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Integral and rational points on algebraic curves of certain types and their Jacobian varieties over number fields

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Integral and rational points on algebraic curves of certain types and their Jacobian varieties over number fields

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Introduction

The logarithmic absolute height function is a fundamental tool when we investigate the distribution of rational points on a projective variety V over the rational number field $\mathbb{Q}$.

Let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. A (logarithmic absolute) height function is a real-valued function on the set $V(\bar{\mathbb{Q}}) = \mathrm{Hom}(\mathrm{Spec}\,\bar{\mathbb{Q}}, V)$ of $\bar{\mathbb{Q}}$-valued points on the variety V. It is defined up to a bounded function for the pair V and a line bundle $\mathcal{L}$ on V. We denote one of the representatives by $h_V(\mathcal{L}, \cdot)$, or simply, by $h(\mathcal{L}, \cdot)$. The set of rational points, or $\mathbb{Q}$-valued points, on V is denoted by $V(\mathbb{Q})$. A standard height function on the N-dimensional projective space $\mathbb{P}^N$ is given on $\mathbb{P}^N(\mathbb{Q})$ by

$$h((1 : x_1 : \ldots : x_N)) = \sum_v - \log \frac{1}{\max\{1, |x_1|_v, \ldots, |x_N|_v\}}, \quad x_i \in \mathbb{Q},$$

where v runs through the set of rational prime numbers and the infinite prime, and $|\cdot|_v$ is the standard absolute value on $\mathbb{Q}$ defined by v. This is a height function attached to the hyperplane section sheaf $\mathcal{O}(1)$ on $\mathbb{P}^N$. The function $1/\max\{1, |x_1|_v, \ldots, |x_N|_v\}$ is a kind of local distance from the point $(1 : x_1 : \ldots : x_N)$ to the hyperplane $\{X_0 = 0\}$ at infinity. Schanuel [19] has shown for $\mathbb{P}^N$ and the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^N$

$$\#\{P \in \mathbb{P}^N(\mathbb{Q}) \mid h(\mathcal{O}(1), P) < H\} = \frac{2^N}{\zeta(N+1)} e^{(N+1)H} + O(1) H e^{NH},$$

$H \in \mathbb{R}$. Here $\#$ is the cardinality function of sets, $\zeta(z)$ is the Riemann zeta function, and $O(1)$ is a bounded function of a suitable variable (H in this case). Given another projective variety W over $\mathbb{Q}$ and a morphism $\phi\colon V \to W$, we have the inverse image $\phi^*\mathcal{L}$ of $\mathcal{L}$ under ϕ and

$$h_V(\phi^*\mathcal{L}, P) = h_W(\mathcal{L}, \phi P) + O(1), \quad P \in V(\bar{\mathbb{Q}}). \tag{0.1}$$

Taking an embedding $\iota\colon V \to \mathbb{P}^N$, we obtain

$$\#\{P \in V(\mathbb{Q}) \mid h_V(\iota^*\mathcal{O}(1), P) < H\} < \text{const.}\ e^{(N+1)H}, \quad H \in \mathbb{R}.$$

A height function is additive with respect to the tensor operation on the group $\operatorname{Pic} V$ of line bundles on the projective variety V. For $\mathcal{L}$ and $\mathcal{M} \in \operatorname{Pic} V$, we have

$$h(\mathcal{L} \otimes \mathcal{M}, P) = h(\mathcal{L}, P) + h(\mathcal{M}, P) + O(1), \quad P \in V(\bar{\mathbb{Q}}). \tag{0.2}$$

Hence for an abelian variety A over $\mathbb{Q}$, Theorem of the cube implies that a height function is essentially a polynomial function of degree at most two on the abelian group $A(\bar{\mathbb{Q}})$ of $\bar{\mathbb{Q}}$-valued points. A Néron-Tate height function [18] is the unique representative of the class of height functions attached to a line bundle that is in reality a function of degree at most two, namely, a quadratic form plus a linear functional on $A(\bar{\mathbb{Q}})$. The quadratic part of a Néron-Tate height function associated with an ample line bundle $\mathcal{N}$ on A gives a non-degenerate non-negative quadratic form on the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} A(\bar{\mathbb{Q}})$. The Mordell-Weil theorem says the group $A(\mathbb{Q})$ of rational points on the abelian variety A is finitely generated, which was substantially proved in [16] and [25]. Let R be the rank of $A(\mathbb{Q})$. The quadratic form attached to the ample line bundle $\mathcal{N}$ on A determines an R-dimensional Euclidean space structure on $\mathbb{R} \otimes_{\mathbb{Z}} A(\mathbb{Q})$ and we see

$$\#\{P \in A(\mathbb{Q}) \mid h_A(\mathcal{N}, P) < H\} = O(1) \cdot H^{R/2}, \quad H \in \mathbb{R}$$

because $A(\mathbb{Q})$ modulo torsion is a lattice in $\mathbb{R} \otimes_{\mathbb{Z}} A(\mathbb{Q})$.

If we start from polynomial equations with coefficients in the rational integer ring $\mathbb{Z}$, which define a quasi-projective variety over $\mathbb{Q}$, we naturally think about the integral solutions of the equations, which we call the integral points of the variety. Progresses in the Baker theory have made it possible to bound the heights of integral points of a curve over $\mathbb{Q}$ of positive genus with several basic quantities of the curve, which the author does not explain here. For more information, see [8], [1, p. 40], [13], [9], etc. Silverman [21] estimated the number of integral points on the so-called Thue curve by mapping them into the Jacobian variety of the curve. The image of rational points on the curve under a morphism over the base field into the Jacobian variety is a set of rational points on the Jacobian variety, therefore it can be considered up to the torsion subgroup as a set of lattice points of a Euclidean space as in the previous paragraph. He gave a lower bound for the norms of lattice points and an explicit (but not effective) upper bound for those of integral points and counted *all* the lattice points with norms smaller than the given upper bound.

THEOREM 0.1 (SILVERMAN). *Let a be a nonzero rational integer and $T(X,Y)$ a homogeneous binary polynomial of degree $n \geq 3$ with coefficients in $\mathbb{Z}$ whose discriminant is not zero. There exists a constant $a_0 = a_0(T)$ such that if a is n-th power-free and $|a| > a_0$, then*

$$\#\{(x,y) \in \mathbb{Z}^2 \mid T(x,y) = a\} < \exp(2n^2 \log n) \cdot (8n^3)^{r(T,a)},$$

where $r(T,a)$ is the rank of the group of rational points on the Jacobian variety of the plane curve over $\mathbb{Q}$ defined by $T(X,Y) = a \cdot Z^n$.

Mumford [17] deduced the following theorem from the property that a height function associated with an effective divisor is bounded below outside the support of the divisor. This point of view is fundamental in Vojta's proof [24] of the Mordell conjecture (see also [3]). The Jacobian variety J of a curve C over $\mathbb{Q}$ of genus $g > 1$ is canonically provided with a Néron-Tate height by the Poincaré sheaf and the principal polarization attached to a theta divisor. The Néron-Tate height gives a non-degenerate non-negative symmetric bilinear form on $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}}) \times \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}})$. We denote it by $\langle\cdot,\cdot\rangle$ and the associated norm by $\|\cdot\|$. Let $\Omega_{C/\mathbb{Q}}$ be the invertible sheaf of holomorphic differentials of C over $\mathbb{Q}$. We have a canonical morphism f of C into J over $\mathbb{Q}$ defined by

$$C(\bar{\mathbb{Q}}) \ni P \mapsto \Omega_{C/\mathbb{Q}} \otimes \mathcal{O}_C(-(2g-2)P) \in \mathrm{Pic}^\circ(C \times_{\mathbb{Q}} \bar{\mathbb{Q}}) \simeq J(\bar{\mathbb{Q}}),$$

where $\mathcal{O}_C(-(2g-2)P)$ is the line bundle on C one of whose rational sections has $-(2g-2)P$ as the corresponding divisor, and Pic° is the functor which associates the group of isomorphism classes of line bundles algebraically equivalent to zero. The function $\|f(\cdot)\|^2$ on $C(\bar{\mathbb{Q}})$ is a height function attached to the invertible sheaf $\Omega_{C/k}^{\otimes 2(2g-2)g}$. We call it the canonical height function on the curve C.

Theorem 0.2 (Mumford). *As a function of (P,Q) on $C(\bar{\mathbb{Q}})\times C(\bar{\mathbb{Q}}) \simeq (C\times C)(\bar{\mathbb{Q}})$,*

$$\frac{\|fP\|^2}{2g} + \frac{\|fQ\|^2}{2g} - \langle fP, fQ\rangle$$

is bounded below outside of the support of the diagonal divisor Δ on $C\times C$

This is because the function is a height function attached to the divisor $(2g-2)^2\Delta$.

Corollary 0.3. *For distinct P and $Q \in C(\bar{\mathbb{Q}})$ with sufficiently large norms, if $\|fP\| \le \|fQ\|$ and $\langle fP, fQ\rangle/\|fP\|\|fQ\| > 3/4$, then*

$$\frac{3g}{5}\|fP\| < \|fQ\|.$$

This implies the discreteness of $f(C(\bar{\mathbb{Q}}))$ outside a big ball in the infinite dimensional normed real vector space $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}})$. Bogomolov [2] conjectured that the *whole* image of $C(\bar{\mathbb{Q}})$ under f should be discrete in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}})$. Zhang [26] gave the affirmative answer in several cases.

Suppose we want to estimate the number of rational points P on the curve C whose canonical heights $\|f(P)\|^2$ are less than a given large upper bound in the Euclidean space $(\mathbb{R} \otimes_{\mathbb{Z}} J(\mathbb{Q}), \langle \cdot, \cdot \rangle)$. Theoretically, thanks to Corollary 0.3, we do not have to take into consideration *all* the lattice points coming from the rational points of the Jacobian variety J. The author [5] refined Theorem 0.1 in the case where the genus is greater than one. As before, let a be a nonzero rational integer and $T(X, Y)$ a homogeneous binary polynomial of degree $n > 3$ with coefficients in $\mathbb{Z}$ whose discriminant is not zero. The set of integral solutions

$$I := \{(x, y) \in \mathbb{Z}^2 \mid T(x, y) = a\}$$

is naturally regarded as a subset of the set of rational points on the plane curve over $\mathbb{Q}$

$$C_a \colon T(X, Y) = a \cdot Z^n.$$

We denote by J_a the Jacobian variety of C_a.

THEOREM 0.4. *There exists a constant* $a_1 = a_1(T)$ *such that if* a *is*

n-th power-free and $|a| > a_1$, then the image $f(I)$ under the canonical morphism $f\colon C_a \to J_a$ contains no torsion points of J_a and, when we take any cone K of the Euclidean space $(\mathbb{R} \otimes_{\mathbb{Z}} J_a(\mathbb{Q}), \langle \cdot, \cdot \rangle)$ that satisfies $\langle v, w \rangle / \|v\| \|w\| > 3/4$ for v and $w \in K$, the number of elements of I which are mapped into K under $f\colon I \subset C_a(\mathbb{Q}) \to \mathbb{R} \otimes_{\mathbb{Z}} J_a(\mathbb{Q})$ is at most four.

COROLLARY 0.5. *If a is n-th power-free and $|a| > a_1$, then*

$$\#I = \#\{(x, y) \in \mathbb{Z}^2 \mid T(x, y) = a\} \leq 4 \cdot 7^{r(T,a)},$$

where $r(T, a)$ is the rank of the group $J_a(\mathbb{Q})$ of rational points on the Jacobian variety J_a of the plane curve C_a.

Manin [12, Propositions 15 and 19] was aware that if the rank of the Néron-Severi group of the Jacobian variety J is larger than one, then the image of $C(\bar{\mathbb{Q}})$ in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}})$ is contained in a region near some quadric *hypersurface* defined by a height function on J. When a projective variety V is nonsingular, boundedness of a height function $h_V(\mathcal{L}, \cdot)$ attached to $\mathcal{L} \in \operatorname{Pic} V$ is equivalent to the condition that the invertible sheaf $\mathcal{L}$ is a torsion sheaf (see [20, Section 3.11]). By the additive nature (0.2), when V is regular, a $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Pic} V$ can be considered as a $\mathbb{Q}$-subspace of the space of real valued functions on $V(\bar{\mathbb{Q}})$ modulo bounded functions under the functor $h_V\colon \mathcal{L} \mapsto h_V(\mathcal{L}, \cdot)$. Property (0.1) assures the functoriality of h_V. Applying this to the canonical morphism $f\colon C \to J$, we see the Néron-Tate height functions associated with nonzero elements of the kernel of the $\mathbb{Q}$-linear map $f^*\colon \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Pic} J \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Pic} C$ are polynomial functions of degree two on $J(\bar{\mathbb{Q}})$, and are bounded on $C(\bar{\mathbb{Q}})$.

THEOREM 0.6 (MANIN). *Let r be the rank of the Néron-Severi group* $\mathrm{NS}(J)$ *of the Jacobian variety J of the curve C. There exist $r-1$ quadric hypersurfaces in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}})$ defined as the zero loci of some Néron-Tate height functions, with $\mathbb{Q}$-linearly independent defining equations such that the image of $\bar{\mathbb{Q}}$-valued points of C under the canonical map $f\colon C \to J$ is in the intersection of the neighborhoods of the hypersurfaces.*

Over the algebraic closure $\bar{\mathbb{Q}}$, the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{NS}(A \times_{\mathbb{Q}} \bar{\mathbb{Q}})$ for an abelian variety A over $\mathbb{Q}$ is identified with the subalgebra of elements in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(A \times_{\mathbb{Q}} \bar{\mathbb{Q}})$ fixed by an involution (called Rosati involution). When the curve C has a nontrivial automorphism over the base field, it induces an automorphism of the Jacobian variety J and this may yield a nontrivial element of the endomorphism algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(J)$, hence $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{NS}(J)$. Of an automorphism of C the author [6] constructed a line bundle on J whose inverse image under the canonical morphism f becomes the structure sheaf on C. The associated Néron-Tate height is fairly explicitly described.

THEOREM 0.7. *Given a nontrivial automorphism* ψ *of the curve* C*, set*

$$D := (1_C, \psi)^* \Delta \in \operatorname{Div} C \quad \textit{and} \quad d := \deg D,$$

where $(1_C, \psi)\colon C \to C \times C$ *is the morphism whose composition with the first projection is the identity map of* C *and the composition with the second is* ψ*, and* Δ *is the diagonal divisor on* $C \times C$*. We denote by* Ψ *the isometric linear transformation of* $(\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}}), \langle \cdot, \cdot \rangle)$ *induced by* ψ*. Then the canonical image of* $C(\bar{\mathbb{Q}})$ *in* $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}})$ *is in a neighborhood of the quadric* hypersurface *given by*

$$\left\langle v, \left(\Psi + \frac{d-2}{2g}\right) v + \mathcal{O}_C((2g-2)D) \otimes \Omega_{C/\mathbb{Q}}^{\otimes(-d)} \right\rangle = 0, \quad v \in \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{\mathbb{Q}}),$$

where g *is the genus of* C *and* $\mathcal{O}_C((2g-2)D) \otimes \Omega_{C/\mathbb{Q}}^{\otimes(-d)}$ *is considered as an element of* $J(\mathbb{Q}) \simeq \operatorname{Pic}^\circ C$*. The phrase "in a neighborhood" means that the function of* v *on the left side of the equation becomes bounded on the image of* $C(\bar{\mathbb{Q}})$*.*

For $P \in C(\bar{\mathbb{Q}})$, we see by the theorem

$$\frac{\langle f(P), f(\psi(P)) \rangle}{\|f(P)\| \|f(\psi(P))\|} \longrightarrow -\frac{d-2}{2g} \quad \text{as} \quad \|f(P)\| = \|f(\psi(P))\| \longrightarrow \infty.$$

This leads to a new proof of a fact which is usually an application of the Riemann-Hurwitz formula.

COROLLARY 0.8. *The number of fixed points of a nontrivial automorphism of a curve over a number field of genus* $g > 1$ *is at most* $2g + 2$*.*

If we estimate the number of integral or rational points of a curve in its Jacobian variety, Theorem 0.7 may be useful. When deducing Corollary 0.5 from Theorem 0.4, we covered the entire space $\mathbb{R} \otimes_{\mathbb{Z}} J_a(\mathbb{Q})$ by cones. If the Thue curve C_a has a nontrivial automorphism, we might be able to manage with fewer number of cones. As an extreme case, we obtain another proof of the theorem of Dem'yanenko.

THEOREM 0.9 (DEM'YANENKO [4, Example 1]). *Let Q be the plane quartic curve over $\mathbb{Q}$ defined by*

$$Q\colon X^4 + Y^4 = aZ^4$$

and E an elliptic curve given by a Weierstrass equation

$$E\colon y^2 = x^3 - ax.$$

If the rank of $E(\mathbb{Q})$ is at most one, then the canonical heights of rational points on the curve Q are bounded by an absolute constant. Especially, the number of rational points is finite.

Making use of the lower bound for the canonical heights on a Thue curve, we derive the following:

COROLLARY 0.10 (SILVERMAN [23, Corollary 1 to Theorem 1]). *Except a finite number of $a \mod (\mathbb{Q}^\times)^4$, where $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$, the condition*

$\operatorname{rank} E(\mathbb{Q}) \leq 1$ *implies*

$$\{(x, y) \in \mathbb{Q}^2 \mid x^4 + y^4 = a\} = \emptyset.$$

NOTATION AND TERMINOLOGY 0.11. For the objects X, Y, Z and the morphisms $f\colon X \to Y$, $g\colon Y \to Z$ in a category, the composition map is denoted by $g \circ f$, or simply by gf. Given another morphism $h\colon Y \to W$, the morphism determined by the pair g and h of Y into $Z \times W$ is denoted as (g, h). If $Y = Y_1 \times Y_2$ and there exist morphisms $g'\colon Y_1 \to Z$ and $h'\colon Y_2 \to W$ such that $g = g' \circ p$ and $h = h' \circ q$, where p and q are the respective projections of $Y_1 \times Y_2$ onto the first and the second factors, then $(g, h)\colon Y_1 \times Y_2 \to Z \times W$ is abbreviated to $g' \times h'$.

We denote respectively by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ the ring of rational integers, the field of rational numbers, and the field of real numbers. A finite extension field of $\mathbb{Q}$ is called a *number field*.

Let k be a number field and $\bar{k}$ an algebraic closure of k. For a scheme V over k, the scheme $V \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ over $\bar{k}$ is denoted by $\bar{V}$. We denote respectively by $V(k)$, $V(\bar{k})$, and $\bar{V}(\bar{k})$ the set $\operatorname{Hom}_k(\operatorname{Spec} k, V)$ of rational points on V, the set $\operatorname{Hom}_k(\operatorname{Spec} \bar{k}, V)$ of $\bar{k}$-valued points on V, and the set $\operatorname{Hom}_{\bar{k}}(\operatorname{Spec} \bar{k}, \bar{V})$ of $\bar{k}$-valued points on $\bar{V}$. We do not distinguish the elements of $V(k)$ or $V(\bar{k}) \simeq \bar{V}(\bar{k})$ from the corresponding closed points of V or $\bar{V}$, respectively. The N-dimensional affine space over $\mathbb{Z}$ is denoted by $\mathbb{A}^N$. When we speak of $\mathbb{A}^N$, coordinate functions are specified and are written by capital letters, for example by $Y_1, Y_2, \ldots, Y_N$. A k or $\bar{k}$-valued point of $\mathbb{A}^N$ is designated with the values of coordinate functions at the point such as $(y_1, y_2, \ldots, y_N) \in \mathbb{A}^N(k)$ or $\mathbb{A}^N(\bar{k})$. The same is valid for the N-dimensional projective space $\mathbb{P}^N$ over $\mathbb{Z}$ with its homogeneous coordinate functions specified. The schemes $\mathbb{A}^N \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} k$ and $\mathbb{P}^N \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} k$ are respectively denoted as $\mathbb{A}_k^N$ and $\mathbb{P}_k^N$.

When V is a nonsingular variety over k or $\bar{k}$, the Weil divisor group on V is denoted by $\operatorname{Div} V$. An element F of $\operatorname{Div} V$ determines an invertible sheaf on V one of whose rational sections has F as its divisor. We denote it by $\mathcal{O}_V(F)$. The structure sheaf on V is denoted as $\mathcal{O}_V$. The set of sections of a sheaf $\mathcal{F}$ on V over an open subset U is denoted by $\Gamma(U, \mathcal{F})$. The set of isomorphism classes, which is designated as $\operatorname{Pic} V$, of invertible sheaves on V is a group under the tensor operation $\otimes$. The subgroup whose elements are algebraically equivalent to zero is denoted by $\operatorname{Pic}^\circ V$. For a morphism $f\colon V \to W$ of varieties and a sheaf $\mathcal{G}$ on W of $\mathcal{O}_W$-modules, we denote the inverse image by $f^*\mathcal{G}$. For a nonsingular complete curve C over k, a divisor F on C is a formal sum of closed points of C:

$$F = \sum_{x \in C\colon \text{ a closed point}} m_x \cdot x, \quad m_x \in \mathbb{Z}.$$

The degree of F is given by

$$\deg F = \sum_x m_x [k(x) : k],$$

where $k(x)$ is the residue field of the local ring at x and $[k(x) : k]$ is the extension degree of $k(x)$ over k.

The symbol $O(1)$ expresses a bounded function of appropriate variables in a given context.

Contents

1. Standard height on a projective space

We specify what we call the standard height function on a projective space over a number field and see a property. Lemma 1.2 is of technical nature. The reader who is familiar with usual height functions can skip this section until it is needed.

Let k be a number field. For a finite extension field K of k, we define $M(K)$ as the set of normalized absolute values on K so that for an Archimedean $v \in M(K)$,

$$|\cdot|_v = \text{the usual absolute value if restricted to } \mathbb{Q};$$

for a non-Archimedean $v \in M(K)$,

$$|p|_v = 1/p \quad \text{for some prime } p \in \mathbb{Q}.$$

Denoting by K_v the completion of K at $v \in M(K)$, set

$$\epsilon(v) := \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]},$$

where $[K_v : \mathbb{Q}_v]$ and $[K : \mathbb{Q}]$ are the respective extension degrees.

For a closed point x of the projective space $\mathbb{P}^N_k = \mathbb{P}^N \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, k$ over k, we denote by $k(x)$ the residue field of the local ring at x. This is the field of definition for x over k. The value of a rational function R on $\mathbb{P}^N_k$ at x is denoted by $R(x)$, which is an element of $k(x)$. If the closed point

x is not on the hyperplane $\{X_0 = 0\}$, let

$$x_i := \frac{X_i}{X_0}(x) \in k(x).$$

We call the *standard height function* on $\mathbb{P}^N_k$ the following function h of the closed points x on $\mathbb{P}^N_k$:

$$h(x) := \sum_{v \in M(k(x))} -\epsilon(v) \log \frac{1}{\max\{|x_0|_v, \dots, |x_N|_v\}}$$

The function $1/\max\{|x_0|_v, \dots, |x_N|_v\}$ can be regarded as a local distance from the closed point x to the hyperplane $\{X_0 = 0\}$ at infinity. By virtue of the product formula

$$\prod_{v \in M(K)} |y|_v^{\epsilon(v)} = 1 \quad \text{for } y \in K \setminus \{0\},$$

where K is a finite extension field of k, the hyperplane $\{X_0 = 0\}$ may be changed to any hyperplane, for example, the hyperplane $\{X_j = 0\}$, and we can well-define $h(x)$ for all closed points x of $\mathbb{P}^N_k$.

LEMMA 1.1 (SILVERMAN [22, Theorem 2]). *For a closed point x on $\mathbb{P}^N_k$, the field $k(x)$ of definition for x over k (the residue field at x) is a finite extension field of k, so we can consider the discriminant ideal $D_k^{k(x)}$. If $\delta = [k(x) : k] > 1$, we have*

$$h(x) \geq \frac{1}{2} \cdot \frac{1}{\delta - 1} \left(\frac{1}{\delta d} \log |N_{\mathbb{Q}}^k D_k^{k(x)}| - \log \delta \right),$$

where $d = [k : \mathbb{Q}]$ and $N_{\mathbb{Q}}^k$ is the norm function of fractional ideals in k.

There is an obvious identification of the field k with the set $\mathbb{A}^1(k)$ of k-valued points on the one-dimensional affine space $\mathbb{A}^1$. The affine space $\mathbb{A}^1$ is naturally isomorphic to the open subscheme $\{X_0 \neq 0\}$ of the one-dimensional projective space $\mathbb{P}^1$. The height function h_k on k is defined by

$$h_k(z) := h((1 : z) \in \mathbb{P}^1(k))$$

for $z \in k$.

Lemma 1.2 (Silverman [21, Proposition 2 (b)]). *There exists a constant c_k depending only on k and satisfying the following property: Let S be a finite set of normalized absolute values on k containing the set $M^\infty(k)$ of normalized Archimedean absolute values on k, and n a positive rational integer. For any non-zero S-integer $a \in \mathfrak{o}_S \setminus \{0\}$, there exists an S-unit $u \in \mathfrak{o}_S^\times$ such that*

$$\begin{aligned} h_k(au^n) &< \left| \sum_{v \in S} \epsilon(v) \log |a|_v \right| + c_k \cdot n \\ &= \left| \frac{1}{d} \log |N^k_{\mathbb{Q}} a| + \sum_{v \in S \setminus M^\infty(k)} \epsilon(v) \log |a|_v \right| + c_k \cdot n, \end{aligned}$$

where $d = [k : \mathbb{Q}]$ and $N^k_{\mathbb{Q}}$ is the norm function on k.

Let $\bar{k}$ be an algebraic closure of k. For a $\bar{k}$-valued point $P \in \mathbb{P}^N_k(\bar{k}) =$ $\mathrm{Hom}(\mathrm{Spec}\, \bar{k},$

$\mathbb{P}^N$), we define the *standard height of* P (denoted also by h) as the standard height of the closed point on $\mathbb{P}_k^N$ determined by P.

2. Canonical height on a curve and the Mumford inequality

In this section, we find a canonically defined height function on a curve over a number field of genus at least two. These height functions are invariant under isomorphisms of curves over the algebraic closure of the ground field. When we use this height function, the Mumford type inequality has a simple form.

Let k be a number field, C a nonsingular complete curve over k of genus $g > 1$, J the Jacobian variety of C over k, and $\bar{k}$ an algebraic closure of k. (For the definition and properties of a Jacobian variety, see, for example, [15].)

Fix a point $P_0 \in C(\bar{k})$. A divisor Θ on $\bar{J} = J \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is defined by

$$\begin{aligned} \Theta(\bar{k}) :&= \{\mathcal{O}_{\bar{C}}(Q_1 + \cdots + Q_{g-1} - (g-1)P_0) \mid Q_j \in C(\bar{k})\} \\ &\subset \bar{J}(\bar{k}) \simeq \operatorname{Pic}^\circ(\bar{C}), \end{aligned} \tag{2.1}$$

where $\bar{C} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$. We denote by s, p and $q\colon \bar{J} \times \bar{J} \to \bar{J}$ the sum, the projections onto the first and the second factors, respectively. Define an invertible sheaf $\mathcal{N}_0$ on $\bar{J} \times \bar{J}$ by

$$\mathcal{N}_0 := s^*\mathcal{O}_{\bar{J}}(\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-\Theta). \tag{2.2}$$

Lemma 2.1. *The isomorphism class of the invertible sheaf $\mathcal{N}_0$ on an* *abelian variety $\bar{J} \times \bar{J}$ over $\bar{k}$ does not depend on the choice of P_0.*

Proof. See Lemma 4.9 (i) below. $\square$

The Néron-Tate height function $\langle \cdot, \cdot \rangle$ attached to $\mathcal{N}_0$ is a symmetric bilinear form on $(J \times J)(\bar{k}) \simeq J(\bar{k}) \times J(\bar{k})$ which is non-degenerate and non-negative on $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}) \times \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$. (For general facts about Néron-Tate height functions, see [10] or [20].) The induced inner product on $\mathbb{R} \otimes J(\bar{k})$ is also denoted by $\langle \cdot, \cdot \rangle$. The norm on $\mathbb{R} \otimes J(\bar{k})$ associated with $\langle \cdot, \cdot \rangle$ is denoted by $\| \cdot \|$. Since Néron-Tate height functions are uniquely determined by invertible sheaves, Lemma 2.1 shows that for a and $b \in J(\bar{k})$, the quantities $\langle a, b \rangle$ and $\|a\|$ are independent of the the base point of Θ and are canonically defined. We say the functions $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are *associated with a theta divisor* or are *attached to a theta divisor*.

Let $\Omega_{C/k}$ be the invertible sheaf of regular differentials of C over k. We have a canonical morphism $f\colon C \to J$ given by

$$(2.3) \qquad C(\bar{k}) \ni P \mapsto \Omega_{C/k} \otimes \mathcal{O}_{\bar{C}}(-(2g-2)P) \in \mathrm{Pic}^{\circ}(\bar{C}) \simeq J(\bar{k}).$$

For another curve C' over k, define J', Θ', and $\mathcal{N}_0'$, similarly. An isomorphism ϕ of $\bar{C}$ onto $\bar{C}'$ over $\bar{k}$ induces an isomorphism Φ of $\bar{J}$ onto $\bar{J}'$ over

$\bar{k}$:

$$J(\bar{k}) \simeq \mathrm{Pic}^\circ(\bar{C}) \ni \mathcal{L} \mapsto \Phi(\mathcal{L}) := \phi_* \mathcal{L} \in \mathrm{Pic}^\circ(\bar{C}') \simeq J'(\bar{k})$$

and the next diagram is commutative:

$$\begin{array}{ccc} \bar{C} & \xrightarrow{f} & \bar{J} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \bar{C}' & \xrightarrow[f]{} & \bar{J}' \end{array}$$

Here, by abuse of notation, $f\colon \bar{C} \to \bar{J}$ denotes the canonical morphism over $\bar{k}$ obtained by the base change of the canonical morphism $f\colon C \to J$ over k. The norm $\|\cdot\|$ on $\mathbb{R} \otimes J(\bar{k})$ (resp. $\mathbb{R} \otimes J'(\bar{k})$) was defined with the help of the Néron-Tate height function associated with the invertible sheaf $\mathcal{N}_0$ (resp. $\mathcal{N}_0'$) on $\bar{J} \times \bar{J}$ (resp. $\bar{J}' \times \bar{J}'$). Since the divisor Θ on $\bar{J}$ is mapped under Φ to a translate of Θ', the sheaf $(\Phi \times \Phi)_* \mathcal{N}_0$ on $\bar{J}' \times \bar{J}'$ is $\mathcal{N}_0'$ with Θ' replaced by the translate of Θ'. Lemma 2.1 says this is isomorphic to the previous $\mathcal{N}_0'$, therefore Φ is norm-preserving by the functoriality of Néron-Tate height functions (cf. Lemma 4.7):

$$\langle \Phi(\mathcal{L}), \Phi(\mathcal{M}) \rangle = \langle \mathcal{L}, \mathcal{M} \rangle;$$

in particular, for elements P and Q of $C(\bar{k})$

(2.4)

$$\langle f(\phi(P)), f(\phi(Q)) \rangle = \langle f(P), f(Q) \rangle \quad \text{and} \quad \|f(\phi(P))\| = \|f(P)\|.$$

PROPOSITION 2.2. *There exists a canonically defined scalar product $\langle \cdot, \cdot \rangle$ on the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ which is preserved from the isomorphisms of Jacobian varieties over $\bar{k}$ induced by isomorphisms of curves over $\bar{k}$. Letting $\| \cdot \|$ be the associated norm, we have*

$$\begin{aligned}\langle f(P), f(Q) \rangle = h_{\bar{C}}(\Omega_{\bar{C}/\bar{k}}^{\otimes(2g-2)}, P) + h_{\bar{C}}(\Omega_{\bar{C}/\bar{k}}^{\otimes(2g-2)}, Q) \\ - h_{\bar{C}\times\bar{C}}\left(\mathcal{O}_{\bar{C}\times\bar{C}}((2g-2)^2\Delta), (P,Q)\right) + O(1) \\ \text{for } (P,Q) \in (C \times C)(\bar{k})\end{aligned}$$

and

$$\|f(P)\|^2 = h_{\bar{C}}(\Omega_{\bar{C}/\bar{k}}^{\otimes 2(2g-2)g}, P) + O(1) \quad \text{for } P \in C(\bar{k}),$$

where $f\colon C \to J$ is the canonical map defined by (2.3), *$h_V(\mathcal{L}, \cdot)$ is a logarithmic absolute height function on a projective variety V over $\bar{k}$ attached to an invertible sheaf $\mathcal{L}$ on V, $\Omega_{\bar{C}/\bar{k}}$ is the sheaf of differentials of $\bar{C}$ over $\bar{k}$, and Δ is the diagonal divisor on $C \times C$.*

PROOF. See Lemma 4.11. □

THEOREM 2.3 (MUMFORD). *The function of (P,Q) on $(C \times C)(\bar{k})$*

$$\frac{1}{2g}\|f(P)\|^2 + \frac{1}{2g}\|f(Q)\|^2 - \langle f(P), f(Q) \rangle$$

is bounded below outside of the diagonal subset $\Delta(\bar{k})$.

Proof. By Proposition 2.2 and the additive property of heights (cf. Lemma 4.6), the expression is a height function attached to the divisor $(2g-2)^2\Delta$ on $C \times C$. The claim is true, for a height function associated with an effective divisor is positive up to a constant outside the support of the divisor (cf. for example, [20, Section 2.10]). $\square$

We call the height function $\|f(\cdot)\|^2$ attached to the invertible sheaf $\Omega_{C/k}^{\otimes 2(2g-2)g}$ on the curve C the *canonical height function* on C.

3. Integral points on a Thue curve

We prove in this section a theorem concerning the distribution of integral points of what we call a Thue curve under a canonical morphism of the curve into its Jacobian variety.

Let k be a number field, $T(X,Y) \in k[X,Y]$ a homogeneous binary polynomial of degree $n > 3$ with non-zero discriminant, and $a \in k \setminus \{0\}$. Let C_a be the nonsingular complete curve on the projective plane $\mathbb{P}^2_k$ over k defined by

$$C_a \colon T(X,Y) = aZ^n \tag{3.1}$$

and J_a the Jacobian variety of C_a over k. We call the curve C_a a *Thue curve*.

Let $\mathfrak{o}_k$ be the ring of integers in k and $a\mathfrak{o}_k = \prod_{\mathfrak{p}} \mathfrak{p}^{s(\mathfrak{p})}$ the prime ideal decomposition of the fractional ideal $a\mathfrak{o}_k$ in k. If $s(\mathfrak{p}) = r(\mathfrak{p}) + q(\mathfrak{p}) \cdot n$ for $r(\mathfrak{p})$ and $q(\mathfrak{p}) \in \mathbb{Z}$ with $0 \leq r(\mathfrak{p}) < n$, then we can so arrange that $a\mathfrak{o}_k = \prod_{\mathfrak{p}} \mathfrak{p}^{r(\mathfrak{p})} (\prod_{\mathfrak{p}} \mathfrak{p}^{q(\mathfrak{p})})^n$. Set $\mathfrak{a} := \prod_{\mathfrak{p}} \mathfrak{p}^{r(\mathfrak{p})}$ and $\mathfrak{b} := \prod_{\mathfrak{p}} \mathfrak{p}^{q(\mathfrak{p})}$. Then

$$a\mathfrak{o}_k = \mathfrak{a}\mathfrak{b}^n, \quad \mathfrak{a} \subset \mathfrak{o}_k,$$

and $\mathfrak{a}$ is n-th power-free. Such a decomposition is easily seen to be unique. Let S be a finite set of normalized absolute values on k which contains the set $M^\infty(k)$ of Archimedean absolute values on k. We define a real-valued function e_S of a by

$$(3.2) \qquad e_S(a) := \frac{(\log|N^k_{\mathbb{Q}}\mathfrak{b}^n|)/d + \sum_{v\in S\setminus M^\infty(k)} \epsilon(v)\log|a|_v}{(\log|N^k_{\mathbb{Q}}\mathfrak{a}|)/d}.$$

The value of e_S is a sort of number which measures the defect of the n-th power freeness. If a is an unramified rational prime number and cannot be divided by the prime ideals corresponding to the absolute values in $S \setminus M^\infty(k)$, then $e_S(a) = 0$.

THEOREM 3.1. *Suppose the coefficients of the polynomial T are in the ring $\mathfrak{o}_S$ of S-integers in k. Let K be a closed cone of the Euclidean space $\mathbb{R}\otimes_{\mathbb{Z}} J_a(k)$ such that $\langle v,w\rangle/\|v\|\,\|w\| \geq 21/(20(n-2)^{1/2})$ for v and $w \in K\setminus\{0\}$. If $|1+e_S(a)| < 2$ and $|N^k_{\mathbb{Q}}\mathfrak{a}|$ is sufficiently large compared with the degree n of the polynomial T and some other quantities not depending on K, then*

$$\#(\{(x,y)\in\mathfrak{o}_S \mid T(x,y)=a\}\cap f^{-1}K) \leq \begin{cases} 1 & \textit{when } n\geq 194 \\ 2 & \textit{when } n\geq 7 \\ 3 & \textit{when } n\geq 5 \\ 4 & \textit{when } n\geq 4, \end{cases}$$

where the set $\{(x,y)\in\mathfrak{o}_S \mid T(x,y)=a\}$ of S-integral solutions is regarded as a subset of $C_a(k)$ in the natural way and $f\colon C_a \to J_a$ is the canonical *map given by*

$$C_a(\bar{k}) \ni P \mapsto \Omega_{C_a/k} \otimes \mathcal{O}_{\bar{C}_a}(-(n-3)nP) \in \operatorname{Pic}^\circ \bar{C}_a \simeq J_a(\bar{k}).$$

COROLLARY 3.2. *If* $|1+e_S(a)| < 2$ *and* $|N_{\mathbb{Q}}^k \mathfrak{a}|$ *is sufficiently large, then*

$$\#\{(x,y) \in \mathfrak{o}_k \mid T(x,y) = a\} \leq 4 \cdot 7^{r(T,a)},$$

where $r(T,a)$ *is the rank of the group* $J_a(k)$ *of rational points of the Jacobian variety* J_a *of the Thue curve* C_a.

PROOF. Notice that $21/(20(n-2)^{1/2}) < 3/4$, because $n > 3$. We know that $\mathbb{R}^N$ can be covered by 7^N closed cones K such that $\langle v, w\rangle/\|v\|\,\|w\| \geq 3/4$ for v and $w \in K \setminus \{0\}$ (cf. [3, § 10]), hence, by the theorem, we obtain the result. $\square$

Here follow many lemmas to show the theorem.

Choose an element α of an algebraic closure $\bar{k}$ of k such that $\alpha^n = a$ and define an isomorphism ϕ of $\bar{C}_a = C_a \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ onto $\bar{C}_1$ over $\bar{k}$ by

$$C_a(\bar{k}) \ni (x : y : z) \mapsto (x : y : \alpha z) \in C_1(\bar{k}). \tag{3.3}$$

We have a canonical map $f\colon C_a \to J_a$ over k given by

$$C_a(\bar{k}) \ni P \mapsto \Omega \otimes \mathcal{O}(-(2g-2)P) \in \operatorname{Pic}^\circ(\bar{C}_a) \simeq J_a(\bar{k}),$$

where Ω is the sheaf of differentials of C_a over k and g substitutes for the genus $(n-1)(n-2)/2$ of C_a.

Lemma 3.3. *There exists a positive constant $L = L(T) = L(T(X,Y))$ such that*

$$-L < \frac{1}{2g}\left(\|fP\|^2 + \|fQ\|^2\right) - \langle fP, fQ\rangle$$

for $(P,Q) \in (C_a \times C_a)(\bar{k}) \setminus \Delta(\bar{k})$.

Proof. Owing to Theorem 2.3, we have a positive number $L = L(T)$ satisfying

$$-L < \frac{1}{2g}\left(\|fP_1\|^2 + \|fQ_1\|^2\right) - \langle fP_1, fQ_1\rangle$$

for $(P_1, Q_1) \in (C_1 \times C_1)(\bar{k}) \setminus \Delta(\bar{k})$. By the invariance (2.4) of the scalar products applied to the twisting map $\phi\colon \bar{C}_a \to \bar{C}_1$, we are done. $\square$

Lemma 3.4. *Let P and $Q \in C_a(\bar{k})$ be distinct points such that $\|fP\| \leq \|fQ\|$. If*

$$\|fP\|^2 > 20(n-2)^{1/2}L$$

for the constant $L = L(T)$ in the previous lemma and

$$\frac{\langle fP, fQ\rangle}{\|fP\|\,\|fQ\|} > \frac{21}{20(n-2)^{1/2}},$$

then we have

$$\|fQ\| > \frac{g}{(n-2)^{1/2}}\|fP\|.$$

In other words, if a $\bar{k}$-valued point $P \in C_a(\bar{k})$ with large norm appears in a cone K in the normed real vector space $\mathbb{R} \otimes_{\mathbb{Z}} J_a(\bar{k})$ such that

$\langle v, w\rangle/\|v\|\,\|w\| > 21/(20(n-2)^{1/2})$ for v and $w \in K$, then another $\bar{k}$-valued point $Q \in C_a(\bar{k})$ with the next smallest norm which appears in the cone K, if any, has the norm at least $g/(n-2)^{1/2}$ times the norm of P.

Proof. By the above lemma, we see that

$$-\frac{L}{\|fP\|\,\|fQ\|} < \frac{1}{2g}\left(\frac{\|fP\|}{\|fQ\|} + \frac{\|fQ\|}{\|fP\|}\right) - \frac{\langle fP, fQ\rangle}{\|fP\|\,\|fQ\|}$$
$$< \frac{1}{g}\cdot\frac{\|fQ\|}{\|fP\|} - \frac{21}{20(n-2)^{1/2}}.$$

From the assumptions $\|fQ\|^2 \geq \|fP\|^2 > 20(n-2)^{1/2}L$, we have

$$-\frac{1}{20(n-2)^{1/2}} < \frac{1}{g}\cdot\frac{\|fQ\|}{\|fP\|} - \frac{21}{20(n-2)^{1/2}}. \quad \square$$

We regard $C_a(\bar{k})$ as a subset of $\mathbb{P}^2(\bar{k})$ in the natural way and denote the standard height on $\mathbb{P}^2_{\bar{k}}$ by h.

Lemma 3.5. *There exist non-negative constants $m = m(T) = m(T(X, Y))$ and $M = M(T)$ such that*

$$-m < \|fP\|^2 - (n-3)^2(n-2)(n-1)n\cdot h(\phi P) < M$$

for $P \in C_a(k)$, where $n = \deg T(X, Y)$ and $\phi\colon \bar{C}_a \to \bar{C}_1$ is the twisting map (3.3).

Proof. By Proposition 2.2, we have for $P_1 \in C_1(\bar{k})$

$$\|fP_1\|^2 = 2(2g-2)g\cdot h(\Omega, P_1) + O(1).$$

Since the canonical sheaf of a nonsingular plane curve of degree n is isomorphic to the pull-back of $n-3$ times the hyperplane section sheaf (see, for example, [7, II 8.20.3]),

$$\|fP_1\|^2 = 2(n-3)(2g-2)g \cdot h(P_1) + O(1), \quad P_1 \in C_1(\bar{k}) \subset \mathbb{P}^2(\bar{k}),$$

that is, there are non-negative numbers $m = m(T)$ and $M = M(T)$ satisfying

$$-m < \|fP_1\|^2 - 2(n-3)(2g-2)g \cdot h(P_1) < M$$

for $P_1 \in C_1(\bar{k}) \subset \mathbb{P}^2(\bar{k})$. Substitute ϕP ($P \in C_a(\bar{k})$) for P_1. Then, the invariance of norms (2.4) means

$$\|fP_1\| = \|f\phi P\| = \|fP\|,$$

and the desired inequalities follow because $g = (n-2)(n-1)/2$. $\square$

Proposition 3.6. *For $P \in C_a(k)$ such that the Z-coordinate of the defining equation* (3.1) *is not zero, we have*

$$\|fP\|^2 > \frac{(n-3)^2}{2}\frac{n-2}{n-1}\frac{\log|N_{\mathbb{Q}}^k\mathfrak{a}|}{d} - \frac{\log 2}{2}\nu n - m,$$

where $d = [k : \mathbb{Q}]$, n is the degree of C_a in the projective plane $\mathbb{P}^2_k$, ν is the integer $(n-3)^2(n-2)(n-1)$, and m is the constant in Proposition 3.5.

Proof. We see from Proposition 3.5 that

$$\|fP\|^2 > (n-3)^2(n-2)(n-1)n \cdot h(\phi P) - m.$$

Since the Z-coordinate of P is not zero and $\phi\colon \bar{C}_a \to \bar{C}_1$ was defined as $(x:y:z) \mapsto (x:y:\alpha z)$, we have $k(\phi P) = k(\alpha)$, where $k(\phi P)$ is the field of definition for the image in C_1 of the $\bar{k}$-valued point $\phi P \in \bar{C}_1(\bar{k})$. Then, if $\delta = [k(\alpha):k] > 1$, we find from Lemma 1.1 and the fact $2 \le \delta \le n$ that

$$
\begin{aligned}
\|fP\|^2 &> \frac{(n-3)^2(n-2)(n-1)n}{2(\delta-1)\delta d}\log|N_{\mathbb{Q}}^k D_k^{k(\alpha)}| \\
&\qquad\qquad - \frac{\log\delta}{2(\delta-1)}(n-3)^2(n-2)(n-1)n - m \\
&> \frac{(n-3)^2(n-2)}{2d}\log|N_{\mathbb{Q}}^k D_k^{k(\alpha)}| - \frac{\log 2}{2}\nu n - m.
\end{aligned}
$$

The right hand side of the last inequality is negative if $k(\alpha) = k$, because we then have $|N_{\mathbb{Q}}^k D_k^{k(\alpha)}| = 1$. Hence this inequality is valid also when $\delta = 1$. If we look at the ramification in the extension $k(\alpha)$ over k, then we find that $\mathfrak{a}$ divides $(D_k^{k(\alpha)})^{n-1}$. Thus $|N_{\mathbb{Q}}^k D_k^{k(\alpha)}|^{n-1} \ge |N_{\mathbb{Q}}^k \mathfrak{a}|$. $\square$

Fix a number λ such that $2 < \lambda < n = \deg T(X,Y)$, where $T(X,Y)$ is the homogeneous polynomial defining C_a.

Lemma 3.7. *When the coefficients of the homogeneous polynomial $T(X,Y)$ are in the S-integer ring $\mathfrak{o}_S$, there exists a constant $c = c(k,S,T(X,Y),\lambda)$ such that*

$$h((x:y:1)) < \frac{1}{n-\lambda}h_k(T(x,y)) + c$$

for x and $y \in \mathfrak{o}_S$. Here h_k is the height function on k defined § 1.

Proof. See [21, Theorem 1]. $\square$

REMARK 3.8. The lemma means in particular that the set of S-integral points of a Thue curve is finite.

From now in this section, we assume that the coefficients of $T(X,Y)$ are in $\mathfrak{o}_S$.

Let $a\mathfrak{o}_k = \mathfrak{a}\mathfrak{b}^n$ be the ideal decomposition such that $\mathfrak{a}$ is integral and n-th power-free, $e_S(a)$ is the real-valued function (3.2), and

$$I_a := \{(x,y) \in \mathfrak{o}_S^2 \mid T(x,y) = a\}.$$

We naturally regard I_a as a subset of $C_a(k)$.

We now bound the norms of S-integral points from above.

LEMMA 3.9. *For an S-integral point P of C_a, the Z-coordinate of P is not zero and*

$$\begin{aligned} \|fP\|^2 \;&<\; \left(1 + \frac{n}{n-\lambda}\right)\nu|1 + e_S(a)|\frac{\log|N_{\mathbb{Q}}^k \mathfrak{a}|}{d} \\ &\qquad + \left(1 + \frac{n}{n-\lambda}\right)\nu n \cdot c_k + \nu n c + M, \end{aligned}$$

where ν is the integer $(n-3)^2(n-2)(n-1)$; λ is a number such that $2 < \lambda < n$; c_k, $c = c(k, S, T(X,Y), \lambda)$, and $M = M(T(X,Y))$ are the constants in Lemma 1.2 and Lemma 3.7 as well as Lemma 3.5, respectively.

PROOF. Note first that for $x, y, z, \alpha \in \bar{k}$ and a normalized absolute value v on $\bar{k}$, we have

$$\max\{|x|_v, |y|_v, |\alpha z|_v\} \leq \max\{|1|_v, |\alpha^n|_v\}^{1/n} \cdot \max\{|x|_v, |y|_v, |z|_v\}.$$

If we use this for $P = (x : y : z)$ and α which was used in defining the twisting map $\phi\colon \bar{C}_a \to \bar{C}_1$ (cf. (3.3)), and take the logarithms of both sides, then we see that

$$h(\phi P) \leq \frac{1}{n} h(a) + h(P). \tag{3.4}$$

To the second inequality

$$\|fP\|^2 < \nu n \cdot h(\phi P) + M$$

of Lemma 3.5 we apply the inequality (3.4) and Lemma 3.7, and we obtain

$$\|fP\|^2 < \left(1 + \frac{n}{n-\lambda}\right) \nu \cdot h(a) + \nu n c + M.$$

Now, for an arbitrary $u \in \mathfrak{o}_S^\times$, let $\psi\colon C_a \to C_{au^n}$ be an isomorphism over k given by $(x : y : z) \mapsto (x : y : u^{-1} z)$. Then we see that $\psi(I_a) = I_{au^n}$, hence by the invariance (2.4) of heights

$$\|fP\|^2 = \|f(\psi P)\|^2 \leq \left(1 + \frac{n}{n-\lambda}\right) \nu \inf_{u \in \mathfrak{o}_S^\times} h(au^n) + \nu n c + M.$$

By Lemma 1.2, we see that

$$\begin{aligned} \|fP\|^2 \;\; < \;\; & \left(1 + \frac{n}{n-\lambda}\right) \nu \left| \frac{\log |N_{\mathbb{Q}}^k a|}{d} + \sum_{v \in S \setminus M^\infty(k)} \epsilon(v) \log |a|_v \right| \\ & + \left(1 + \frac{n}{n-\lambda}\right) \nu n \cdot c_k + \nu n c + M. \end{aligned}$$

By the definition of e_S, we are done. $\square$

Let K be a closed cone of $\mathbb{R} \otimes_{\mathbb{Z}} J_a(k)$ such that $\langle v, w\rangle / \|v\| \, \|w\| \geq 21/(20(n-2)^{1/2})$ for v and $w \in V \setminus \{0\}$.

LEMMA 3.10. *Let* $t = \#(I_a \cap f^{-1}K) - 1$ *and assume* $t \geq 0$. *If* $|N_{\mathbb{Q}}^k \mathfrak{a}|$ *is sufficiently large, then we have*

$$\left(\frac{(n-1)^2(n-2)}{4}\right)^t < 4(1 + |1 + e_S(a)|)\left(1 + \frac{n}{n-\lambda}\right)(n-1)^2.$$

Proof. Let $I_a \cap f^{-1}K = \{P_0, P_1, \ldots, P_t\}$ and $\|fP_0\| \leq \|fP_1\| \leq \cdots \leq \|fP_t\|$. When

$$\log |N_{\mathbb{Q}}^k \mathfrak{a}| > \frac{2d}{(n-3)^2}\frac{n-1}{n-2}\left(\frac{\log 2}{2}\nu n + m + 20(n-2)^{1/2}L\right),$$

we see by Proposition 3.6 that $\|fP_i\|^2 > 20(n-2)^{1/2}L$ for any i. So, by Lemma 3.4, we have

$$\left(\frac{g^2}{n-2}\right)^t \cdot \|fP_0\|^2 \leq \left(\frac{g^2}{n-2}\right)^{t-1} \cdot \|fP_1\|^2 \leq \cdots \leq \|fP_t\|^2.$$

Applying the previous lemma to the extreme right hand side and again Proposition 3.6 to the extreme left hand side, we find

$$\begin{aligned}\left(\frac{g^2}{n-2}\right)^t &< \left[\left(1 + \frac{n}{n-\lambda}\right)\nu |1 + e_S(a)| \frac{\log |N_{\mathbb{Q}}^k \mathfrak{a}|}{d}\right. \\ &\qquad\qquad \left. + \left(1 + \frac{n}{n-\lambda}\right)\nu n \cdot c_k + \nu n c + M\right] \\ &\qquad \cdot \left[\frac{(n-3)^2}{2d}\frac{n-2}{n-1}\log |N_{\mathbb{Q}}^k \mathfrak{a}| - \frac{\log 2}{2}\nu n - m\right]^{-1}.\end{aligned}$$

Further if

$$\begin{aligned}\log|N^k_{\mathbb{Q}}\mathfrak{a}| \quad > \quad \max\Bigg\{&\left[\left(1+\frac{n}{n-\lambda}\right)\nu n\cdot c_k+\nu nc+M\right]\left[\left(1+\frac{n}{n-\lambda}\right)\frac{\nu}{d}\right]^{-1},\\ &\left[\frac{\log 2}{2}\nu n+m\right]\left[\frac{(n-3)^2}{4d}\frac{n-2}{n-1}\right]^{-1}\Bigg\},\end{aligned}$$

then, substituting $(n-1)(n-2)/2$ for g and estimating the right hand side of the above inequality, we obtain

$$\begin{aligned}\left(\frac{(n-2)(n-1)^2}{4}\right)^t \quad &< \quad \left(1+\frac{n}{n-\lambda}\right)\nu\frac{|1+e_S(a)|+1}{d}\cdot\frac{4d}{(n-3)^2}\frac{n-1}{n-2}\\ &= \quad 4(n-1)^2\left(1+\frac{n}{n-\lambda}\right)(|1+e_S(a)|+1)\,. \quad \square\end{aligned}$$

Proof of the theorem. In the last lemma, take $\lambda = 2n/3$ when $n \geq 5$; $\lambda = 5/2$, $n = 4$. $\square$

4. Rational points of the curves with automorphisms

In this section, we give the description of a neighborhood of a hypersurface in a normed real vector space containing the canonical image of rational points of a curve with the genus greater than one which has a nontrivial automorphism.

Let k be a number field, C a nonsingular complete curve over k of genus $g > 1$, J the Jacobian variety of C, and $\bar{k}$ an algebraic closure of k. We have a canonical morphism f of C into J over k (cf. (2.3)) and the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ attached to a theta divisor.

Let Δ be the diagonal divisor on $C \times C$. Suppose we have an automorphism ψ of C over k different from the identity map, and set

$$D := (1_C, \psi)^* \Delta \in \operatorname{Div} C \quad \text{and} \quad d := \deg D.$$

In a sense, D is the divisor of fixed points with multiplicities of the morphism ψ. The orthogonal transformation of $(\mathbb{R} \otimes J(\bar{k}), \langle \cdot, \cdot \rangle)$ induced by ψ is denoted as Ψ.

Theorem 4.1. *The image $f(C(\bar{k}))$ under the canonical map $f\colon C \to J$ in the normed real vector space $(\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}), \langle \cdot, \cdot \rangle)$ is contained in the neighborhood of a quadric hypersurface defined as*

$$\left| \left\langle v, \left(\Psi + \frac{d-2}{2g} \right) v + \mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)} \right\rangle \right| \leq \text{const.},$$

$v \in \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$, where $\Omega_{C/k}$ is the sheaf of differentials of C over k and $\mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)}$ is considered as an element of $J(k) \simeq \operatorname{Pic}^\circ C$.

Remark 4.2. The function of degree two on the left is what was called a *null height* by Manin [12, p. 339] (cf. Remark 4.13).

Corollary 4.3. *For $P \in C(\bar{k})$, we have $\|f(\psi(P))\| = \|f(P)\|$. The angle made by $\psi(P)$ and P under f in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ is* $\arccos((2-d)/(2g) + O(1) \cdot \|f(P)\|^{-1})$.

Proof. The former part is already confirmed true (cf. (2.4)). Here is another proof. There exists a finite extension field K of k such that P can be considered as in $C(K)$. The automorphism Ψ induced by ψ of the finite dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} J(K)$ is of finite order, hence is a Euclidean motion. Therefore we obtain $\|f(\psi(P))\| = \|\Psi(f(P))\| = \|f(P)\|$. The latter half is easy because by the theorem, we see

$$\frac{\langle f(P), f(\psi(P))\rangle}{\|f(P)\|\|f(\psi(P))\|} = \frac{\langle f(P), \Psi(f(P))\rangle}{\|f(P)\|\|\Psi(f(P))\|} \longrightarrow \frac{2-d}{2g}$$

as $\|f(P)\| \longrightarrow \infty$. $\square$

Corollary 4.4. *The number of fixed points of a nontrivial automorphism of a curve whose genus g is at least two is not larger than $2g+2$.*

Remark 4.5. Ordinarily, this follows from the Riemann-Hurwitz formula.

PROOF. Values on $\mathbb{R}$ of the cosine function are not less than -1. So by the previous corollary, the degree d of the divisor of fixed points must not be greater than $2g+2$. $\square$

To prove the theorem, we need some notation and several lemmas.

For a projective variety V over $\bar{k}$ and an invertible sheaf $\mathcal{L}$ on V, we denote by $h_V(\mathcal{L},\cdot)$ a height function on V attached to $\mathcal{L}$. We have the ambiguity of bounded functions in choosing a height function. When V is abelian, we agree to choose the canonical height *all the time* and for $u \in V(\bar{k})$, the translation-by-u-map is denoted by $t_u\colon V \to V$.

LEMMA 4.6 (ADDITIVE PROPERTY). *For a projective variety V over $\bar{k}$ and invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on V, we have*

$$h_V(\mathcal{L}\otimes\mathcal{M},x) = h_V(\mathcal{L},x) + h_V(\mathcal{M},x) + O(1), \quad x \in V(\bar{k}).$$

PROOF. See, for example, [20, Theorem of § 2.8]. $\square$

LEMMA 4.7 (FUNCTORIALITY). *For a morphism $\phi\colon V \to W$ of projective varieties over $\bar{k}$ and an invertible sheaf $\mathcal{L}$ on W, we have*

$$h_V(\phi^*\mathcal{L},x) = h_W(\mathcal{L},\phi x) + O(1), \quad x \in V(\bar{k}).$$

For a homomorphism $\chi\colon A \to B$ of abelian varieties over $\bar{k}$ and an invertible sheaf $\mathcal{M}$ on B, we have

$$h_A(\chi^*\mathcal{M},x) = h_B(\mathcal{M},\chi x), \quad x \in A(\bar{k}).$$

For $u \in B(\bar{k})$ and the translation-by-u-map $t_u \colon B(\bar{k}) \ni x \mapsto x+u \in B(\bar{k})$,

$$h_B(t_u^*\mathcal{M}, x) = h_B(\mathcal{M}, x+u) - h_B(\mathcal{M}, u).$$

PROOF. For the first part, see, for example, [20, § 2.8]. For the second, see [20, § 3.2] and for the third, [20, Lemma of § 3.4]. □

LEMMA 4.8. *For an invertible sheaf $\mathcal{L}$ on an abelian variety A and a rational integer n,*

$$n^*\mathcal{L} \simeq \mathcal{L}^{\otimes (n+1)n/2} \otimes (-1)^*\mathcal{L}^{\otimes (n-1)n/2}.$$

PROOF. See, for example, [14, Corollary 6.6]. □

Fix a point $P_0 \in C(\bar{k})$. Define a divisor Θ on $\bar{J}$ as (2.1), an invertible sheaf $\mathcal{N}_0$ on $\bar{J} \times \bar{J}$ by (2.2), and a morphism $f_0 \colon \bar{C} \to \bar{J}$ over $\bar{k}$ as

$$C(\bar{k}) \ni P \mapsto \mathcal{O}_{\bar{C}}(P - P_0) \in \operatorname{Pic}^\circ \bar{C} \simeq J(\bar{k}).$$

Let $\mathcal{M}_0 \in \operatorname{Pic}(\bar{C} \times \bar{J})$ be the universal divisorial correspondence between (C, P_0) and $(J, 0)$ (cf. [15, § 1]).

LEMMA 4.9. (i) *For $u \in J(\bar{k})$,*

$$s^*\mathcal{O}_{\bar{J}}(t_u^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-t_u^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-t_u^*\Theta) \simeq \mathcal{N}_0,$$

where s, p and $q \colon \bar{J} \times \bar{J} \to \bar{J}$ are respectively the sum, the projections onto the first and the second factors.

(ii) *For u and $v \in J(\bar{k})$, we have $t_u^*\mathcal{O}_{\bar{J}}(t_v^*\Theta - \Theta) \simeq \mathcal{O}_{\bar{J}}(t_v^*\Theta - \Theta)$.*

(iii) *For* $u \in J(\bar{k})$, *we have* $\mathcal{O}_{\bar{J}}(t_u^*\Theta - \Theta) \simeq (1_J, u)^*\mathcal{N}_0 \simeq (u, 1_J)^*\mathcal{N}_0$.

(iv) $(f_0 \times 1_J)^*\mathcal{N}_0 \simeq \mathcal{M}_0^{\otimes(-1)}$.

(v) $(1_C \times f_0)^*\mathcal{M}_0 \simeq \mathcal{O}_{\bar{C}\times\bar{C}}(\Delta - P_0 \times C - C \times P_0)$.

Proof. (i) An application of Theorem of the cube [14, Corollary 6.4]. (ii) Theorem of the square [14, Theorem 6.7]. (iii) Easily follows from the definition. (iv)(v) See [15, Summary 6.11]. □

Lemma 4.10. (i) $(-1)^*\mathcal{N}_0 \simeq \mathcal{N}_0$.

(ii) *For* $u \in J(\bar{k})$, *we have* $(-1)^*\mathcal{O}_{\bar{J}}(t_u^*\Theta - \Theta) \simeq \mathcal{O}_{\bar{J}}(\Theta - t_u^*\Theta)$.

Proof. (i) By the Riemann-Roch theorem, we know

$$(-1)^*\Theta = t_w^*\Theta \quad \text{for some } w \in J(\bar{k})$$

(see, for example, [20, § 5.6 (1)]). Therefore, from the previous lemma,

$$\begin{aligned}
(-1)^*\mathcal{N}_0 &\simeq (-1)^*\left[s^*\mathcal{O}_{\bar{J}}(\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-\Theta)\right] \\
&\simeq s^*\mathcal{O}_{\bar{J}}((-1)^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-(-1)^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-(-1)^*\Theta) \\
&\simeq s^*\mathcal{O}_{\bar{J}}(t_w^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-t_w^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-t_w^*\Theta) \\
&\simeq \mathcal{N}_0.
\end{aligned}$$

(ii) See [14, § 9]. □

Lemma 4.11. *We have for* $(P,Q) \in (C \times C)(\bar{k})$

$$\langle fP, fQ\rangle = h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, P) + h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, Q)$$
$$- h_{\bar{C}\times\bar{C}}\left(\mathcal{O}_{C\times C}((2g-2)^2\Delta), (P,Q)\right) + O(1),$$

in particular,

$$\|fP\|^2 = h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)g}, P) + O(1), \quad P \in C(\bar{k}).$$

Proof. We are going to show

$$(f\times f)^*\mathcal{N}_0 \simeq p^*\Omega_{C/k}^{\otimes(2g-2)} \otimes q^*\Omega_{C/k}^{\otimes(2g-2)} \otimes \mathcal{O}_{C\times C}(-(2g-2)^2\Delta).$$

Here p and $q\colon C\times C \to C$ are respectively the projections onto the first and the second factors. Then by the functoriality of heights, we obtain the first relation. The second relation is an immediate consequence of the first because as well-known

$$(1_C, 1_C)^*\mathcal{O}_{C\times C}(-\Delta) \simeq \Omega_{C/k},$$

where $(1_C, 1_C)\colon C \to C\times C$ is the diagonal map.

Since $fP = fP_0 - (2g-2)f_0P$ for $P \in C(\bar{k})$, setting $a := fP_0 \in J(\bar{k})$, we have $f = t_a \circ (2-2g)\circ f_0$ hence

$$(f\times f)^*\mathcal{N}_0 \simeq (f_0\times f_0)^*(2g-2)^*(t_a\times t_a)^*\mathcal{N}_0.$$

By definition and Lemmas 4.9 (i) and 4.9 (ii),

$$\begin{aligned}
(t_a \times t_a)^* \mathcal{N}_0 &\simeq (t_a \times t_a)^* \left[s^* \mathcal{O}_{\bar{J}}(\Theta) \otimes p^* \mathcal{O}_{\bar{J}}(-\Theta) \otimes q^* \mathcal{O}_{\bar{J}}(-\Theta) \right] \\
&\simeq s^* \mathcal{O}_{\bar{J}}(t_{2a}^* \Theta) \otimes p^* \mathcal{O}_{\bar{J}}(-t_a^* \Theta) \otimes q^* \mathcal{O}_{\bar{J}}(-t_a^* \Theta) \\
&\simeq \left[s^* \mathcal{O}_{\bar{J}}(t_{2a}^* \Theta) \otimes p^* \mathcal{O}_{\bar{J}}(-t_{2a}^* \Theta) \otimes q^* \mathcal{O}_{\bar{J}}(-t_{2a}^* \Theta) \right] \\
&\qquad \otimes p^* \mathcal{O}_{\bar{J}}(t_{2a}^* \Theta - t_a^* \Theta) \otimes q^* \mathcal{O}_{\bar{J}}(t_{2a}^* \Theta - t_a^* \Theta) \\
&\simeq \mathcal{N}_0 \otimes p^* t_a^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \otimes q^* t_a^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \\
&\simeq \mathcal{N}_0 \otimes p^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \otimes q^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta),
\end{aligned}$$

where by abuse of notation, the morphisms p and q are the projections of $\bar{J} \times \bar{J}$. By Lemma 4.8 and Lemma 4.10,

$$\begin{aligned}
&(2-2g)^*(t_a \times t_a)^* \mathcal{N}_0 \\
&\simeq (2-2g)^* \left[\mathcal{N}_0 \otimes p^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \otimes q^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \right] \\
&\simeq (2-2g)^* \mathcal{N}_0 \otimes p^* (2-2g)^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \otimes q^* (2-2g)^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta) \\
&\simeq \mathcal{N}_0^{\otimes (2-2g)^2} \otimes p^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta)^{\otimes (2-2g)} \otimes q^* \mathcal{O}_{\bar{J}}(t_a^* \Theta - \Theta)^{\otimes (2-2g)} \\
&\simeq \mathcal{N}_0^{\otimes (2-2g)^2} \otimes p^* (1_J, a)^* \mathcal{N}_0^{\otimes (2-2g)} \otimes q^* (a, 1_J)^* \mathcal{N}_0^{\otimes (2-2g)}.
\end{aligned}$$

We further pull back this invertible sheaf by $f_0 \times f_0$.

$$
\begin{aligned}
&(f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq (f_0 \times f_0)^* \left[\mathcal{N}_0^{\otimes(2-2g)^2} \otimes p^*(1_J, a)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_J)^*\mathcal{N}_0^{\otimes(2-2g)}\right] \\
&\simeq (f_0 \times f_0)^*\mathcal{N}_0^{\otimes(2-2g)^2} \\
&\quad \otimes p^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_C)^*(1_J \times f_0)^*\mathcal{N}_0^{\otimes(2-2g)} \\
&\simeq (1_C \times f_0)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)^2} \\
&\quad \otimes p^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_C)^*(1_J \times f_0)^*\mathcal{N}_0^{\otimes(2-2g)} \\
&\simeq (1_C \times f_0)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)^2} \\
&\quad \otimes p^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)}.
\end{aligned}
$$

The last transformation is permitted due to the symmetry of $\mathcal{N}_0$. By Lemma 4.9 (iv),

$$
\begin{aligned}
&(f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq (1_C \times f_0)^*\mathcal{M}_0^{\otimes-(2-2g)^2} \otimes p^*(1_C, a)^*\mathcal{M}_0^{\otimes(2g-2)} \otimes q^*(1_C, a)^*\mathcal{M}_0^{\otimes(2g-2)}.
\end{aligned}
$$

We have Lemma 4.9 (v). And, since the universality of $\mathcal{M}_0$ tells us

$(1_C, y)^*\mathcal{M}_0 \simeq y$ for $y \in J(\bar{k}) \simeq \mathrm{Pic}^\circ(\bar{C})$,

$$
\begin{aligned}
&(f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq \mathcal{O}_{C\times C}(P_0 \times C + C \times P_0 - \Delta)^{\otimes(2-2g)^2} \otimes p^*a^{\otimes(2g-2)} \otimes q^*a^{\otimes(2g-2)} \\
&\simeq p^*\mathcal{O}_C(P_0)^{\otimes(2g-2)^2} \otimes q^*\mathcal{O}_C(P_0)^{\otimes(2g-2)^2} \otimes \mathcal{O}_{C\times C}(-\Delta)^{\otimes(2g-2)^2} \\
&\qquad \otimes p^*\left[\Omega_{C/k} \otimes \mathcal{O}_C(-(2g-2)P_0)\right]^{\otimes(2g-2)} \\
&\qquad \otimes q^*\left[\Omega_{C/k} \otimes \mathcal{O}_C(-(2g-2)P_0)\right]^{\otimes(2g-2)} \\
&\simeq \mathcal{O}_{C\times C}(-\Delta)^{\otimes(2-2g)^2} \otimes p^*\Omega_{C/k}^{\otimes(2g-2)} \otimes q^*\Omega_{C/k}^{\otimes(2g-2)}. \quad \square
\end{aligned}
$$

LEMMA 4.12. *Let $\mathcal{L} \in J(\bar{k}) \simeq \mathrm{Pic}^\circ(\bar{C})$. We have*

$$\langle fP, \mathcal{L}\rangle = h_{\bar{C}}(\mathcal{L}^{\otimes(2g-2)}, P) + O(1), \quad P \in C(\bar{k}).$$

Proof. We have only to show

$$(f, \mathcal{L})^*\mathcal{N}_0 \simeq \mathcal{L}^{\otimes(2g-2)}.$$

Since $f = t_a \circ (2-2g) \circ f_0$, where $a = fP_0$, and $(f, \mathcal{L}) = (1_J, \mathcal{L}) \circ f$, we see

$$(f, \mathcal{L})^*\mathcal{N}_0 \simeq f_0^*(2-2g)^*t_a^*(1_J, \mathcal{L})^*\mathcal{N}_0.$$

By Lemma 4.9 (iii), Lemma 4.9 (ii), Lemma 4.8, and Lemma 4.10 (ii), we gain

$$\begin{aligned}(2-2g)^*t_a^*(1_J,\mathcal{L})^*\mathcal{N}_0 &\simeq (2-2g)^*t_a^*\mathcal{O}_{\bar{J}}(t_{\mathcal{L}}^*\Theta-\Theta)\\ &\simeq (2-2g)^*\mathcal{O}_{\bar{J}}(t_{\mathcal{L}}^*\Theta-\Theta)\\ &\simeq \mathcal{O}_{\bar{J}}(t_{\mathcal{L}}^*\Theta-\Theta)^{\otimes(2-2g)}\\ &\simeq (1_J,\mathcal{L})^*\mathcal{N}_0^{\otimes(2-2g)}.\end{aligned}$$

Pulling this back by f_0, we have from Lemma 4.9 (iv) and the property of the universal divisorial correspondence $\mathcal{M}_0$,

$$\begin{aligned}f_0^*(2-2g)^*t_a^*(1_J,\mathcal{L})^*\mathcal{N}_0 &\simeq f_0^*(1_J,\mathcal{L})^*\mathcal{N}_0^{\otimes(2-2g)}\\ &\simeq (1_C,\mathcal{L})^*(f_0\times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)}\\ &\simeq (1_C,\mathcal{L})^*\mathcal{M}_0^{\otimes(2g-2)}\\ &\simeq \mathcal{L}^{\otimes(2g-2)}. \quad \square\end{aligned}$$

Proof of the theorem. By Lemma 4.11 and the functoriality of heights, we see

$$\begin{aligned}\langle fP,\Psi(fP)\rangle &= \langle fP, f(\psi P)\rangle\\ &= h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)},P)+h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)},\psi P)\\ &\qquad - h_{\bar{C}\times\bar{C}}(\mathcal{O}_{C\times C}((2g-2)^2\Delta),(P,\psi P))+O(1)\\ &= h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)},P)+h_{\bar{C}}(\psi^*\Omega_{C/k}^{\otimes(2g-2)},P)\\ &\qquad - h_{\bar{C}}((1_C,\psi)^*\mathcal{O}_{C\times C}((2g-2)^2\Delta),P)+O(1)\end{aligned}$$

as functions of the $\bar{k}$-valued points P on $C(\bar{k})$. Since $\psi^*\Omega_{C/k} \simeq \Omega_{C/k}$,

$$\langle fP, \Psi(fP)\rangle = h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)} \otimes \mathcal{O}_C(-(2g-2)^2 D), P) + O(1).$$

By the second equality of Lemma 4.11 and Lemma 4.12,

$$\begin{aligned}
&2g\langle fP, \Psi(fP)\rangle + (d-2)\|fP\|^2 \\
&= h_{\bar{C}}(\Omega_{C/k}^{\otimes 4(2g-2)g} \otimes \mathcal{O}_C(-2(2g-2)^2 gD), P) \\
&\qquad + h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)(d-2)g}, P) + O(1) \\
&= h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)gd} \otimes \mathcal{O}_C(-2(2g-2)^2 gD), P) + O(1) \\
&= 2g \cdot h_{\bar{C}}(\left[\Omega_{C/k}^{\otimes d} \otimes \mathcal{O}_C(-(2g-2)D)\right]^{\otimes(2g-2)}, P) + O(1) \\
&= 2g\langle fP, \Omega_{C/k}^{\otimes d} \otimes \mathcal{O}_C(-(2g-2)D)\rangle + O(1). \quad \square
\end{aligned}$$

Remark 4.13. Using the additive property and functoriality of height functions, and setting $\mathcal{L}_\psi := \mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)}$, we get for $v \in J(\bar{k})$,

$$\begin{aligned}
&2g\langle v, \Psi v\rangle + (d-2)\|v\|^2 + 2g\langle v, \mathcal{L}_\psi\rangle \\
&= h_{\bar{J}}((1_J, \Psi)^*\mathcal{N}_0^{\otimes 2g} \otimes (1_J, 1_J)^*\mathcal{N}_0^{\otimes(d-2)} \otimes (1_J, \mathcal{L}_\psi)^*\mathcal{N}_0^{\otimes 2g}, v).
\end{aligned}$$

This is a canonical height on J and after all, we have proved

$$f^*\left[(1_J, \Psi)^*\mathcal{N}_0^{\otimes 2g} \otimes (1_J, 1_J)^*\mathcal{N}_0^{\otimes(d-2)} \otimes (1_J, \mathcal{L}_\psi)^*\mathcal{N}_0^{\otimes 2g}\right] \simeq \mathcal{O}_{\bar{C}}.$$

5. Twisted Fermat curve of degree four

It is probable that the action of an automorphism of a curve on the Jacobian variety is of a simple form in terms of its isogeneous components. This is the case for the so-called twisted Fermat curve of degree four. As an application, we obtain another proof of a certain well-known finiteness result (Theorem 0.9).

Let k be a number field and a, b, c elements of k different from zero. We call the curve Q in the projective plane $\mathbb{P}^2_k$ over k defined by the homogeneous equation

$$Q\colon aX^4 + bY^4 + cZ^4 = 0 \tag{5.1}$$

a twisted Fermat curve of degree four. We define also an elliptic curve E_X over k given by a Weierstrass equation

$$E_X\colon S^2T = R^3 + a^2bcT^2R,$$

where R, S, and T are the homogeneous coordinates of $\mathbb{P}^2_k$. This is a quotient curve of Q by a subgroup of order four of the automorphism group of $\bar{Q} = Q \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ over $\bar{k}$, where $\bar{k}$ is an algebraic closure of k. The quotient map $\phi_X\colon Q \to E_X$ is given over k by

$$Q(\bar{k}) \ni (x : y : z) \mapsto (r : s : t) = (-aby^2z : a^2bx^2y : z^3) \in E_X(\bar{k}).$$

There exists a homomorphism Ψ_X over k of E_X into the Jacobian variety J of Q induced by ϕ_X such that

$$\Psi_X\colon E_X(\bar{k}) \simeq \mathrm{Pic}^\circ(\bar{E}_X) \ni \mathcal{L} \mapsto \phi_X^*\mathcal{L} \in \mathrm{Pic}^\circ(\bar{Q}) \simeq J(\bar{k}).$$

The image of Ψ_X is one-dimensional, for we have a natural homomorphism $\Phi_X\colon J \to E_X$ satisfying $\Phi_X \circ \Psi_X = 4$ (cf. [15, Proposition 6.1]).

We denote by μ the group of square roots of unity in k. Then $\mu\times\mu\times\mu$ acts on Q as follows: For $(\xi,\eta,\zeta) \in \mu\times\mu\times\mu$, the action is given by

$$Q(\bar{k}) \ni (x:y:z) \mapsto (\xi x:\eta y:\zeta z) \in Q(\bar{k}).$$

This action yields an action on E_X compatible with the quotient map ϕ_X of Q onto E_X. Let γ_X, γ_Y, and γ_Z be the respective automorphisms of Q over k corresponding to the elements $(-1,1,1), (1,-1,1)$, and $(1,1,-1)$ of $\mu\times\mu\times\mu$. The next diagrams are commutative:

$$\begin{tikzcd} Q \arrow[r, "\phi_X"] \arrow[d, "\gamma_X"'] & E_X \arrow[d, equal] \\ Q \arrow[r] & E_X \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d, "\gamma_Y"'] & E_X \arrow[d, "-1"] \\ Q \arrow[r] & E_X \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d, "\gamma_Z"'] & E_X \arrow[d, "-1"] \\ Q \arrow[r] & E_X \end{tikzcd}$$

Denoting the induced automorphisms of J over k respectively by Γ_X, Γ_Y, and Γ_Z, we obtain the following commutative diagrams:

$$\begin{tikzcd} J \arrow[d, "\Gamma_X"'] & E_X \arrow[l, "\Psi_X"'] \arrow[d, equal] \\ J & E_X \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} J \arrow[d, "\Gamma_Y"'] & E_X \arrow[l] \arrow[d, "-1"] \\ J & E_X \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} J \arrow[d, "\Gamma_Z"'] & E_X \arrow[l] \arrow[d, "-1"] \\ J & E_X \arrow[l] \end{tikzcd}$$

We define in a cyclic manner E_Y, $\Psi_Y\colon E_Y \to J$, E_Z, and $\Psi_Z\colon E_Z \to J$. We see for i and $j \in \{X, Y, Z\}$

$$\Gamma_i \circ \Psi_j = (-1)^{\delta_{ij}-1}\Psi_j,$$

where δ_{ij} is Kronecker's delta function.

Now consider the map $\Psi := \Psi_X \circ p_1 + \Psi_Y \circ p_2 + \Psi_Z \circ p_3$ of $E_X \times E_Y \times E_Z$ into J, where p_i is the projection onto the i-th factor. The subvariety $\Psi(E_X \times E_Y \times 0)$ of J includes a curve $\Psi(E_X \times 0 \times 0) = \Psi_X(E_X)$, and the action of γ_X on $\Psi(E_X \times 0 \times 0)$ is trivial but not so on $\Psi(E_X \times E_Y \times 0) = \Psi_X(E_X) + \Psi_Y(E_Y)$. Therefore $\Psi(E_X \times E_Y \times 0)$ must be two-dimensional. By the same sort of reasoning, $\Psi(E_X \times E_Y \times E_Z)$ is three-dimensional. Since the dimension of J is also three, Ψ is an isogeny. Accordingly, we get an isomorphism of $\mathbb{R}$-vector spaces

$$\Psi\colon \bigoplus_{i=X,Y,Z} \mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k}) \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}).$$

We identify $\mathbb{R} \otimes E_i(\bar{k})$ with the corresponding subspace of $\mathbb{R} \otimes J(\bar{k})$ by this isomorphism.

Provide $\mathbb{R} \otimes J(\bar{k})$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ attached to a theta divisor. The elements of $\mathbb{R} \otimes E_i(\bar{k})$ are simultaneous eigenvectors of $\mu \times \mu \times \mu$ and each eigenspace $\mathbb{R} \otimes E_i(\bar{k})$ corresponds to a different character of $\mu \times \mu \times \mu$. Since eigenvectors of an orthogonal transformation with different eigenvalues are orthogonal to each other, the above decomposition of $\mathbb{R} \otimes J(\bar{k})$ into $\mathbb{R} \otimes E_i(\bar{k})$ is in addition orthogonal with respect to $\langle \cdot, \cdot \rangle$.

For $v = v_X + v_Y + v_Z \in \mathbb{R} \otimes J(\bar{k})$, where $v_i \in \mathbb{R} \otimes E_i(\bar{k})$, we have

$$\begin{aligned}\langle v, \Gamma_X v \rangle &= \langle v, \Gamma_X v_X + \Gamma_X v_Y + \Gamma_X v_Z \rangle \\ &= \langle v_X + v_Y + v_Z, v_X - v_Y - v_Z \rangle \\ &= \|v_X\|^2 - \|v_Y\|^2 - \|v_Z\|^2.\end{aligned}$$

Similarly, we gain

$$\langle v, \Gamma_Y v \rangle = -\|v_X\|^2 + \|v_Y\|^2 - \|v_Z\|^2$$

and

$$\langle v, \Gamma_Z v \rangle = -\|v_X\|^2 - \|v_Y\|^2 + \|v_Z\|^2.$$

Proposition 5.1. *Let Q be a twisted Fermat curve of degree four, J its Jacobian variety, and $f\colon Q \to J$ the canonical morphism given as* (2.3). *We equip $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ with the norm $\|\cdot\|$ associated with a theta divisor. Then there exist absolute constants c_1 and c_2 and an orthogonal decomposition $\mathbb{R} \otimes J(\bar{k}) = V_X \oplus V_Y \oplus V_Z$ into subspaces such that the* *image $f(Q(\bar{k}))$ is contained in the region of $\mathbb{R} \otimes J(\bar{k})$ defined by*

$$\begin{cases} |\|v_X\|^2 - \|v_Y\|^2| & \leq c_1 \\ |\|v_Z\|^2 - \|v_X\|^2| & \leq c_2, \end{cases}$$

where $v_X \in V_X$*,* $v_Y \in V_Y$*, and* $v_Z \in V_Z$*.*

PROOF. For $P \in Q(\bar{k})$, let $f(P) = v_X + v_Y + v_Z$, $v_i \in V_i := \mathbb{R} \otimes E_i(\bar{k})$. By Proposition 5.3 below, we see

$$\begin{aligned} \langle f(P), \Gamma_X(f(P)) \rangle = \langle f(P), f(\gamma_X(P)) \rangle &= -\frac{1}{3}\|f(P)\|^2 + O(1) \\ &= -\frac{1}{3}(\|v_X\|^2 + \|v_Y\|^2 + \|v_Z\|^2) + O(1) \end{aligned}$$

and

$$\langle f(P), \Gamma_Y(f(P)) \rangle = -\frac{1}{3}(\|v_X\|^2 + \|v_Y\|^2 + \|v_Z\|^2) + O(1)$$

with $O(1)$ terms bounded by absolute constants. Combining these with the equalities before Proposition 5.1, we obtain

$$2\|v_X\|^2 - \|v_Y\|^2 - \|v_Z\|^2 = O(1)$$

and

$$-\|v_X\|^2 + 2\|v_Y\|^2 - \|v_Z\|^2 = O(1).$$

Eliminations of appropriate terms give the result. □

LEMMA 5.2. *Let F be a plane curve of degree four over k defined as*

$$F\colon -X^4 - Y^4 + Z^4 = 0 \tag{5.2}$$

and γ an automorphism of F over k which acts as multiplications by -1 or 1 of X, Y, and Z coordinates. If γ is different from the identity map, then for $P \in F(\bar{k})$,

$$\langle f(P), f(\gamma(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\gamma(P))\| + O(1).$$

PROOF. Similar to the proof of Lemma 5.5 below (cf. [5, Proposition 6.4]). $\square$

PROPOSITION 5.3. *Let Q be a twisted Fermat curve of degree four given by (5.1) and γ an automorphism of Q over k which acts as multiplications by -1 or 1 of X, Y, and Z coordinates. If γ is not the identity map, then for $P \in Q(\bar{k})$,*

$$\langle f(P), f(\gamma(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\gamma(P))\| + O(1)$$

with $O(1)$ bounded by absolute constants. Here $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are respectively the scalar product and the norm on $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ attached to a theta divisor.

PROOF. Choose elements ξ, η, and ζ of $\bar{k}$ such that $a = -\xi^4$, $b = -\eta^4$, and $c = \zeta^4$. There exists an isomorphism ϕ of $\bar{Q}$ onto $\bar{F}$, where F is the curve (5.2), given by

$$Q(\bar{k}) \ni (x : y : z) \mapsto (\xi x : \eta y : \zeta z) \in F(\bar{k}).$$

The morphism ϕ is compatible with the respective automorphisms γ of Q and F, that is, $\gamma \circ \phi = \phi \circ \gamma$. By the invariance of heights (2.4), for $P \in Q(\bar{k})$

$$\begin{aligned}\langle f(P), f(\gamma(P))\rangle &= \langle f(\phi(P)), f(\phi(\gamma(P)))\rangle \\ &= \langle f(\phi(P)), f(\gamma(\phi(P)))\rangle \\ &= -\frac{1}{3}\|f(\phi(P))\|^2 + O(1) \\ &= -\frac{1}{3}\|f(P)\|^2 + O(1).\end{aligned}$$

The last $O(1)$ is the composition of ϕ with $O(1)$ in the previous lemma hence is absolutely bounded. $\square$

When the coefficients of X^4 and Y^4 of the defining equation (5.1) are the same, we receive some more information about the distribution of rational points of the curve in the Jacobian variety.

Assume $a = b = -1$. In this case, there is another automorphism τ of Q given by the exchange of X and Y coordinates. The automorphism τ yields isomorphisms $\tau_X \colon E_Y \to E_X$, $\tau_Y \colon E_X \to E_Y$ (not as groups), and an automorphism τ_Z of E_Z (not as a group) compatible with the quotient maps $\phi_i \colon Q \to E_i$. Explicitly, these morphisms are defined as

$$\tau_X \colon E_Y(\bar{k}) \ni (x : y : z) \mapsto (-czx : -cyz : x^2) \in E_X(\bar{k}),$$
$$\tau_Y \colon E_X(\bar{k}) \ni (x : y : z) \mapsto (-czx : -cyz : x^2) \in E_Y(\bar{k}),$$

and

$$\tau_Z \colon E_Z(\bar{k}) \ni (x : y : z) \mapsto (c^2zx : c^2yz : x^2) \in E_Z(\bar{k}).$$

They induce group isomorphisms $T \colon J \to J$, $T_X \colon E_Y \to E_X$, $T_Y \colon E_X \to E_Y$, and $T_Z \colon E_Z \to E_Z$ all compatible with $\Psi_i \colon E_i \to J$. Since τ_X and τ_Y are inverse to each other, so are T_X and T_Y. On the other hand, $T_Z = -1$, because τ_Z is the multiplication-by-(-1)-map plus a two-torsion point $(0 : 0 : 1)$.

Proposition 5.4. *Let Q be a twisted Fermat curve of degree four whose coefficients of X^4 and Y^4 of the defining equation* (5.1) *are the same, τ the automorphism of Q exchanging the X and Y coordinates, and f the canonical map of Q into the Jacobian variety J of Q defined as* (2.3). *For $P \in Q(\bar{k})$, we have*

$$\langle f(P), f(\tau(P)) \rangle = -\frac{1}{3}\|f(P)\|\|f(\tau(P))\| + O(1),$$

where $O(1)$ is bounded by absolute constants.

Proof. Follows from the next lemma in the same way as Proposition 5.3 followed from Lemma 5.2 $\square$

Lemma 5.5. *Let F be the plane curve* (5.2)*, τ the automorphism of F exchanging the X and Y coordinates, and f the canonical map of F into the Jacobian variety of F defined as* (2.3)*. For $P \in F(\bar{k})$, we have*

$$\langle f(P), f(\tau(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\tau(P))\| + O(1).$$

Proof. Since F is a plane curve of degree four, the canonical sheaf $\Omega_{F/k}$ is isomorphic to the inverse image of $\mathcal{O}_{\mathbb{P}^2}(1)$ (cf. [7, II 8.20.3]), where $\mathbb{P}^2$ denotes the ambient projective plane. According to Theorem 4.1, we have only to show for the diagonal Δ on $F \times F$ the divisor $(1_F, \tau)^*\Delta$ is linearly equivalent to a hyperplane section.

We compute the inverse image under $(1_F, \tau)$ of the ideal sheaf $\mathcal{O}_{F\times F}(-\Delta)$ of the diagonal subvariety of $F \times F$. Let $X_1, Y_1, Z_1; X_2, Y_2, Z_2$ be the bihomogeneous coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$ and X, Y, Z the homogeneous coordinates of $\mathbb{P}^2$. We naturally regard $F \times F$ as in $\mathbb{P}^2_k \times \mathbb{P}^2_k$ and F, in $\mathbb{P}^2_k$. Fixed points of τ are not mapped to the closed subscheme $\{Z_1Z_2 = 0\}$ by the map $(1_F, \tau)\colon F \to F \times F$, hence it suffices to see the affine open subvariety $\{Z_1Z_2 \neq 0\}$. Since

$$\Gamma\left(\{Z_1Z_2 \neq 0\}, \mathcal{O}_{F\times F}(-\Delta)\right) \simeq (X_1/Z_1 - X_2/Z_2, Y_1/Z_1 - Y_2/Z_2)$$
$$\bmod \left(-(X_1/Z_1)^4 - (Y_1/Z_1)^4 + 1, -(X_2/Z_2)^4 - (Y_2/Z_2)^4 + 1\right),$$

where the right hand side is an ideal of $\Gamma(\{Z_1Z_2 \neq 0\}, \mathcal{O}_{F\times F})$, we have

$$\Gamma\left(\{Z \neq 0\}, (1_F, \tau)^*\mathcal{O}_{F\times F}(-\Delta)\right) \simeq (X/Z - Y/Z)$$
$$\bmod \left(-(X/Z)^4 - (Y/Z)^4 + 1\right).$$

From this, we see $(1_F, \tau)^*\mathcal{O}_{F\times F}(-\Delta)$ is naturally isomorphic to the ideal sheaf of a hyperplane $\{X - Y = 0\}$, which is the desired result. $\square$

Proposition 5.6. *Notation being the same as in Proposition* 5.1*, suppose the coefficients of X^4 and Y^4 of the defining equation* (5.1) *are both minus one. Then, besides the neighborhoods of hypersurfaces in Proposition* 5.1*, the image $f(Q(\bar{k}))$ is included in the region near another quadric hypersurface in $\mathbb{R} \otimes J(\bar{k})$ given by*

$$|\langle v_X, T_X v_Y\rangle| \leq c_3$$

with an absolute constant c_3, where $T_X\colon V_Y \to V_X$ is a metric linear isomorphism. Let E be an elliptic curve defined by a Weierstrass equation

$$y^2 = x^3 - cx. \tag{5.3}$$

Then we can take V_X and V_Y as $\mathbb{R} \otimes_{\mathbb{Z}} E(\bar{k})$, and T_X is induced by an automorphism of E over k.

PROOF. For $P \in Q(\bar{k})$, let $f(P) = v_X + v_Y + v_Z$, $v_i \in \mathbb{R} \otimes E_i(\bar{k})$. Then we see

$$\begin{aligned}\langle f(P), T(f(P))\rangle &= \langle v_X + v_Y + v_Z, T_Y v_X + T_X v_Y + T_Z v_Z\rangle \\ &= \langle v_X, T_X v_Y\rangle + \langle v_Y, T_Y v_X\rangle + \langle v_Z, T_Z v_Z\rangle \\ &= \langle v_X, T_X v_Y\rangle + \langle T_X v_Y, v_X\rangle - \|v_Z\|^2,\end{aligned}$$

because $T_X = T_Y^{-1}$ does not change norm and $T_Z = -1$. From Proposition 5.4, we know

$$\langle f(P), T(f(P))\rangle = -\frac{1}{3}(\|v_X\|^2 + \|v_Y\|^2 + \|v_Z\|^2) + O(1)$$

with an absolutely bounded function $O(1)$ of P on $Q(\bar{k})$. Therefore we have

$$6\langle v_X, T_X v_Y\rangle + \|v_X\|^2 + \|v_Y\|^2 - 2\|v_Z\|^2 = O(1).$$

Add appropriate times the inequalities in Proposition 5.1 to this. $\square$

COROLLARY 5.7 (DEM'YANENKO [4, Example 1] [20, § 5.3]). *If the rank of $E(k)$ is not larger than one, then the canonical heights of rational points on Q are bounded by an absolute constant.*

Proof. Note first that the whole story was occurring over the base field k. So in Proposition 5.1 and Proposition 5.6, we can replace $\bar{k}$ with k.

For $P \in Q(k)$, let $f(P) = v_X + v_Y + v_Z$, $v_i \in V_i$. When $\dim V_X = \dim V_Y = 1$, we see $v_X = 0$ or $T_X v_Y = r v_X$ for some $r \in \mathbb{Q}$. In the latter situation, we have $\|v_Y\| = |r| \cdot \|v_X\|$, for T_X preserves the norm. If $|r| < 1/2$, then Proposition 5.1 says $\|v_X\|$ is absolutely bounded. If $|r| \geq 1/2$, then Proposition 5.6 still asserts $\|v_X\|$ is absolutely bounded. Anyway, by Proposition 5.1, $\|v_i\|$'s are all absolutely bounded, hence $\|f(P)\|$, too. $\square$

The twisted Fermat curve is a Thue curve. Using the lower bound for the canonical heights in § 3 we obtain a souped-up version of the previous corollary:

Corollary 5.8 (Silverman [23, Corollary 1 to Theorem 1]). *Let* $k^\times := k \setminus \{0\}$. *Except a finite number of* $c \bmod (k^\times)^4$, *if the rank of the group of rational points of the elliptic curve* (5.3) *over* k *is at most one, then*

$$\{(x, y) \in k^2 \mid x^4 + y^4 = c\} = \emptyset.$$

Proof. Note that for any constant G, the number of $c \bmod (k^\times)^4$

satisfying

$$|N_{\mathbb{Q}}^{k}\mathfrak{a}| < G,$$

where $\mathfrak{a}$ is the integral ideal of the ring $\mathfrak{o}_k$ of integers in k which is not divided by a fourth power of an ideal in $\mathfrak{o}_k$ and such that $c\mathfrak{o}_k = \mathfrak{a}\mathfrak{b}^4$ is an ideal decomposition of the principal ideal $c\mathfrak{o}_k$ into $\mathfrak{a}$ and the fourth power of a fractional ideal $\mathfrak{b}$ in k, is finite because of the finiteness of the ideal class group of k. Use the above corollary and Proposition 3.6. $\square$

REMARK 5.9. Manin [11, Example 1] got the same kind of result as Dem'yanenko's. He has constructed a null height on the product of two copies of an elliptic curve on which lies a special twisted Fermat curve.

REMARK 5.10. The quantity $\langle v_X, T_X v_Y \rangle$ for a point in $f(Q(k))$ does not necessarily vanish. Consider the example $a = b = -1$ and $c = 2$, in other words, $Q\colon X^4+Y^4 = 2Z^4$ and $E_X, E_Y : y^2 = x^3-2x$. In this case, we know the ranks of $E_X(\mathbb{Q})$ and $E_Y(\mathbb{Q})$ are one. For $P = (1:1:1) \in Q(\mathbb{Q})$, let $f(P) = v_X + v_Y + v_Z$ as above. We can see

$$v_X = \frac{1}{4} \otimes \phi_X(P) \in \mathbb{R} \otimes E_X(\mathbb{Q}) \quad \text{and} \quad v_Y = \frac{1}{4} \otimes \phi_Y(P) \in \mathbb{R} \otimes E_Y(\mathbb{Q}).$$

Since P is a fixed point of τ,

$$\begin{aligned}
T_X v_Y &= \frac{1}{4} \otimes [\tau_X(\phi_Y(P)) - \tau_X(\infty)] \\
&= \frac{1}{4} \otimes \phi_X(\tau(P)) - \frac{1}{4} \otimes (0,0) \\
&= \frac{1}{4} \otimes \phi_X(P) \\
&= v_X.
\end{aligned}$$

Consequently, $\langle v_X, T_X v_Y \rangle = \|v_X\|^2$. On the other hand, $\phi_X(P) = (-1,-1) \in E_X(\mathbb{Q})$ is not a torsion point, therefore $v_X \neq 0$.

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