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Saturation of the approximation by spectral decompositions associated with the Schrödinger operator

by

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Saturation of the approximation by spectral decompositions associated with the Schrödinger operator

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Contents

Chapter 1

Introduction

Let Ω be an open domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$ and Δ be the Laplacian $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. We say $\Delta f \in L^\infty_{loc}(\Omega)$ if for every compact set K in Ω there is a constant C_K such that

$$\left| \int_K f(x)\, \Delta g(x)\, dx \right| \le C_K ||g||_{L^1(K)}$$

for any infinitely differentiable function g whose support is contained in K. Let $\{\varphi_\varepsilon\}$ be infinitely differentiable approximate identity with supports contained in $\{\, x;\ |x| < \varepsilon \,\}$. For a function f on Ω and $x \in \Omega$, f is said to be regulated at x if $f * \varphi_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$. In 1970, Igari proved the following Theorem ([5]).

Theorem 1. *Suppose that there exist a complete orthonormal system $\{u_k\}$ of smooth functions in $L^2(\Omega)$ and a numerical sequence $\{\lambda_k\}$ for which $-\Delta u_k = \lambda_k u_k$ in Ω. Let*

$$f_k = \int_\Omega f(x)\, \overline{u_k(x)}\, dx, \qquad f \in L^2(\Omega)$$

and

$$s^\delta_\lambda f = \sum_{\lambda_k \le \lambda} \left(1 - \frac{\lambda_k}{\lambda} \right)^\delta f_k u_k, \qquad f \in L^2(\Omega).$$

Let $\delta \ge (n+3)/2$ and $f \in L^2(\Omega)$ be regulated in Ω.

(i) *It holds that*

$$\left\| s^\delta_\lambda f - f \right\|_{L^\infty(K)} = O\left(\frac{1}{\lambda} \right)$$

as $\lambda \to \infty$ for every compact set K in Ω if and only if $\Delta f \in L^\infty_{loc}(\Omega)$.

(ii) *It holds that*

$$\left\| s^\delta_\lambda f - f \right\|_{L^\infty(K)} = o\left(\frac{1}{\lambda} \right)$$

as $\lambda \to \infty$ for every compact set K in Ω if and only if Δf vanishes in Ω.

We shall give a generalization of Theorem 1. Let $L^2(\Omega)$ be a Hilbert space with inner product

$$(f, g) = (f, g)_{L^2(\Omega)} = \int_\Omega f(x)\, \overline{g(x)}\, dx, \qquad f, g \in L^2(\Omega).$$

Let V be a nonnegative function in $L^\infty_{loc}(\Omega)$. Consider the operator $A=-\Delta+V(x)$ in $L^2(\Omega)$ with the domain of definition $D(A)=C_c^\infty(\Omega)$. Let $\hat{A}$ be a nonnegative selfadjoint extension of A, that is, $D\left(\hat{A}\right)\supset D(A)$, $\hat{A}f=Af$ for any $f\in D(A)$, $\left(\hat{A}f,f\right)\geq 0$ for any $f\in D\left(\hat{A}\right)$ and if $g,h\in L^2(\Omega)$ and $\left(\hat{A}f,g\right)=(f,h)$ for any $f\in D\left(\hat{A}\right)$ then $g\in D\left(\hat{A}\right)$ and $h=\hat{A}g$. Such an extension exists by Friedrichs' theorem. By von Neumann's theorem, there is a unique spectral measure E corresponding to $\hat{A}$ (see [3]). E is a map from the Borel field on $\boldsymbol{R}$ into orthogonal projection operators in $L^2(\Omega)$. For fixed $f,g\in L^2(\Omega)$, $(E(\cdot)f,g)$ is a Radon measure whose support contained in $[\,0,\infty)$,

$$D\left(\hat{A}\right)=\left\{f\in L^2(\Omega);\int_{\boldsymbol{R}}t^2(E(dt)f,f)<\infty\right\}$$

and

$$\left(\hat{A}f,g\right)=\int_{\boldsymbol{R}}t\,(E(dt)f,g),\qquad f\in D\left(\hat{A}\right),\ g\in L^2(\Omega).$$

Let $\delta\geq 0$ and $\lambda>0$. Since the function $\left(1-\frac{t}{\lambda}\right)_+^\delta$ is bounded in $[\,0,\infty)$, we can define the bounded operator s_λ^δ in $L^2(\Omega)$ by

$$\left(s_\lambda^\delta f,g\right)=\int_{\boldsymbol{R}}\left(1-\frac{t}{\lambda}\right)_+^\delta(E(dt)f,g),\qquad f,g\in L^2(\Omega).$$

s_λ^δ is called the Riesz mean of order δ with respect to $\hat{A}$. Let $E_t=E(\,(-\infty,t\,]\,)$.

Example 1. Suppose that there exist a complete orthonormal system $\{u_k\}$ of smooth functions in $L^2(\Omega)$ and a sequence $\{\lambda_k\}\subset\boldsymbol{C}$ such that $(-\Delta+V)u_k=\lambda_k u_k$ in Ω. Let

$$f_k=\int_\Omega f(x)\,\overline{u_k(x)}\,dx,\qquad f\in L^2(\Omega).$$

Let $\hat{A}$ be the selfadjoint extension of $-\Delta+V(x)$ defined by

$$D\left(\hat{A}\right)=\left\{f\in L^2(\Omega);\ \sum_{k=1}^\infty\lambda_k^2\,|f_k|^2<\infty\right\}$$

and

$$\hat{A}f=\sum_{k=1}^\infty\lambda_k f_k u_k,\qquad f\in D\left(\hat{A}\right).$$

The spectral decomposition of any $f\in L^2(\Omega)$ is given by

$$E_t f=\sum_{\lambda_k\leq t}f_k u_k$$

and the Riesz mean of order δ by

$$s_\lambda^\delta f=\sum_{\lambda_k\leq\lambda}\left(1-\frac{\lambda_k}{\lambda}\right)^\delta f_k u_k,\qquad f\in L^2(\Omega).$$

Example 2. Let $\Omega = \boldsymbol{R}^n$ and V be a nonnegative constant on $\boldsymbol{R}^n$. Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int f(x)\, e^{-i\,\xi\cdot x} dx, \qquad f \in L^2(\boldsymbol{R}^n).$$

In this case, there is a unique nonnegative selfadjoint extension $\hat{A}$ of $-\Delta + V$ defined by

$$D\left(\hat{A}\right) = \left\{\, f \in L^2(\boldsymbol{R}^n);\ \left(|\xi|^2 + V\right) \hat{f}(\xi) \in L^2(\boldsymbol{R}^n) \,\right\}$$

and

$$\hat{A}f(x) = \frac{1}{\sqrt{2\pi}^n} \int \left(|\xi|^2 + V\right) \hat{f}(\xi)\, e^{i\,x\cdot\xi} d\xi, \qquad f \in D\left(\hat{A}\right).$$

The spectral decomposition of any $f \in L^2(\boldsymbol{R}^n)$ is given by

$$E_t f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{|\xi|^2 + V \le t} \hat{f}(\xi)\, e^{i\,x\cdot\xi} d\xi$$

and the Riesz mean of order δ is

$$s^\delta_\lambda f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{|\xi|^2 + V \le \lambda} \left(1 - \frac{|\xi|^2 + V}{\lambda}\right)^\delta \hat{f}(\xi)\, e^{i\,x\cdot\xi} d\xi, \qquad f \in L^2(\boldsymbol{R}^n).$$

Let V be a nonnegative function in $L^\infty_{loc}(\Omega)$. For $1 < p \le \infty$, we say $(-\Delta + V)f$ belongs to $L^p_{loc}(\Omega)$ if for every compact set K in Ω there is a constant C_K such that

$$\left| \int_K f(x)\, (-\Delta + V(x)) g(x)\, dx \right| \le C_K ||g||_{L^{p'}(K)}$$

for any infinitely differentiable function g whose support is contained in K and $1/p + 1/p' = 1$. Our results are as follows.

Theorem 2. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $V(x)$ be a nonnegative function in $L^\infty_{loc}(\Omega)$. Let s^δ_λ be the Riesz mean of order δ with respect to a nonnegative selfadjoint extension of $-\Delta + V$ in Ω. Suppose that f is a function in $L^2(\Omega)$ regulated in Ω. Let $1 < p \le \infty$ and $\delta \ge (n+3)/2$.*

(i) *Let $\alpha > n/4$ and $f \in D\left(\hat{A}^\alpha\right)$. Suppose that K is a compact set in Ω and K' is a closed subset of K with* dist $(K', K^c) > 0$. *If f and $(-\Delta + V)f \in L^p(K)$, then*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K')} = O\left(\lambda^{-1}\right) \qquad as \quad \lambda \to \infty.$$

(ii) *Suppose that K is a compact set in Ω. If*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K)} = O\left(\lambda^{-1}\right)$$

as $\lambda \to \infty$, then $(-\Delta + V)f \in L^p(K)$.

(iii) *Let $\alpha > n/4$ and $f \in D\left(\hat{A}^\alpha\right)$. Suppose that K is a compact set in Ω and K' is a closed subset of K with* dist $(K', K^c) > 0$. *If f vanishes in K, then*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K')} = o\left(\lambda^{-1}\right) \qquad as \quad \lambda \to \infty.$$

(iv) *Suppose that K is a compact set in Ω. If*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K)} = o\left(\lambda^{-1}\right)$$

as $\lambda \to \infty$, then $(-\Delta + V)f$ vanishes in K.

Theorem 3. *Let* Ω *be an open domain in* $\boldsymbol{R}^n$ *and* s_λ^δ *be the Riesz mean of order* δ *with respect to a nonnegative selfadjoint extension of* $-\Delta$ *in* Ω*. Let* f *be a function in* $L^2(\Omega)$ *regulated in* Ω*. Suppose that* $1 < p \leq \infty$ *and* $f \in L^p_{loc}(\Omega)$*. Let* $\delta \geq (n+3)/2$*.*

(i) *It holds that*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K)} = O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to \infty$ *for every compact set* K *in* Ω *if and only if* $\Delta f \in L^p_{loc}(\Omega)$*.*

(ii) *It holds that*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K)} = o\left(\frac{1}{\lambda}\right)$$

as $\lambda \to \infty$ *for every compact set* K *in* Ω *if and only if* Δf *vanishes in* Ω*.*

Theorem 2 follows from Theorem 4 in chapter 2 and Corollary 1 in chapter 3. Theorem 3 follows from Theorem 4 in chapter 2 and Corollary 2 in chapter 3.

Chapter 2

Saturation of the approximation

2.1 Preliminary lemma

Let H be a Hilbert space. Let S be a selfadjoint operator in H. By definition if $g, h \in H$ and $(Sf, g) = (f, h)$ for any $f \in D(S)$ then $g \in D(S)$ and $h = Sg$. By von Neumann's theorem, there is a unique spectral measure E corresponding to S. E is a map from the Borel field on $\boldsymbol{R}$ into orthogonal projection operators in H. For arbitrarily fixed $f, g \in H$, $(E(\cdot)f, g)$ is a Radon measure,

$$D(S) = \left\{ f \in H; \int_{\boldsymbol{R}} t^2 (E(dt)f, f) < \infty \right\}$$

and

$$(Sf, g) = \int_{\boldsymbol{R}} t\,(E(dt)f, g), \qquad f \in D(S),\ g \in H.$$

For a Borel function φ on $\boldsymbol{R}$, we define the operator $\varphi(S)$ in H by

$$(\varphi(S)\,f, g) = \int_{\boldsymbol{R}} \varphi(t)\,(E(dt)f, g)$$

for $f \in D(\varphi(S))$ and $g \in H$, where

$$D(\varphi(S)) = \left\{ f \in H; \int_{\boldsymbol{R}} |\varphi(t)|^2 (E(dt)f, f) < \infty \right\}.$$

Let $\{k_\lambda(t)\}$ be a sequence of bounded Borel functions on $\boldsymbol{R}$. Let $\{\theta_\lambda\}$ be a positive numerical sequence, $\varphi(t)$ be a Borel function on $\boldsymbol{R}$ and

$$\psi_\lambda(t) := \frac{k_\lambda(t) - 1}{\theta_\lambda\,\varphi(t)}.$$

Suppose that

$$(1) \quad \sup_{\lambda, t} |\psi_\lambda(t)| < \infty. \qquad (2) \quad \lim_{\lambda \to \infty} \psi_\lambda(t) = 1 \quad \text{for any } t \in \boldsymbol{R}.$$

We define the bounded operator L_λ in H by

$$(L_\lambda f, g) = \int_{\boldsymbol{R}} k_\lambda(t)\,(E(dt)f, g), \qquad f, g \in H. \tag{2.1.1}$$

We prove the following lemma.

Lemma 1. *If* $f \in H$ *and* $g \in D(\overline{\varphi}(S))$, *then* $\theta_\lambda^{-1}(L_\lambda f - f, g) \longrightarrow (f, \overline{\varphi}(S)g)$ *as* $\lambda \to \infty$.

We shall use the following lemma to prove Lemma 1.

Lemma 2. ([3: p.1199, Cor.XII.2.7 (c)]) *Let* ψ *and* φ *be Borel functions on the real line. If* $g \in D(\varphi(S))$ *and* $\varphi(S)g \in D(\psi(S))$, *then* g *belongs to* $D((\psi\varphi)(S))$ *and* $(\psi\varphi)(S)g = \psi(S)\varphi(S)g$.

Proof of Lemma 1. By the definition of L_λ (see (2.1.1))

$$\begin{aligned}\theta_\lambda^{-1}(L_\lambda f - f, g) &= \theta_\lambda^{-1}\int_{\boldsymbol{R}}[\,k_\lambda(t) - 1\,]\,(E(dt)f, g)\\ &= \int_{\boldsymbol{R}} \frac{k_\lambda(t)-1}{\theta_\lambda}\,(f, E(dt)g) = \int_{\boldsymbol{R}} \frac{k_\lambda(t)-1}{\theta_\lambda\,\varphi(t)}\,\varphi(t)\,(f, E(dt)g).\end{aligned}$$

Therefore by Lemma 2

$$\begin{aligned}\theta_\lambda^{-1}(L_\lambda f - f, g) &= \int_{\boldsymbol{R}}(\psi_\lambda\varphi)(t)\,(f, E(dt)g) = \Big(f, \overline{\psi_\lambda\varphi}(S)\,g\Big)\\ &= \Big(f, \overline{\psi_\lambda}(S)\,\overline{\varphi}(S)\,g\Big) = \int_{\boldsymbol{R}}\psi_\lambda(t)\,(f, E(dt)\,\overline{\varphi}(S)\,g).\end{aligned}$$

Let $\rho = (f, E(\cdot)\,\overline{\varphi}(S)\,g)$ and $|\rho|$ be the total variation of ρ. Then

$$\int_{\boldsymbol{R}}|\rho|(dt) \le ||f||_H||\overline{\varphi}(S)\,g||_H < \infty.$$

Therefore by (1), (2) and Lebesgue's dominated convergence theorem

$$\begin{aligned}\lim_{\lambda\to\infty}\theta_\lambda^{-1}(L_\lambda f - f, g) &= \lim_{\lambda\to\infty}\int_{\boldsymbol{R}}\psi_\lambda(t)\,\rho(dt)\\ &= \int_{\boldsymbol{R}}\lim_{\lambda\to\infty}\psi_\lambda(t)\,\rho(dt) = \int_{\boldsymbol{R}}\rho\,(dt) = (f, \overline{\varphi}(S)\,g).\end{aligned}$$

Thus Lemma 1 is proved.

2.2 Saturation of the approximation

Let (Ω, F, μ) be a σ-finite measure space. Let S be a selfadjoint operator in $L^2(\Omega, F, \mu)$. By von Neumann's theorem, there is a unique spectral measure E corresponding to S. Let $\{k_\lambda(t)\}$ be a sequence of bounded Borel functions on $\boldsymbol{R}$, $\{\theta_\lambda\}$ be a positive numerical sequence, $\varphi(t)$ be a Borel function on $\boldsymbol{R}$ and

$$\psi_\lambda(t) := \frac{k_\lambda(t)-1}{\theta_\lambda\,\varphi(t)}.$$

Suppose that $\sup_{\lambda,t}|\psi_\lambda(t)| < \infty$ and $\lim_{\lambda\to\infty}\psi_\lambda(t) = 1$ for any $t \in \boldsymbol{R}$. We define the bounded operator L_λ in $L^2(\Omega, F, \mu)$ by

$$(L_\lambda f, g) = \int_{\boldsymbol{R}} k_\lambda(t)\,(E(dt)f, g), \qquad f, g \in L^2(\Omega, F, \mu).$$

Let K be any subset in Ω, $1 < p \le \infty$, $1/p + 1/p' = 1$ and

$$M := \left\{ g \in L^2(\Omega, F, \mu);\ g \in L^{p'}(K) \cap D(\overline{\varphi}(S)) \ \text{ and } \ \operatorname{supp} g \subset K \right\}.$$

We prove the following lemma.

Lemma 3. *Let* $f \in L^2(\Omega, F, \mu)$.

(i) *If* $\|L_\lambda f - f\|_{L^p(K)} = O(\theta_\lambda)$ *as* $\lambda \to \infty$, *then there exists* $h \in L^p(K)$ *such that*

$$\int_K h\, \overline{g}\, d\mu = (f, \overline{\varphi}(S)g) \qquad \text{for any} \quad g \in M.$$

(ii) *If* $\|L_\lambda f - f\|_{L^p(K)} = o(\theta_\lambda)$ *as* $\lambda \to \infty$, *then* $(f, \overline{\varphi}(S)g) = 0$ *for any* $g \in M$.

Proof of Lemma 3. Let $g \in M$. Then by Lemma 1

$$\theta_\lambda^{-1} (L_\lambda f - f, g) \longrightarrow (f, \overline{\varphi}(S)g) \qquad \text{as} \quad \lambda \to \infty. \tag{2.2.1}$$

On the other hand, we have

$$\left| \theta_\lambda^{-1} (L_\lambda f - f, g) \right| \le \theta_\lambda^{-1} \|L_\lambda f - f\|_{L^p(K)} ||g||_{L^{p'}(K)}. \tag{2.2.2}$$

If $\|L_\lambda f - f\|_{L^p(K)} = O(\theta_\lambda)$ as $\lambda \to \infty$, then $\left|\theta_\lambda^{-1} (L_\lambda f - f, g)\right| \le C||g||_{L^{p'}(K)}$ for any λ with some constant $C > 0$ by (2.2.2). Therefore $|\,(f, \overline{\varphi}(S)g)\,| \le C||g||_{L^{p'}(K)}$ for any $g \in M$ by (2.2.1). By the Hahn-Banach extension theorem, there exists a bounded linear functional Φ on $L^{p'}(K)$ such that $\Phi(g) = (f, \overline{\varphi}(S)g)$ for any $g \in M$. By the F. Riesz representation theorem, there exists $h \in L^p(K)$ such that

$$\int_K h\, \overline{g}\, d\mu = \Phi(g)$$

for any $g \in L^{p'}(K)$. Thus (i) is proved.

If $\|L_\lambda f - f\|_{L^p(K)} = o(\theta_\lambda)$ as $\lambda \to \infty$, then (ii) is proved by the same way as in (i).

2.3 Saturation theorem

Let Ω be an open domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Let

$$A = \sum_{|j| \le m} a_j(x) D^j \tag{2.3.1}$$

be a differential operator, where $j = (j_1, j_2, \ldots, j_n)$, $|j| = j_1 + j_2 + \cdots + j_n$, $D^j = (-i)^{|j|}(\partial/\partial x_1)^{j_1} \cdots (\partial/\partial x_n)^{j_n}$ and $a_j \in L^\infty_{loc}(\Omega)$. We consider A as an operator in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$. Suppose that A is formally selfadjoint, that is, $(Af, g) = (f, Ag)$ for any $f, g \in D(A)$. Suppose that $\hat{A}$ is a selfadjoint extension of A, that is, $D\left(\hat{A}\right) \supset D(A)$, $\hat{A}f = Af$ for any $f \in D(A)$

and if $g, h \in L^2(\Omega)$ and $\left(\hat{A}f, g\right) = (f, h)$ for any $f \in D\left(\hat{A}\right)$ then $g \in D\left(\hat{A}\right)$ and $h = \hat{A}\, g$. By von Neumann's theorem there is a unique spectral measure E corresponding to $\hat{A}$. Let $\{k_\lambda(t)\}$ be a sequence of bounded Borel functions on $\boldsymbol{R}$, $\{\theta_\lambda\}$ be a positive numerical sequence, $\varphi(t)$ be a polynomial on $\boldsymbol{R}$ and

$$\psi_\lambda(t) := \frac{k_\lambda(t) - 1}{\theta_\lambda\, \varphi(t)}. \tag{2.3.2}$$

Suppose that

$$(1) \quad \sup_{\lambda, t} |\psi_\lambda(t)| < \infty. \qquad (2) \quad \lim_{\lambda \to \infty} \psi_\lambda(t) = 1 \quad \text{for any } t \in \boldsymbol{R}.$$

We define the bounded operator L_λ in $L^2(\Omega)$ by

$$(L_\lambda f, g) = \int_{\boldsymbol{R}} k_\lambda(t)\, (E(dt)f, g), \qquad f, g \in L^2(\Omega). \tag{2.3.3}$$

Let K be any compact subset in Ω and $1 < p \leq \infty$. We say $\varphi(A)f \in L^p(K)$ if

$$||\varphi(A)f||_{L^p(K)} := \sup_{||g||_{L^{p'}(K)}=1} \left| \int_K f(x)\, \overline{\varphi\left(\hat{A}\right) g(x)}\, dx \right| < \infty,$$

where $1/p + 1/p' = 1$ and g is infinitely differentiable function whose support is contained in K. The following theorem is a simple consequence of Lemma 3.

Theorem 4. *Let Ω be an open domain in $\boldsymbol{R}^n$ and A be a formally selfadjoint differential operator with coefficients in $L^\infty_{loc}(\Omega)$ given by* (2.3.1)*. Suppose that $\hat{A}$ is a selfadjoint extension of A and E is the spectral measure corresponding to $\hat{A}$. Let $\{k_\lambda(t)\}$ be a sequence of bounded Borel functions on $\boldsymbol{R}$, $\{\theta_\lambda\}$ be a positive numerical sequence and $\varphi(t)$ be a polynomial on $\boldsymbol{R}$ such that the sequence $\{\psi_\lambda(t)\}$ of Borel functions on $\boldsymbol{R}$ given by* (2.3.2) *satisfies* (1) *and* (2)*. Let $\{L_\lambda\}$ be the sequence of bounded operators in $L^2(\Omega)$ given by* (2.3.3)*. Let $f \in L^2(\Omega)$, $1 < p \leq \infty$ and K be any compact set in Ω.*

(i) *If*

$$\|L_\lambda f - f\,\|_{L^p(K)} = O\left(\theta_\lambda\right)$$

as $\lambda \to \infty$, then $\varphi(A)f \in L^p(K)$.

(ii) *If*

$$\|L_\lambda f - f\,\|_{L^p(K)} = o\left(\theta_\lambda\right)$$

as $\lambda \to \infty$, then $\varphi(A)f$ vanishes in K.

Proof of Theorem 4. If $\|L_\lambda f - f\|_{L^p(K)} = O(\theta_\lambda)$ as $\lambda \to \infty$, then by Lemma 3 there exists $h \in L^p(K)$ such that

$$\int_K h(x)\, \overline{g(x)}\, dx = \left(f, \overline{\varphi}\left(\hat{A}\right) g\right)$$

for any $g \in M := \left\{ g \in L^2(\Omega);\, g \in L^{p'}(K) \cap D\left(\overline{\varphi}\left(\hat{A}\right)\right) \text{ and } \operatorname{supp} g \subset K \right\}$ where $1/p + 1/p' = 1$. If g is a infinitely differentiable function and $\operatorname{supp} g \subset K$, then $g \in M$ and

$$\left(f, \overline{\varphi}\left(\hat{A}\right) g\right) = \int_K f(x) \overline{\overline{\varphi}\left(\hat{A}\right) g(x)}\, dx.$$

Therefore

$$\left| \int_K f(x) \overline{\overline{\varphi}\left(\hat{A}\right) g(x)}\, dx \right| = \left| \int_K h(x) \overline{g(x)}\, dx \right| \le ||h||_{L^p(K)} ||g||_{L^{p'}(K)}.$$

Thus (i) is proved. If $\|L_\lambda f - f\|_{L^p(K)} = o(\theta_\lambda)$ as $\lambda \to \infty$, then by the same way as in (i), (ii) is proved.

Examples. Riesz summation. For $\kappa > 0$ and $\alpha > 0$, let $k_\lambda(t) = \left(1 - \left|\frac{t}{\lambda}\right|^\kappa\right)_+^\alpha$, $\theta_\lambda = \lambda^{-\kappa}$ and $\varphi(t) = -\alpha |t|^\kappa$.

Fejer-Korovkin summation.

$$k_\lambda(t) = \begin{cases} \left(1 - \frac{|t|}{\lambda}\right) \cos \frac{\pi |t|}{\lambda} + \frac{1}{\lambda} \cot \frac{\pi}{\lambda} \sin \frac{\pi |t|}{\lambda} & |t| < \lambda, \\ 0 & |t| \ge \lambda, \end{cases}$$

$\theta_\lambda = \lambda^{-2}$ and $\varphi(t) = -\frac{(\pi t)^2}{2}$.

Rogosinski summation.

$$k_\lambda(t) = \begin{cases} \cos \frac{\pi |t|}{2\lambda} & |t| < \lambda, \\ 0 & |t| \ge \lambda, \end{cases}$$

$\theta_\lambda = \lambda^{-2}$ and $\varphi(t) = -\frac{(\pi t)^2}{8}$.

Jackson summation.

$$k_\lambda(t) = \begin{cases} 1 - \frac{3}{2}\left|\frac{t}{\lambda}\right|^2 + \frac{3}{4}\left|\frac{t}{\lambda}\right|^3 & |t| < \lambda, \\ \frac{1}{4}\left(2 - \frac{|t|}{\lambda}\right)^3 & \lambda \le |t| < 2\lambda, \\ 0 & |t| \ge 2\lambda, \end{cases}$$

$\theta_\lambda = \lambda^{-2}$ and $\varphi(t) = -\frac{3}{2} t^2$.

Poisson summation. We consider $P_r(t) := r^{|t|}$ as $r \to 1^-$. Then let $\lambda := \frac{1}{1-r}$, $k_\lambda(t) = \left(1 - \frac{1}{\lambda}\right)^{|t|}$, $\theta_\lambda = \lambda^{-1}$ and $\varphi(t) = -|t|$.

Weierstrass summation. For $\kappa > 0$, we consider $W_s(t) := e^{-s|t|^\kappa}$ as $s \to 0^+$. Then let $\lambda := \frac{1}{s}$, $k_\lambda(t) = e^{-\frac{|t|^\kappa}{\lambda}}$, $\theta_\lambda = \lambda^{-1}$ and $\varphi(t) = -|t|^\kappa$.

Chapter 3

Estimates of $s_\lambda^\delta f - f$

The aim of this section is to prove the following propositions.

Proposition 1. *Under the notations of* Theorem 2, *let* $\alpha > n/4$ *and* f *be a function in* $D\left(\hat{A}^\alpha\right)$ *regulated in* Ω. *Suppose that* $1 < p \leq \infty$, $\delta > (n-3)/2$, K *is a compact set in* Ω *and* K' *is a closed subset of* K *with* $\mathrm{dist}(K', K^c) > 0$.

(i) *If* f *and* $(-\Delta + V)f \in L^p(K)$, *then*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = \begin{cases} o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) & \text{if } \frac{n-3}{2} < \delta < \frac{n+3}{2} \\ O\left(\lambda^{-1}\right) & \text{if } \delta \geq \frac{n+3}{2} \end{cases} \quad \text{as} \quad \lambda \to \infty.$$

(ii) *If* f *vanishes in* K, *then*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) \qquad \text{as} \quad \lambda \to \infty.$$

Proposition 2. *Under the notations of* Theorem 3, *let* f *be a function in* $L^2(\Omega)$ *regulated in* Ω. *Let* $1 < p \leq \infty$ *and* $\delta > (n-3)/2$. *Suppose that* K *is a compact set in* Ω, K' *is a closed subset of* K *with* $\mathrm{dist}(K', K^c) > 0$ *and* $f \in L^p(K)$.

(i) *If* $\Delta f \in L^p(K)$, *then*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = \begin{cases} o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) & \text{if } \frac{n-3}{2} < \delta < \frac{n+3}{2} \\ O\left(\lambda^{-1}\right) & \text{if } \delta \geq \frac{n+3}{2} \end{cases} \quad \text{as} \quad \lambda \to \infty.$$

(ii) *If* Δf *vanishes in* K, *then*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) \qquad \text{as} \quad \lambda \to \infty.$$

If $\delta \geq (n+3)/2$ and $\lambda \geq 1$, then $\sqrt{\lambda}^{(n-1)/2-\delta} \leq \lambda^{-1}$. Therefore we have the followings:

Corollary 1. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $V(x)$ be a nonnegative function in $L^\infty_{loc}(\Omega)$. Let s^δ_λ be the Riesz mean of order δ with respect to a nonnegative selfadjoint extension of $-\Delta + V(x)$ in Ω where $\delta \geq (n+3)/2$. Let $\alpha > n/4$ and f be a function in $D\left(\hat{A}^\alpha\right)$ regulated in Ω. Suppose that K is a compact set in Ω and K' is a closed subset of K with $\operatorname{dist}(K', K^c) > 0$. Let $1 < p \leq \infty$.*

(i) *If f and $(-\Delta + V)f \in L^p(K)$, then*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K')} = O\left(\lambda^{-1}\right) \qquad \text{as} \quad \lambda \to \infty.$$

(ii) *If f vanishes in K, then*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K')} = o\left(\lambda^{-1}\right) \qquad \text{as} \quad \lambda \to \infty.$$

Corollary 2. *Let Ω be an open domain in $\boldsymbol{R}^n$. Let $\delta \geq (n+3)/2$ and s^δ_λ be the Riesz mean of order δ with respect to a nonnegative selfadjoint extension of $-\Delta$ in Ω. Let f be a function in $L^2(\Omega)$ regulated in Ω. Suppose that K is a compact set in Ω, K' is a closed set in K with $\operatorname{dist}(K', K^c) > 0$ and $f \in L^p(K)$ where $1 < p \leq \infty$.*

(i) *If $\Delta f \in L^p(K)$, then*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K')} = O\left(\lambda^{-1}\right) \qquad \text{as} \quad \lambda \to \infty.$$

(ii) *If Δf vanishes in K, then*

$$\left\| s^\delta_\lambda f - f \right\|_{L^p(K')} = o\left(\lambda^{-1}\right) \qquad \text{as} \quad \lambda \to \infty.$$

3.1 Generalized eigenfunction system

In the proof of these propositions, we shall use the generalized eigenfunction system corresponding to an ordered representation of $L^2(\Omega)$ associated with the Schrödinger operator. We shall begin with definitions.

3.1.1 Ordered representation of $L^2(\Omega)$

Let Ω be an open domain in $\boldsymbol{R}^n$ and A be a differential operator with coefficients in $L^\infty_{loc}(\Omega)$. We consider A as the operator in $L^2(\Omega)$ with the domain of definition $D(A) = C^\infty_c(\Omega)$. Suppose that A is formally selfadjoint and $\hat{A}$ is a selfadjoint extension of A. Let $\mathfrak{B}$ be the Borel field on $\boldsymbol{R}$ and E be the unique spectral measure corresponding to $\hat{A}$. For $h \in L^2(\Omega)$, we define the following closed subspace of $L^2(\Omega)$:

$$\begin{aligned} H(h) &:= \left\{ F\left(\hat{A}\right) h;\ F \text{ is a Borel function on } \boldsymbol{R} \text{ and } h \in D\left(F\left(\hat{A}\right)\right) \right\} \\ &= \left\{ F\left(\hat{A}\right) h;\ F \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h)) \right\}. \end{aligned}$$

For $f \in H(h)$ we can write uniquely $f = F\left(\hat{A}\right)h$ where $F \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))$ and

$$||f||_{L^2(\Omega)} = \left(\int_{\boldsymbol{R}} |F(t)|^2 (E(dt)h, h)\right)^{\frac{1}{2}}.$$

Therefore we can define an isomorphism U_h from $H(h)$ onto $L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))$ by $U_h f := F$. Moreover, this mapping preserves inner products.

There exist a sequence of functions $\{h_k\} \subset L^2(\Omega)$ and a sequence of sets $\{e_k\} \subset \mathfrak{B}$ with the following properties (see [3: XII.3.16] or [4: Chap.14]):

(I)

$$L^2(\Omega) = \bigoplus_k H(h_k).$$

That is, $H(h_k)$ are mutually orthogonal and span $L^2(\Omega)$.

(II) $\boldsymbol{R} = e_1 \supseteq e_2 \supseteq \ldots.$ $\{e_k\}$ is called the set of multiplicity.

(III) $(E(e)h_k, h_k) = (E(e \cap e_k)h_1, h_1)$ for any $e \in \mathfrak{B}$.

By (I), for $f \in L^2(\Omega)$ we can write uniquely

$$f = \sum_k F_k\left(\hat{A}\right)h_k$$

where $F_k \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h_k, h_k))$ and

$$\left(\sum_k \int_{\boldsymbol{R}} |F_k(t)|^2 (E(dt)h_k, h_k)\right)^{\frac{1}{2}} = \left(\sum_k \left\|F_k\left(\hat{A}\right)h_k\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = ||f||_{L^2(\Omega)} < \infty.$$

Therefore we can define an isometry U from $L^2(\Omega)$ onto

$$\begin{aligned}&\bigoplus_k L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h_k, h_k))\\ &= \left\{ \{F_k\};\ F_k \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h_k, h_k)) \text{ and } \sum_k \int_{\boldsymbol{R}} |F_k(t)|^2 (E(dt)h_k, h_k) < \infty \right\}\end{aligned}$$

by $Uf := \{F_k\}$. We denote $F_k =: (Uf)_k$. By (III) we have

$$\bigoplus_k L^2(\boldsymbol{R}, (E(\cdot)h_k, h_k)) = \bigoplus_k L^2(e_k, (E(\cdot)h_1, h_1)).$$

Let $\rho := (E(\cdot)h_1, h_1)$. Then U is an isomorphism from $L^2(\Omega)$ onto $\bigoplus_k L^2(e_k, \rho)$ which preserves inner products, that is, for any $f, g \in L^2(\Omega)$ it holds that

$$(f, g)_{L^2(\Omega)} = \sum_k \int_{e_k} (Uf)_k(t)\, \overline{(Ug)_k(t)}\, \rho(dt). \tag{3.1.1}$$

U is called an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$.

3.1.2 Generalized eigenfunction system of elliptic operators

Let Ω be an open domain in $\boldsymbol{R}^n$ and let

$$A = \sum_{|j|\le m} a_j(x) D^j$$

be a differential operator with $a_j \in L^\infty_{loc}(\Omega)$ where $j = (j_1, \ldots, j_n)$, $|j| = j_1 + \cdots + j_n$ and $D^j = (-i)^{|j|}(\partial/\partial x_1)^{j_1} \cdots (\partial/\partial x_n)^{j_n}$. Suppose that A is elliptic, that is

$$\sum_{|j|=m} a_j(x)\,\xi^j \ne 0 \qquad \text{for} \quad x \in \Omega \quad \text{and} \quad \xi \in \boldsymbol{R}^n - \{0\}.$$

We consider a formally differential operator A as the operator in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$. Suppose that A is formally selfadjoint and $\hat{A}$ is a selfadjoint extension of A. Let $\mathfrak{B}$ be the Borel field on $\boldsymbol{R}$ and E be the spectral measure corresponding to $\hat{A}$. Let U be an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$, $\{e_k\}$ be the sets of multiplicity and ρ be the measure with respect to U. Then there exists a sequence of functions $\{u_k(x,t)\}$ defined on the product space of $\Omega \times \boldsymbol{R}$ such that the following conditions are satisfied (see [3: XII.3 and XIV.6] or [4: Chap.15]):

(i) $u_k(x,t)$ are $dx \times d\rho(t)$-measurable and vanish on $\Omega \times (\boldsymbol{R} - e_k)$, where dx is the Lebesgue measure.

(ii) For any fixed $\lambda \in \boldsymbol{R}$, $u_k(x,\lambda) \in C^\infty(\Omega)$ and satisfy

$$A\,u_k(x,\lambda) = \lambda\,u_k(x,\lambda), \qquad x \in \Omega. \tag{3.1.2}$$

(iii) For each compact subset K of Ω and each bounded Borel set e in $\boldsymbol{R}$

$$\operatorname*{ess\,sup}_{x\in K} \int_e |u_k(x,t)|^2 \rho(dt) < \infty.$$

(iv) For each $f \in L^2(\Omega)$

$$(Uf)_k(t) = \int_\Omega f(x)\,\overline{u_k(x,t)}\,dx, \tag{3.1.3}$$

where the integral exists in the sense of $L^2(e_k, \rho)$.

(v) For each $f \in L^2(\Omega)$ and each $e \in \mathfrak{B}$

$$E(e)f(x) = \sum_k \int_e (Uf)_k(t)\,u_k(x,t)\,\rho(dt) \tag{3.1.4}$$

where the integral exists and the series converges in the sense of $L^2(\Omega)$.

$\{u_k\}$ is called the generalized eigenfunction system of $\hat{A}$ corresponding to U. By (v), for $\delta \ge 0$, $\lambda > 0$ and $f \in L^2(\Omega)$ we have

$$s_\lambda^\delta f(x) = \sum_k \int_{\boldsymbol{R}} \left(1 - \frac{t}{\lambda}\right)_+^\delta (Uf)_k(t)\,u_k(x,t)\,\rho(dt) \tag{3.1.5}$$

and for $\alpha \in \boldsymbol{R}$ and $f \in D\left(\hat{A}^{\alpha}\right)$ we have

$$\hat{A}^{\alpha} f(x) = \sum_k \int_{\boldsymbol{R}} t^{\alpha} (Uf)_k(t)\, u_k(x,t)\, \rho(dt). \tag{3.1.6}$$

3.1.3 An example

The ordered representation U and the generalized eigenfunction system $\{u_k\}$ are not unique.

For example, let $\Omega = \boldsymbol{R}^n$ and $A = -\Delta$. H_k denotes the space of spherical harmonics of degree k. S^{n-1} denotes the unit sphere and σ the Lebesgue measure on S^{n-1}. Let $\left\{Y_1^k, \ldots, Y_{d_k}^k\right\}$ be an orthonormal basis of H_k as a subspace of $L^2(S^{n-1}, \sigma)$ where $d_k = \dim H_k$. We choose an arbitrary Schwartz function Φ on the real line which is not vanishing everywhere. We define a sequence of functions $\left\{h_l^k\right\} \subset L^2(\boldsymbol{R}^n)$ by the Fourier transform

$$\widehat{h_l^k}(\xi) = \Phi(|\xi|)\, Y_l^k\left(\frac{\xi}{|\xi|}\right), \qquad k = 0, 1, \ldots, \quad l = 1, \ldots, d_k.$$

Let E be the spectral measure corresponding to the nonnegative selfadjoint extension of $-\Delta$. Then we have for $f \in L^2(\boldsymbol{R}^n)$

$$E(e)f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\{\xi; |\xi|^2 \in e\}} \hat{f}(\xi)\, e^{i x\cdot\xi} d\xi, \qquad e \in \mathfrak{B}.$$

Therefore for $k = 0, 1, \ldots$, $l = 1, \ldots, d_k$ and $e \in \mathfrak{B}$

$$\begin{aligned} \left(E(e)h_l^k, h_l^k\right)_{L^2} &= \int_{\{\xi; |\xi|^2 \in e\}} \left|\Phi(|\xi|)\, Y_l^k\left(\frac{\xi}{|\xi|}\right)\right|^2 d\xi \\ &= \int_{\{0 \le r < \infty;\, r^2 \in e\}} |\Phi(r)|^2 r^{n-1} dr = \frac{1}{2}\int_{[0,\infty)\cap e} \left|\Phi\left(\sqrt{t}\right)\right|^2 t^{\frac{n}{2}-1} dt. \end{aligned} \tag{3.1.7}$$

Thus for a Borel function F on $\boldsymbol{R}$, $F \in L^2\left(\boldsymbol{R}, \mathfrak{B}, \left(E(\cdot)h_l^k, h_l^k\right)\right)$ if and only if

$$\int_0^{\infty} \left|F\left(r^2\right)\Phi(r)\right|^2 r^{n-1} dr < \infty.$$

In this case

$$\begin{aligned} \left(F(-\Delta)h_l^k\right)(x) &= \left(\int F(t)E(dt)h_l^k\right)(x) = \frac{1}{\sqrt{2\pi}^n}\int F\left(|\xi|^2\right)\widehat{h_l^k}(\xi)\, e^{i x\cdot\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}^n}\int F\left(|\xi|^2\right)\Phi(|\xi|)\, Y_l^k\left(\frac{\xi}{|\xi|}\right) e^{i x\cdot\xi} d\xi, \end{aligned}$$

that is, $\left(F(-\Delta)h_l^k\right)\hat{}(\xi) = F\left(|\xi|^2\right)\Phi(|\xi|)\, Y_l^k\left(\dfrac{\xi}{|\xi|}\right)$. Therefore we have

$$\begin{aligned} H\left(h_l^k\right) &= \left\{F(-\Delta)h_l^k; F \in L^2\left(\boldsymbol{R}, \mathfrak{B}, \left(E(\cdot)h_l^k, h_l^k\right)\right)\right\} \\ &= \left\{f \in L^2(\boldsymbol{R}^n); \hat{f}(\xi) = G(|\xi|)\, Y_l^k\left(\frac{\xi}{|\xi|}\right) \text{ and} \int_0^{\infty} |G(r)|^2 r^{n-1} dr < \infty\right\}. \end{aligned}$$

Thus $L^2(\boldsymbol{R}^n) = \bigoplus_{k,l} H\left(h_l^k\right)$ and for $f \in L^2(\boldsymbol{R}^n)$, we can write $f = \sum_{k,l} (Uf)_l^k(-\Delta)h_l^k$. We have

$$\int_{S^{n-1}} \hat{f}\,(r\omega)\,\overline{Y_l^k(\omega)}\,\sigma(d\omega) = \sum_{k',l'} \int_{S^{n-1}} \left((Uf)_{l'}^{k'}(-\Delta)h_{l'}^{k'}\right)\hat{}\,(r\omega)\,\overline{Y_l^k(\omega)}\,\sigma(d\omega)$$

$$= \Phi(r) \sum_{k',l'} (Uf)_{l'}^{k'}\left(r^2\right) \int_{S^{n-1}} Y_{l'}^{k'}(\omega)\,\overline{Y_l^k(\omega)}\,\sigma(d\omega) = \Phi(r)\,(Uf)_l^k\left(r^2\right).$$

Therefore we have

$$(Uf)_l^k(t) = \frac{1}{\Phi\left(\sqrt{t}\right)} \int_{S^{n-1}} \hat{f}\left(\sqrt{t}\,\omega\right)\overline{Y_l^k(\omega)}\,\sigma(d\omega).$$

Let $e_l^k := \boldsymbol{R}$. Then $\left(E(e)h_l^k, h_l^k\right)_{L^2} = (E(e)h_1^0, h_1^0)_{L^2} = \left(E(e \cap e_l^k)h_1^0, h_1^0\right)_{L^2}$ for all $e \in \mathfrak{B}$ by (3.1.7). Therefore $\left\{e_l^k\right\}$ is the set of multiplicity. Furthermore by (3.1.7)

$$\rho(e) = \left(E(e)h_1^0, h_1^0\right)_{L^2} = \frac{1}{2}\int_{[0,\infty)\cap e} \left|\Phi\left(\sqrt{t}\right)\right|^2 t^{\frac{n}{2}-1}dt, \qquad e \in \mathfrak{B}.$$

Let $\left\{u_l^k\right\}$ be the generalized eigenfunction system corresponding to U. Then

$$u_l^k(x,t) = \frac{1}{\sqrt{2\pi}^n\,\overline{\Phi\left(\sqrt{t}\right)}} \int_{S^{n-1}} e^{i\sqrt{t}\,x\cdot\omega}\,Y_l^k(\omega)\,\sigma(d\omega).$$

Indeed, (i) is easily shown. On the other hand,

$$\begin{aligned} -\Delta_x u_l^k(x,\lambda) &= \frac{1}{\sqrt{2\pi}^n\,\overline{\Phi\left(\sqrt{\lambda}\right)}} \int_{S^{n-1}} \left(-\Delta_x e^{i\sqrt{\lambda}\,x\cdot\omega}\right) Y_l^k(\omega)\,\sigma(d\omega) \\ &= \frac{\lambda}{\sqrt{2\pi}^n\,\overline{\Phi\left(\sqrt{\lambda}\right)}} \int_{S^{n-1}} e^{i\sqrt{\lambda}\,x\cdot\omega}\,Y_l^k(\omega)\,\sigma(d\omega) = \lambda\,u_l^k(x,\lambda). \end{aligned}$$

Thus (ii) holds. For each $f \in L^2(\boldsymbol{R}^n)$, we have

$$\begin{aligned} \int f(x)\,\overline{u_l^k(x,t)}\,dx &= \frac{1}{\sqrt{2\pi}^n\,\Phi\left(\sqrt{t}\right)} \int_{S^{n-1}} \overline{Y_l^k(\omega)} \int f(x)\,e^{-i\sqrt{t}\,x\cdot\omega}dx\,\sigma(d\omega) \\ &= \frac{1}{\Phi\left(\sqrt{t}\right)} \int_{S^{n-1}} \overline{Y_l^k(\omega)}\,\hat{f}\left(\sqrt{t}\,\omega\right)\sigma(d\omega) = (Uf)_l^k(t). \end{aligned}$$

Thus (iv) holds. For each $f \in L^2(\boldsymbol{R}^n)$ and each $e \in \mathfrak{B}$, we have

$$\begin{aligned} &\sum_{k,l} \int_e (Uf)_l^k(t)\,u_l^k(x,t)\,\rho(dt) \\ &= \sum_{k,l} \int_{\{0\le r<\infty;\, r^2\in e\}} (Uf)_l^k\left(r^2\right) u_l^k\left(x, r^2\right) |\Phi(r)|^2 r^{n-1}dr \\ &= \frac{1}{\sqrt{2\pi}^n} \sum_{k,l} \int_{\{0\le r<\infty;\, r^2\in e\}} (Uf)_l^k\left(r^2\right) \int_{S^{n-1}} e^{irx\cdot\omega}\,Y_l^k(\omega)\,\sigma(d\omega)\,\Phi(r)\,r^{n-1}dr \\ &= \frac{1}{\sqrt{2\pi}^n} \sum_{k,l} \int_{\{\xi;|\xi|^2\in e\}} (Uf)_l^k\left(|\xi|^2\right) e^{ix\cdot\xi}\,Y_l^k\left(\frac{\xi}{|\xi|}\right) \Phi(|\xi|)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}^n} \int_{\{\xi;|\xi|^2\in e\}} \hat{f}(\xi)\,e^{ix\cdot\xi}d\xi = E(e)f(x). \end{aligned}$$

Thus (v) holds. To see (iii), we observe that

$$u_l^k(x,t) = \frac{i^k J_{k+\frac{n}{2}-1}\left(\sqrt{t}|x|\right)}{\Phi\left(\sqrt{t}\right)\left(\sqrt{t}|x|\right)^{\frac{n}{2}-1}} Y_l^k\left(\frac{x}{|x|}\right)$$

where J_β denotes the Bessel function of order β.

Let $0 \le a < b < \infty$ and $e = [a,b]$. We have

$$\begin{aligned}
&\int_e \left|u_l^k(x,t)\right|^2 \rho(dt) = \left|Y_l^k\left(\frac{x}{|x|}\right)\right|^2 \times \int_{\sqrt{a}\le r\le\sqrt{b}} \left|\frac{J_{k+\frac{n}{2}-1}(|x|r)}{(|x|r)^{\frac{n}{2}-1}}\right|^2 r^{n-1}dr \\
&= \left|Y_l^k\left(\frac{x}{|x|}\right)\right|^2 \times \frac{1}{2|x|^{n-2}}\left[b\left\{J_{k+\frac{n}{2}-1}^2\left(\sqrt{b}|x|\right) - J_{k+\frac{n}{2}-2}\left(\sqrt{b}|x|\right)J_{k+\frac{n}{2}}\left(\sqrt{b}|x|\right)\right\}\right. \\
&\qquad \left. -a\left\{J_{k+\frac{n}{2}-1}^2\left(\sqrt{a}|x|\right) - J_{k+\frac{n}{2}-2}\left(\sqrt{a}|x|\right)J_{k+\frac{n}{2}}\left(\sqrt{a}|x|\right)\right\}\right] \\
&\le C_{n,k,a,b}\left||x|^k Y_l^k\left(\frac{x}{|x|}\right)\right|^2
\end{aligned}$$

where $C_{n,k,a,b}$ is a constant independent of x. Therefore for each compact subset K of Ω

$$\operatorname*{ess\,sup}_{x\in K}\int_e \left|u_l^k(x,t)\right|^2 \rho(dt) \le C_{n,k,a,b} \operatorname*{ess\,sup}_{x\in K}\left||x|^k Y_l^k\left(\frac{x}{|x|}\right)\right|^2 < \infty.$$

Thus (iii) holds.

3.1.4 Formulas for the Bessel functions

J_β denotes the Bessel function and Y_β the Bessel function of the second kind of order β. That is,

$$J_\beta(s) = \left(\frac{s}{2}\right)^\beta \sum_{j=0}^\infty \frac{(-1)^j}{j!\,\Gamma(j+\beta+1)}\left(\frac{s}{2}\right)^{2j},$$

$$Y_\beta(s) = \frac{\cos\beta\pi\, J_\beta(s) - J_{-\beta}(s)}{\sin\beta\pi} \qquad (\beta \ne \text{integer})$$

and

$$\begin{aligned}
Y_N(s) \;=\; & \lim_{\beta\to N} Y_\beta(s) = \frac{2}{\pi}J_N(s)\left(\gamma + \log\frac{s}{2}\right) - \frac{1}{\pi}\left(\frac{s}{2}\right)^{-N}\sum_{j=0}^{N-1}\frac{(N-j-1)!}{j!}\left(\frac{s}{2}\right)^{2j} \\
& - \frac{1}{\pi}\left(\frac{s}{2}\right)^N \sum_{j=0}^\infty \frac{(-1)^j}{j!\,(N+j)!}\left(\frac{s}{2}\right)^{2j}(h_{j+N}+h_j) \qquad (N = 0,1,\dots) \qquad (3.1.8)
\end{aligned}$$

where Γ is the Gamma function, γ is the Euler constant,

$$h_j = \sum_{k=1}^j \frac{1}{k} \quad j = 1,2,\dots \qquad \text{and} \qquad h_0 = 0.$$

If $\beta = 0$, the finite sum in (3.1.8) is to be omitted. The following recurrence formulas hold:

$$\frac{d}{ds}\frac{J_\beta(s)}{s^\beta} = -\frac{J_{\beta+1}(s)}{s^\beta}, \qquad \frac{d}{ds}\frac{Y_\beta(s)}{s^\beta} = -\frac{Y_{\beta+1}(s)}{s^\beta}, \tag{3.1.9}$$

$$Y_\beta(s)\, J_{\beta+1}(s) - J_\beta(s)\, Y_{\beta+1}(s) = \frac{2}{\pi s}, \tag{3.1.10}$$

$$\frac{d}{ds}J_\beta(s) = J_{\beta-1}(s) - \frac{\beta J_\beta(s)}{s} \qquad \text{and} \qquad \frac{d}{ds}Y_\beta(s) = Y_{\beta-1}(s) - \frac{\beta Y_\beta(s)}{s}. \tag{3.1.11}$$

On the other hand, the following asymptotic formulas hold:

$$J_\beta(s) = \sqrt{\frac{2}{\pi s}}\cos\left[s - (2\beta+1)\frac{\pi}{4}\right] + O\left(\frac{1}{s^{\frac{3}{2}}}\right) \qquad (s \to \infty), \tag{3.1.12}$$

$$Y_\beta(s) = \sqrt{\frac{2}{\pi s}}\sin\left[s - (2\beta+1)\frac{\pi}{4}\right] + O\left(\frac{1}{s^{\frac{3}{2}}}\right) \qquad (s \to \infty), \tag{3.1.13}$$

$$J_\beta(s) = \frac{s^\beta}{2^\beta\Gamma(\beta+1)} + O\left(s^{\beta+2}\right) \qquad (s \to 0^+), \tag{3.1.14}$$

$$Y_\beta(s) = -\frac{2^\beta\Gamma(\beta)}{\pi s^\beta} + O\left(s^{-\beta+2} + s^\beta\right) \qquad (s \to 0^+, \quad \beta > 0), \tag{3.1.15}$$

and

$$Y_0(s) = \frac{2}{\pi}\log s + O(1) \qquad (s \to 0^+). \tag{3.1.16}$$

We shall use the letter C to denote constants, different in different contexts. $C_{a,b,\ldots}$ denotes constants dependent on $a, b, \ldots$.

3.2 Decomposition of $s_\lambda^\delta f - f$

Throughout what follows, Ω denotes an open domain in $\boldsymbol{R}^n$, V a nonnegative function in $L^\infty_{loc}(\Omega)$ and $\hat{A}$ a nonnegative selfadjoint extension of $-\Delta + V$. U denotes an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$, $\{u_k\}$ the generalized eigenfunction system and ρ the measure associated with U. Let $\lambda > 0$. s_λ^δ denotes the Riesz mean of order δ with respect to $\hat{A}$. We denote the gamma function by Γ, the unit sphere in $\boldsymbol{R}^n$ by S^{n-1}, the Lebesgue measure on the unit sphere S^{n-1} by σ and the surface area $2\sqrt{\pi}^n/\Gamma(\frac{n}{2})$ of S^{n-1} by ω_n. Let $\beta_0 = \frac{n}{2} - 1$.

3.2.1 Mean-value formula

Lemma 4. *Let $u \in C^2(\Omega)$ be a solution of the equation $(-\Delta + V)\,u = \lambda\, u$ in Ω. For $x \in \Omega$, we choose $r > 0$ such that $\{\, y \in \boldsymbol{R}^n;\ |\, y - x\,| \le r\,\} \subset \Omega$. Then*

$$\int_{S^{n-1}} u(x - r\,w)\,\sigma(dw) = \sqrt{2\pi}^n\,\frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right)}{\left(\sqrt{\lambda}\,r\right)^{\beta_0}} \times u(x) + \frac{\pi}{2\,r^{\beta_0}}$$

$$\times \int_{|y|<r} u(x-y)\,V(x-y)\,\frac{Y_{\beta_0}\left(\sqrt{\lambda}\,r\right)J_{\beta_0}\left(\sqrt{\lambda}|y|\right) - J_{\beta_0}\left(\sqrt{\lambda}\,r\right)Y_{\beta_0}\left(\sqrt{\lambda}|y|\right)}{|y|^{\beta_0}}\,dy.$$

Proof. Let us consider the function

$$v_\lambda^r(y) := \frac{\pi}{2\,r^{\beta_0}} \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0}\left(\sqrt{\lambda}|y|\right) - Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0}\left(\sqrt{\lambda}|y|\right)}{|y|^{\beta_0}}.$$

Then by Green's formula, for any $0 < \varepsilon < r$

$$\begin{aligned}
&\int_{\varepsilon<|y|<r} [\,\lambda\, u(x-y)\, v_\lambda^r(y) - u(x-y)\,(-\Delta_y + V(x-y))\, v_\lambda^r(y)\,]\; dy \\
&= r^{n-1}\int_{S^{n-1}} \left[u(x-rw) \sum_{i=1}^n w_i \left.\frac{\partial v_\lambda^r}{\partial y_i}\right|_{y=rw} - v_\lambda^r(rw) \sum_{i=1}^n w_i \left.\frac{\partial u}{\partial z_i}\right|_{z=x-rw} \right] \sigma(dw) \\
&\quad -\varepsilon^{n-1}\int_{S^{n-1}} \left[u(x-\varepsilon w) \sum_{i=1}^n w_i \left.\frac{\partial v_\lambda^r}{\partial y_i}\right|_{y=\varepsilon w} - v_\lambda^r(\varepsilon w) \sum_{i=1}^n w_i \left.\frac{\partial u}{\partial z_i}\right|_{z=x-\varepsilon w} \right] \sigma(dw) \quad (3.2.1)
\end{aligned}$$

where $w = (w_1, \ldots, w_n) \in S^{n-1}$. For $w \in S^{n-1}$, $v_\lambda^r(rw) = 0$. On the other hand, by (3.1.9), we have

$$\frac{\partial v_\lambda^r}{\partial y_i} = -\frac{\pi\sqrt{\lambda}\,y_i}{2\,r^{\beta_0}} \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0+1}\left(\sqrt{\lambda}|y|\right) - Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0+1}\left(\sqrt{\lambda}|y|\right)}{|y|^{\beta_0+1}}. \quad (3.2.2)$$

Therefore

$$\sum_{i=1}^n w_i \left.\frac{\partial v_\lambda^r}{\partial y_i}\right|_{y=\varepsilon w} = -\frac{\pi\sqrt{\lambda}}{2\,r^{\beta_0}} \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0+1}\left(\sqrt{\lambda}\,\varepsilon\right) - Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0+1}\left(\sqrt{\lambda}\,\varepsilon\right)}{\varepsilon^{\beta_0}} \quad (3.2.3)$$

and by (3.1.10),

$$\sum_{i=1}^n w_i \left.\frac{\partial v_\lambda^r}{\partial y_i}\right|_{y=rw} = \frac{1}{r^{n-1}}. \quad (3.2.4)$$

By (3.2.2) and (3.1.11), we have

$$\begin{aligned}
\frac{\partial^2 v_\lambda^r}{\partial y_i^2} &= -\lambda\, v_\lambda^r(y) \frac{y_i^2}{|y|^2} \\
&\quad + \frac{\pi\sqrt{\lambda}}{2\,r^{\beta_0}} \left(n\,\frac{y_i^2}{|y|^2} - 1 \right) \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0+1}\left(\sqrt{\lambda}|y|\right) - Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0+1}\left(\sqrt{\lambda}|y|\right)}{|y|^{\beta_0+1}}.
\end{aligned}$$

Therefore $-\Delta v_\lambda^r = \lambda\, v_\lambda^r$. Thus by (3.2.1), (3.2.3) and (3.2.4), we have

$$\begin{aligned}
&\frac{\pi}{2r^{\beta_0}} \int_{\varepsilon<|y|<r} u(x-y)\, V(x-y)\, \frac{Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0}\left(\sqrt{\lambda}|y|\right) - J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0}\left(\sqrt{\lambda}|y|\right)}{|y|^{\beta_0}}\, dy \\
&= \int_{S^{n-1}} u(x-rw)\, \sigma(dw) \\
&\quad + \frac{\pi\sqrt{\lambda}\,\varepsilon^{\frac{n}{2}}}{2\,r^{\beta_0}} \left[J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0+1}\left(\sqrt{\lambda}\,\varepsilon\right) - Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0+1}\left(\sqrt{\lambda}\,\varepsilon\right) \right] \\
&\qquad\qquad \times \int_{S^{n-1}} u(x-\varepsilon w)\, \sigma(dw) \\
&\quad + \frac{\pi\varepsilon^{\frac{n}{2}}}{2\,r^{\beta_0}} \left[J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0}\left(\sqrt{\lambda}\,\varepsilon\right) - Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0}\left(\sqrt{\lambda}\,\varepsilon\right) \right] \\
&\qquad\qquad \times \int_{S^{n-1}} \sum_{i=1}^n w_i \left.\frac{\partial u}{\partial z_i}\right|_{z=x-\varepsilon w} \sigma(dw). \quad (3.2.5)
\end{aligned}$$

By (3.1.14), we have

$$\varepsilon^{\frac{n}{2}} J_{\beta_0+1}\left(\sqrt{\lambda}\,\varepsilon\right) \int_{S^{n-1}} u(x-\varepsilon w)\,\sigma(dw) \to 0 \tag{3.2.6}$$

and

$$\varepsilon^{\frac{n}{2}} J_{\beta_0}\left(\sqrt{\lambda}\,\varepsilon\right) \int_{S^{n-1}} \sum_{i=1}^{n} w_i \left.\frac{\partial u}{\partial z_i}\right|_{z=x-\varepsilon w} \sigma(dw) \to 0 \tag{3.2.7}$$

as $\varepsilon \to 0^+$. On the other hand, by (3.1.15) and (3.1.16), we have

$$\begin{aligned} &\frac{\pi\sqrt{\lambda}\,\varepsilon^{\frac{n}{2}}}{2\,r^{\beta_0}} J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0+1}\left(\sqrt{\lambda}\,\varepsilon\right) \int_{S^{n-1}} u(x-\varepsilon w)\,\sigma(dw) \\ &\qquad \to -2^{\beta_0}\Gamma\left(\frac{n}{2}\right) \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right)}{\left(\sqrt{\lambda}\,r\right)^{\beta_0}}\,\omega_n\,u(x) = -\sqrt{2\pi}^{\,n} \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right)}{\left(\sqrt{\lambda}\,r\right)^{\beta_0}}\,u(x) \end{aligned} \tag{3.2.8}$$

and

$$\varepsilon^{\frac{n}{2}} Y_{\beta_0}\left(\sqrt{\lambda}\,\varepsilon\right) \int_{S^{n-1}} \sum_{i=1}^{n} w_i \left.\frac{\partial u}{\partial z_i}\right|_{z=x-\varepsilon w} \sigma(dw) \to 0 \tag{3.2.9}$$

as $\varepsilon \to 0^+$. Therefore by (3.2.5), (3.2.6), (3.2.7), (3.2.8) and (3.2.9), we have

$$\begin{aligned} &\frac{\pi}{2r^{\beta_0}} \int_{|y|<r} u(x-y)\,V(x-y)\, \frac{Y_{\beta_0}\left(\sqrt{\lambda}\,r\right) J_{\beta_0}\left(\sqrt{\lambda}|y|\right) - J_{\beta_0}\left(\sqrt{\lambda}\,r\right) Y_{\beta_0}\left(\sqrt{\lambda}|y|\right)}{|y|^{\beta_0}}\,dy \\ &\quad = \int_{S^{n-1}} u(x-rw)\,\sigma(dw) - \sqrt{2\pi}^{\,n} \frac{J_{\beta_0}\left(\sqrt{\lambda}\,r\right)}{\left(\sqrt{\lambda}\,r\right)^{\beta_0}}\,u(x). \end{aligned}$$

Thus Lemma 4 is proved.

3.2.2 Fourier image of radial functions

Lemma 5. *Let $x \in \Omega$ and $\{\, y \in \boldsymbol{R}^n;\ |\,y-x\,| \le R\,\} \subset \Omega$. Suppose that G is a radial function in $L^2_{loc}(\boldsymbol{R}^n)$ and define $g_x^R(y)$ by*

$$g_x^R(y) = \begin{cases} G(\,|y-x|\,) & \text{for } |y-x| < R \\ 0 & \text{for } |y-x| \ge R. \end{cases}$$

Then $g_x^R \in L^2(\Omega)$ and for $t > 0$

$$\begin{aligned} \left(U g_x^R\right)_k(t) &= \overline{u_k(x,t)} \times \frac{\sqrt{2\pi}^{\,n}}{\sqrt{t}^{\,\beta_0}} \int_0^R G(r)\,J_{\beta_0}\left(\sqrt{t}\,r\right) r^{\frac{n}{2}}\,dr \\ &\quad + \frac{\pi}{2} \int_0^R G(r)\,r^{\frac{n}{2}} \int_{|y|<r} \overline{u_k(x-y,t)}\,V(x-y) \\ &\qquad \times \frac{Y_{\beta_0}\left(\sqrt{t}\,r\right) J_{\beta_0}\left(\sqrt{t}|y|\right) - J_{\beta_0}\left(\sqrt{t}\,r\right) Y_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\,dy\,dr. \end{aligned}$$

Proof. By (3.1.3) $\left(Ug_x^R\right)_k(t) = \int_\Omega g_x^R(y)\,\overline{u_k(y,t)}\,dy$. Thus we have

$$\begin{aligned}\left(Ug_x^R\right)_k(t) &= \int_{|y-x|<R} G(\,|y-x|\,)\,\overline{u_k(y,t)}\,dy \\ &= \int_0^R G(r)\,r^{n-1}\int_{S^{n-1}}\overline{u_k(x-rw,t)}\,\sigma(dw)\,dr. \qquad (3.2.10)\end{aligned}$$

On the other hand, by (3.1.2), $u_k(x,t)\in C^\infty(\Omega)$ and $(-\Delta+V)\,u_k(x,t) = t\,u_k(x,t)$ for $x\in\Omega$. Furthermore by Lemma 4, we have

$$\int_{S^{n-1}} u_k(x-rw,t)\,\sigma(dw) = \sqrt{2\pi}^{\,n}\,\frac{J_{\beta_0}\left(\sqrt{t}\,r\right)}{\left(\sqrt{t}\,r\right)^{\beta_0}}\times u_k(x,t)$$

$$+\frac{\pi}{2r^{\beta_0}}\int_{|y|<r} u_k(x-y,t)\,V(x-y)\,\frac{Y_{\beta_0}\left(\sqrt{t}\,r\right)J_{\beta_0}\left(\sqrt{t}|y|\right)-J_{\beta_0}\left(\sqrt{t}\,r\right)Y_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\,dy.$$

Combining last equation with (3.2.10) we get Lemma 5.

3.2.3 Lemma 6

Let Ω be an open domain in $\boldsymbol{R}^n$ and V be a nonnegative function in $L^\infty_{loc}(\Omega)$. Let $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta+V$, U be an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$, ρ be the measure and $\{u_k\}$ be the generalized eigenfunction system with respect to U. We shall use the following lemma later ([1: p.655] and [6: p.42]).

Lemma 6. *Under the assumptions above, if K is a compact set in Ω then*

$$\left(\sum_k\int_{N\le\sqrt{t}\le N+1}|u_k(x,t)|^2\rho(dt)\right)^{\frac{1}{2}}\le C_K\,(N+1)^{\frac{n-1}{2}},$$

where C_K is a constant independent of $N\ge 0$ and $x\in K$.

Corollary 2. *Under the assumptions above, if K is a compact set in Ω then*

$$\left(\sum_k\int_1^\lambda|u_k(x,t)|^2\,t^{\alpha-\frac{n}{2}}\rho(dt)\right)^{\frac{1}{2}}\le\begin{cases}C_{K,\alpha}\,\lambda^{\frac{\alpha}{2}} & \text{for}\quad \alpha>0\\ C_K\,(\log\lambda)^{\frac{1}{2}} & \text{for}\quad \alpha=0\\ C_{K,\alpha} & \text{for}\quad \alpha<0\end{cases}$$

and

$$\left(\sum_k\int_\lambda^\infty|u_k(x,t)|^2\,t^{-\alpha-\frac{n}{2}}\rho(dt)\right)^{\frac{1}{2}}\le C_{K,\alpha}\,\lambda^{-\frac{\alpha}{2}}\qquad\text{for}\quad\alpha>0,$$

where $C_{K,\alpha}$ and C_K are constants independent of $\lambda\ge 1$ and $x\in K$.

Our proofs depend strongly on Lemma 6. For the completeness we reproduce a proof of Lemma 6 following [6].

Proof of Lemma 6 We start by establishing the rough estimate

$$\left(\sum_k\int_{N\le\sqrt{t}\le N+1}|u_k(x,t)|^2\rho(dt)\right)^{\frac{1}{2}}\le C_K(N+1)^{\frac{n}{2}+2}. \qquad (3.2.11)$$

In order to prove (3.2.11), we shall establish a stronger result:

$$\left(\sum_k \int_0^\infty \frac{|u_k(x,t)|^2}{(1+t)^{\frac{n}{2}+2}} \rho(dt) \right)^{\frac{1}{2}} \leq C_K. \tag{3.2.12}$$

Let I be the unit operator and L be an elliptic operator defined by $L = -\Delta + V + I$. Let us put:

$$H(x,y) = \begin{cases} \dfrac{1}{(n-2)\,\omega_n} |y-x|^{2-n} & \text{for} \quad n > 2, \\ \dfrac{1}{2\,\pi} \log \dfrac{1}{|y-x|} & \text{for} \quad n = 2. \end{cases}$$

A function $G(x,y)$, which is continuous in the variables x and y in Ω with $y \neq x$, together with its first and second derivatives with respect to the y_i's, is called a Levi function if it satisfies the estimates of the following type for some $0 < \varepsilon \leq 1$:

$$\begin{aligned} G - H &= O\left(|y-x|^{\varepsilon+2-n}\right), \\ \frac{\partial[\,G-H\,]}{\partial y_i} &= O\left(|y-x|^{\varepsilon+1-n}\right), \\ \frac{\partial^2[\,G-H\,]}{\partial y_i \partial y_k} &= O\left(|y-x|^{\varepsilon-n}\right), \end{aligned}$$

where the bounds are required to be uniform in every bounded domain contained in Ω. A Levi function $G(x,y)$ which is a solution of the equation $L_y G = 0$ is called a fundamental solution of the equation $Lu = 0$. A function $K(x,y)$ continuous in the variables x and y for x and y in Ω and for $y \neq x$, is called a kernel of class N^α, with $\alpha < n$, if it satisfies the bound $K = O\left(|x-y|^{\alpha-n}\right)$ uniformly in every bounded domain contained in Ω.

Lemma 7. ([7: p.27,12,VII]) *Suppose that* $0 < \alpha < n$, $K \in N^\alpha$, $1 \leq p < n/\alpha$ *and* $f \in L^p_{loc}(\Omega)$. *If we put*

$$g(x) = \int_\Omega K(x,y)\, f(y)\, dy,$$

then we have $g \in L^q_{loc}(\Omega)$ *for* $n/q \geq n/p - \alpha$.

$K_1(x,y)$ denotes the fundamental solution for the operator L. Concerning the existence of the fundamental solution, refer to [7: p.64,19,VIII]. If $n = 2$, we have $K_1(x,\cdot) \in L^2_{loc}(\Omega)$. If $n > 2$, $K_1 \in N^2$ and $K_1(x,\cdot) \in L^p_{loc}(\Omega)$ for $1/p > 1 - 2/n$. Put

$$K_s(x,y) = \int_\Omega K_{s-1}(x,z)\, K_1(z,y)\, dz \qquad s = 2, 3, \ldots.$$

Then by Lemma 7 $K_s(x,\cdot) \in L^q_{loc}(\Omega)$ for $\frac{1}{q} = \frac{1}{p} - \frac{2}{n}(s-1)$. Let $s_0 = [n/4] + 1$ and $\frac{1}{p_0} = \frac{2}{n}\,(s_0 - 1) + \frac{1}{2}$. Then $1/p_0 > 1 - 2/n$. Therefore we have $K_{s_0}(x,\cdot) \in L^2_{loc}(\Omega)$.

We fix a compact subset K of Ω and a point x of K. Next, we fix an R such that $\{\, y; \mathrm{dist}(y,K) \leq R \,\} \subset \Omega$, and define a cut off function $\eta_x(y)$ which is unity in $|y-x| \leq R/2$, zero in $|y-x| \geq R$, and which is infinitely differentiable over $\boldsymbol{R}^n$. We consider the function $v_x(y) = K_{s_0}(x,y)\, \eta_x(y)$.

By (3.1.2), each $u_k(y,\lambda)$ is a regular solution of the equation

$$Lu_k(y,\lambda) = (\lambda+1)\,u_k(y,\lambda).$$

Therefore, applying successively Green's formula, we have

$$\begin{aligned}\int \overline{u_k(y,\lambda)}\,L^{s_0-1}v_x(y)\,dy \;&=\; (\lambda+1)\int \overline{u_k(y,\lambda)}\,L^{s_0-2}v_x(y)\,dy \\ =\cdots \;&=\; (\lambda+1)^{s_0-1}\int \overline{u_k(y,\lambda)}\,v_x(y)\,dy. \end{aligned} \tag{3.2.13}$$

Now we take into account that

$$L^{s_0-1}v_x(y) = \begin{cases} L^{s_0-1}K_{s_0}(x,y) = K_1(x,y) & \text{for} \quad |y-x| \le R/2, \\ 0 & \text{for} \quad |y-x| \ge R, \\ \in C^\infty & \text{for} \quad |y-x| > R/2. \end{cases}$$

Applying Green's formula once again, by the singularity of the fundamental solution and (3.2.13) we obtain

$$\begin{aligned}\int \overline{u_k(y,\lambda)}\,L^{s_0}v_x(y)\,dy \;&=\; \overline{u_k(x,\lambda)} + (\lambda+1)\int \overline{u_k(y,\lambda)}\,L^{s_0-1}v_x(y)\,dy \\ &=\; \overline{u_k(x,\lambda)} + (\lambda+1)^{s_0}\int \overline{u_k(y,\lambda)}\,v_x(y)\,dy, \end{aligned}$$

which implies

$$\frac{|u_k(x,\lambda)|}{(\lambda+1)^{s_0}} \le \left|\int v_x(y)\,\overline{u_k(y,\lambda)}\,dy\right| + \frac{1}{(\lambda+1)^{s_0}}\left|\int L^{s_0}v_x(y)\,\overline{u_k(y,\lambda)}\,dy\right|.$$

Thus

$$\begin{aligned}\sum_k \int_{e_k} \frac{|u_k(x,t)|^2}{(t+1)^{2s_0}}\,\rho(dt) \le 2\sum_k \int_{e_k} \left|\int v_x(y)\,\overline{u_k(y,t)}\,dy\right|^2 \rho(dt) \\ +2\sum_k \int_{e_k} \left|\int L^{s_0}v_x(y)\,\overline{u_k(y,t)}\,dy\right|^2 \rho(dt). \end{aligned} \tag{3.2.14}$$

The iterated kernel $K_{s_0}(x,\cdot)$ has a square-integrable singularity, and therefore we have $v_x \in L^2(\Omega)$. On the other hand, $L^{s_0}v_x(y)$ becomes zero in $|y-x| \le R/2$ and $|y-x| \ge R$ and belongs to $L^2(\Omega)$. By (3.1.1) and (3.1.3) we have

$$\sum_k \int_{e_k} \left|\int_\Omega v_x(y)\,\overline{u_k(y,t)}\,dy\right|^2 \rho(dt) = \int_\Omega |v_x(y)|^2\,dy \tag{3.2.15}$$

and

$$\sum_k \int_{e_k} \left|\int_\Omega L^s v_x(y)\,\overline{u_k(y,t)}\,dy\right|^2 \rho(dt) = \int_\Omega |L^s v_x(y)|^2\,dy. \tag{3.2.16}$$

Thus by (3.2.15) and (3.2.16), the left-hand side of (3.2.14) is bounded for any $x \in K$.

Since $2s = 2[N/4]+2 \le N/2+2$, the above reasoning leads to inequality (3.2.12) and, consequently, to (3.2.11).

We fix a compact subset K of Ω, a point x of K and a small $R > 0$ so that $\{\, y; \mathrm{dist}(y, K) \le R \,\} \subset \Omega$. Put

$$w_x^{R,N}(y) = \begin{cases} W^N(|y-x|) & \text{for } R/2 < |y-x| < R, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$W^N(r) = \frac{\sqrt{N}^n J_{\frac{n}{2}-1}(Nr)}{\sqrt{2\pi}^n r^{\frac{n}{2}-1}}.$$

According to Lemma 5, we have

$$\left(U w_x^{R,N}\right)_k (t) = \overline{w}_k^{R,N}(x,t) + \tilde{w}_k^{R,N}(x,t),$$

where

$$\overline{w}_k^{R,N}(x,t) = \overline{u_k(x,t)} \times \frac{\sqrt{N}^n}{\sqrt{t}^{\frac{n}{2}-1}} \int_{\frac{R}{2}}^{R} r\, J_{\frac{n}{2}-1}(Nr)\, J_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) dr,$$

$$\tilde{w}_k^{R,N}(x,t) = \frac{\pi \sqrt{N}^n}{2\sqrt{2\pi}^n} \int_{\frac{R}{2}}^{R} r\, J_{\frac{n}{2}-1}(Nr)\, dr \int_{|y|<r} \overline{u_k(x-y,t)}\, V(x-y)\, v(t,r,|y|)\, dy$$

and

$$v(t,r,\tau) = \frac{Y_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) J_{\frac{n}{2}-1}\left(\sqrt{t}\, \tau\right) - J_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) Y_{\frac{n}{2}-1}\left(\sqrt{t}\, \tau\right)}{\tau^{\frac{n}{2}-1}}.$$

Let N_0 be a sufficiently large but a fixed number. There is a constant $\alpha > 0$ such that for all N and t with $N_0 < N \le \sqrt{t} \le N+1$, the inequality

$$\left| \frac{\sqrt{N}^n}{\sqrt{t}^{\frac{n}{2}-1}} \int_{\frac{R}{2}}^{R} r\, J_{\frac{n}{2}-1}(Nr)\, J_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) dr \right| \ge \alpha \tag{3.2.17}$$

holds. On the other hand, applying (3.1.1), the estimate

$$\begin{aligned} \sum_k \int_0^\infty \left| \left(U w_x^{R,N}\right)_k (t) \right|^2 \rho(dt) &= \int_\Omega \left| w_x^{R,N}(y) \right|^2 dy \\ &= \frac{N^n \omega_n}{(2\pi)^n} \int_{\frac{R}{2}}^{R} r\, J_{\frac{n}{2}-1}^2(Nr)\, dr = O\left(N^{n-1}\right) \end{aligned} \tag{3.2.18}$$

is valid. Making use of (3.2.17) and (3.2.18), we obtain

$$\begin{aligned} &\sum_k \int_{N \le \sqrt{t} \le N+1} |u_k(x,t)|^2 \rho(dt) \le \frac{1}{\alpha^2} \sum_k \int_{N \le \sqrt{t} \le N+1} \left| \overline{w}_k^{R,N}(x,t) \right|^2 \rho(dt) \\ &\le \frac{2}{\alpha^2} \sum_k \int_0^\infty \left| \left(U w_x^{R,N}\right)_k (t) \right|^2 \rho(dt) + \frac{2}{\alpha^2} \sum_k \int_{N \le \sqrt{t} \le N+1} \left| \tilde{w}_k^{R,N}(x,t) \right|^2 \rho(dt) \\ &= O\left(N^{n-1}\right) + \frac{2}{\alpha^2} \sum_k \int_{N \le \sqrt{t} \le N+1} \left| \tilde{w}_k^{R,N}(x,t) \right|^2 \rho(dt). \end{aligned} \tag{3.2.19}$$

To prove Lemma 6, we show that if the estimate

$$\sum_k \int_{N\le\sqrt{t}\le N+1} |u_k(x,t)|^2 \rho(dt) = O(N^s) \tag{3.2.20}$$

holds uniformly with respect to x in a compact subset K of Ω for an $s \ge n-1$, then the estimate

$$\sum_k \int_{N\le\sqrt{t}\le N+1} |u_k(x,t)|^2 \rho(dt) = O\left(N^{\max(n-1,s-4)}\right)$$

holds uniformly with respect to x in any closed set K' in K such that dist$(K', K^c) > 0$. To this end it suffices to show, by virtue of inequality (3.2.19), that if (3.2.20) is valid uniformly with respect to x in an arbitrary compact subset K of Ω for a certain $s \ge n-1$, then the estimate

$$\sum_k \int_{N\le\sqrt{t}\le N+1} \left|\tilde{w}_k^{R,N}(x,t)\right|^2 \rho(dt) = O\left(N^{n-1}\right) + O\left(N^{s-4}\right)$$

holds uniformly with respect to x in an arbitrary closed subset K' of K such that dist$(K', K^c) > 0$. Fix $0 < R <$dist(K', K^c). Then we have

$$\begin{aligned}
&\sum_k \int_{N\le\sqrt{t}\le N+1} \left|\tilde{w}_k^{R,N}(x,t)\right|^2 \rho(dt) \\
&\le \frac{\pi^2}{2}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \left| \int_{\frac{R}{2}}^R r\, J_{\frac{n}{2}-1}(Nr)\, dr \right.\\
&\qquad \left. \times \int_0^{\frac{1}{N}} v(t,r,\tau)\, \tau^{n-1} d\tau \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\, \sigma(d\omega) \right|^2 \\
&+ \frac{\pi^2}{2}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \left| \int_{\frac{R}{2}}^R r\, J_{\frac{n}{2}-1}(Nr)\, dr \right.\\
&\qquad \left. \times \int_{\frac{1}{N}}^r v(t,r,\tau)\, \tau^{n-1} d\tau \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\, \sigma(d\omega) \right|^2 \\
&= K_1^{R,N}(x) + K_2^{R,N}(x), \qquad \text{say.}
\end{aligned} \tag{3.2.21}$$

It suffices to show that $K_1^{R,N}$ is of order $O(N^{s-4})$, and $K_2^{R,N}$ is of order $O(N^{n-1})$.

First, we estimate $K_1^{R,N}(x)$. We have, by virtue of Schwarz's inequality (also taking into account that $V \in L^\infty_{loc}(\Omega)$),

$$\begin{aligned}
K_1^{R,N}(x) &\le \frac{\pi^2}{2}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt)\, \frac{R}{2} \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\, dr \\
&\quad \times \left| \int_0^{\frac{1}{N}} v(t,r,\tau)\, \tau^{n-1} d\tau \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\, \sigma(d\omega) \right|^2 \\
&\le \frac{\pi^2 R}{4}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\, dr \\
&\quad \times \frac{1}{N} \int_0^{\frac{1}{N}} v^2(t,r,\tau)\, \tau^{2(n-1)} d\tau \left| \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\, \sigma(d\omega) \right|^2
\end{aligned}$$

$$\le \frac{\pi^2 R}{4N}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\,dr$$
$$\times \int_0^{\frac{1}{N}} v^2(t,r,\tau)\,\tau^{2(n-1)} d\tau\, \omega_n ||V||^2_{L^\infty(K)} \int_{S^{n-1}} |u_k(x-\tau\omega,t)|^2\,\sigma(d\omega).$$

Since

$$v^2(t,r,\tau) \le 2\,\tau^{2-n} Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\,r\right) J^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) + 2\,\tau^{2-n} J^2_{\frac{n}{2}-1}\left(\sqrt{t}\,r\right) Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right),$$

we obtain

$$K_1^{R,N}(x) \le C_{V,K} N^{n-1} \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\, Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\,r\right) dr$$
$$\times \int_0^{\frac{1}{N}} \tau^n\, J^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) d\tau \int_{S^{n-1}} |u_k(x-\tau\omega,t)|^2\,\sigma(d\omega)$$
$$+ C_{V,K} N^{n-1} \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\, J^2_{\frac{n}{2}-1}\left(\sqrt{t}\,r\right) dr$$
$$\times \int_0^{\frac{1}{N}} \tau^n\, Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) d\tau \int_{S^{n-1}} |u_k(x-\tau\omega,t)|^2\,\sigma(d\omega). \qquad (3.2.22)$$

The estimates

$$J_\nu^2(Nr) \le C(Nr)^{-1}, \quad Y_\nu^2\left(\sqrt{t}\,r\right) \le C(Nr)^{-1} \text{ and } J_\nu^2\left(\sqrt{t}\,r\right) \le C(Nr)^{-1} \qquad (3.2.23)$$

hold for all $N \le \sqrt{t} \le N+1$ and for all $R/2 \le r \le R$, whereas the estimates

$$J^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \le C(N\tau)^{n-2} \qquad \text{and} \qquad Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \le C(N\tau)^{2-n} \qquad (3.2.24)$$

hold for all $0 < \tau \le 1/N$. Invoking (3.2.23) and (3.2.24) to estimating the right-hand side of (3.2.22), we obtain

$$K_1^{R,N} \le C_{V,K,R}\, N^{2n-5} \int_0^{\frac{1}{N}} \tau^{2n-2} d\tau \int_{S^{n-1}} \sigma(d\omega) \sum_k \int_{N\le\sqrt{t}\le N+1} |u_k(x-\tau\omega,t)|^2 \rho(dt)$$
$$+ C_{V,K,R}\, N^{-1} \int_0^{\frac{1}{N}} \tau^2 d\tau \int_{S^{n-1}} \sigma(d\omega) \sum_k \int_{N\le\sqrt{t}\le N+1} |u_k(x-\tau\omega,t)|^2 \rho(dt). \qquad (3.2.25)$$

By (3.2.20), the right-hand side of (3.2.25) does not exceed $C_{V,K,R}\, N^{s-4}$.

Next, we estimate $K_2^{R,N}(x)$. By Schwarz's inequality we have

$$K_2^{R,N}(x) \le \frac{\pi^2}{2}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt)\, \frac{R}{2} \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\,dr$$
$$\times \left| \int_{\frac{1}{N}}^r v(t,r,\tau)\,\tau^{n-1} d\tau \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega) \right|^2$$
$$\le \frac{\pi^2 R}{2}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \int_{\frac{R}{2}}^R r^2 J^2_{\frac{n}{2}-1}(Nr)\, Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\,r\right) dr$$
$$\times \left| \int_{\frac{1}{N}}^r \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) d\tau \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega) \right|^2$$

$$+\frac{\pi^2 R}{2}\left(\frac{N}{2\pi}\right)^n \sum_k \int_{N\le\sqrt{t}\le N+1} \rho(dt) \int_{\frac{R}{2}}^{R} r^2 J^2_{\frac{n}{2}-1}(Nr)\, J^2_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) dr$$

$$\times \left| \int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) d\tau \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\right|^2. \quad (3.2.26)$$

Carrying out integration by parts with respect to τ, we get

$$\int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\, d\tau$$
$$= \int_{\frac{1}{N}}^{r} \tau^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \frac{d}{d\tau}\left\{ \int_{\frac{1}{N}}^{\tau} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\right\} d\tau$$
$$= r^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) \int_{\frac{1}{N}}^{r} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)$$
$$- \int_{\frac{1}{N}}^{r} \frac{d}{d\tau}\left\{ \tau^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\} \int_{\frac{1}{N}}^{\tau} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\, d\tau$$

and

$$\int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\, d\tau$$
$$= r^{\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) \int_{\frac{1}{N}}^{r} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)$$
$$- \int_{\frac{1}{N}}^{r} \frac{d}{d\tau}\left\{ \tau^{\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\} \int_{\frac{1}{N}}^{\tau} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\, d\tau.$$

From these equalities, with Schwarz's inequality, we obtain

$$\left| \int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\, d\tau\right|^2$$
$$\le 2\, r J^2_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) \left| \int_{\frac{1}{N}}^{r} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\right|^2$$
$$+2 \int_{\frac{1}{N}}^{r} \left[\frac{d}{d\tau}\left\{ \tau^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\}\right]^2 d\tau$$
$$\times \int_{\frac{1}{N}}^{r} \left| \int_{\frac{1}{N}}^{\tau} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\right|^2 d\tau \quad (3.2.27)$$

and

$$\left| \int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\, d\tau\right|^2$$
$$\le 2\, r Y^2_{\frac{n}{2}-1}\left(\sqrt{t}\, r\right) \left| \int_{\frac{1}{N}}^{r} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\right|^2$$
$$+2 \int_{\frac{1}{N}}^{r} \left[\frac{d}{d\tau}\left\{ \tau^{\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\}\right]^2 d\tau$$
$$\times \int_{\frac{1}{N}}^{r} \left| \int_{\frac{1}{N}}^{\tau} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega)\right|^2 d\tau. \quad (3.2.28)$$

We remark now that

$$\int_{\frac{1}{N}}^{r} s^{\frac{n-1}{2}} ds \int_{S^{n-1}} \overline{u_k(x-s\omega,t)}\, V(x-s\omega)\,\sigma(d\omega) = \int_{\frac{1}{N}<|y-x|<r} \frac{\overline{u_k(y,t)}\, V(y)}{|y-x|^{\frac{n-1}{2}}}\, dy. \quad (3.2.29)$$

Next from the recurrence formulas $[t^\nu J_\nu(t)]' = t^\nu J_{\nu-1}(t)$ and $[t^\nu Y_\nu(t)]' = t^\nu Y_{\nu-1}(t)$, it follows

$$\frac{d}{d\tau}\left\{\tau^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\} = \frac{d}{d\tau}\left\{\tau^{\frac{1}{2}-\left(\frac{n}{2}-1\right)}\left[\tau^{\frac{n}{2}-1} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right]\right\}$$
$$= \sqrt{t}\,\tau^{\frac{1}{2}} J_{\frac{n}{2}-2}\left(\sqrt{t}\,\tau\right) - \frac{n-3}{2}\,\tau^{-\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)$$

and

$$\frac{d}{d\tau}\left\{\tau^{\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\} = \sqrt{t}\,\tau^{\frac{1}{2}} Y_{\frac{n}{2}-2}\left(\sqrt{t}\,\tau\right) - \frac{n-3}{2}\,\tau^{-\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right).$$

From (3.2.23) we obtain

$$\left[\frac{d}{d\tau}\left\{\tau^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\}\right]^2 \le 2\,(N+1)^2 \tau J^2_{\frac{n}{2}-2}\left(\sqrt{t}\,\tau\right) + 2\left(\frac{n-3}{2}\right)^2 \tau^{-1} J^2_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)$$
$$\le C\left(N + N^{-1}\tau^{-2}\right) \le CN$$

and

$$\left[\frac{d}{d\tau}\left\{\tau^{\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\}\right]^2 \le CN$$

for $N \le \sqrt{t} \le N+1$ and $\tau \ge 1/N$. Therefore

$$\int_{\frac{1}{N}}^{r}\left[\frac{d}{d\tau}\left\{\tau^{\frac{1}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\}\right]^2 d\tau \le CNr \quad (3.2.30)$$

and

$$\int_{\frac{1}{N}}^{r}\left[\frac{d}{d\tau}\left\{\tau^{\frac{1}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right)\right\}\right]^2 d\tau \le CNr. \quad (3.2.31)$$

From relations (3.2.27)-(3.2.31) and (3.2.23) it follows that

$$\left|\int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\, d\tau\right|^2$$
$$\le CN^{-1}\left|\int_{\frac{1}{N}<|y-x|<r} \frac{\overline{u_k(y,t)}\, V(y)}{|y-x|^{\frac{n-1}{2}}} dy\right|^2$$
$$+CNr\int_{\frac{1}{N}}^{r}\left|\int_{\frac{1}{N}<|y-x|<\tau} \frac{\overline{u_k(y,t)}\, V(y)}{|y-x|^{\frac{n-1}{2}}} dy\right|^2 d\tau \quad (3.2.32)$$

and

$$\left|\int_{\frac{1}{N}}^{r} \tau^{\frac{n}{2}} Y_{\frac{n}{2}-1}\left(\sqrt{t}\,\tau\right) \int_{S^{n-1}} \overline{u_k(x-\tau\omega,t)}\, V(x-\tau\omega)\,\sigma(d\omega)\, d\tau\right|^2$$
$$\le CN^{-1}\left|\int_{\frac{1}{N}<|y-x|<r} \frac{\overline{u_k(y,t)}\, V(y)}{|y-x|^{\frac{n-1}{2}}} dy\right|^2$$
$$+CNr\int_{\frac{1}{N}}^{r}\left|\int_{\frac{1}{N}<|y-x|<\tau} \frac{\overline{u_k(y,t)}\, V(y)}{|y-x|^{\frac{n-1}{2}}} dy\right|^2 d\tau \quad (3.2.33)$$

for $N \le \sqrt{t} \le N+1$ and $R/2 \le r \le R$. Applying (3.2.32), (3.2.33) and (3.2.23) to the right-hand side of (3.2.26) we get

$$K_2^{R,N}(x) \le CN^{n-3}\int_{\frac{R}{2}}^{R} dr \sum_k \int_0^\infty \rho(dt)\left|\int_{\frac{1}{N}<|y-x|<r} \frac{\overline{u_k(y,t)}\,V(y)}{|y-x|^{\frac{n-1}{2}}}\,dy\right|^2$$
$$+CN^{n-1}\int_{\frac{R}{2}}^{R} r\,dr\int_{\frac{1}{N}}^{r} d\tau \sum_k \int_0^\infty \rho(dt)\left|\int_{\frac{1}{N}<|y-x|<\tau} \frac{\overline{u_k(y,t)}\,V(y)}{|y-x|^{\frac{n-1}{2}}}\,dy\right|^2. \quad (3.2.34)$$

Since for any point x of K' and for any $\tau \le r \le R$ the function defined by

$$f_x^N(y) = \begin{cases} \dfrac{V(y)}{|y-x|^{\frac{n-1}{2}}} & \text{for } \frac{1}{N} < |y-x| < r \\ 0 & \text{otherwise,} \end{cases}$$

belongs to L^2 and the L^2-norm is bounded by the same constant for $x \in K'$. We have

$$\sum_k \int_0^\infty \left|\int_{\frac{1}{N}<|y-x|<r} \frac{\overline{u_k(y,t)}\,V(y)}{|y-x|^{\frac{n-1}{2}}}\,dy\right|^2 \rho(dt) = \int_\Omega \left|f_x^N(y)\right|^2 dy \le \omega_n R\,||V||^2_{L^\infty(K)}.$$

Thus the inequality (3.2.34) implies the estimate

$$K_2^{R,N}(x) \le C_{V,K}N^{n-3}\int_{\frac{R}{2}}^{R} dr + C_{V,K}N^{n-1}\int_{\frac{R}{2}}^{R} r\,dr\int_{\frac{1}{N}}^{r} d\tau \le C_{V,K}N^{n-1}$$

uniformly in $x \in K'$.

Thus the proof of Lemma 6 is complete.

3.2.4 Decomposition of $s_\lambda^\delta f$

Lemma 8. *Let $\delta \ge 0$. Let $f \in L^2(\Omega)$, $x \in \Omega$ and suppose $\{\, y;\ |\, y - x \,| \le R \} \subset \Omega$. Then*

$$s_\lambda^\delta f(x) = \frac{2^\delta \Gamma(\delta+1)}{\sqrt{2\pi}^{\,n}\sqrt{\lambda}^{\,\delta-\frac{n}{2}}}\int_{|y|<R} \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}|y|\right)}{|y|^{\frac{n}{2}+\delta}} f(x-y)\,dy$$
$$+\frac{2^\delta \Gamma(\delta+1)}{\sqrt{\lambda}^{\,\delta-\frac{n}{2}}}\sum_k \int_0^\infty \frac{I_\lambda^{\delta,R}(t)}{\sqrt{t}^{\,\beta_0}}\,u_k(x,t)\,(Uf)_k(t)\,\rho(dt)$$
$$+\frac{2^{\delta-\frac{n}{2}-1}\Gamma(\delta+1)}{\pi^{\beta_0}\sqrt{\lambda}^{\,\delta-\frac{n}{2}}}\sum_k \int_0^\infty (Uf)_k(t)\,\rho(dt)\int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\int_{|y|<r} V(x-y)$$
$$\times\, u_k(x-y,t)\,\frac{J_{\beta_0}\left(\sqrt{t}\,r\right)Y_{\beta_0}\left(\sqrt{t}|y|\right) - Y_{\beta_0}\left(\sqrt{t}\,r\right)J_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\,dy\,dr$$

where

$$I_\lambda^{\delta,R}(t) = \int_R^\infty \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)J_{\beta_0}\left(\sqrt{t}\,r\right)}{r^\delta}\,dr.$$

Proof. Put

$$g_{\lambda,x}^{\delta,R}(y) = \begin{cases} G_\lambda^\delta(\,|y-x|\,) & \text{for} \quad |y-x| < R, \\ 0 & \text{for} \quad |y-x| \geq R, \end{cases}$$

where

$$G_\lambda^\delta(r) = \frac{2^\delta \Gamma(\delta+1)}{\sqrt{2\pi}^{\,n} \sqrt{\lambda}^{\,\delta-\frac{n}{2}}} \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right)}{r^{\frac{n}{2}+\delta}} \qquad \text{for} \quad r > 0.$$

Then by Lemma 5 we have

$$\begin{aligned}\left(Ug_{\lambda,x}^{\delta,R}\right)_k(t) &= \overline{u_k(x,t)} \times \frac{2^\delta \Gamma(\delta+1)}{\sqrt{\lambda}^{\,\delta-\frac{n}{2}} \sqrt{t}^{\,\beta_0}} \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right) J_{\beta_0}\left(\sqrt{t}\, r\right)}{r^\delta}\, dr \\ &+ \frac{2^{\delta-\frac{n}{2}-1}\Gamma(\delta+1)}{\pi^{\beta_0}\sqrt{\lambda}^{\,\delta-\frac{n}{2}}} \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \int_{|y|<r} \overline{u_k(x-y,t)}\, V(x-y) \\ &\times \frac{Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}|y|\right) - J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\, dy\, dr. \qquad (3.2.35)\end{aligned}$$

By the formula (see [2: p.92,(34)])

$$\frac{2^\delta \Gamma(\delta+1)}{\sqrt{\lambda}^{\,\delta-\frac{n}{2}} \sqrt{t}^{\,\beta_0}} \int_0^\infty \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right) J_{\beta_0}\left(\sqrt{t}\, r\right)}{r^\delta}\, dr = \left(1 - \frac{t}{\lambda}\right)_+^\delta,$$

the first term on the right-hand side of (3.2.36) equals

$$\left(1 - \frac{t}{\lambda}\right)_+^\delta \overline{u_k(x,t)} - \frac{2^\delta \Gamma(\delta+1)}{\sqrt{\lambda}^{\,\delta-\frac{n}{2}} \sqrt{t}^{\,\beta_0}}\, I_\lambda^{\delta,R}(t)\, \overline{u_k(x,t)}. \qquad (3.2.36)$$

By (3.2.36) and (3.2.37) we have

$$\begin{aligned}\left(1 - \frac{t}{\lambda}\right)_+^\delta u_k(x,t) \;&=\; \overline{\left(Ug_{\lambda,x}^{\delta,R}\right)_k(t)} + \frac{2^\delta \Gamma(\delta+1)}{\sqrt{\lambda}^{\,\delta-\frac{n}{2}} \sqrt{t}^{\,\beta_0}}\, I_\lambda^{\delta,R}(t)\, u_k(x,t) \\ &+\; \frac{2^{\delta-\frac{n}{2}-1}\Gamma(\delta+1)}{\pi^{\beta_0}\sqrt{\lambda}^{\,\delta-\frac{n}{2}}} \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \int_{|y|<r} u_k(x-y,t)\, V(x-y) \\ &\times\; \frac{J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}|y|\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\, dy\, dr.\end{aligned}$$

Let $\{e_k\}$ be the set of multiplicity with respect to U. Then by (3.1.5)

$$\begin{aligned}s_\lambda^\delta f(x) &= \sum_k \int_0^\infty \left(1 - \frac{t}{\lambda}\right)_+^\delta u_k(x,t)\, (Uf)_k(t)\, \rho(dt) \\ &= \sum_k \int_{e_k} \overline{\left(Ug_{\lambda,x}^{\delta,R}\right)_k(t)}\, (Uf)_k(t)\, \rho(dt) \\ &+ \frac{2^\delta \Gamma(\delta+1)}{\sqrt{\lambda}^{\,\delta-\frac{n}{2}}} \sum_k \int_0^\infty \frac{I_\lambda^{\delta,R}(t)}{\sqrt{t}^{\,\beta_0}}\, u_k(x,t)\, (Uf)_k(t)\, \rho(dt)\end{aligned}$$

$$+\frac{2^{\delta-\frac{n}{2}-1}\Gamma(\delta+1)}{\pi^{\beta_0}\sqrt{\lambda}^{\delta-\frac{n}{2}}}\sum_k\int_0^\infty (Uf)_k(t)\,\rho(dt)\int_0^R\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\int_{|y|<r}u_k(x-y,t)$$
$$\times V(x-y)\,\frac{J_{\beta_0}\left(\sqrt{t}\,r\right)Y_{\beta_0}\left(\sqrt{t}|y|\right)-Y_{\beta_0}\left(\sqrt{t}\,r\right)J_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\,dy\,dr. \quad (3.2.37)$$

Since U preserves the inner products ((3.1.1)), we have

$$\sum_k\int_{e_k}\overline{\left(Ug_{\lambda,x}^{\delta,R}\right)_k(t)}\,(Uf)_k(t)\,\rho(dt)=\int_\Omega\overline{g_{\lambda,x}^{\delta,R}(y)}\,f(y)\,dy$$
$$=\int_{|y-x|<R}\overline{G_\lambda^\delta(\,|y-x|\,)}\,f(y)\,dy$$
$$=\frac{2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n\sqrt{\lambda}^{\delta-\frac{n}{2}}}\int_{|y|<R}\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}|y|\right)}{|y|^{\frac{n}{2}+\delta}}f(x-y)\,dy. \quad (3.2.38)$$

By (3.2.38) and (3.2.39), we get Lemma 8.

3.2.5 Decomposition of $s_\lambda^\delta f-f$

To prove Propositions 1 and 2, we shall use the following representation of $s_\lambda^\delta f-f$:

Lemma 9. *Assume* $\delta>\max\,(0,(n-3)/2)$. *Let* $f\in L^2(\Omega)$ *and* $x\in\Omega$. *We choose* $R>0$ *so that* $\{\,y\in\boldsymbol{R}^n;\,|\,y-x\,|\le R\,\}\subset\Omega$. *Then*

$$s_\lambda^\delta f(x)-f(x)=\frac{2^\delta\Gamma(\delta+1)}{\sqrt{\lambda}^{\delta-\frac{n}{2}}}\sum_k\int_0^\infty\frac{I_\lambda^{\delta,R}(t)}{\sqrt{t}^{\beta_0}}\,u_k(x,t)\,(Uf)_k(t)\,\rho(dt)$$
$$-f(x)\times\frac{\omega_n2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n\sqrt{\lambda}^{\delta-\frac{n}{2}}}\int_R^\infty\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr$$
$$+\frac{\omega_n2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n\sqrt{\lambda}^{\delta-\frac{n}{2}}}\int_0^R\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\left[\frac{1}{\omega_n}\int_{S^{n-1}}f(x-rw)\,\sigma(dw)-f(x)\right]dr$$
$$+\frac{2^{\delta-\frac{n}{2}-1}\Gamma(\delta+1)}{\pi^{\beta_0}\sqrt{\lambda}^{\delta-\frac{n}{2}}}\sum_k\int_0^\infty(Uf)_k(t)\,\rho(dt)\int_0^R\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\int_{|y|<r}V(x-y)$$
$$\times u_k(x-y,t)\,\frac{J_{\beta_0}\left(\sqrt{t}\,r\right)Y_{\beta_0}\left(\sqrt{t}|y|\right)-Y_{\beta_0}\left(\sqrt{t}\,r\right)J_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\,dy\,dr.$$

Proof. Let $\delta>(n-3)/2$. We have (see [2: p.49,(19)])

$$\frac{\omega_n2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n\sqrt{\lambda}^{\delta-\frac{n}{2}}}\int_0^\infty\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr=1.$$

Therefore

$$\frac{2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n\sqrt{\lambda}^{\delta-\frac{n}{2}}}\int_{|y|<R}\frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}|y|\right)}{|y|^{\frac{n}{2}+\delta}}f(x-y)\,dy-f(x)$$

$$= \frac{\omega_n 2^\delta \Gamma(\delta+1)}{\sqrt{2\pi}^n \sqrt{\lambda}^{\delta-\frac{n}{2}}} \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}} \left[\frac{1}{\omega_n}\int_{S^{n-1}} f(x-rw)\,\sigma(dw) - f(x)\right] dr$$

$$-f(x) \times \frac{\omega_n 2^\delta \Gamma(\delta+1)}{\sqrt{2\pi}^n \sqrt{\lambda}^{\delta-\frac{n}{2}}} \int_R^\infty \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\, dr. \qquad (3.2.39)$$

Lemma 9 follows from (3.2.39) and Lemma 8.

3.3 Estimate of $V_\lambda^{\delta,R} f$

Let Ω be an open domain in $\boldsymbol{R}^n$ and $f \in L^1_{loc}(\Omega)$. Let $\beta_0 = \frac{n}{2} - 1$, $\delta > (n-3)/2$ and $R > 0$. Put

$$V_\lambda^{\delta,R} f(x) = f(x) \times \sqrt{\lambda}^{\frac{n}{2}-\delta} \int_R^\infty \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\, dr.$$

By (3.1.12), there exists a constant $C_{\delta,R}$ such that

$$\left| \int_R^\infty \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\, dr \right| \leq C_{\delta,R} \sqrt{\lambda}^{-\frac{3}{2}}.$$

Therefore

$$\left| V_\lambda^{\delta,R} f(x) \right| \leq C_{\delta,R} \sqrt{\lambda}^{\frac{n-3}{2}-\delta} |f(x)|.$$

Let K be any compact set in Ω and $f \in L^p(K)$ where $0 < p \leq \infty$. Then we have

$$\left\| V_\lambda^{\delta,R} f \right\|_{L^p(K)} \leq C_{\delta,R} \sqrt{\lambda}^{\frac{n-3}{2}-\delta} ||f||_{L^p(K)}. \qquad (3.3.1)$$

3.4 Estimate of $w_\lambda^{\delta,R} f$

We use notations in §3.2. Let $\delta > (n-3)/2$, $R > 0$ and $\beta_0 = \frac{n}{2} - 1$. For $f \in L^2(\Omega)$ and $x \in \Omega$, we define

$$w_\lambda^{\delta,R} f(x) = \sqrt{\lambda}^{\frac{n}{2}-\delta} \sum_k \int_0^\infty \frac{I_\lambda^{\delta,R}(t)}{\sqrt{t}^{\beta_0}}\, u_k(x,t)\,(Uf)_k(t)\,\rho(dt),$$

where

$$I_\lambda^{\delta,R}(t) = \int_R^\infty \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right) J_{\beta_0}\left(\sqrt{t}\,r\right)}{r^\delta}\, dr.$$

Lemma 10. *If K is a compact set in Ω, then*

$$\left\| w_\lambda^{\delta,R} f \right\|_{L^\infty(K)} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) \qquad as \quad \lambda \to \infty.$$

Proof. We shall use the following estimates (see [8: p.202, Lemma 18.10 a]) to prove Lemma 10: For $\delta > 0$

$$\left|I_\lambda^{\delta,R}(t)\right| \le C_{\delta,R}\sqrt{\lambda}^{-\frac{1}{2}}\sqrt{t}^{-\frac{1}{2}} \qquad (\,t,\lambda > 0\,), \qquad (3.4.1)$$

$$\left|I_\lambda^{\delta,R}(t)\right| \le C_{\delta,R}\frac{\sqrt{\lambda}^{-\frac{3}{2}}\sqrt{t}^{\frac{1}{2}}}{\sqrt{\lambda}-\sqrt{t}} + C_{\delta,R}\sqrt{\lambda}^{-\frac{3}{2}}\sqrt{t}^{-\frac{1}{2}} \qquad (\,0 < t < \lambda\,), \qquad (3.4.2)$$

$$\left|I_\lambda^{\delta,R}(t)\right| \le C_{\delta,R}\frac{\sqrt{\lambda}^{\frac{1}{2}}\sqrt{t}^{-\frac{3}{2}}}{\sqrt{t}-\sqrt{\lambda}} + C_{\delta,R}\sqrt{\lambda}^{-\frac{1}{2}}\sqrt{t}^{-\frac{3}{2}} \qquad (\,0 < \lambda < t\,). \qquad (3.4.3)$$

We have

$$\begin{aligned} w_\lambda^{\delta,R}f(x) &= \sqrt{\lambda}^{\frac{n}{2}-\delta}\sum_k\int_0^\infty \frac{I_\lambda^{\delta,R}(t)}{\sqrt{t}^{\beta_0}}\, u_k(x,t)\,(Uf)_k(t)\,\rho(dt) \\ &= \int_{0\le\sqrt{t}\le\sqrt{\lambda}-1} + \int_{|\sqrt{t}-\sqrt{\lambda}|<1} + \int_{\sqrt{t}\ge\sqrt{\lambda}+1} = I_{\lambda,1}^{\delta,R}f(x) + I_{\lambda,2}^{\delta,R}f(x) + I_{\lambda,3}^{\delta,R}f(x), \end{aligned}$$

say. Let E be the spectral measure corresponding to $\hat{A}$.

3.4.1 Estimate of $I_{\lambda,2}^{\delta,R}f$

We have by (3.4.1)

$$\begin{aligned} \left|I_{\lambda,2}^{\delta,R}f(x)\right| &\le \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta-\frac{n-1}{2}}}\sum_k\int_{|\sqrt{t}-\sqrt{\lambda}|\le 1}\frac{|(Uf)_k(t)\,u_k(x,t)|}{\sqrt{t}^{\frac{n-1}{2}}}\,\rho(dt) \\ &\le \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta}}\sum_k\int_{|\sqrt{t}-\sqrt{\lambda}|\le 1}|(Uf)_k(t)\,u_k(x,t)|\,\rho(dt) \\ &\le \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta}}\left(\sum_k\int_{e_k\cap\{|\sqrt{t}-\sqrt{\lambda}|\le 1\}}|(Uf)_k(t)|^2\rho(dt)\right)^{\frac{1}{2}} \\ &\quad\times\left(\sum_k\int_{|\sqrt{t}-\sqrt{\lambda}|\le 1}|u_k(x,t)|^2\rho(dt)\right)^{\frac{1}{2}}. \end{aligned}$$

Now by (3.1.1) and (3.1.4) we have

$$\left(\sum_k\int_{e_k\cap\{|\sqrt{t}-\sqrt{\lambda}|\le 1\}}|(Uf)_k(t)|^2\rho(dt)\right)^{\frac{1}{2}} = \left\|\,E\left(\left\{t;\left|\sqrt{t}-\sqrt{\lambda}\right|\le 1\right\}\right)f\,\right\|_{L^2(\Omega)} \longrightarrow 0$$

as $\lambda\to\infty$. By Lemma 6, there exists a constant C_K such that

$$\left(\sum_k\int_{|\sqrt{t}-\sqrt{\lambda}|\le 1}|u_k(x,t)|^2\rho(dt)\right)^{\frac{1}{2}} \le C_K\sqrt{\lambda}^{(n-1)/2}$$

for $x\in K$ and $\lambda\ge 1$. Thus we have $\left\|I_{\lambda,2}^{\delta,R}f\right\|_{L^\infty(K)} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right)$ as $\lambda\to\infty$.

3.4.2 Estimate of $I_{\lambda,3}^{\delta,R} f$

By (3.4.3) we have

$$\begin{aligned}
&\left|I_{\lambda,3}^{\delta,R} f(x)\right| \\
&\leq \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta-\frac{n}{2}}} \sum_k \int_{\sqrt{t}\geq\sqrt{\lambda}+1} \left(\frac{\sqrt{\lambda}^{\frac{1}{2}}}{\sqrt{t}^{\frac{n+1}{2}}\left(\sqrt{t}-\sqrt{\lambda}\right)} + \frac{\sqrt{\lambda}^{-\frac{1}{2}}}{\sqrt{t}^{\frac{n+1}{2}}} \right) |(Uf)_k(t)\, u_k(x,t)|\, \rho(dt) \\
&= I_{\lambda,3,1}^{\delta,R} f(x) + I_{\lambda,3,2}^{\delta,R} f(x), \qquad \text{say.}
\end{aligned}$$

First we have, by Schwarz's inequality

$$\begin{aligned}
I_{\lambda,3,1}^{\delta,R} f(x) \leq \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta-\frac{n+1}{2}}} &\left(\sum_k \int_{e_k\cap\left\{\sqrt{t}\geq\sqrt{\lambda}+1\right\}} |(Uf)_k(t)|^2 \rho(dt) \right)^{\frac{1}{2}} \\
\times &\left(\sum_k \int_{\sqrt{t}\geq\sqrt{\lambda}+1} \frac{|u_k(x,t)|^2}{\sqrt{t}^{n+1}\left(\sqrt{t}-\sqrt{\lambda}\right)^2} \rho(dt) \right)^{\frac{1}{2}}. \qquad (3.4.4)
\end{aligned}$$

By Lemma 6, there exists a constant C_K such that for $x \in K$

$$\begin{aligned}
&\left(\sum_k \int_{\sqrt{t}\geq\sqrt{\lambda}+1} \frac{|u_k(x,t)|^2}{\sqrt{t}^{n+1}\left(\sqrt{t}-\sqrt{\lambda}\right)^2} \rho(dt) \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{N=\sqrt{\lambda}+1}^{\infty} \frac{1}{N^{n+1}\left(N-\sqrt{\lambda}\right)^2} \sum_k \int_{N\leq\sqrt{t}\leq N+1} |u_k(x,t)|^2 \rho(dt) \right)^{\frac{1}{2}} \leq \frac{C_K}{\sqrt{\lambda}}. \quad (3.4.5)
\end{aligned}$$

Next we have

$$\begin{aligned}
I_{\lambda,3,2}^{\delta,R} f(x) \leq \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta-\frac{n-1}{2}}} &\left(\sum_k \int_{e_k\cap\left\{\sqrt{t}\geq\sqrt{\lambda}+1\right\}} |(Uf)_k(t)|^2 \rho(dt) \right)^{\frac{1}{2}} \\
\times &\left(\sum_k \int_{\sqrt{t}\geq\sqrt{\lambda}+1} \frac{|u_k(x,t)|^2}{\sqrt{t}^{n+1}} \rho(dt) \right)^{\frac{1}{2}}. \qquad (3.4.6)
\end{aligned}$$

Applying Lemma 6, again

$$\begin{aligned}
&\left(\sum_k \int_{\sqrt{t}\geq\sqrt{\lambda}+1} \frac{|u_k(x,t)|^2}{\sqrt{t}^{n+1}} \rho(dt) \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{N=\sqrt{\lambda}+1}^{\infty} \frac{1}{N^{n+1}} \sum_k \int_{N\leq\sqrt{t}\leq N+1} |u_k(x,t)|^2 \rho(dt) \right)^{\frac{1}{2}} \leq C_K \sqrt{\lambda}^{-\frac{1}{2}}. \quad (3.4.7)
\end{aligned}$$

On the other hand by (3.1.1) and (3.1.4)

$$\begin{aligned}
&\left(\sum_k \int_{e_k\cap\left\{\sqrt{t}\geq\sqrt{\lambda}+1\right\}} |(Uf)_k(t)|^2 \rho(dt) \right)^{\frac{1}{2}} \\
&= \left\| E\left(\left\{t; \sqrt{t}\geq\sqrt{\lambda}+1\right\}\right) f \right\|_{L^2(\Omega)} \longrightarrow 0 \qquad \text{as} \quad \lambda\to\infty.
\end{aligned}$$

By (3.4.4), (3.4.5), (3.4.6) and (3.4.7), we have

$$\left\|I_{\lambda,3}^{\delta,R}f\right\|_{L^\infty(K)} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) + o\left(\sqrt{\lambda}^{(n-2)/2-\delta}\right) = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) \quad \text{as } \lambda \to \infty.$$

3.4.3 Estimate of $I_{\lambda,1}^{\delta,R}f$

By (3.4.2) we have

$$\begin{aligned}
\left|I_{\lambda,1}^{\delta,R}f(x)\right| &\le \frac{C_{\delta,R}}{\sqrt{\lambda}^{\delta-\frac{n-3}{2}}} \sum_k \int_{0\le\sqrt{t}\le\sqrt{\lambda}-1} \left(\frac{\sqrt{t}^{-\frac{n-3}{2}}}{\sqrt{\lambda}-\sqrt{t}} + \sqrt{t}^{-\frac{n-1}{2}}\right) \\
&\qquad \times |u_k(x,t)\,(Uf)_k(t)|\,\rho(dt) \\
&\le C_{\delta,R}\sqrt{\lambda}^{\frac{n-3}{2}-\delta} \sum_{0\le N\le\sqrt{\lambda}-2} \sum_k \int_{N\le\sqrt{t}\le N+1} N^{-\frac{n-1}{2}} \left(\frac{N+1}{\sqrt{\lambda}-N-1}+1\right) \\
&\qquad \times |(Uf)_k(t)\,u_k(x,t)|\,\rho(dt) \\
&\le C_{\delta,R}\sqrt{\lambda}^{\frac{n-1}{2}-\delta} \frac{1}{\sqrt{\lambda}} \sum_{0\le N\le\sqrt{\lambda}-2} \frac{\sqrt{\lambda}}{\sqrt{\lambda}-N-1} N^{-\frac{n-1}{2}} \\
&\times \left(\sum_k \int_{N\le\sqrt{t}\le N+1} |u_k(x,t)|^2\,\rho(dt)\right)^{\frac{1}{2}} \left(\sum_k \int_{\left\{N\le\sqrt{t}\le N+1\right\}\cap e_k} |(Uf)_k(t)|^2\,\rho(dt)\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 6

$$\left(\sum_k \int_{N\le\sqrt{t}\le N+1} |u_k(x,t)|^2\,\rho(dt)\right)^{\frac{1}{2}} \le C_K\, N^{\frac{n-1}{2}}$$

and by (3.1.1) and (3.1.4)

$$\begin{aligned}
&\left(\sum_k \int_{\left\{N\le\sqrt{t}\le N+1\right\}\cap e_k} |(Uf)_k(t)|^2\,\rho(dt)\right)^{\frac{1}{2}} \\
&= \left\|E\left(\left\{t\,;N\le\sqrt{t}\le N+1\right\}\right)f\right\|_{L^2(\Omega)} =: A_N.
\end{aligned}$$

We have that $\sum_{N=M}^{\infty} A_N^2 \longrightarrow 0$ as $M\to\infty$. Thus

$$\left\|I_{\lambda,1}^{\delta,R}f\right\|_{L^\infty(K)} \le C_{\delta,R,K}\sqrt{\lambda}^{\frac{n-1}{2}-\delta} \sum_{0\le N\le\sqrt{\lambda}-2} \frac{A_N}{\sqrt{\lambda}-N-1} = o\left(\sqrt{\lambda}^{\frac{n-1}{2}-\delta}\right).$$

3.5 Estimate of $v_\lambda^{\delta,R}f$

We use notations in §3.2. Let $\delta > (n-3)/2$, $\beta_0 = \frac{n}{2}-1$ and $f \in L^1_{loc}(\Omega)$. Suppose that K is a compact set in Ω, K' is a closed subset of K such that $\operatorname{dist}(K',K^c) > 0$ and $0 < R < \operatorname{dist}(K',K^c)$. For $x \in K'$ put

$$v_\lambda^{\delta,R}f(x) = \frac{\omega_n\sqrt{\lambda}^{\frac{n}{2}-\delta}}{\sqrt{2\pi}^n} \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}} \left[\frac{1}{\omega_n}\int_{S^{n-1}} f(x-rw)\,\sigma(dw) - f(x)\right] dr.$$

Lemma 11. *Assume* $(-\Delta + V)f \in L^p(K)$ *where* $1 < p \leq \infty$. *Let* $\lambda > R^{-2}$.

(i) *If* $f \in L^p(K)$, *then*

$$\left\| v_\lambda^{\delta,R} f \right\|_{L^p(K')} \leq C_{V,K} c_\delta(\lambda) \sqrt{\lambda}^{\frac{n}{2}-\delta} \left(||f||_{L^p(K)} + ||(-\Delta + V)f||_{L^p(K)} \right)$$

where

$$c_\delta(\lambda) = \begin{cases} C_\delta \sqrt{\lambda}^{-\frac{3}{2}} & \text{if} \quad \frac{n-3}{2} < \delta < \frac{n+1}{2}, \\ C_\delta \sqrt{\lambda}^{-\frac{3}{2}} \log \lambda & \text{if} \quad \delta = \frac{n+1}{2}, \\ C_\delta \sqrt{\lambda}^{\delta - \frac{n}{2} - 2} & \text{if} \quad \delta > \frac{n+1}{2}. \end{cases}$$

(ii) *In the case of* $V \equiv 0$,

$$\left\| v_\lambda^{\delta,R} f \right\|_{L^p(K')} \leq c_\delta(\lambda) \sqrt{\lambda}^{\frac{n}{2}-\delta} ||\Delta f||_{L^p(K)}.$$

3.5.1 Proof of Lemma 11 (step 1)

Assume $f \in C_c^\infty(\boldsymbol{R}^n)$. Let $x \in K'$ and $0 < r < R$. By the Fourier inversion formula

$$\begin{aligned} &\frac{1}{\omega_n} \int_{S^{n-1}} f(x - rw)\, \sigma(dw) - f(x) \\ &= \frac{1}{\sqrt{2\pi}^n} \int \hat{f}(\xi) \left[\frac{1}{\omega_n} \int_{S^{n-1}} e^{-ir\xi\cdot w} \sigma(dw) - 1 \right] e^{ix\cdot\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}^n} \int \hat{f}(\xi) \left[2^{\beta_0}\, \Gamma\left(\frac{n}{2}\right) \frac{J_{\beta_0}(r|\xi|)}{(r|\xi|)^{\beta_0}} - 1 \right] e^{ix\cdot\xi} d\xi. \end{aligned} \tag{3.5.1}$$

By formulas (3.1.9) and (3.1.14), for $\xi \in \boldsymbol{R}^n - \{0\}$

$$2^{\beta_0}\, \Gamma\left(\frac{n}{2}\right) \frac{J_{\beta_0}(r|\xi|)}{(r|\xi|)^{\beta_0}} - 1 = -2^{\beta_0}\, \Gamma\left(\frac{n}{2}\right) |\xi|^2 \int_0^r \frac{J_{\frac{n}{2}}(s|\xi|)}{(s|\xi|)^{\frac{n}{2}}}\, s\, ds. \tag{3.5.2}$$

Therefore by (3.5.1) and (3.5.2), we have

$$\begin{aligned} &\frac{1}{\omega_n} \int_{S^{n-1}} f(x - rw)\, \sigma(dw) - f(x) \\ &= -2^{\beta_0} \Gamma\left(\frac{n}{2}\right) \frac{1}{\sqrt{2\pi}^n} \int |\xi|^2 \hat{f}(\xi) \left[\int_0^r \frac{J_{\frac{n}{2}}(s|\xi|)}{(s|\xi|)^{\frac{n}{2}}}\, s\, ds \right] e^{ix\cdot\xi} d\xi. \end{aligned} \tag{3.5.3}$$

On the other hand

$$\frac{1}{\sqrt{2\pi}^n} \int |\xi|^2 \hat{f}(\xi)\, e^{ix\cdot\xi} d\xi = -\Delta f(x) \tag{3.5.4}$$

and

$$\frac{1}{\sqrt{2\pi}^n} \int \left[\int_0^r \frac{J_{\frac{n}{2}}(s|\xi|)}{(s|\xi|)^{\frac{n}{2}}}\, s\, ds \right] e^{ix\cdot\xi} d\xi = \int_0^r \chi_s(x)\, s^{-n+1} ds \tag{3.5.5}$$

where χ_s is the characteristic function of $\{ x \in \boldsymbol{R}^n;\ |x| < s \}$. Therefore by (3.5.3), (3.5.4) and (3.5.5), we have

$$\begin{aligned}
&\frac{1}{\omega_n}\int_{S^{n-1}} f(x-rw)\,\sigma(dw) - f(x)\\
&= 2^{\beta_0}\Gamma\left(\frac{n}{2}\right)\int \Delta f(x-y)\left(\int_0^r \chi_s(y)\,s^{-n+1}ds\right)dy\\
&= 2^{\beta_0}\Gamma\left(\frac{n}{2}\right)\int_0^r \left(\int \Delta f(x-y)\,\chi_s(y)\,dy\right)s^{-n+1}ds.
\end{aligned}$$

Thus we have

$$\begin{aligned}
v_\lambda^{\delta,R} f(x) &= \sqrt{\lambda}^{\frac{n}{2}-\delta}\int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\left[\int_0^r \left(\int_{|y|<s} \Delta f(x-y)\,dy\right)s^{-n+1}ds\right]dr\\
&= \sqrt{\lambda}^{\frac{n}{2}-\delta}\int_0^R \left(\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr\right)\left(\int_{|y|<s} \Delta f(x-y)\,dy\right)s^{-n+1}ds.
\end{aligned}$$

By Minkowski's inequality for integral we have

$$\begin{aligned}
\left\|v_\lambda^{\delta,R} f\right\|_{L^p(K')} &\leq \sqrt{\lambda}^{\frac{n}{2}-\delta}\int_0^R \left|\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr\right|\int_{|y|\leq s} ||\Delta f(\cdot-y)||_{L^p(K')}\,dy\,\frac{ds}{s^{n-1}}\\
&\leq \sqrt{\lambda}^{\frac{n}{2}-\delta}||\Delta f||_{L^p(K)}\int_0^R \left|\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr\right| s\,ds. \qquad (3.5.6)
\end{aligned}$$

To estimate the right-hand side of (3.5.6) we divide the integral with respect to s into two integrals over $\left(0, 1/\sqrt{\lambda}\right)$ and $\left(1/\sqrt{\lambda}, R\right)$. For $s \geq \lambda^{-\frac{1}{2}}$, by formula (3.1.12)

$$\left|\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr\right| \leq C_\delta \frac{s^{-\delta+\frac{n-3}{2}}}{\sqrt{\lambda}^{\frac{3}{2}}}.$$

Therefore we have

$$\int_{\frac{1}{\sqrt{\lambda}}}^R \left|\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}r\right)}{r^{\delta-\beta_0}}dr\right| s\,ds \leq c_\delta(\lambda). \qquad (3.5.7)$$

For $0 < s \leq \lambda^{-\frac{1}{2}}$, put

$$\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr = \int_{\frac{1}{\sqrt{\lambda}}}^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr + \int_s^{\frac{1}{\sqrt{\lambda}}} \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr.$$

To the first integral on the right-hand side we apply formula (3.1.12) and to the second integral formula (3.1.14). Then we get

$$\left|\int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\,r\right)}{r^{\delta-\beta_0}}\,dr\right| \leq C_\delta \sqrt{\lambda}^{\delta-\frac{n}{2}}.$$

Therefore we have

$$\int_0^{\frac{1}{\sqrt{\lambda}}} \left| \int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda} r\right)}{r^{\delta-\beta_0}} dr \right| s\, ds \le C_\delta \sqrt{\lambda}^{\delta-\frac{n}{2}-2}. \tag{3.5.8}$$

By (3.5.7) and (3.5.8), we have

$$\int_0^{R} \left| \int_s^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right)}{r^{\delta-\beta_0}}\, dr \right| s\, ds \le c_\delta(\lambda). \tag{3.5.9}$$

On the other hand, we have

$$||\Delta f||_{L^p(K)} \le ||(-\Delta + V) f||_{L^p(K)} + ||V||_{L^\infty(K)} ||f||_{L^p(K)}. \tag{3.5.10}$$

Therefore by (3.5.6), (3.5.9) and (3.5.10), we have

$$\left\| v_\lambda^{\delta,R} f \right\|_{L^p(K')} \le c_\delta(\lambda) \sqrt{\lambda}^{\frac{n}{2}-\delta} \left(||(-\Delta + V) f||_{L^p(K)} + ||V||_{L^\infty(K)} ||f||_{L^p(K)} \right). \tag{3.5.11}$$

Thus Lemma 11 is proved in this case.

3.5.2 Proof of Lemma 11 (step 2)

For $x \in K'$, we can write

$$\begin{aligned} v_\lambda^{\delta,R} f(x) &= \frac{\omega_n \sqrt{\lambda}^{\frac{n}{2}-\delta}}{\sqrt{2\pi}^n} \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right)}{r^{\delta-\beta_0}} \left[\frac{1}{\omega_n} \int_{S^{n-1}} f(x - rw)\, \sigma(dw) - f(x) \right] dr \\ &= \frac{\sqrt{\lambda}^{\frac{n}{2}-\delta}}{\sqrt{2\pi}^n} \int_{|y|<R} \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}|y|\right)}{|y|^{\delta+\frac{n}{2}}}\, f(x-y)\, dy \\ &\qquad -f(x) \times \frac{\omega_n \sqrt{\lambda}^{\frac{n}{2}-\delta}}{\sqrt{2\pi}^n} \int_0^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^{\delta-\beta_0}}\, dr. \end{aligned} \tag{3.5.12}$$

Let χ_K be the characteristic function of K, $f_K = f \cdot \chi_K$ and $\{\varphi_\varepsilon\}$ be an infinitely differentiable approximate identity with supports contained in $\{\, x \in \boldsymbol{R}^n;\ |x| < \varepsilon \,\}$. We choose a closed set K'' in K so that $\operatorname{dist}(K'', K^c) > 0$ and $\operatorname{dist}(K', (K'')^c) > R$. Let $0 < \varepsilon < \operatorname{dist}(K'', K^c)$. By (3.5.12) we have

$$\begin{aligned} v_\lambda^{\delta,R}(f_K * \varphi_\varepsilon)(x) &= \frac{\sqrt{\lambda}^{\frac{n}{2}-\delta}}{\sqrt{2\pi}^n} \int_{|y|<R} \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}|y|\right)}{|y|^{\delta+\frac{n}{2}}} \left(f_K * \varphi_\varepsilon\right)(x-y)\, dy \\ &\quad - \left(f_K * \varphi_\varepsilon\right)(x) \times \frac{\omega_n \sqrt{\lambda}^{\frac{n}{2}-\delta}}{\sqrt{2\pi}^n} \int_0^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^{\delta-\beta_0}}\, dr. \end{aligned}$$

We observe that $v_\lambda^{\delta,R}\left(f_K * \varphi_\varepsilon\right)(x)$ converges in K' to $v_\lambda^{\delta,R} f(x)$. Indeed, by asymptotic formula (3.1.14),

$$\left| \int_{|y|<R} \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}|y|\right)}{|y|^{\delta+\frac{n}{2}}} \left(f_K * \varphi_\varepsilon\right)(x-y)\, dy - \int_{|y|<R} \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}|y|\right)}{|y|^{\delta+\frac{n}{2}}} f_K(x-y)\, dy \right|$$

$$
\begin{aligned}
&\leq \int \frac{\left|J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}|y|\right)\right|}{|y|^{\delta+\frac{n}{2}}} \left|(f_K * \varphi_\varepsilon)(x-y) - f_K(x-y)\right| dy \\
&\leq C_\delta \sqrt{\lambda}^{\delta+\frac{n}{2}} \int \left| (f_K * \varphi_\varepsilon)(x-y) - f_K(x-y)\right| dy \\
&= C_\delta \sqrt{\lambda}^{\delta+\frac{n}{2}} \left\|(f_K * \varphi_\varepsilon)(x-\cdot) - f_K(x-\cdot)\right\|_{L^1} \longrightarrow 0
\end{aligned}
$$

as $\varepsilon \to 0^+$ since $f_K \in L^1(\boldsymbol{R}^n)$. By $\lim_{\varepsilon\to 0^+} (f_K * \varphi_\varepsilon)(x) = f_K(x)$, $v_\lambda^{\delta,R}(f_K * \varphi_\varepsilon)(x)$ converges to $v_\lambda^{\delta,R}(f_K)(x)$. Since $R < \operatorname{dist}(K', (K'')^c)$, we have $v_\lambda^{\delta,R}(f_K) = v_\lambda^{\delta,R} f$ in K'.

Let $u \in C_c^\infty(K'')$. We have

$$
\begin{aligned}
&\int (-\Delta_x + V(x))\,(f_K * \varphi_\varepsilon)(x)\, u(x)\, dx \\
= &\int_K (-\Delta_y + V(y))\, f(y)\, (u * \varphi_\varepsilon)(y)\, dy \\
+ &\int_{K''} \int_{|y|<\varepsilon} [V(x) - V(x-y)]\, f(x-y)\, \varphi_\varepsilon(y)\, u(x)\, dx\, dy.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left|\int (-\Delta_x + V(x))\,(f_K * \varphi_\varepsilon)(x)\, u(x)\, dx\right| \\
&\leq ||(-\Delta + V) f||_{L^p(K)} \left\|u * \varphi_\varepsilon\right\|_{L^{p'}(K)} + 2||V||_{L^\infty(K)} \left\|f * \varphi_\varepsilon\right\|_{L^p(K'')} ||u||_{L^{p'}(K'')}.
\end{aligned}
$$

Thus

$$
\left\|(-\Delta + V)\,(f_K * \varphi_\varepsilon)\right\|_{L^p(K'')} \leq ||(-\Delta + V) f||_{L^p(K)} + 2||V||_{L^\infty(K)} ||f||_{L^p(K)}.
$$

On the other hand, since $f_K * \varphi_\varepsilon \in C_c^\infty(\boldsymbol{R}^n)$, by (3.5.11)

$$
\begin{aligned}
&\left\|v_\lambda^{\delta,R}(f_K * \varphi_\varepsilon)\right\|_{L^p(K')} \\
&\leq c_\delta(\lambda) \sqrt{\lambda}^{\frac{n}{2}-\delta} \left(\left\|(-\Delta + V)\,(f_K * \varphi_\varepsilon)\right\|_{L^p(K'')} + ||V||_{L^\infty(K'')} \left\|f_K * \varphi_\varepsilon\right\|_{L^p(K'')} \right).
\end{aligned}
$$

Therefore by Fatou's lemma, we have

$$
\begin{aligned}
&\left\|v_\lambda^{\delta,R} f\right\|_{L^p(K')} \leq \liminf_{\varepsilon \to 0^+} \left\|v_\lambda^{\delta,R}(f_K * \varphi_\varepsilon)\right\|_{L^p(K')} \\
&\leq c_\delta(\lambda) \sqrt{\lambda}^{\frac{n}{2}-\delta} \liminf_{\varepsilon \to 0^+} \left(\left\|(-\Delta + V)\,(f_K * \varphi_\varepsilon)\right\|_{L^p(K'')} + ||V||_{L^\infty(K'')} \left\|f_K * \varphi_\varepsilon\right\|_{L^p(K'')} \right) \\
&\leq c_\delta(\lambda) \sqrt{\lambda}^{\frac{n}{2}-\delta} \left(\left\|(-\Delta + V) f\right\|_{L^p(K)} + 3||V||_{L^\infty(K)} ||f||_{L^p(K)} \right).
\end{aligned}
$$

Thus Lemma 11 is proved.

3.6 Estimate of $W_\lambda^{\delta,R} f$

We use notations in §3.2. Let $\delta > (n-3)/2$ and $\beta_0 = \frac{n}{2} - 1$. Let $\alpha > n/4$ and $f \in D\left(\hat{A}^\alpha\right)$. Suppose that K is a compact set in Ω, K' is a closed subset of K such that $\operatorname{dist}(K', K^c) > 0$ and $0 < R < \operatorname{dist}(K', K^c)$. For $x \in K'$ let

$$W_\lambda^{\delta,R} f(x) = \sqrt{\lambda}^{\frac{n}{2}-\delta} \sum_k \int_0^\infty (Uf)_k(t)\, \rho(dt) \int_0^R \frac{J_{\frac{n}{2}+\delta}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \int_{|y|<r} V(x-y)$$
$$\times\, u_k(x-y,t)\, \frac{J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}|y|\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\, dy\, dr.$$

Lemma 12. *We have*

$$W_\lambda^{\delta,R} f(x) = \sqrt{\lambda}^{\frac{n}{2}-\delta} \sum_k \int_0^\infty (Uf)_k(t)\, \rho(dt) \int_0^R ds \int_{|y|<s} V(x-y)\, u_k(x-y,t)\, dy$$
$$\times \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, w(t,s,r)\, dr,$$

where

$$w(t,s,r) = \sqrt{t}\, s^{-\beta_0} \left\{ J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0+1}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0+1}\left(\sqrt{t}\, s\right) \right\}.$$

Proof. For $x \in K'$ and $0 < t < \infty$, we have

$$\int_0^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, dr \int_{|y|<r} V(x-y)\, u_k(x-y,t)$$
$$\times \frac{J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}|y|\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}|y|\right)}{|y|^{\beta_0}}\, dy$$
$$= \int_0^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, dr \int_0^r \int_{S^{n-1}} V(x-sw)\, u_k(x-sw,t)\, \sigma(dw)$$
$$\times \left\{ J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}\, s\right) \right\} s^{\frac{n}{2}}\, ds.$$

Reversing the order of integration with respect to r and s, the last expression equals

$$\int_0^R ds \int_{S^{n-1}} V(x-sw)\, u_k(x-sw,t)\, \sigma(dw)$$
$$\times \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \left\{ J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}\, s\right) \right\} dr\, s^{\frac{n}{2}}$$
$$= \int_0^R ds\, \frac{d}{ds} \left[\int_0^s \tau^{n-1} \int_{S^{n-1}} V(x-\tau w)\, u_k(x-\tau w,t)\, \sigma(dw)\, d\tau \right]$$
$$\times s^{-\beta_0} \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \left\{ J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}\, s\right) \right\} dr.$$

Integration by parts with respect to s yields the following expression of the right-hand side:

$$-\int_0^R ds \int_0^s \tau^{n-1} \int_{S^{n-1}} V(x-\tau w)\, u_k(x-\tau w,t)\, \sigma(dw)\, d\tau$$

$$\times \frac{d}{ds}\left[s^{-\beta_0} \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \left\{ J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}\, s\right)\right\} dr \right]$$

$$= -\int_0^R ds \int_{|y|<s} V(x-y)\, u_k(x-y,t)\, dy$$

$$\times \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \frac{d}{ds}\left[\frac{J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}\, s\right)}{s^{\beta_0}} \right] dr. \quad (3.6.1)$$

On the other hand, by recurrence formulas (3.1.9),

$$\frac{d}{ds}\left[s^{-\beta_0} \left\{ J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0}\left(\sqrt{t}\, s\right) - Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0}\left(\sqrt{t}\, s\right)\right\}\right] = -w(t,s,r).$$

Combining the last equation with (3.6.1) we get Lemma 12.

Lemma 13.

$$\left\| W_\lambda^{\delta,R} f \right\|_{L^\infty(K')} = \begin{cases} O\left(\sqrt{\lambda}^{(n-3)/2-\delta} \right) & \text{if} \quad \frac{n-3}{2} < \delta < \frac{n+1}{2}, \\ O\left(\lambda^{-1}\log \lambda\right) & \text{if} \quad \delta = \frac{n+1}{2}, \\ O\left(\lambda^{-1}\right) & \text{if} \quad \delta > \frac{n+1}{2} \end{cases}$$

as $\lambda \to \infty$. Moreover, if f vanishes in K, then $W_\lambda^{\delta,R} f$ vanishes in K'.

For $x \in K'$ put

$$W_\lambda^{\delta,R} f(x) = \sqrt{\lambda}^{\frac{n}{2}-\delta} \sum_k \left(\int_0^1 + \int_1^\lambda + \int_\lambda^\infty \right) (Uf)_k(t)\, \rho(dt)$$

$$\times \int_0^R ds \int_{|y|<s} V(x-y)\, u_k(x-y,t)\, dy \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} w(t,s,r)\, dr$$

$$= \sqrt{\lambda}^{\frac{n}{2}-\delta} \left(K_{\lambda,1}^{\delta,R} f(x) + K_{\lambda,2}^{\delta,R} f(x) + K_{\lambda,3}^{\delta,R} f(x) \right),$$

say.

3.6.1 Estimate of $K_{\lambda,1}^{\delta,R} f$

Let $\lambda > R^{-2}$. By Schwarz's inequality and (3.1.1) we have

$$\left| K_{\lambda,1}^{\delta,R} f(x) \right| \le \left[\sum_k \int_{e_k} |(Uf)_k(t)|^2 \rho(dt) \right]^{\frac{1}{2}}$$

$$\times \left[\sum_k \int_0^1 \left| \int_0^R ds \int_{|y|<s} V(x-y)\, u_k(x-y,t)\, dy \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} w(t,s,r)\, dr \right|^2 \rho(dt) \right]^{\frac{1}{2}}$$

$$\le ||f||_{L^2(\Omega)} \left[\left(\sum_k \int_0^1 \left| \int_0^{\frac{1}{\sqrt{\lambda}}} \cdots ds \right|^2 \rho(dt) \right)^{\frac{1}{2}} + \left(\sum_k \int_0^1 \left| \int_{\frac{1}{\sqrt{\lambda}}}^R \cdots ds \right|^2 \rho(dt) \right)^{\frac{1}{2}} \right]$$

$$= ||f||_{L^2(\Omega)} \times \left[k_{1,1}(x) + k_{1,2}(x) \right],$$

say. For $0 < t \le 1$ and $0 < s \le \sqrt{\lambda}^{-1}$, put

$$\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr = \left(\int_s^{\frac{1}{\sqrt{\lambda}}} + \int_{\frac{1}{\sqrt{\lambda}}}^R\right) \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr.$$

To the first integral on the right-hand side apply formulas (3.1.14)–(3.1.16) and to the second formulas (3.1.12) and (3.1.14)–(3.1.16). Since $\delta > (n-3)/2$, we get

$$\left|\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr\right| \le C_\delta \sqrt{\lambda}^{\delta-\frac{n}{2}} s^{1-n}.$$

Therefore we have

$$k_{1,1}(x) \le C_\delta ||V||_{L^\infty(K)} \sqrt{\lambda}^{\delta-\frac{n}{2}} \left[\sum_k \int_0^1 \rho(dt) \left(\int_0^{\frac{1}{\sqrt{\lambda}}} s^{1-n} ds \int_{|y|<s} |u_k(x-y,t)|\,dy\right)^2\right]^{\frac{1}{2}}.$$

By Schwarz's inequality

$$k_{1,1}(x) \le C_{\delta,V,K} \sqrt{\lambda}^{\delta-\frac{n}{2}} \left[\sum_k \int_0^1 \rho(dt) \left(\int_0^{\frac{1}{\sqrt{\lambda}}} \frac{ds}{s^{n-2}} \int_{|y|<s} dy\right)\right.$$
$$\left. \times \left(\int_0^{\frac{1}{\sqrt{\lambda}}} \frac{ds}{s^n} \int_{|y|<s} |u_k(x-y,t)|^2 dy\right)\right]^{\frac{1}{2}}.$$

If we change the order of integrations, then the right-hand side becomes

$$C_{\delta,V,K} \sqrt{\lambda}^{\delta-\frac{n}{2}-\frac{3}{2}} \left[\int_{|y|<\frac{1}{\sqrt{\lambda}}} |y|^{1-n} dy \sum_k \int_0^1 |u_k(x-y,t)|^2 \rho(dt)\right]^{\frac{1}{2}}.$$

By Lemma 6 we have

$$\sum_k \int_0^1 |u_k(x-y,t)|^2 \rho(dt) \le C_K \tag{3.6.2}$$

for $x \in K'$ and $|y| < R$. Thus we have

$$k_{1,1}(x) \le C_{\delta,V,K} \sqrt{\lambda}^{\delta-\frac{n}{2}-2}. \tag{3.6.3}$$

Let $0 < t \le 1$ and $\sqrt{\lambda}^{-1} \le s \le R$. Since $\delta > (n-3)/2$, by formulas (3.1.12) and (3.1.14)–(3.1.16) we have

$$\left|\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr\right| \le C_\delta \sqrt{\lambda}^{-\frac{3}{2}} s^{-\delta-\frac{n}{2}-\frac{1}{2}}.$$

Therefore we have

$$k_{1,2}(x) \le C_\delta ||V||_{L^\infty(K)} \sqrt{\lambda}^{-\frac{3}{2}}$$
$$\times \left[\sum_k \int_0^1 \rho(dt) \left(\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-\delta-\frac{n}{2}-\frac{1}{2}} ds \int_{|y|<s} |u_k(x-y,t)|\,dy\right)^2\right]^{\frac{1}{2}}.$$

If $\delta < (n+1)/2$, let $\delta < a < (n+1)/2$. By Schwarz's inequality

$$k_{1,2}(x) \le \frac{C_{\delta,V,K}}{\sqrt{\lambda}^{\frac{3}{2}}} \left[\sum_k \int_0^1 \rho(dt) \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{2\delta-2a+n+1}} \int_{|y|<s} dy \right) \right. \\ \left. \times \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{2a}} \int_{|y|<s} |u_k(x-y,t)|^2 \, dy \right) \right]^{\frac{1}{2}}. \tag{3.6.4}$$

By interchanging the order of integrations, we find that the right-hand side becomes

$$C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} \left[\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-2a} ds \int_{|y|<s} dy \sum_k \int_0^1 |u_k(x-y,t)|^2 \rho(dt) \right]^{\frac{1}{2}}.$$

By (3.6.2) we have

$$k_{1,2}(x) \le C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}}. \tag{3.6.5}$$

If $\delta = (n+1)/2$, by Schwarz's inequality

$$k_{1,2}(x) \le \frac{C_{\delta,V,K}}{\sqrt{\lambda}^{\frac{3}{2}}} \left[\sum_k \int_0^1 \rho(dt) \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{n+1}} \int_{|y|<s} dy \right) \right. \\ \left. \times \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{n+1}} \int_{|y|<s} |u_k(x-y,t)|^2 \, dy \right) \right]^{\frac{1}{2}}.$$

Interchanging the order of integrations yields that the right-hand side is equal to

$$C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} (\log \lambda)^{\frac{1}{2}} \left[\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-n-1} ds \int_{|y|<s} dy \sum_k \int_0^1 |u_k(x-y,t)|^2 \rho(dt) \right]^{\frac{1}{2}}.$$

By (3.6.2) we have

$$k_{1,2}(x) \le C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} \log \lambda. \tag{3.6.6}$$

If $\delta > (n+1)/2$, let $(n+1)/2 < a < \delta$. By Schwarz's inequality (3.6.4) holds. By the interchange of the order of integrations, the right-hand side turns out to be equal to

$$C_{\delta,V,K} \sqrt{\lambda}^{\delta-a-\frac{3}{2}} \left[\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-2a} ds \int_{|y|<s} dy \sum_k \int_0^1 |u_k(x-y,t)|^2 \rho(dt) \right]^{\frac{1}{2}}.$$

By (3.6.2) we have

$$k_{1,2}(x) \le C_{\delta,V,K} \sqrt{\lambda}^{\delta-\frac{n}{2}-2}. \tag{3.6.7}$$

3.6.2 Estimate of $K_{\lambda,3}^{\delta,R} f$

Let $\lambda > R^{-2}$. By Schwarz's inequality, (3.1.1) and (3.1.6) we have

$$\left| K_{\lambda,3}^{\delta,R} f(x) \right| \le \left[\sum_k \int_{e_k} |t^\alpha (Uf)_k(t)|^2 \rho(dt) \right]^{\frac{1}{2}}$$

$$\times\left[\sum_k\int_\lambda^\infty\left|\int_0^R ds\int_{|y|<s}V(x-y)\,u_k(x-y,t)\,dy\int_s^R\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right]^{\frac{1}{2}}$$

$$\leq\left\|\hat{A}^\alpha f\right\|_{L^2(\Omega)}\left[\left(\sum_k\int_\lambda^\infty\left|\int_0^{\frac{1}{\sqrt{t}}}\cdots ds\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\sum_k\int_\lambda^\infty\left|\int_{\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{\lambda}}}\cdots ds\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right)^{\frac{1}{2}}+\left(\sum_k\int_\lambda^\infty\left|\int_{\frac{1}{\sqrt{\lambda}}}^R\cdots ds\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right)^{\frac{1}{2}}\right]$$

$$=\left\|\hat{A}^\alpha f\right\|_{L^2(\Omega)}\times\left[k_{3,1}(x)+k_{3,2}(x)+k_{3,3}(x)\right],\qquad\text{say.}$$

For $\lambda\leq t<\infty$ and $0<s\leq\sqrt{t}^{\,-1}$, put

$$\int_s^R\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr=\left(\int_s^{\frac{1}{\sqrt{t}}}+\int_{\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{\lambda}}}+\int_{\frac{1}{\sqrt{\lambda}}}^R\right)\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr.$$

To the first integral on the right-hand side apply formulas (3.1.14)–(3.1.16) and to the second integral and the third integral formulas (3.1.12)–(3.1.16). Then we get

$$\left|\int_s^R\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr\right|\leq C_\delta\sqrt{\lambda}^{\,\delta-\frac{1}{2}}\sqrt{t}^{\,\frac{1-n}{2}}s^{1-n}.$$

Therefore we have

$$k_{3,1}(x)\leq C_\delta||V||_{L^\infty(K)}\sqrt{\lambda}^{\,\delta-\frac{1}{2}}\left[\sum_k\int_\lambda^\infty\frac{\rho(dt)}{t^{2\alpha+\frac{n-1}{2}}}\left(\int_0^{\frac{1}{\sqrt{t}}}s^{1-n}ds\int_{|y|<s}|u_k(x-y,t)|\,dy\right)^2\right]^{\frac{1}{2}}.$$

By Schwarz's inequality

$$k_{3,1}(x)\leq C_{\delta,V,K}\sqrt{\lambda}^{\,\delta-\frac{1}{2}}\left[\sum_k\int_\lambda^\infty\frac{\rho(dt)}{t^{2\alpha+\frac{n-1}{2}}}\left(\int_0^{\frac{1}{\sqrt{t}}}\frac{ds}{s^{n-2}}\int_{|y|<s}dy\right)\right.$$

$$\left.\times\left(\int_0^{\frac{1}{\sqrt{t}}}\frac{ds}{s^n}\int_{|y|<s}|u_k(x-y,t)|^2dy\right)\right]^{\frac{1}{2}}.$$

After changing the order of integrations, the right-hand side becomes

$$C_{\delta,V,K}\sqrt{\lambda}^{\,\delta-\frac{1}{2}}\left[\int_{|y|<\frac{1}{\sqrt{\lambda}}}|y|^{1-n}dy\sum_k\int_\lambda^{\frac{1}{|y|^2}}|u_k(x-y,t)|^2\,\frac{\rho(dt)}{t^{2\alpha+\frac{n}{2}+1}}\right]^{\frac{1}{2}}.$$

Since $\alpha>n/4$, by Corollary 3 we have

$$\sum_k\int_\lambda^\infty|u_k(x-y,t)|^2\,t^{-(2\alpha+1)-\frac{n}{2}}\rho(dt)\leq C_{K,\alpha}\lambda^{-(2\alpha+1)}<C_{K,\alpha}\lambda^{-\frac{n}{2}-1}$$

for $x\in K'$ and $|y|<R$. Thus we have

$$k_{3,1}(x)<C_{\delta,V,K,\alpha}\sqrt{\lambda}^{\,\delta-\frac{n}{2}-2}. \tag{3.6.8}$$

For $\lambda \le t < \infty$ and $\sqrt{t}^{-1} \le s \le \sqrt{\lambda}^{-1}$, put

$$\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} w(t,s,r)\,dr = \left(\int_s^{\frac{1}{\sqrt{\lambda}}} + \int_{\frac{1}{\sqrt{\lambda}}}^R\right) \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} w(t,s,r)\,dr.$$

To the first integral on the right-hand side apply formulas (3.1.12)–(3.1.14) and to the second integral formulas (3.1.12) and (3.1.13). We get

$$\left|\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} w(t,s,r)\,dr\right| \le C_\delta \sqrt{\lambda}^{\delta-\frac{1}{2}} s^{\frac{1-n}{2}}.$$

Therefore we have

$$k_{3,2}(x) \le C_\delta ||V||_{L^\infty(K)} \sqrt{\lambda}^{\delta-\frac{1}{2}} \left[\sum_k \int_\lambda^\infty \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{\lambda}}} s^{\frac{1-n}{2}} ds \int_{|y|<s} |u_k(x-y,t)|\,dy\right)^2\right]^{\frac{1}{2}}.$$

By Schwarz's inequality

$$k_{3,2}(x) \le C_{\delta,V,K} \sqrt{\lambda}^{\delta-\frac{1}{2}} \left[\sum_k \int_\lambda^\infty \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{\lambda}}} \frac{ds}{s^{n-1}} \int_{|y|<s} dy\right)\right.$$
$$\left.\times \left(\int_{\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{\lambda}}} ds \int_{|y|<s} |u_k(x-y,t)|^2 dy\right)\right]^{\frac{1}{2}}.$$

If we change the order of integrations, then the right-hand side becomes

$$C_{\delta,V,K} \sqrt{\lambda}^{\delta-2} \left[\int_{|y|<\frac{1}{\sqrt{\lambda}}} dy \sum_k \int_\lambda^\infty |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}}\right]^{\frac{1}{2}}.$$

Since $\alpha > n/4$, by Corollary 3 we have

$$\int_\lambda^\infty |u_k(x-y,t)|^2\, t^{-\left(2\alpha-\frac{n}{2}\right)-\frac{n}{2}} \rho(dt) \le C_{K,\alpha} \lambda^{-\left(2\alpha-\frac{n}{2}\right)} < C_{K,\alpha} \tag{3.6.9}$$

for $x \in K'$ and $|y| < R$. Thus we have

$$k_{3,2}(x) < C_{\delta,V,K,\alpha} \sqrt{\lambda}^{\delta-\frac{n}{2}-2}. \tag{3.6.10}$$

Let $\lambda \le t < \infty$ and $\sqrt{\lambda}^{-1} \le s \le R$. By formulas (3.1.12) and (3.1.13) we get

$$\left|\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} w(t,s,r)\,dr\right| \le C_\delta \sqrt{\lambda}^{-\frac{3}{2}} s^{-\delta-\frac{n}{2}-\frac{1}{2}}.$$

Therefore we have

$$k_{3,3}(x) \le C_\delta ||V||_{L^\infty(K)} \sqrt{\lambda}^{-\frac{3}{2}} \left[\sum_k \int_\lambda^\infty \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-\delta-\frac{n}{2}-\frac{1}{2}} ds \int_{|y|<s} |u_k(x-y,t)|\,dy\right)^2\right]^{\frac{1}{2}}.$$

If $\delta < (n+1)/2$, let $\delta < a < (n+1)/2$. By Schwarz's inequality

$$k_{3,3}(x) \leq \frac{C_{\delta,V,K}}{\sqrt{\lambda}^{\frac{3}{2}}} \left[\sum_k \int_\lambda^\infty \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{2\delta-2a+n+1}} \int_{|y|<s} dy \right) \right.$$
$$\left. \times \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{2a}} \int_{|y|<s} |u_k(x-y,t)|^2 \, dy \right) \right]^{\frac{1}{2}}. \tag{3.6.11}$$

By interchanging the order of integrations, we find that the right-hand side becomes

$$C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} \left[\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-2a} ds \int_{|y|<s} dy \sum_k \int_\lambda^\infty |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}} \right]^{\frac{1}{2}}.$$

By (3.6.9) we have

$$k_{3,3}(x) < C_{\delta,V,K,\alpha} \sqrt{\lambda}^{-\frac{3}{2}}. \tag{3.6.12}$$

If $\delta = (n+1)/2$, by Schwarz's inequality

$$k_{3,3}(x) \leq \frac{C_{\delta,V,K}}{\sqrt{\lambda}^{\frac{3}{2}}} \left[\sum_k \int_\lambda^\infty \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{n+1}} \int_{|y|<s} dy \right) \right.$$
$$\left. \times \left(\int_{\frac{1}{\sqrt{\lambda}}}^R \frac{ds}{s^{n+1}} \int_{|y|<s} |u_k(x-y,t)|^2 \, dy \right) \right]^{\frac{1}{2}}.$$

Interchanging the order of integrations yields that the right-hand side is equal to

$$C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} (\log \lambda)^{\frac{1}{2}} \left[\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-n-1} ds \int_{|y|<s} dy \sum_k \int_\lambda^\infty |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}} \right]^{\frac{1}{2}}.$$

By (3.6.9) we have

$$k_{3,3}(x) < C_{\delta,V,K,\alpha} \sqrt{\lambda}^{-\frac{3}{2}} \log \lambda. \tag{3.6.13}$$

If $\delta > (n+1)/2$, let $(n+1)/2 < a < \delta$. By Schwarz's inequality (3.6.11) holds. By the interchange of the order of integrations, the right-hand side turns out to be equal to

$$C_{\delta,V,K} \sqrt{\lambda}^{\delta-a-\frac{3}{2}} \left[\int_{\frac{1}{\sqrt{\lambda}}}^R s^{-2a} ds \int_{|y|<s} dy \sum_k \int_\lambda^\infty |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}} \right]^{\frac{1}{2}}.$$

By (3.6.9) we have

$$k_{3,3}(x) < C_{\delta,V,K,\alpha} \sqrt{\lambda}^{\delta-\frac{n}{2}-2}. \tag{3.6.14}$$

3.6.3 Estimate of $K_{\lambda,2}^{\delta,R} f$

By Schwarz's inequality, (3.1.1) and (3.1.6) we have

$$\left| K_{\lambda,2}^{\delta,R} f(x) \right| \leq \left[\sum_k \int_{e_k} |t^\alpha (Uf)_k(t)|^2 \rho(dt) \right]^{\frac{1}{2}}$$

$$\times\left[\sum_k\int_1^\lambda\left|\int_0^R\!\!\int_{|y|<s}V(x-y)\,u_k(x-y,t)\,dy\int_s^R\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}w(t,s,r)\,dr\,ds\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right]^{\frac{1}{2}}$$

$$\leq\left\|\hat{A}^\alpha f\right\|_{L^2(\Omega)}\left[\left(\sum_k\int_1^\lambda\left|\int_0^{\frac{R}{\sqrt{\lambda}}}\cdots ds\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right)^{\frac{1}{2}}+\left(\sum_k\int_1^\lambda\left|\int_{\frac{R}{\sqrt{\lambda}}}^R\cdots ds\right|^2\frac{\rho(dt)}{t^{2\alpha}}\right)^{\frac{1}{2}}\right]$$

$$=\left\|\hat{A}^\alpha f\right\|_{L^2(\Omega)}\times\left[k_{2,1}(x)+k_{2,2}(x)\right],\qquad \text{say.}$$

For $1\leq t\leq\lambda$ and $0<s\leq R/\sqrt{\lambda}$, put

$$\int_s^R\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr=\left(\int_s^{\frac{R}{\sqrt{\lambda}}}+\int_{\frac{R}{\sqrt{\lambda}}}^{\frac{R}{\sqrt{t}}}+\int_{\frac{R}{\sqrt{t}}}^R\right)\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr.$$

To the first integral on the right-hand side apply formulas (3.1.14)–(3.1.16), to the second integral formulas (3.1.12) and (3.1.14)–(3.1.16) and to the third integral formulas (3.1.12)–(3.1.16). Since $\delta>(n-3)/2$, we get

$$\left|\int_s^R\frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\,r\right)}{r^\delta}\,w(t,s,r)\,dr\right|\leq C_\delta\sqrt{\lambda}^{\,\delta-\frac{n}{2}}s^{1-n}.$$

Therefore we have

$$k_{2,1}(x)\leq C_\delta||V||_{L^\infty(K)}\sqrt{\lambda}^{\,\delta-\frac{n}{2}}\left[\sum_k\int_1^\lambda\frac{\rho(dt)}{t^{2\alpha}}\left(\int_0^{\frac{R}{\sqrt{\lambda}}}s^{1-n}ds\int_{|y|<s}|u_k(x-y,t)|\,dy\right)^2\right]^{\frac{1}{2}}.$$

By Schwarz's inequality

$$k_{2,1}(x)\leq C_{\delta,V,K}\sqrt{\lambda}^{\,\delta-\frac{n}{2}}\left[\sum_k\int_1^\lambda\frac{\rho(dt)}{t^{2\alpha}}\left(\int_0^{\frac{R}{\sqrt{\lambda}}}\frac{ds}{s^{n-2}}\int_{|y|<s}dy\right)\right.$$
$$\left.\times\left(\int_0^{\frac{R}{\sqrt{\lambda}}}\frac{ds}{s^n}\int_{|y|<s}|u_k(x-y,t)|^2dy\right)\right]^{\frac{1}{2}}.$$

After changing the order of integrations, the right-hand side becomes

$$C_{\delta,V,K}\sqrt{\lambda}^{\,\delta-\frac{n}{2}-\frac{3}{2}}\left[\int_{|y|<\frac{R}{\sqrt{\lambda}}}|y|^{1-n}dy\sum_k\int_1^\lambda|u_k(x-y,t)|^2\,\frac{\rho(dt)}{t^{2\alpha}}\right]^{\frac{1}{2}}.$$

Since $\alpha>n/4$, by Corollary 3 we have

$$\sum_k\int_1^\lambda|u_k(x-y,t)|^2\,t^{\left(\frac{n}{2}-2\alpha\right)-\frac{n}{2}}\rho(dt)\leq C_{K,\alpha}\tag{3.6.15}$$

for $x\in K'$ and $|y|<R$. Thus we have

$$k_{2,1}(x)\leq C_{\delta,V,K,\alpha}\sqrt{\lambda}^{\,\delta-\frac{n}{2}-2}.\tag{3.6.16}$$

Let $1 \leq t \leq \lambda$ and $R/\sqrt{\lambda} \leq s \leq R/\sqrt{t}$ and put

$$\int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, w(t,s,r)\, dr = \left(\int_s^{\frac{R}{\sqrt{t}}} + \int_{\frac{R}{\sqrt{t}}}^R\right) \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, w(t,s,r)\, dr.$$

To the first integral on the right-hand side apply formulas (3.1.12) and (3.1.14)–(3.1.16) and to the second formulas (3.1.12)–(3.1.16). Since $\delta > (n-3)/2$, we get

$$\left| \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, w(t,s,r)\, dr \right| \leq C_\delta \sqrt{\lambda}^{-\frac{3}{2}} s^{-\delta-\frac{n}{2}-\frac{1}{2}}. \tag{3.6.17}$$

Let $1 \leq t \leq \lambda$ and $R/\sqrt{t} \leq s \leq R$. By formulas (3.1.12) and (3.1.13) we get

$$\left| \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta}\, w(t,s,r)\, dr \right| \leq C_\delta \sqrt{\lambda}^{-\frac{3}{2}} s^{-\delta-\frac{n}{2}-\frac{1}{2}}. \tag{3.6.18}$$

Therefore by (3.6.17) and (3.6.18) we have

$$k_{2,2}(x) \leq C_\delta ||V||_{L^\infty(K)} \sqrt{\lambda}^{-\frac{3}{2}} \left[\sum_k \int_1^\lambda \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{R}{\sqrt{\lambda}}}^R s^{-\delta-\frac{n}{2}-\frac{1}{2}} ds \int_{|y|<s} |u_k(x-y,t)|\, dy \right)^2 \right]^{\frac{1}{2}}.$$

If $\delta < (n+1)/2$, let $\delta < a < (n+1)/2$. By Schwarz's inequality

$$\begin{aligned} k_{2,2}(x) \leq \frac{C_{\delta,V,K}}{\sqrt{\lambda}^{\frac{3}{2}}} &\left[\sum_k \int_1^\lambda \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{R}{\sqrt{\lambda}}}^R \frac{ds}{s^{2\delta-2a+n+1}} \int_{|y|<s} dy \right) \right. \\ &\times \left. \left(\int_{\frac{R}{\sqrt{\lambda}}}^R \frac{ds}{s^{2a}} \int_{|y|<s} |u_k(x-y,t)|^2 dy \right) \right]^{\frac{1}{2}}. \end{aligned} \tag{3.6.19}$$

By interchanging the order of integrations, we find that the right-hand side becomes

$$C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} \left[\int_{\frac{R}{\sqrt{\lambda}}}^R s^{-2a} ds \int_{|y|<s} dy \sum_k \int_1^\lambda |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}} \right]^{\frac{1}{2}}.$$

By (3.6.15) we have

$$k_{2,2}(x) \leq C_{\delta,V,K,\alpha} \sqrt{\lambda}^{-\frac{3}{2}}. \tag{3.6.20}$$

If $\delta = (n+1)/2$, by Schwarz's inequality

$$\begin{aligned} k_{2,2}(x) \leq \frac{C_{\delta,V,K}}{\sqrt{\lambda}^{\frac{3}{2}}} &\left[\sum_k \int_1^\lambda \frac{\rho(dt)}{t^{2\alpha}} \left(\int_{\frac{R}{\sqrt{\lambda}}}^R \frac{ds}{s^{n+1}} \int_{|y|<s} dy \right) \right. \\ &\times \left. \left(\int_{\frac{R}{\sqrt{\lambda}}}^R \frac{ds}{s^{n+1}} \int_{|y|<s} |u_k(x-y,t)|^2 dy \right) \right]^{\frac{1}{2}}. \end{aligned}$$

Interchanging the order of integrations yields that the right-hand side is equal to

$$C_{\delta,V,K} \sqrt{\lambda}^{-\frac{3}{2}} (\log \lambda)^{\frac{1}{2}} \left[\int_{\frac{R}{\sqrt{\lambda}}}^R s^{-n-1} ds \int_{|y|<s} dy \sum_k \int_1^\lambda |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}} \right]^{\frac{1}{2}}.$$

By (3.6.15) we have

$$k_{2,2}(x) \le C_{\delta,V,K,\alpha} \sqrt{\lambda}^{-\frac{3}{2}} \log \lambda. \tag{3.6.21}$$

If $\delta > (n+1)/2$, let $(n+1)/2 < a < \delta$. By Schwarz's inequality (3.6.19) holds. By the interchange of the order of integrations the right-hand side turns out to be equal to

$$C_{\delta,V,K} \sqrt{\lambda}^{\delta-a-\frac{3}{2}} \left[\int_{\frac{R}{\sqrt{\lambda}}}^{R} s^{-2a} ds \int_{|y|<s} dy \sum_k \int_1^\lambda |u_k(x-y,t)|^2 \frac{\rho(dt)}{t^{2\alpha}} \right]^{\frac{1}{2}}.$$

By (3.6.15) we have

$$k_{2,2}(x) \le C_{\delta,V,K,\alpha} \sqrt{\lambda}^{\delta-\frac{n}{2}-2}. \tag{3.6.22}$$

Thus by (3.6.3), (3.6.5)–(3.6.8), (3.6.10), (3.6.12)–(3.6.14), (3.6.16) and (3.6.20)–(3.6.22)

$$\left\| W_\lambda^{\delta,R} f \right\|_{L^\infty(K')} = \begin{cases} O\left(\sqrt{\lambda}^{(n-3)/2-\delta} \right) & \text{if } \frac{n-3}{2} < \delta < \frac{n+1}{2}, \\ O\left(\lambda^{-1} \log \lambda\right) & \text{if } \delta = \frac{n+1}{2}, \\ O\left(\lambda^{-1}\right) & \text{if } \delta > \frac{n+1}{2} \end{cases}$$

as $\lambda \to \infty$.

If f vanishes in K, then by (3.1.1), (3.1.3) and (3.1.6) we have

$$\begin{aligned} &\int_\Omega \overline{g(z)} \left[\sum_k \int_0^\infty t^j \, (Uf)_k(t) \, u_k(z,t) \, \rho(dt) \right] dz \\ &= \sum_k \int_0^\infty t^j \, (Uf)_k(t) \, \overline{(Ug)_k(t)} \, \rho(dt) = \int_\Omega f(z) \, \overline{(-\Delta + V)^j g(z)} \, dz = 0 \end{aligned}$$

for $g \in C_c^\infty(K)$ and $j = 0, 1, 2, \ldots$. Therefore for $s, r > 0$ we have

$$\begin{aligned} &\int_\Omega \overline{g(z)} \left[\sum_k \int_0^\infty w(t,s,r) \, (Uf)_k(t) \, u_k(z,t) \, \rho(dt) \right] dz \\ &= s^{-\beta_0} \int_\Omega \overline{g(z)} \left[\sum_k \int_0^\infty \sqrt{t} \, J_{\beta_0}\left(\sqrt{t}\, r\right) Y_{\beta_0+1}\left(\sqrt{t}\, s\right) (Uf)_k(t) \, u_k(z,t) \, \rho(dt) \right] dz \\ &- s^{-\beta_0} \int_\Omega \overline{g(z)} \left[\sum_k \int_0^\infty \sqrt{t} \, Y_{\beta_0}\left(\sqrt{t}\, r\right) J_{\beta_0+1}\left(\sqrt{t}\, s\right) (Uf)_k(t) \, u_k(z,t) \, \rho(dt) \right] dz \\ &= 0. \end{aligned}$$

Thus for $x \in K'$ we have

$$\begin{aligned} W_\lambda^{\delta,R} f(x) = \sqrt{\lambda}^{\frac{n}{2}-\delta} \int_0^R ds \int_s^R \frac{J_{\delta+\frac{n}{2}}\left(\sqrt{\lambda}\, r\right)}{r^\delta} \int_{|y|<s} V(x-y) \\ \times \left[\sum_k \int_0^\infty w(t,s,r) \, (Uf)_k(t) \, u_k(x-y,t) \, \rho(dt) \right] dy \, dr = 0. \end{aligned}$$

Thus Lemma 13 is proved.

3.7 Proof of Propositions 1 and 2

Let K be a compact set in Ω and K' be a closed set in K with $\operatorname{dist}(K', K^c) > 0$. We choose $0 < R < \operatorname{dist}(K', K^c)$. Let $1 < p \leq \infty$ and $\delta > (n-3)/2$.

Let $\alpha > n/4$ and f be a function in $D\left(\hat{A}^\alpha\right)$ regulated in Ω. We decompose $s_\lambda^\delta f - f$ into four terms: $V_\lambda^{\delta,R} f$, $w_\lambda^{\delta,R} f$, $v_\lambda^{\delta,R} f$ and $W_\lambda^{\delta,R} f$, and estimate by (3.3.1) and Lemma 9, 10 and 12 respectively.

If f and $(-\Delta + V)f \in L^p(K)$, then

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = \begin{cases} o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) & \text{if } \frac{n-3}{2} < \delta < \frac{n+3}{2} \\ O\left(\lambda^{-1}\right) & \text{if } \delta \geq \frac{n+3}{2} \end{cases} \quad \text{as} \quad \lambda \to \infty.$$

If f vanishes in K, then

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right)$$

as $\lambda \to \infty$. Thus Proposition 1 is proved.

Let $V \equiv 0$ and f be a function in $L^2(\Omega)$ regulated in Ω. Then we have $W_\lambda^{\delta,R} f \equiv 0$. We estimate $V_\lambda^{\delta,R} f$, $w_\lambda^{\delta,R} f$ and $v_\lambda^{\delta,R} f$ by (3.32), Lemma 9 and 10.

If f and $\Delta f \in L^p(K)$, then

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = \begin{cases} o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) & \text{if } \frac{n-3}{2} < \delta < \frac{n+3}{2} \\ O\left(\lambda^{-1}\right) & \text{if } \delta \geq \frac{n+3}{2} \end{cases}$$

as $\lambda \to \infty$. If $f \in L^p(K)$ and Δf vanishes in K, then

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K')} = o\left(\sqrt{\lambda}^{(n-1)/2-\delta}\right) \quad \text{as} \quad \lambda \to \infty.$$

Thus Proposition 2 is proved.

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Bibliography

[1] S. A. Alimov and V. A. Il'in, *Conditions for the convergence of spectral expansions corresponding to selfadjoint extensions of elliptic operators* II (*selfadjoint extensions of Laplace's operator with arbitrary spectra*), Differential Equations **7** (1971), 651-670.

[2] H. Bateman, *Higher transcendental functions.* Volume II, McGraw-Hill Book Company, Inc. New York 1953.

[3] N. Dunford and J. T. Schwartz, *Linear operators.* Part II (*Spectral theory*), Pure and Appl. Math. VII, Interscience Publishers, New York 1963.

[4] H. Fujita, S.-T. Kuroda and S. Itô, *Kansû kaiseki*, Iwanami kiso sûgaku sensyo, Iwanami syoten, Tôkyô 1991.

[5] S. Igari, *Saturation of the approximation by eigenfunction expansions associated with the Laplace operator*, Tôhoku Math. J. **22** (1970), 231-239.

[6] V. A. Il'in, *Conditions for the convergence of spectral expansions corresponding to selfadjoint extensions of elliptic operators* IV (*Theorems of negative type for an arbitrary extension of a general selfadjoint second order elliptic operator*), Differential Equations **9** (1973), 38-54.

[7] C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer-Verlag, Berlin 1970.

[8] E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations.* Part II, Oxford Univ. Press, Oxford 1958.

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[1: p.655] S. A. Alimov and V. A. Il'in, *Conditions for the convergence of spectral expansions corresponding to selfadjoint extensions of elliptic operators* II (*self-adjoint extensions of Laplace's operator with arbitrary spectra*), Differential equations **7** (1971), 651-670.

[2: p.92,(34)] H. Bateman,*Higher transcendental functions.* Volume II, McGraw-Hill Book Company, Inc. New York 1953.

[2: p.49,(19)] H. Bateman,*Higher transcendental functions.* Volume II, McGraw-Hill Book Company, Inc. New York 1953.

[3: p.1199, Cor.XII.2.7 (c)] N. Dunford and J. T. Schwartz, *Linear operators.* Part II (*Spectral theory*), Pure and Appl. Math. VII, Interscience Publishers, New York 1963.

[3: XII.3.16] N. Dunford and J. T. Schwartz, *Linear operators.* Part II (*Spectral theory*), Pure and Appl. Math. VII, Interscience Publishers, New York 1963.

[3: XII.3 and XIV.6] N. Dunford and J. T. Schwartz,*Linear operators.* Part II (*Spectral theory*), Pure and Appl. Math. VII, Interscience Publishers, New York 1963.

[4: Chap.14] H. Fujita, S. Kuroda and S. Itô, *Kansû kaiseki,* Iwanami kiso sûgaku sensyo, Iwanami syoten, Tôkyô 1991.

[4: Chap.15] H. Fujita, S. Kuroda and S. Itô, *Kansû kaiseki,* Iwanami kiso sûgaku sensyo, Iwanami syoten, Tôkyô 1991.

[6: p.42] V. A. Il'in, *Conditions for the convergence of spectral expansions corresponding to selfadjoint extensions of elliptic operators* IV (*Theorems of negative type for an arbitrary extension of a general selfadjoint second order elliptic operator*), Differential equations **9** (1973), 38-54.

[7: p.64,19,VIII] C.Milanda,*Partial Differential Equations of Elliptic Type*,Springer-Verlag, Berlin 1970.

[7: p.27,12,VII] C.Milanda,*Partial Differential Equations of Elliptic Type*,Springer-Verlag, Berlin 1970.

[8: p.202, Lemma 18.10 a] E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations.* Part II, Oxford Univ. Press, Oxford 1958.