=1}^{s_1}(h_2 - h_{r_1+r_3-2i+1,s_1+s_3-2j+1}(t)\,)^{\,2}.$$

Proof. This follows from [FFu,Theorem 1.12]. (Note that L_n is denoted by e_{-n} in [FFu]. Then our $f_{r,s}(h_1, h_2, h_3; t)$ is $p_{r,s}(-h_2, h_1 - h_2 - h_3, t)$ in [FFu].)

Proposition 5.2.5. *Let $\Phi(z) : V_{c,h_1} \to \hat{V}_{c,h_3}$ be a primary field of type $\boldsymbol{h}$. Then there is a unique conformal block $\widetilde{\Phi}(z) : V_{c,h_2} \otimes V_{c,h_1} \to \hat{V}_{c,h_3}$ such that $\widetilde{\Phi}(z)(v_{h_2} \otimes u) = \Phi(z)(u)$ for any $u \in V_{c,h_1}$. Moreover, this map $\mathcal{V}^{\mathrm{pr}}_{c,\boldsymbol{h}} \ni \Phi(z) \mapsto \widetilde{\Phi}(z) \in \mathcal{V}^{\nabla}_{c,\boldsymbol{h}}$ is an isomorphism of local systems.*

Proof. ([Ku, Appendix B]) We may assume $\Phi(z) \neq 0$. By the same method of [TK2], we can uniquely extend $\Phi(z)$ to $\Phi'(z) : M_{c,h_2} \otimes V_{c,h_1} \to \hat{V}_{c,h_3}$ with $\Phi'(z)(v_{h_2} \otimes u) = \Phi(z)(u)$ ($u \in V_{c,h_1}$) such that $\Phi'(z)$ satisfy the condition of conformal block with M_{c,h_2} instead of V_{c,h_2}.

Let $h_2 = h_{r_2,s_2}$. We know that the maximal proper submodule of M_{c,h_2} is generated by $S_{r_2,s_2} v_{h_2}$ and $S_{q-r_2,p-s_2} v_{h_2}$ (see [FFu]), where we put $S_{r,s} := S_{r,s}(-q/p)$ for short. We shall show that

$$\Phi'(z)(U(Vir)S_{r_2,s_2}\, v_{h_2} \otimes u) = \Phi'(z)(U(Vir)S_{q-r_2,p-s_2}\, v_{h_2} \otimes u) = 0$$

for all $u \in V_{c,h_1}$, and hence $\Phi'(z)$ induces the conformal block $\widetilde{\Phi}(z) : V_{c,h_2} \otimes V_{c,h_1} \to \hat{V}_{c,h_3}$. By the condition of conformal block, it is sufficient to show

$$\langle v_{h_3} \,|\, \Phi'(z)(S_{r_2,s_2}\, v_{h_2} \otimes v_{h_1})\,\rangle = \langle v_{h_3} \,|\, \Phi'(z)(S_{q-r_2,p-s_2}\, v_{h_2} \otimes v_{h_1})\,\rangle = 0.$$

If $m > 0$, we have, by the conditions of conformal blocks,

$$\langle v_{h_3} | \Phi'(z)(L_{-m} v \otimes v_{h_1}) \rangle = (-1)^m \mathcal{L}^{(h_1)}_{-m}\, \langle v_{h_3} | \Phi'(z)(v \otimes v_{h_1}) \rangle \quad (\text{for all } v \in M_{c,h_2}).$$

On the other hand, by the relation $[L_0, \Phi(z)] = -\mathcal{L}^{(h_2)}_0 \Phi(z)$, we have

$$\langle v_{h_3} | \Phi'(z)(v_{h_2} \otimes v_{h_1}) \rangle = z^{-h_1-h_2+h_3}$$

up to constant multiples. Hence, we have

$$\langle v_{h_3} | \Phi'(z)(S_{r_2,s_2}\, v_{h_2} \otimes v_{h_1}) \rangle = (-1)^{r_2 s_2} \mathcal{D}^{(h_1)}_{r_2,s_2}\; z^{-h_1-h_2+h_3},$$

$$\langle v_{h_3} | \Phi'(z)(S_{q-r_2,p-s_2}\, v_{h_2} \otimes v_{h_1}) \rangle = (-1)^{(q-r_2)(p-s_2)} \mathcal{D}^{(h_1)}_{q-r_2,p-s_2}\; z^{-h_1-h_2+h_3},$$

where $\mathcal{D}^{(h_1)}_{r,s} = \mathcal{D}^{(h_1)}_{r,s}(-q/p)$.

Since (h_1, h_2, h_3) is a fusion triple (Theorem 3.3.13), we have, by Lemma 5.2.4,

$$\mathcal{D}^{(h_1)}_{r_2,s_2}\; z^{-h_1-h_2+h_3} = \mathcal{D}^{(h_1)}_{q-r_2,p-s_2}\; z^{-h_1-h_2+h_3} = 0.$$

Hence $\langle v_{h_3} | \Phi'(z)(S_{r_2,s_2}\, v_{h_2} \otimes v_{h_1}) \rangle = \langle v_{h_3} | \Phi'(z)(S_{q-r_2,p-s_2}\, v_{h_2} \otimes v_{h_1}) \rangle = 0$.

For any $\widetilde{\Phi}(z) \in \mathcal{V}^{\nabla}_{c,\boldsymbol{h}}$, we have $\Phi(z) \in \mathcal{V}^{\mathrm{pr}}_{c,\boldsymbol{h}}$ by the restriction i.e. $\Phi(z)(u) := \widetilde{\Phi}(z)(v_{h_2} \otimes u)$ ($u \in V_{h_1}$). This gives the inverse of the map $\Phi(z) \mapsto \widetilde{\Phi}(z)$. ∎

Remark 3.2.6. By an argument similar to the proof of the above proposition, we can prove Lemma 3.3.13 (see [Ku, Appendix B]). We note that the formula in Lemma 5.2.4 is the key.

We call $\Phi(z)$ the primary part of $\widetilde{\Phi}(z)$. In the following we just write $\Phi(z)$ for $\widetilde{\Phi}(z)$, and identify the primary field and the corresponding conformal block.

6. Spinor realizations

We prepare some results on representation theory which are used in the next section to prove Theorem 4.4.

Let Cl_δ $(\delta = 0, 1/2)$ be the $\mathbb{C}$-algebras with the generators $\chi[n], \bar{\chi}[n], \psi[n], \bar{\psi}[n]$ $(n \in \mathbb{Z} + \delta)$, and the relations:

$$\{\chi[m], \bar{\chi}[n]\} = \{\psi[m], \bar{\psi}[n]\} = \delta_{m+n,0},$$
$$\text{all other pairs of generators anti-commute,}$$

where we defined $\{a, b\} := ab + ba$. We define the Fock representations $\mathcal{F}_\delta$ of Cl_δ as follows. $\mathcal{F}_\delta$ is generated by the vector $|\,\delta\,\rangle$ satisfying

$$\chi[n]|\,\delta\,\rangle = \psi[n]|\,\delta\,\rangle = 0 \quad \text{for} \quad n \geq 0,$$
$$\bar{\chi}[n]|\,\delta\,\rangle = \bar{\psi}[n]|\,\delta\,\rangle = 0 \quad \text{for} \quad n > 0.$$

Denote by $\mathcal{F}_\delta^+, \mathcal{F}_\delta^-$ the even and odd part respectively with respect to the number of the generators.

Let us introduce the following formal generating functions called fermions

$$\chi(z) = \sum_{n \in \mathbb{Z}+\delta} \chi[n] z^{-n-1/2}, \ \bar{\chi}(z) = \sum_{n \in \mathbb{Z}+\delta} \bar{\chi}[n] z^{-n-1/2},$$
$$\psi(z) = \sum_{n \in \mathbb{Z}+\delta} \psi[n] z^{-n-1/2}, \ \bar{\psi}(z) = \sum_{n \in \mathbb{Z}+\delta} \bar{\psi}[n] z^{-n-1/2}.$$

We also put $\psi_\pm[n] := \psi[n] \pm \bar{\psi}[n] \in Cl_\delta$ and

$$\psi_\pm(z) := \psi(z) \pm \bar{\psi}(z).$$

We define the normal order $:\cdot:$ by

$$: a[m]\, b[n] := \begin{cases} a[m]\, b[n] & (m < n) \\ \frac{1}{2}(a[m]\, b[n] - b[n]\, a[m]) & (m = n) \\ -b[n]\, a[m] & (m > n) \end{cases},$$

where $a, b = \chi, \bar{\chi}, \psi, \bar{\psi}$.

Let E_i, H_i, F_i $(i = 1, 2)$ denote the copy of the matrices E, H, F respectively. Then we have the following realizations of $\hat{\mathfrak{sl}}_2 \times \hat{\mathfrak{sl}}_2$-modules of the level $(1, 1)$.

Lemma 6.1 ([FFr], [F], [KP]). *The following set of generating functions:*

$$E_1(z) = \chi(z)\, \bar{\psi}(z)\,,\ F_1(z) = \psi(z)\, \bar{\chi}(z)\,,\ H_1(z) = :\chi(z)\, \bar{\chi}(z): - :\psi(z)\, \bar{\psi}(z):,$$
$$E_2(z) = \chi(z)\, \psi(z)\,,\ F_2(z) = \bar{\psi}(z)\, \bar{\chi}(z)\,,\ H_2(z) = :\chi(z)\, \bar{\chi}(z): + :\psi(z)\, \bar{\psi}(z):$$

gives a $\hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$ -module structure on $\mathcal{F}_\delta$ and we have

$$\mathcal{F}_{1/2}^+ \cong L_{1,0} \otimes L_{1,0}, \quad \mathcal{F}_{1/2}^- \cong L_{1,1} \otimes L_{1,1}, \quad \mathcal{F}_0^+ \cong L_{1,0} \otimes L_{1,1}, \quad \mathcal{F}_0^- \cong L_{1,1} \otimes L_{1,0}.$$

The highest weight vectors are given by $|\,1/2\,\rangle$, $\chi[-1/2]\,|\,1/2\,\rangle$, $|\,0\,\rangle$, $\bar{\psi}\,[0]\,|\,0\,\rangle$ respectively.

By means of the last assertion of the above lemma, we can easily verify the next lemma.

Lemma 6.2. *The underlying $\mathfrak{g}\times\mathfrak{g}$ modules are given respectively as follows:*

$$\mathbb{C}|\,1/2\,\rangle \quad \textit{for} \quad \mathcal{F}^{+}_{1/2},$$

$$\mathbb{C}\chi[-1/2]|\,1/2\,\rangle\oplus\mathbb{C}\bar{\chi}[-1/2]|\,1/2\,\rangle\oplus\mathbb{C}\psi[-1/2]|\,1/2\,\rangle\oplus\mathbb{C}\bar{\psi}[-1/2]|\,1/2\,\rangle \quad \textit{for} \quad \mathcal{F}^{-}_{1/2},$$

$$\mathbb{C}|\,0\,\rangle\oplus\mathbb{C}\bar{\psi}[0]\bar{\chi}[0]|\,0\,\rangle \quad \textit{for} \quad \mathcal{F}^{+}_{0},$$

$$\mathbb{C}\bar{\psi}[0]|\,0\,\rangle\oplus\mathbb{C}\bar{\chi}[0]|\,0\,\rangle \quad \textit{for} \quad \mathcal{F}^{-}_{0}.$$

The action of the diagonal subalgebra is given by

$$E(z)=\chi(z)\,\psi_+(z)\,,\ F(z)=\psi_+(z)\,\bar{\chi}(z)\,,\ H(z)=\ 2:\chi(z)\,\bar{\chi}(z):.$$

On the other hand, the operators $T^{\cos}[m]$ are given as follows.

Lemma 6.3. *We have the formula:*

$$T^{\cos}(z)=\frac{1}{4}:\,\psi_-(z)\frac{d}{dz}\psi_-(z)\,:+\frac{1-2\delta}{16}z^{-2}.$$

Proof. By [FFr, Prop. 13], we have

$$T^{\mathrm{tot}}(z)=-\frac{1}{2}:\chi(z)\overset{\leftrightarrow}{\frac{d}{dz}}\bar{\chi}(z):-\frac{1}{2}:\psi(z)\overset{\leftrightarrow}{\frac{d}{dz}}\bar{\psi}(z):+\frac{1-2\delta}{4}z^{-2},$$

where we used the notation

$$a\overset{\leftrightarrow}{\frac{d}{dz}}b:=a\frac{d}{dz}b-\left(\frac{d}{dz}a\right)b.$$

Straightforward calculation using this leads to the formula. ∎

The irreducible decomposition of $\mathcal{F}^{\pm}_{1/2}$ as a $\triangle(\hat{\mathfrak{g}})\times Vir$-module (see Example 2.4) can be described as follows.

Lemma 6.4. *For $a=\chi,\psi$, we denote by $\mathcal{F}_{a,\bar{a}}$ the subspaces of $\mathcal{F}_{1/2}$ generated from $|\,1/2\,\rangle$ by $\{a[n],\bar{a}[n]\,|\,n\in\mathbb{Z}+1/2\}$. Let $\mathcal{F}_{\psi_\pm}$ be the subspace of $\mathcal{F}_{\psi,\bar{\psi}}$ generated from $|\,1/2\,\rangle$ by $\{\psi_\pm[n]\,|\,n\in\mathbb{Z}+1/2,n<0\}$. Then we have a linear isomorphism $\mathcal{F}_{\psi,\bar{\psi}}=\mathcal{F}_{\psi_+}\otimes\mathcal{F}_{\psi_-}$. Put also $\mathcal{F}_\triangle:=\mathcal{F}_{\chi,\bar{\chi}}\otimes\mathcal{F}_{\psi_+}$. Then we have $\mathcal{F}_{1/2}=\mathcal{F}_\triangle\otimes\mathcal{F}_{\psi_-}$ and*

$$\mathcal{F}^{+}_{\triangle}\otimes\mathcal{F}^{+}_{\psi_-}\cong L_{2,0}\otimes V_{1/2,0},\quad \mathcal{F}^{-}_{\triangle}\otimes\mathcal{F}^{-}_{\psi_-}\cong L_{2,2}\otimes V_{1/2,1/2},$$

$$\mathcal{F}^{-}_{\triangle}\otimes\mathcal{F}^{+}_{\psi_-}\cong L_{2,2}\otimes V_{1/2,0},\quad \mathcal{F}^{+}_{\triangle}\otimes\mathcal{F}^{-}_{\psi_-}\cong L_{2,0}\otimes V_{1/2,1/2},$$

where the superscripts mean the even (odd) part with respect to the generators.

Proof. This is verified by a calculation of the characters. ∎

7. Proof of Theorem 4.4.

7.1. Conformal blocks for $\hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$.

By Lemma 6.1, we shall identify the set $P_1 \times P_1$ parametrizing the isomorphism classes of the integrable $\hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$-module of the level $(1,1)$ with the set $\{(\rho, \delta) \mid \rho = \pm,\ \delta = 1/2, 0\}$. We can summarize the existence of conformal blocks in the following form.

Proposition 7.1.1. $(\mathcal{F}_{\delta_1}^{\rho_1}, \mathcal{F}_{\delta_2}^{\rho_2}, \mathcal{F}_{\delta_3}^{\rho_3})$ *is a fusion triple if and only if* $\rho_1 = \rho_2\rho_3$, $\delta_1 = \delta_2\delta_3$. *Here* $\rho_2\rho_3$ *is understood as the product of signs, and the product* $\delta_2\delta_3$ *has the same meaning identifying* $1/2$ *with a plus sign and* 0 *with a minus sign.*

Proof. This is easily verified by Lemma 3.2.4. ∎

Let $\hat{\mathcal{F}}_\delta^{\pm}$ denote the direct product of all the eigenspaces of $T^{\mathrm{tot}}[0]$. In the following, the fermions shall be considered as $\mathrm{Hom}(\mathcal{F}_\delta, \hat{\mathcal{F}}_\delta)$-valued holomorphic functions on $\mathbb{C}^\times$, which are double valued when $\delta = 0$.

In the next subsection, we shall use the following explicit forms of the primary fields (cf. Lemma 6.2).

Lemma 7.1.2. (1) *The primary part of any nonzero conformal block* $\Phi(z) : \mathcal{F}_{1/2}^{+} \otimes \mathcal{F}_\delta^{\pm} \to \hat{\mathcal{F}}_\delta^{\pm}$ *is given (up to a nonzero factor) by* $\Phi(|\,1/2\,\rangle, z) = \mathrm{Id}_{\mathcal{F}_\delta^{\pm}}$,

(2) *The primary part of any nonzero conformal block* $\Phi(z) : \mathcal{F}_{1/2}^{-} \otimes \mathcal{F}_\delta^{\pm} \to \hat{\mathcal{F}}_\delta^{\mp}$ *is given (up to a nonzero factor) by* $\Phi(a[-1/2]|\,1/2\,\rangle, z) = a(z) : \mathcal{F}_\delta^{\pm} \to \hat{\mathcal{F}}_\delta^{\mp}, (a = \chi, \bar{\chi}, \psi, \bar{\psi})$.

Note that the fermions in the right hand sides are those for $\mathcal{C}l_\delta$.

Proof. In view of Proposition 5.2.2, it is enough to prove that they form primary fields. In fact, we can show this by straightforward calculations using the operator product expansion. ∎

Lemma 7.1.3. *Let* $\Phi(z) : \mathcal{F}_{1/2}^{+} \otimes \mathcal{F}_\delta^{\pm} \to \hat{\mathcal{F}}_\delta^{\pm}$ *be the conformal block corresponding to the primary field* $\Phi(z)$ *in Lemma 7.1.2 (1). Then we have*

$$\Phi(a[-1/2]\psi_-[-1/2]|\,1/2\,\rangle, z) = a(z)\psi_-(z) : \mathcal{F}_\delta^{\pm} \to \hat{\mathcal{F}}_\delta^{\pm}$$

for $a = \chi, \bar{\chi}, \psi_+$.

Note that the product of $a(z)$ and $\psi_-(z)$ is well-defined since they anti-commute with each other.

Proof. This follows from the general formula in [TK2, §2.5]. ∎

Remark 7.1.4. The coefficients of the operators $\chi(z)\psi_-(z)$ etc. with respect to the valuable z belong to $\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_\delta^{\pm}, \mathcal{F}_\delta^{\pm})$. Hence we shall denote $\chi(z)\psi_-(z) : \mathcal{F}_\delta^{\pm} \to \mathcal{F}_\delta^{\pm}$ etc. for simplicity of the notation.

7.2. Decomposition of conformal blocks. Let $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be a fusion triple of Fock modules (cf. Proposition 7.1.1), and take a nonzero conformal block $\Phi(z)$: $\mathcal{F}_2 \otimes \mathcal{F}_1 \to \hat{\mathcal{F}}_3$. As a $\hat{\mathfrak{g}} \times Vir$-module, each $\mathcal{F}_i$ is irreducible or decomposed into two components (cf. Example 2.4). We shall calculate the corresponding components with respect to this decomposition. If a component corresponds to a fusion triple, we shall call it a fusion component. We know that the rank of the sheaves of conformal blocks are at most one for $N = 3$. Therefore, for the proof of Theorem 4.1, it is enough to show that all the fusion components do not vanish. We prepare the following lemma which is easily verified.

Lemma 7.2.1. *In the Fock module* $\mathcal{F}_{1/2} \cong L_{1,0} \otimes L_{1,0} \oplus L_{1,1} \otimes L_{1,1}$, *we have*

$$\mathbb{C}|\,1/2\,\rangle \cong L_{2,0}[0] \otimes V_{1/2,0}[0],$$

$$\bigoplus_{a=\chi,\psi_+,\bar{\chi}} \mathbb{C}a[-1/2]\psi_-[-1/2]|\,1/2\,\rangle \cong L_{2,2}[0] \otimes V_{1/2,1/2}[0],$$

$$\mathbb{C}\psi_-[-1/2]|\,1/2\,\rangle \cong L_{2,0}[0] \otimes V_{1/2,1/2}[0],$$

$$\bigoplus_{a=\chi,\psi_+,\bar{\chi}} \mathbb{C}a[-1/2]|\,1/2\,\rangle \cong L_{2,2}[0] \otimes V_{1/2,0}[0].$$

Proof of Theorem 4.4. We check type by type that every fusion component does not vanish.

We first observe that the type $\mathcal{F}_2 \otimes \mathcal{F}_1 \to \hat{\mathcal{F}}_3$ and the type $\mathcal{F}_1 \otimes \mathcal{F}_2 \to \hat{\mathcal{F}}_3$ are equivalent, since the point $0 \in \mathbb{P}^1$ and the point $z \in \mathbb{P}^1 - \{0, \infty\}$ are interchangeable by an automorphism of $\mathbb{P}^1$ with the point ∞ fixed (cf. Remark 3.3.2). Moreover, by the use of the transformation $z \mapsto z^{-1}$ and the anti-automorphism τ, it is easy to see that the proof for the type $\mathcal{F}_2 \otimes \mathcal{F}_1 \to \hat{\mathcal{F}}_3$ reduces to the type $\mathcal{F}_2 \otimes \mathcal{F}_3 \to \hat{\mathcal{F}}_1$. Because of these symmetries, it suffices to consider the types $\mathcal{F}_2 \otimes \mathcal{F}_1 \to \hat{\mathcal{F}}_3$ such that $\mathcal{F}_2 = \mathcal{F}^{\pm}_{1/2}$.

Let us begin with the type $\mathcal{F}^+_{1/2} \otimes \mathcal{F}^+_{1/2} \to \hat{\mathcal{F}}^+_{1/2}$. Let $\Phi(z)$ be a nonzero conformal block of this type (cf. Proposition 7.1.1). We first consider the restriction of $\Phi(z)$ to $(L_{2,0} \otimes V_{1/2,0}) \otimes \mathcal{F}^+_{1/2}$. By Lemma 7.1.2 (1), we have $\Phi(|\,1/2\,\rangle, z) = \mathrm{Id}_{\mathcal{F}^+_{1/2}}$. Clearly its nonzero components are $L_{2,0} \otimes V_{1/2,0} \to L_{2,0} \otimes V_{1/2,0}$ and $L_{2,2} \otimes V_{1/2,1/2} \to L_{2,2} \otimes V_{1/2,1/2}$. These are exactly the fusion components (Lemmas 3.2.4 and 3.3.13). Next we consider the restriction to $(L_{2,2} \otimes V_{1/2,1/2}) \otimes \mathcal{F}^+_{1/2}$. By Lemmas 7.1.2 (1), 7.1.3 (cf. Lemma 7.2.1), its primary part is given by $a(z)\psi_-(z) : \mathcal{F}^+_{1/2} \to \mathcal{F}^+_{1/2}$, $a = \chi, \bar{\chi}, \psi_+$ (see Remark 7.1.4 for the notation). The nonzero components of these operators are

$$\mathcal{F}^+_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \to \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \text{ and } \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \to \mathcal{F}^+_{\triangle} \otimes \mathcal{F}^+_{\psi_-}.$$

These are exactly the fusion components (cf. Lemma 6.4). Hence the theorem is proved in this type.

The type $\Phi(z) : \mathcal{F}^+_{1/2} \otimes \mathcal{F}^-_{1/2} \to \hat{\mathcal{F}}^-_{1/2}$ can be treated in an almost the same manner as in the preceding type, and hence we omit the proof.

Next we consider the types $\Phi(z) : \mathcal{F}^-_{1/2} \otimes \mathcal{F}^{\pm}_{1/2} \to \hat{\mathcal{F}}^{\mp}_{1/2}$. The primary parts of the restrictions to $(L_{2,0} \otimes V_{1/2,1/2}) \otimes \mathcal{F}^{\pm}_{1/2}$ are given by $\psi_-(z) : \mathcal{F}^{\pm}_{\psi_-} \to \mathcal{F}^{\mp}_{\psi_-}$ (cf. Lemmas 7.1.2 (2) and 7.2.1). The nonzero components of this operator are

$$\mathcal{F}^+_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \to \mathcal{F}^+_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \text{ and } \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \to \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \text{ for } \mathcal{F}^+_{1/2} \to \mathcal{F}^-_{1/2},$$

and

$$\mathcal{F}^+_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \to \mathcal{F}^+_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \text{ and } \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \to \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \text{ for } \mathcal{F}^-_{1/2} \to \mathcal{F}^+_{1/2}.$$

These are exactly the fusion components. The primary parts of the restrictions to $(L_{2,2} \otimes V_{1/2,0}) \otimes \mathcal{F}^{\pm}_{1/2}$ are given by $a(z)$, $a = \chi, \psi_+, \bar{\chi}$ (cf. Lemmas 7.1.2 (2) and 7.2.1). The nonzero components of these operators are

$$\mathcal{F}^+_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \to \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \text{ and } \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \to \mathcal{F}^+_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \text{ for } \mathcal{F}^+_{1/2} \to \mathcal{F}^-_{1/2},$$

and

$$\mathcal{F}^+_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \to \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^-_{\psi_-} \text{ and } \mathcal{F}^-_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \to \mathcal{F}^+_{\triangle} \otimes \mathcal{F}^+_{\psi_-} \text{ for } \mathcal{F}^-_{1/2} \to \mathcal{F}^+_{1/2}.$$

These are in agreement with the fusion components.

It remains to check the types $\mathcal{F}^+_{1/2} \otimes \mathcal{F}^{\pm}_0 \to \mathcal{F}^{\pm}_0$ and $\mathcal{F}^-_{1/2} \otimes \mathcal{F}^{\pm}_0 \to \mathcal{F}^{\mp}_0$. For each of these types, all the components are fusion components. We shall now show that all the components are nonzero. To see this we use the following explicit forms of the conformal blocks given by Lemmas 7.1.2 and 7.1.3. For the types $\Phi(z) : \mathcal{F}^+_{1/2} \otimes \mathcal{F}^{\pm}_0 \to \mathcal{F}^{\pm}_0$, we have $\Phi(|\,1/2\,\rangle, z) = \mathrm{Id}_{\mathcal{F}^{\pm}_{1/2}}$ and $\Phi(a[-1/2]\psi_-[-1/2]|\,1/2\,\rangle, z) = a(z)\psi_-(z)$, $a = \chi, \psi_+, \bar{\chi}$. For the types $\Phi(z) : \mathcal{F}^-_{1/2} \otimes \mathcal{F}^{\pm}_0 \to \mathcal{F}^{\mp}_0$, we have $\Phi(\psi_-[-1/2]|\,1/2\,\rangle, z) = \psi_-(z)$ and $\Phi(a[-1/2]|\,1/2\,\rangle, z) = a(z)$ ($a = \chi, \psi_+$ and $\bar{\chi}$). These operators are clearly nonzero. Thus the proof completes. ■

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