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Non-isotropic harmonic tori in complex projective spaces and configurations of points on Riemann surfaces

by

Tetsuya TANIGUCHI

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A thesis presented

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Tetsuya TANIGUCHI

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Contents

1. Introduction

It is one of the profound problems in differential geometry to classify and construct harmonic maps from a Riemann surface M into a compact Lie group G or a symmetric space G/K. These objects are related with many fundamental examples in differential geometry, in particular, with minimal surfaces, which have been studied for a long time by geometers. In the late 1970's, such harmonic maps also appeared as non-linear sigma models or chiral models in mathematical physics. Since then, in the study of harmonic maps, many interesting results have been established but major open problems still remain. For example, when the genus of a Riemann surface M is greater than 1, the classification of such harmonic maps is not obtained yet.

When M is the Riemann sphere, the well-established twistor theory of harmonic maps is useful to describe harmonic maps of M into G/K. In fact, this idea was first used by Calabi [11], [12] in his study of minimal 2-spheres in S^n. Moreover, harmonic maps of a two-sphere into a complex Grassmann manifold have been studied and classified in [3], [30] and [31].

In this case, any harmonic map is covered by a horizontal holomorphic map into an auxiliary complex manifold, a twistor space, and the study of such harmonic maps is therefore reduced to a problem in algebraic geometry. In this sense, the case of genus 0 has been accomplished.

In general, each harmonic map of a Riemann surface M into a sphere S^n or a complex projective space $\mathbb{C}P^n$ has a sequence of invariants, which are given as holomorphic differentials on M measuring the lack of orthogonality of iterated derivatives of the map. In particular, when all these invariants vanish, such harmonic map is obtained from a holomorphic curve in a twistor space. These harmonic maps are called isotropic. Since every holomorphic differential on the Riemann sphere vanishes, any harmonic 2-sphere in S^n or $\mathbb{C}P^n$ is isotropic.

For the case of harmonic tori in a sphere S^n or a complex projective space $\mathbb{C}P^n$, we have the following two possibilities:

(1) All invariants vanish, that is, the harmonic torus is isotropic.

(2) The harmonic torus is not isotropic.

In the former case, such torus is again covered by a holomorphic curve in a twistor space. In the latter case, these harmonic tori are called non-isotropic. In 1995 Burstall [1] proved that any non-isotropic harmonic torus in a sphere or a complex projective space is covered by a primitive harmonic map of finite type into a certain generalized flag manifold. Subsequently, Udagawa [28] generalized Burstall's result to those harmonic tori into a complex Grassmann manifold $G_2(\mathbb{C}^4)$ of 2-dimensional complex linear subspaces in $\mathbb{C}^4$ and also constructed, by using a Symes formula, weakly conformal non-superminimal harmonic maps from the complex line to $G_2(\mathbb{C}^4)$. Employing these facts, as well as algebro-geometric methods, McIntosh proved that every non-isotropic harmonic torus in a complex projective space corresponds to a map constructed from a triplet $(X, \pi, \mathcal{L})$, consisting of an auxiliary algebraic curve X, and a rational function π and a line bundle $\mathcal{L}$ on X. Such triplet is called a spectral data. Thus McIntosh realized the moduli space of non-isotropic harmonic tori in complex projective spaces as a subset of the moduli space of these spectral data.

Therefore it seems natural to ask the following: *Which spectral data corresponds to a harmonic torus in a complex projective space?*

In this thesis, we give a partial answer to this problem. More precisely, we prove a criterion on the periodicity of harmonic maps constructed from the spectral data whose spectral curves are smooth rational or elliptic curves.

Before describing the plan of this thesis, we now review briefly McIntosh's results and state our main theorems.

McIntosh [17], [18] has constructed a significant correspondence between the following two spaces: the space of non-isotropic, linearly full harmonic maps $\psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ of finite type, up to isometries, and that of triplets $(X, \pi, \mathcal{L})$ consisting of a real, complete, connected algebraic curve X(which we call the spectral curve for ψ), a rational function π on X and a line bundle $\mathcal{L}$ over X, which are required to satisfy certain conditions.

This correspondence yields a harmonic map from a spectral data in the following fashion. Take a spectral data $(X, \pi, \mathcal{L})$. On the Jacobian variety $J(X)$ of the spectral curve X, we consider a real 2-dimensional linear flow $L\colon \mathbb{R}^2 \to J(X), z \mapsto L(z)$. Then we know that each line bundle contained in this flow has the following properties. Denoting by $H^0(X, \mathcal{L} \otimes L(z))$ the space of global holomorphic sections of $\mathcal{L} \otimes L(z)$, we see that the dimension of $H^0(X, \mathcal{L} \otimes L(z))$ is $n+1$ if the degree of π is $n+1$. Let R be the ramification divisor of π. Then, since $(\mathcal{L} \otimes L(z)) \otimes \overline{\rho_X^*(\mathcal{L} \otimes L(z))}$ is isomorphic to the divisor line bundle $\mathcal{O}_X(R)$, each line bundle $\mathcal{L} \otimes L(z)$ has a natural bilinear form h via a trace map $H^0(X, \mathcal{O}_X(R)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}$, which is induced from π. Thus we obtain a vector bundle W of rank $n+1$ over $\mathbb{R}^2$ with the fiber metric h, where the fiber of W at $z \in \mathbb{R}^2$ is given by $H^0(X, \mathcal{L} \otimes L(z))$.

Next, we consider subbundles $W_0, \dots, W_{n-m}$ and a connection ∇ of W, which enjoy the following property: The rank of W_i is $i+1$ and

$$W_0 \subset \cdots \subset W_{n-m} \subset W, \quad \nabla_{\partial/\partial z} W_i \subset W_{i+1}.$$

Then these subbundles satisfy the weak form of Griffiths transversality, which is almost the condition for the corresponding map to be harmonic (primitive harmonic). For each $z \in \mathbb{R}^2$, by using the above connection, we can identify the fiber of W_0 at z with a complex line in the $(n+1)$-dimensional complex vector space $H^0(X, \mathcal{L} \otimes L(0))$. In this way, we get a desired harmonic map $\psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$.

With these understood, for spectral data with rational spectral curves, we shall prove the following

Theorem 9. *Let X be the smooth rational curve $\mathbb{P}^1$. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if the following conditions are satisfied:*

(1) *(X, ρ_X) is real isomorphic to $(\mathbb{P}^1, \rho)$. By an affine coordinate λ of $\mathbb{P}^1$, ρ is*

given by $\lambda \mapsto 1/\bar{\lambda}$ *and* π *is expressed as*

$$\pi(\lambda) = \alpha_0 \lambda^{m+1} \frac{\prod_{j=1}^{n-m}(\lambda - P_j)}{\prod_{j=1}^{n-m}(\lambda - Q_j)}, \quad P_0 = 0, \quad \alpha_0 = \frac{\prod_{j=1}^{n-m}(1 - Q_j)}{\prod_{j=1}^{n-m}(1 - P_j)}$$

for some m *and* n *with* $1 \leqq m \leqq n-1$*. Here* $P_j \in X^S = \{\lambda \in X \mid 0 < |\lambda| < 1\}$ *and* $Q_j = 1/\overline{P_j}$ *for any* $1 \leqq j \leqq n-m$*.*

(2) $\mathcal{L}$ *is a line bundle of degree* n*.*

Theorem 10. *Choosing a complex coordinate on the source suitably, the harmonic map* $\Psi \colon \mathbb{R}^2 \to \mathbb{C}P^n$ *corresponding to the above spectral data* $(X_\tau, \pi, \mathcal{L})$ *is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i(z)$ *is a function defined by*

$$\Psi_i(z) = \exp\left(\eta_i^{-1} z - \eta_i \overline{z}\right) \cdot \frac{\prod_{j=1}^{n-m}(\eta_i - P_j)}{\prod_{j=1}^{n-m}(\eta_i - R_j)}. \tag{1.1}$$

Here $\{\eta_0, \ldots, \eta_n\}$ *is the inverse image* $\pi^{-1}(1)$ *of* 1 *by* π *and* $R_+ = \sum_{j=1}^{n} R_j$ *is a divisor given by the intersection of* X^S *with* R*, that is,* $R_+ = X^S \cap R$*.*

As for spectral data with elliptic spectral curves, we shall prove

Theorem 12. *Let* X *be a smooth elliptic curve. Then* $(X, \pi, \mathcal{L})$ *is a spectral data if and only if the following conditions are satisfied:*

(1) X *is an elliptic curve* $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$*, where* τ *is a pure imaginary number* $\sqrt{-1}t$ *with* $t > 0$*.* ρ_X *is an anti-holomorhic involution induced by the usual conjugation of* $\mathbb{C}$*. Regarded as a doubly periodic meromorphic function on* $\mathbb{C}$*,* π *is expressed as*

$$\pi(u) = C \frac{\theta_1(u - P_0)^{m+1} \prod_{j=1}^{n-m-1} \theta_1(u - P_j) \cdot \theta_1(u - P_{n-m} - W)}{\theta_1(u - Q_0)^{m+1} \prod_{j=1}^{n-m} \theta_1(u - Q_j)}$$

for some m *and* n *with* $1 \leqq m \leqq n-1$*. Here* $P_i \in X^S = \{x \in X \mid 0 < \mathrm{Im}\,\mathrm{x} < \mathrm{Im}\,\tau/2 \pmod{\mathrm{Im}\,\tau\mathbb{Z}}\}$ *and* $Q_i = \overline{P_i} \pmod{\mathbb{Z} \oplus \mathbb{Z}\tau}$ *for any* $0 \leqq i \leqq n-m$*;* $W = (m+1)P_0 + \sum_{i=1}^{n-m} P_i - (m+1)Q_0 - \sum_{i=1}^{n-m} Q_i$*;* $P_0 \neq P_i$ *for* $i \neq 0$*;* W *belongs to* $\mathbb{Z} \oplus \mathbb{Z}\tau$*; and* C *is the unique constant such that* $\pi(0) = 1$*.*

(2) *Let* $r : \mathrm{Pic^{n+1}(X)} \to \mathrm{Pic^0(X)}$ *be a map defined by* $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_X(-R_+)$, *where* $R_+ = \sum_{j=0}^n R_j$ *is a divisor of degree* $n+1$ *given by the intersection of* X^S *with* R, *that is,* $R_+ = X^S \cap R$. *Then,* $\mathcal{L}$ *is an element of the inverse image of* $\left(\mathbb{Z} \oplus \sqrt{-1}\mathbb{R}\right) / (\mathbb{Z} \oplus \tau\mathbb{Z})$ *by the composition* $J \circ r$. *Here* J *is a biholomorphic map from* $\mathrm{Pic^{n+1}(X)}$ *to* $J(X)$.

Theorem 13. *Choosing a complex coordinate on the source suitably, the harmonic map* $\Psi \colon \mathbb{R}^2 \to \mathbb{C}P^n$ *corresponding to the above spectral data* $(X_\tau,\, \pi,\, \mathcal{L} = \mathcal{O}_X(D))$ *is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i(z)$ *is a function defined by*

$$\begin{aligned}\Psi_i(z) =& \mu_i \exp\left(z[\zeta_w(\eta_i - P_0) - A\,\eta_i] - \overline{z}[\zeta_w(\eta_i - Q_0) - A\,\eta_i]\right)\\ &\cdot \frac{\theta_1(\eta_i - P_0)^m \prod_{j=1}^{n-m} \theta_1(\eta_i - P_j)\theta_1(\eta_i + mP_0 - \sum_{j=1}^{n-m} P_j - D - z + \bar{z})}{\prod_{j=0}^n \theta_1(\eta_i - R_j)}.\end{aligned} \tag{1.2}$$

Here ζ_w *is Weierstrass's zeta function,* $\{\eta_0,\, \dots\,,\, \eta_n\}$ *is the inverse image* $\pi^{-1}(1)$ *of* 1 *by* π, μ_i *is a constant given by* $\mu_i = \exp\left(2\pi\sqrt{-1}(D - R_+)\mathrm{Im}\,\eta_\mathrm{i}/\mathrm{t}\right)$, *and* A *is a constant depending only on the complex structure of* X.

Next, we review McIntosh's construction of spectral data from these harmonic maps. To this end, we need a recent work of Burstall and Pedit [5] on dressing orbits of harmonic maps, who studied an action of a certain loop group on the space of primitive harmonic maps of $\mathbb{R}^2$ into a k-symmetric space. Using their results and the fact that a flag manifold $F^r(\mathbb{C}P^n)$ is a rank one $(r+2)$-symmetric space, McIntosh proved that possibly after an isometry, every primitive lift of finite type lies in a single dressing orbit $\mathcal{O}_\Lambda$. A distinctive feature of these orbits $\mathcal{O}_\Lambda$ is that they admit a hierarchy of commuting flows (conservation laws). We show that this hierarchy can be used to characterize those harmonic maps of finite type, that is, a harmonic map in $\mathcal{O}_\Lambda$ is of finite type if and only if its orbit under the hierarchy is finite-dimensional. Moreover, on this orbit, there exists a dynamical system whose flows are generated by the action of an abelian Lie subalgebra $\mathfrak{C}_R$ of the loop algebra. In terms of this subalgbra, a harmonic map of finite type is described as a map with finite dimensional $\mathfrak{C}_R$-orbit. The stabilizer of such a point determines a maximal abelian subalgebra $\mathfrak{R}$ of the Lie algebra of polynomial Killing fields for the map. Then $\mathfrak{R}$ is a commutative one-dimensional unital $\mathbb{C}$-algebra and the spectrum $\operatorname{Spec}\mathfrak{R}$ of $\mathfrak{R}$ is an affine curve whose completion by smooth points yields X. Since the Killing fields are elements of the loop algebra, $\mathfrak{R}$ comes equipped with a representation, which provides X with $\mathcal{L}$. Furthermore, the tangent space to the $\mathfrak{C}_R$-orbit is identified with the tangent space to a real subgroup $J_R(X)$ of $J(X)$, whose dimension gives the arithmetic genus of X.

In connection with the periodicity, McIntosh observed that the harmonic map of $\mathbb{R}^2$ into a complex projective space $\mathbb{C}P^n$ associated to a spectral data $(X, \pi, \mathcal{L})$ is doubly periodic if and only if a certain homomorphism from $\mathbb{R}^2$ to a generalized Jacobian $J(X_{\mathfrak{o}})$ is doubly periodic. However, generally it is hard to compute this homomorphism. In the case of spectral data with spectral curves of genus 0 or 1, we shall explicitly construct these homomorphisms and prove the following

Theorem 11. *Let $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ be the harmonic map in Theorem 10. Then Ψ is doubly periodic with periods v_1, $v_2 \in \mathbb{C}$ if and only if the set*

$$V = \bigcap_{1 \leqq i \leqq n} \frac{\pi}{\beta_i} (\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}) \tag{1.3}$$

contains the 2-dimensional lattice $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, where $\beta_1, \ldots, \beta_n$ are complex numbers defined by $\beta_i = \eta_i^{-1} - \eta_0^{-1}$.

For the case of an elliptic spectral curve X, we shall also prove the following

Theorem 14. *The harmonic map $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ in Theorem 13 is doubly periodic with periods v_1, $v_2 \in \mathbb{C}$ if and only if the set $V = \bigcap_{0 \leqq i \leqq n} V_i$ contains the 2-dimensional lattice $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, where $V_0, \ldots, V_n$ are the sets defined by*

$$V_i = \begin{cases} \pi\beta_i^{-1}\left(\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}\right) & \text{if } \beta_i \neq 0, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

Here $\beta_0, \beta_1, \ldots, \beta_n$ *are complex numbers defined by*

$$\beta_0 = -2\pi/t, \quad \beta_i = [\zeta_w(\eta_0 - P_0) - \zeta_w(\eta_i - P_0) - B\,(\eta_0 - \eta_i)\tau^{-1}] \quad (1 \leqq i \leqq n).$$

Now we summarize the content of each section.

In Section 2, we recall the definition of the spectral data, and review, with a slight improvement, McIntosh's construction of harmonic maps in terms of these spectral data.

In Sections 3 and 4, we shall review fundamental results obtained by McIntosh. The harmonicity of the maps constructed in the previous sections is shown in Section 3. We also describe in Section 4 the construction of spectral data conversely from non-isotropic harmonic tori in complex projective spaces.

In Section 5, we discuss the properties of spectral data whose spectral curves are compact connected Riemann surfaces.

In Section 6, all spectral data with smooth rational or elliptic spectral curves are classified (Theorems 9 and 12), and corresponding harmonic maps are explicitly constructed (Theorems 10 and 13). Moreover, we prove a necessary and sufficient condition for a constructed harmonic map to be doubly periodic (Theorems 11 and 14). We also construct some examples of harmonic tori by using the method developed in this section. In Sections 6.3 and 6.4, the proofs of Theorems 9 and 12 are given respectively. Sections 6.5 and 6.6 are devoted to proving Theorems 10 and 13. Finally, in Section 6.7 we introduce certain homomorphisms into generalized Jacobians of spectral curves and prove Theorem 14.

2. Construction of harmonic maps into complex projective spaces from spectral data

2.1. **Spectral data.** Let $\mathbb{P}^1$ be the smooth rational curve and λ an affine coordinate on it. Let ρ be an anti-holomorphic involution on $\mathbb{P}^1$ defined by $\lambda \mapsto 1/\overline{\lambda}$. Then the fixed point set of ρ consists of the equator S^1 defined by $\{\lambda \in \mathbb{P}^1 \mid |\lambda| = 1\}$.

First we recall the definition of a spectral data introduced by McIntosh (cf. §2.1 in [18]).

Definition 1. *A spectral data is a triplet $(X, \pi, \mathcal{L})$ of isomorphism classes which satisfies the following conditions:*

(1) *X is a complete, connected, algebraic curve of arithmetic genus p, with a real involution ρ_X.*

(2) *π is a meromorphic function on X of degree $N = n+1$ satisfying $\pi \circ \rho_X = 1/\overline{\pi}$, with a distinguished zero P_0 of degree $m+1$ ($m \geqq 1$) and a pole $P_\infty = \rho_X(P_0)$. We regard X as a covering of degree $n+1$ of the rational curve $\mathbb{P}^1$ via π.*

(3) *$\mathcal{L}$ is a line bundle over X of degree $p+n$ satisfying*

$$\mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \cong \mathcal{O}_X(R), \tag{2.1}$$

where R is the ramification divisor for π. By identifying $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D)$, we can find a meromorphic function f on X which satisfies the following conditions:

(a) *The divisor (f) of f is given by $D + \rho_{*X}D - R$ and $\overline{\rho_X^* f} = f$.*

(b) *Let $X_\mathbb{R}$ be the preimage of S^1 by π. Then f is non-negative on $X_\mathbb{R}$.*

(4) *π has no branch points on S^1 and ρ_X fixes every point of $X_\mathbb{R}$.*

Two triplets are the same if there exists a biholomorphic map between spectral curves which carries the real structure, the meromorphic function and the isomorphism class of the line bundle each other.

When X is a compact conneted Riemann surface, the above definition of spectral data becomes simpler.

Theorem 1. *Let X be a compact conneted Riemann surface. A triplet $(X, \pi, \mathcal{L})$ is a spectral data if and only if it satisfies the following conditions:*

(1) *X is a compact connected Riemann surface of genus p, with real involution ρ_X. The set $X \setminus X^\rho$ consists of two connected components X^N, X^S, where X^ρ is the fixed points of ρ_X. Moreover, X^ρ decomposes into the disjoint union $X^\rho = \coprod_{i=1}^{\nu(X)} S_i^1$ with $S_i^1 = S^1$, that is, $\nu(X)$ copies of a loop.*

(2) *π is a meromorphic function on X of degree $N = n+1$, which satisfies either that all poles are contained in X^N and all zeros are contained in X^S, or that all poles are contained in X^S and all zeros are contained in X^N. Moreover, π has a zero P_0 of order $\geqq 2$ and a point $x \in X^\rho$ such that $|\pi(x)| = 1$, and the set of poles coincides with the image of the set of zeros by ρ_X.*

(3) *$\mathcal{L}$ is a line bundle over X of degree $p + n$ satisfying*

$$D + \rho_{X*}(D) \cong R, \quad \delta(\mathcal{L}) = 0,$$

where R is the ramification divisor for π, D is a divisor such that $\mathcal{L} \cong \mathcal{O}_X(D)$, and $\delta(\mathcal{L})$ is a number defined as follows:

$$\delta(\mathcal{L}) = \nu(X) - |\sharp\{s_i \in \Lambda \mid g(s_i)/g(s_1) > 0\} - \sharp\{s_i \in \Lambda \mid g(s_i)/g(s_1) < 0\}|,$$

where g is a meromorphic function with the divisor $(g) = D + \rho_{X*}D - R$ and Λ is the set of points $s_1, s_2, \ldots, s_{\nu(X)}$ such that $s_i \in S_i^1$ and $g(s_i) \neq 0, \infty$.

(The proof of this theorem will be given in Section 5.)

2.2. **Construction of harmonic maps into complex projective spaces.** By applying McIntosh's method of constructing harmonic maps in terms of spectral data, we shall construct harmonic maps which correspond to spectral data having smooth rational or elliptic spectral curves. We also prove Theorems 10 and 13 in this section.

From now on, for a Riemann surface X and a sheaf $\mathcal{F}$ on X, we denote by $H^i(X, \mathcal{F})$ and $H^i(Y, \mathcal{F})$ the i-th cohomology of the sheaf of holomorphic sections of $\mathcal{F}$ and its restriction to an open subset Y of X, respectively. We also denote the dimension of $H^i(X, \mathcal{F})$ by $h^i(X, \mathcal{F})$. Let $(X, \pi, \mathcal{L})$ be a spectral data as in Definition 1. By identifying $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D)$, we equip $H^0(X, \mathcal{L})$ with a positive definite Hermitian form h as follows.

For given $u, v \in H^0(X, \mathcal{L})$, we define a rational function $h(u, v)$ on $\mathbb{P}^1$ by

$$h(u,v)(p) = \sum_{x \in \pi^{-1}(p)} f(x)u(x)\overline{(v \circ \rho_X)(x)}, \tag{2.2}$$

where p is a point of $\mathbb{P}^1$. Then it is known that $h(u, v)$ is a constant function and the following holds.

Theorem 2. ([18]) *The Hermitian form h is positive definite on $H^0(X, \mathcal{L})$. Moreover, $\pi_*\mathcal{L}$ is a trivial vector bundle of rank $(n+1)$ over $\mathbb{P}^1$, where $n+1$ is the degree of π.*

Let $\pi^{-1}(1) = \{\eta_0, \ldots, \eta_n\}$, the inverse image of 1 by π, and $\theta_i (0 \leqq i \leqq n)$ a local trivialization for $\mathcal{L}$ over a neighbourhood of η_i. Using these local trivializations, the Hermitian form h in (2.2) has also the following expression. For $u \in H^0(X, \mathcal{L})$, let $u_0, \ldots, u_n$ be the complex numbers defined by $u(\eta_i) = u_i\theta_i(\eta_i)$. For $v \in H^0(X, \mathcal{L})$, we define the complex numbers $v_0, \ldots, v_n$ in a similar way. Then (2.2) becomes

$$h(u,v) = \sum_{i=0}^{n} a_i u_i \overline{v_i}, \tag{2.3}$$

where $a_0, \ldots, a_n$ are positive real numbers depending only on the choice of $\theta_0, \ldots, \theta_n$.

Next we construct a line bundle $L(z)$ with a complex parameter z. Let $U(P_0)$ be a neighbourhood of P_0 and $U(P_\infty)$ a neighbourhood of P_∞ defined by $U(P_\infty) = \rho_X(U(P_0))$. Let ζ be a meromorphic function on $U(P_0) \cup U(P_\infty)$ satisfying $\pi = \zeta^{m+1}$ and $\zeta \circ \rho_X = 1/\bar{\zeta}$. We fix an open cover $X_A \cup X_I$ of X, where $X_A = X \setminus \{P_0, P_\infty\}$ and $X_I = U(P_0) \cup U(P_\infty)$. Let $L(z)$ be the unique line bundle with local trivializations

θ_A^z and θ_I^z over X_A and X_I respectively, such that

$$\theta_I^z = \exp(z\zeta^{-1} - \bar{z}\zeta)\,\theta_A^z \quad \text{on } X_A \cap X_I. \tag{2.4}$$

Let $\mathcal{L}_0$ be an ideal sheaf of $\mathcal{L}$ defined by $\mathcal{L}_0 = \mathcal{L}(-mP_0 - E_0)$, where E_0 is the restriction of the zero divisor of π to X_A, that is, $E_0 = P_1 + P_2 + \cdots + P_{n-m}$. Then it is known that $H^0(X, \mathcal{L}_0 \otimes L(z))$ is a one-dimensional complex vector space. For each $z \in \mathbb{C}$, fix a non-zero global section τ of $\mathcal{L}_0 \otimes L(z)$. Then $\tau \otimes {\theta_A^z}^{-1}$ belongs to $H^0(X_A, \mathcal{L})$ and we can find holomorphic functions $\psi_0^z, \ldots, \psi_n^z$ over $\mathbb{P}^1 \setminus \{0, \infty\}$ such that

$$\tau_A \otimes {\theta_A^z}^{-1} = (\psi_0^z \circ \pi)\sigma_0 + \cdots + (\psi_n^z \circ \pi)\sigma_n, \tag{2.5}$$

where $\{\sigma_0, \ldots, \sigma_n\}$ is an orthonormal basis of $H^0(X, \mathcal{L})$ with respect to the Hermitian form h.

Now we are going to construct a harmonic map corresponding to the spectral data $(X, \pi, \mathcal{L})$. Let $\psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ be a map defined by

$$z = x + \sqrt{-1}y \mapsto [\psi_0^z(1) : \cdots : \psi_n^z(1)].$$

Then it is known that ψ is a harmonic map corresponding to the spectral data $(X, \pi, \mathcal{L})$. This construction is due to McIntosh, which is described in detail in [17] and [18]. However, in general it seems difficult to compute $\psi_0^z, \ldots, \psi_n^z$.

We shall now present a method which determines the values of $\psi_0^z(\lambda), \ldots, \psi_n^z(\lambda)$ at $\lambda = 1$. We define a complex $(n+1) \times (n+1)$ matrix $M = (M_{ij})$ by

$$M_{ij}\,\theta_i(\eta_i) = \sigma_j(\eta_i). \tag{2.6}$$

Let t_j^z be complex numbers defined by

$$\tau \otimes {\theta_A^z}^{-1}(\eta_j) = t_j^z\,\theta_j(\eta_j). \tag{2.7}$$

Substituting (2.6) and (2.7) to (2.5), we obtain

$${}^t(t_0^z, \ldots, t_n^z) = M\,{}^t(\psi_0^z(1), \ldots, \psi_n^z(1)). \tag{2.8}$$

Lemma 1. *The determinant of M does not vanish.*

Proof. Since $\{\sigma_0, \dots, \sigma_n\}$ is an orthonormal basis with respect to h, we have $h(\sigma_i, \sigma_j) = \delta_{ij}$. From this and the identity (2.3), it is easy to see that the following identity holds:

$$M \operatorname{diag}(a_0, \dots, a_n) M^* = I_{n+1},$$

where $\operatorname{diag}(a_0, \dots, a_n)$ denotes the diagonal matrix with diagonal components $a_0, \dots, a_n$, and I_{n+1} is the unit matrix of degree $n+1$. In particular, we see that the determinant of M does not vanish. □

Hence the inverse matrix M^{-1} of M exists, and $\psi_0^z(1), \dots, \psi_n^z(1)$ are determined as

$$(2.9) \qquad {}^t(\psi_0^z(1), \dots, \psi_n^z(1)) = M^{-1}\,{}^t(t_0^z, \dots, t_n^z).$$

Moreover, it is known that the components of the matrix M and $t_0^z, \dots, t_n^z$ can be expressed by using theta functions and Baker-Akhizer functions (cf. [16]).

Constructing a special orthonormal basis, the above formula takes a much simpler form. In fact, for $0 \leqq i \leqq n$, take a non-zero element $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$. Rescaling σ_i, we obtain an orthonormal basis $\{\sigma_i\}$ of $\mathcal{L}$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$. Then the matrix M is diagonal and M_{ii} is given by

$$M_{ii} = \left.\frac{\sigma_i}{\theta_i}\right|_{\eta_i}.$$

Therefore the right hand side of the equation (2.9) becomes

(2.10)
$${}^t\left(\left.\frac{\tau \otimes \theta_A(z)^{-1}}{\sigma_0}\right|_{u=\eta_0}, \left.\frac{\tau \otimes \theta_A(z)^{-1}}{\sigma_1}\right|_{u=\eta_1}, \dots, \left.\frac{\tau \otimes \theta_A(z)^{-1}}{\sigma_n}\right|_{u=\eta_n} \right).$$

Let $\psi(z, \bar{z}, u)$ be a function on X such that $\psi(z, \bar{z}, u)\theta_A(z)$ is an element of $H^0(X, \mathcal{L}_0 \otimes L(z))$. Setting $\tau = \psi(z, \bar{z}, u)\theta_A(z)$ and substituting τ into (2.10), we get

$$(2.11) \qquad \psi_i^z(1) = \left.\frac{\psi(z, \bar{z}, u)}{\sigma_i}\right|_{u=\eta_i} \quad \text{for } 0 \leqq i \leqq n.$$

Before closing this subsection, we prove the following lemma for later use.

Lemma 2. *Given a function $\phi(z, \bar{z}, u)$ on X with the parameter z, let U and V be neighbourhoods of the set of the points $\{P_0, P_\infty\}$ which satisfy the following conditions:*

(1) $\{P_0, P_\infty\} \subset U \subset V \subset X_I$.

(2) *$\phi(z, \bar{z}, u)$ is a holomorphic section of $\mathcal{O}_X(M)$ on $X \setminus U$ for any $z \in \mathbb{C}$, where M is a divisor on $X \setminus V$.*

(3) *$\phi(z, \bar{z}, u) \exp(-z\zeta^{-1} + \bar{z}\zeta)$ is a holomorphic section of $\mathcal{O}_X(N)$ on V for any $z \in \mathbb{C}$, where N is a divisor on U.*

Then $\phi(z, \bar{z}, u)\theta_A(z)$ belongs to $H^0(X, \mathcal{F} \otimes L(z))$ for any $z \in \mathbb{C}$, where $\mathcal{F} \cong \mathcal{O}_X(M + N)$.

Proof. From the condition (2), $\phi(z, \bar{z}, u) \otimes \theta_A(z)$ clearly belongs to $H^0(X \setminus U, \mathcal{O}_X(M) \otimes L(z)) = H^0(X \setminus U, \mathcal{F} \otimes L(z))$. It suffices to show that $\phi(z, \bar{z}, u) \otimes \theta_A(z)$ belongs to $H^0(V, \mathcal{O}_X(N) \otimes L(z)) = H^0(V, \mathcal{F} \otimes L(z))$. By using (2.4), we see that $\phi(z, \bar{z}, u) \otimes \theta_A(z) = \phi(z, \bar{z}, u) \exp(-z\zeta^{-1} + \bar{z}\zeta) \otimes \theta_I(z)$ on $V (\subset X_I)$. On the other hand, from the condition (3) it follows that $\phi(z, \bar{z}, u) \exp(-z\zeta^{-1} + \bar{z}\zeta)$ is an element of $H^0(V, \mathcal{F})$ and hence $\phi(z, \bar{z}, u) \otimes \theta_A(z)$ belongs to $H^0(V, \mathcal{F} \otimes L(z))$. Thus $\phi(z, \bar{z}, u)\theta_A(z)$ is a global holomorphic section of $\mathcal{F} \otimes L(z)$ on X. □

3. Proof of the harmonicity of corresponding maps

In this section, we shall prove the harmonicity of those maps constructed in the previous section. For this purpose, we shall show that a primitive map from a Riemann surface M into a certain flag bundle $F^r(\mathbb{C}P^n)$ over $\mathbb{C}P^n$ is harmonic. Moreover, a map from M to $\mathbb{C}P^n$ obtained as the projection of the above map is also harmonic. Since the maps constructed in the previous section conicide with such projections, this completes the proof.

3.1. **Primitive maps.** First, we will recall the definition of primitive maps. Let $pr\colon F^r(\mathbb{C}P^n) \to \mathbb{C}P^n$ donote the bundle of flags in the holomorphic tangent bundle $T^{1,0}\mathbb{C}P^n$ with fiber

$$F^r_x(\mathbb{C}P^n) = \{w_1 \subset \cdots \subset w_r \subset T^{1,0}_x\mathbb{C}P^n \mid \dim \mathrm{w_j} = \mathrm{j}\}.$$

Let U_i denote the unitary group of degree i. For convenience, set $m = r+1$. We denote by G and H the groups U_{n+1} and $\underbrace{U_1 \times \cdots \times U_1}_{m+1\,\text{times}} \times U_{n-m}$, respectively. Then, we can represent $F^r(\mathbb{C}P^n) = G/H$ as a homogeneous space. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Then we have the canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

It is known that $F^r(\mathbb{C}P^n)$ have the structure of $(r+2)$-symmetric space in the sense of Kowalski [15]. In fact, let ν be an automorphism on G defined by $g \mapsto \mathrm{Ad}(\sigma)\mathrm{g}$ with σ the diagonal matrix $\mathrm{diag}(1, \omega, \ldots, \omega^{-\mathrm{r}}, \omega^{-\mathrm{r}-1}, \ldots, \omega^{-\mathrm{r}-1})$ for $\omega = \exp(2\pi\sqrt{-1}/(r+2))$. Let τ be the automorphism indueced by ν of order $(r+2)$ on G/H which gives the $(r+2)$-symmetric structure on G/H. Let $\mathfrak{g}_i$ be the ω^i- eigenspace of τ, where $\omega = \exp(2\pi\sqrt{-1}/(r+2))$. Then we have

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{r} \mathfrak{g}_i, \quad \mathfrak{h}^{\mathbb{C}} = \mathfrak{g}_0, \quad \mathfrak{m}^{\mathbb{C}} = \sum_{i=1}^{r+1} \mathfrak{g}_i,$$
$$\mathfrak{g}_{-i} = \overline{\mathfrak{g}_i}, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

The map $\mathfrak{g} \longrightarrow T_x(G/H)$ given by $\xi \mapsto d/dt\,|_{t=0} \exp t\xi \cdot x$ restricts to an isomorphism

$\mathrm{Ad(g)}\cdot\mathfrak{m} \longrightarrow \mathrm{T_x(G/H)}$. We denote its inverse map by $\beta : T_x(G/H) \longrightarrow \mathrm{Ad(g)}\cdot\mathfrak{m} \subset \mathfrak{g}$ and we may regard β as a $\mathfrak{g}$-valued 1-form on G/H, which is called the *Maurer-Cartan form* for G/H. Denote by $[\mathfrak{g}_i]$ the vector bundle over G/H, for which the fiber at $x = g \cdot o \in G/H$ is given by $\mathrm{Ad(g)}\mathfrak{g}_\mathrm{i}$.

Definition 2. *The map* $\psi\colon \mathbb{C} \to F^r(\mathbb{C}P^n)$ *is said to be primitive if* $(\psi^*\beta)(\partial/\partial z)$ *is* $[\mathfrak{g}_{-1}]$*-valued, where* β *is the Maurer-Cartan form for* G/H*.*

3.2. **A parallel transport.** In order to construct a desired map from $\mathbb{R}^2$ and a connection for a line bundle over $\mathbb{R}^2$, we need to define a parallel transport of a section of $\mathcal{L}$ to that of a line bundle over $\mathbb{R}^2$. Let $J(X)$ denote the Jacobian variety of the spectral curve X, i.e., $J(X) = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$, which is a p-dimensional complex torus, p being the genus of X, and defined by the long exact sequence induced from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

The set of all line bundles $L \in J(X)$ which satisfy $\overline{\rho_{X*}L} \cong L^{-1}$ forms a subgroup of $J(X)$ by tensor product. We denote by $J_\mathbb{R}(X)$ the connected component of the identity of this subgroup (the identity is given by trivial line bundles). Then, it is known that $J_\mathbb{R}(X)$ is a p-dimensional real torus. For any $L \in J_\mathbb{R}(X)$, we see that a line bundle $\mathcal{L} \otimes L$ satisfies $(\mathcal{L} \otimes L) \otimes \overline{\rho_{X*}(\mathcal{L} \otimes L)} \cong \mathcal{O}_X(R)$. In this case, we say that $\mathcal{L} \otimes L$ is *real*. Note that when we replace $\mathcal{L}$ by $\mathcal{L} \otimes L$ for $L \in J_\mathbb{R}(X)$, we see that f is still non-negative on the preimage $X_\mathbb{R}$ of the equator S^1. In fact, f is independent of L. Since $\mathrm{deg}(\mathcal{L} \otimes \mathrm{L}) = \mathrm{deg}(\mathcal{L}) = \mathrm{n} + \mathrm{p}$, it follows from Theorem 2 that $\pi_*(\mathcal{L} \otimes L)$ is a trivial bundle of rank $n + 1$ and $h^0(X, \mathcal{L} \otimes L) = n + 1$.

Now, consider a complex vector bundle $H^0(X) \mapsto J_\mathbb{R}(X)$ for which the fiber at $L \in J_\mathbb{R}(X)$ is given by a $(n + 1)$-dimensional complex vector space $H^0(X, \mathcal{L} \otimes L)$. Recall that $X = X_A \cup X_I$. A given line bundle $L \in J(X)$ can be trivialized over X_A or X_I. We denote by θ_A and θ_I its trivializing sections over X_A and X_I, respectively, i.e.,

$$L\,|_{X_A} \overset{\theta_A}{\cong} X_A \times \mathbb{C}, \quad L\,|_{X_I} \overset{\theta_I}{\cong} X_I \times \mathbb{C}.$$

Over $X_A \cap X_I$, we have a transition relation $\theta_I = e^a\theta_A$. Thus, for $L \in J_{\mathbb{R}}(X)$, we have a 1-cocycle (e^a, X_A, X_I). Conversely, each 1-cocycle (e^a, X_A, X_I) defines a line bundle L with e^a as a transition function. Then, consider a map $L : \mathcal{G} = H^0(X_A \cap X_I, \mathcal{O}_X) \longrightarrow J(X)$ defined by $a \mapsto L(a)$, where $L(a)$ denotes a line bundle with a transition function e^a. Set

$$\mathcal{G}_{\mathbb{R}} = \{a \in \mathcal{G} \mid \overline{\rho_{X*}a} = -a\}.$$

Then, we see that $\mathrm{Im}(\mathrm{L}\,|_{\mathcal{G}_{\mathbb{R}}}) = \mathrm{J}_{\mathbb{R}}(\mathrm{X})$.

Now, fix a trivializing section θ for $\mathcal{L}$ over X_I such that $\mathrm{Tr}(\mathrm{f} \cdot \theta \otimes \overline{\rho_{\mathrm{X}*}\theta}) = 1$. Here Tr is the trace homomorphism, which sends each element of $H^0(X, \mathcal{O}_X(R))$ to those of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$. For $a \in \mathcal{G}_{\mathbb{R}}$, set $\theta_a = \theta \otimes \theta_I$, which gives a trivializing section for $\mathcal{L} \otimes L(a)$ over X_I. We now want to define a map

$$\iota_a : H^0(X_A, \mathcal{L} \otimes L(a)) \longrightarrow H^0(X_A, \mathcal{L}).$$

Lemma 3. *For $\sigma_a \in H^0(X_A, \mathcal{L} \otimes L(a))$, define $\iota_a(\sigma_a)$ by*

$$\iota_a(\sigma_a) = e^a(\sigma_a\theta_a^{-1})\theta.$$

Then, we have $\iota_a(\sigma_a) \in H^0(X_A, \mathcal{L})$.

Proof. Let τ be a trivializing section of $\mathcal{L}$ over X_A. We may write $\theta = e^c\tau$. Hence we have $\theta_a = e^{a+c}\tau \otimes \theta_A$. Now, we calculate

$$\begin{aligned} \iota_a(\sigma_a) &= e^a(\sigma_a\theta_a^{-1})\theta \\ &= e^{-c}\sigma_a(\tau \otimes \theta_A)^{-1}\theta = \sigma_a(\tau \otimes \theta_A)^{-1}\tau, \end{aligned} \tag{3.1}$$

where $\sigma_a(\tau \otimes \theta_a)^{-1}$ is a holomorphic function over X_A and τ is a trivializing section of $\mathcal{L}$ over X_A. Therefore we have $\iota_a(\sigma_a) \in H^0(X_A, \mathcal{L})$. □

In fact, $\iota_a : H^0(X_A, \mathcal{L} \otimes L(a)) \longrightarrow H^0(X_A, \mathcal{L})$ is an isomorphism. The injectivity of ι_a is obvious. To show the surjectivity of ι_a, take an arbitrary $\sigma \in H^0(X_A, \mathcal{L})$. Then, we may write $\sigma = b\tau$ for some $b \in H^0(X_A, \mathcal{O})$. Choose $\sigma_a = b(\tau \otimes \theta_A)$. Then we have $\iota_a(\sigma_a) = b\tau = \sigma$ by (3.1), proving the surjectivity of ι_a.

Let $L^*H^0(X) \to \mathcal{G}_{\mathbb{R}}$ deonte the pull-back bundle of the bundle $H^0(X) \to J_{\mathbb{R}}(X)$ by $L : \mathcal{G}_{\mathbb{R}} \longrightarrow J_{\mathbb{R}}(X)$. Let $\{\tau_0, \cdots, \tau_n\}$ be an orthonormal frame of global sections of $\mathcal{L}$, and denote by B the algebra of holomorphic maps $A \to \mathbb{C}$. Then we have $H^0(X_A, \mathcal{L}) = \mathrm{Span}\{\tau_0, \cdots, \tau_n\}\,|_{X_A}$, since $H^0(X_A, \mathcal{L})$ is a free B-module of rank $(n+1)$ by the fact that $H^0(X_A, \mathcal{L}) = H^0(A, \pi_*\mathcal{L})$ and $\pi_*\mathcal{L}$ is a trivial bundle of rank $(n+1)$. Any element of $H^0(X_A, \mathcal{L})$ is expressed as $\sum \sigma_j(\lambda)\tau_j$. Define an evaluation map $ev_1 : H^0(X_A, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L})$ by $\sum \sigma_j(\lambda)\tau_j \mapsto \sum \sigma_j(1)\tau_j$, where $\sigma_j(1)$ is the value of $\sigma_j(\lambda)$ at $\lambda = 1$. Then the composition $ev_1 \circ \iota_a\,|_{H^0(X, \mathcal{L} \otimes L(a))}$: $H^0(X, \mathcal{L} \otimes L(a)) \longrightarrow H^0(X, \mathcal{L})$ gives rise to an isomorphism. Indeed, clearly it is surjective by its construction and is injective by the fact that $h^0(X, \mathcal{L} \otimes L(a)) = h^0(X, \mathcal{L}) = n + 1$.

Lemma 4. *Let $\sigma_1, \sigma_2 \in H^0(X, \mathcal{L} \otimes L(a))$, and set $s_j = \iota_a(\sigma_j)$ for $j = 1, 2$. Then $h(s_1, s_2)$ is constant.*

Proof. For simplicity, set $\mathcal{L}(a) = \mathcal{L} \otimes L(a)$.

We first note that the map $\iota_a : H^0(X_A, \mathcal{L}(a)) \longrightarrow H^0(X_A, \mathcal{L})$ induces an isomorphism $\kappa_a : H^0(X_A, \mathcal{L}(a) \otimes \overline{\rho_{X*}\mathcal{L}(a)}) \longrightarrow H^0(X_A, \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}})$. In fact, $\kappa_a(\sigma) = \sigma(\theta_a \otimes \overline{\rho_{X*}\theta_a})^{-1}\theta \otimes \overline{\rho_{X*}\theta}$, because the transition functions e^a for $L(a)$ and e^{-a} for $\overline{\rho_{X*}L(a)}$ cancel out each other. Set $s_{12} = \kappa_a(\sigma_1 \otimes \overline{\rho_{X*}\sigma_2})$. We claim that s_{12} is a globally defined holomorphic section of $\mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}}$. Indeed, s_{12} is holomorphic over X_A. In order to see that it is also holomorphic over X_I, set $f_j = \sigma_j\theta_a^{-1}$ for $j = 1, 2$, which is a holomorphic function on X_I. Then we have

$$\begin{aligned} s_{12} &= \sigma_1 \otimes \overline{\rho_{X*}\sigma_2}(\theta_a \otimes \overline{\rho_{X*}\theta_a})^{-1}\theta \otimes \overline{\rho_{X*}\theta} \\ &= f_1\theta \otimes \overline{\rho_{X*}(f_2\theta)}, \end{aligned}$$

which shows that s_{12} is also holomorphic over X_I, since θ is a holomorphic frame field over X_I. Thus, s_{12} is a globally holomorphic section as claimed.

Now we have

$$\begin{aligned} h(s_1,s_2) &= h(\iota_a(\sigma_1),\iota_a(\sigma_2)) \\ &= \mathrm{Tr}(\mathrm{f}\cdot\iota_{\mathrm{a}}(\sigma_1)\otimes\overline{\rho_{\mathrm{X}*}\iota_{\mathrm{a}}(\sigma_2)}) \\ &= \mathrm{Tr}(\mathrm{f}\cdot\kappa_{\mathrm{a}}(\sigma_1\otimes\overline{\rho_{\mathrm{X}*}\sigma_2})) = \mathrm{Tr}(\mathrm{f}\cdot\mathrm{s}_{12}). \end{aligned}$$

Since $\mathrm{Tr}(\mathrm{f}\cdot\mathrm{s}_{12})\in\mathrm{H}^0(\mathbb{P}^1,\mathcal{O})$ (notice that $f\cdot s_{12}\in H^0(X,\mathcal{O}_X(R))$), we see that $h(s_1,s_2)$ is constant. □

3.3. **Ideal sheaves on spectral curves.** Define a map $a:\mathbb{R}^2\longrightarrow\mathcal{G}_{\mathbb{R}}$ by $z\mapsto a(z,\overline{z})=z\zeta^{-1}-\overline{z}\zeta$, where ζ is considered only on $X_A\cap(U_0\cup U_\infty)$, U_0 (resp. U_∞) being a connected component of X_0 (resp. X_∞) which contains P_0 (resp. Q_0). Then $L(a)=L(z\zeta^{-1}-\overline{z}\zeta)$ is a 2-parameter subgroup of $J_{\mathbb{R}}(X)$. We have the following diagram:

$$\begin{array}{ccccc} L(a)^*H^0(X) & \longrightarrow & L^*H^0(X) & \longrightarrow & H^0(X)\supset H^0(X,\mathcal{L}\otimes L) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^2 & \xrightarrow{a} & \mathcal{G}_{\mathbb{R}} & \xrightarrow{L} & J_{\mathbb{R}}(X)\ni L \end{array}$$

We also write $L(a)^*H^0(X)$ as $H^0(X)$ if there is no confusion. Fix a h-orthonormal basis $\{\tau_j\}$ for $H^0(X,\mathcal{L})$ so that $(H^0(X,\mathcal{L}),h)\longrightarrow(\mathbb{C}^{n+1},\langle\ ,\ \rangle)$ is an isometry. We want to decompose the vector bundle $H^0(X)\to\mathbb{R}^2$ into line subbundles which are orthogonal to each other. To this end, we first define the following line bundles, whose sheaves of germs of holomorphic sections are subsheaves of the sheaf of germs of holomorphic sections for $\mathcal{L}$:

(3.2)

$$\begin{cases} \mathcal{L}_j=\mathcal{L}\otimes\mathcal{O}_X(-(m-j)P_0-jQ_0-\displaystyle\sum_{i=1}^{n-m}P_i) \quad \text{for } \mathrm{j}=0,1,\cdots,\mathrm{m}-1, \\ \mathcal{L}_m=\mathcal{L}\otimes\mathcal{O}_X(-mQ_0). \end{cases}$$

Lemma 5. *For $j=0,1,\cdots,m$, each $\mathcal{L}_j$ is non-special, i.e., $h^1(X,\mathcal{L}_j)=0$.*

Proof. Set $\mathcal{I}_j = \mathcal{L} \otimes \mathcal{O}_X(jP_0 - jQ_0)$ for $j = 0, 1, \cdots, m$. Note that $\mathcal{I}_j$ is a real line bundle, i.e., it satisfies $\mathcal{I}_j \otimes \overline{\rho_{X*}\mathcal{I}_j} \cong \mathcal{O}_X(R)$. It follows from Lemma 2 that $\pi_*\mathcal{I}_j$ is a trivial bundle of rank $n+1$. Define $\mathcal{F}_j$ by $\mathcal{F}_j = \mathcal{I}_j \otimes \mathcal{O}_X(-(m+1)P_0 - \sum_{i=1}^{n-m} P_i)$. Then we obtain

$$\mathcal{F}_j = \begin{cases} \mathcal{L}_j(-P_0) & \text{for j} = 0, \cdots, \text{m} - 1, \\ \\ \mathcal{L}_m(-P_0 - \displaystyle\sum_{i=1}^{n-m} P_i) & \text{for j} = \text{m}. \end{cases}$$

Note that $\deg(\mathcal{F}_\mathrm{j}) = \mathrm{p} - 1$ for $j = 0, 1, \cdots, m$.

In general, for any non-special line bundle L we know that $H^0(X, L(-P)) \cong \{s \in H^0(X, L) \mid s(P) = 0\}$, where $L(D) = L \otimes \mathcal{O}_X(D)$ for a divisor D on X. In fact, if we fix a meromorphic section τ with the divisor $(\tau) = (-P)$, then taking the tensor product of each element with τ or τ^{-1} gives an isomorphism. Now, suppose that $\mathcal{F}_j$ has a non-trivial global section. Then there is a global section of $\pi_*\mathcal{I}_j$ which vanishes at $\lambda = 0$, since any global holomorphic section of $\mathcal{F}_j$ gives rise to a global holomorphic section of $\mathcal{I}_j$ with divisor $(m+1)P_0 + \sum P_i$. However, since $\pi_*\mathcal{I}_j$ is a trivial bundle, it must be identically zero. Thus, we see that $h^0(X, \mathcal{F}_j) = 0$. Now, the Riemann-Roch formula implies that $h^1(X, \mathcal{F}_j) = 0$, because $\deg(\mathcal{F}_\mathrm{j}) = \mathrm{p} - 1$ for $j = 0, 1, \cdots, m$.

In general, for any line bundle L and any point $P \in X$, $h^1(X, L) = 0$ implies that $h^1(X, L(P)) = 0$. Indeed, it follows from the Serre duality theorem that $0 = h^1(X, \mathcal{L}) = h^0(X, \Omega_X^{1,0} \otimes L^{-1})$, where $\Omega_X^{1,0}$ is the holomorphic cotangent bundle (= the canonical bundle) of X. Again, it follows from the Serre duality that $h^1(X, L(P)) = h^0(X, \Omega_X^{1,0} \otimes L^{-1}(-P))$. Therefore, if there is a non-trivial element of $H^1(X, L(P))$, then there is a global section of $\Omega_X^{1,0} \otimes L^{-1}$ vanishing at P. However, it must be identically zero, because $h^0(X, \Omega_X^{1,0} \otimes L^{-1}) = 0$.

To complete of the proof, it suffices to notice that $\mathcal{L}_j = \mathcal{F}_j(P_0)$ for $j = 0, 1, \cdots m-1$ and $\mathcal{L}_m = \mathcal{F}_m(P_0 + \sum P_i)$, and then apply the general theory above to these line bundles once or successively. □

Corollary 1. *Given any $a \in \mathcal{G}_{\mathbb{R}}$, we have $h^1(X, \mathcal{L}_j \otimes L(a)) = 0$ for $j = 0, 1, \cdots, m$.*

Proof. We only need to replace $\mathcal{L}$ by $\mathcal{L} \otimes L(a)$ in the definition of $\mathcal{I}_j$ in the proof of Lemma 5. □

Corollary 1, together with the Riemann-Roch theorem, yields that

$$h^0(X, \mathcal{L}_j \otimes L(a)) = \begin{cases} 1 & \text{for} \quad \mathrm{j = 0, 1, \cdots, m - 1,} \\ n + 1 - m & \text{for} \quad \mathrm{j = m.} \end{cases}$$

Then, obviously we obtain:

$$H^0(X, \mathcal{L} \otimes L(a)) = \bigoplus_{j=0}^{m} H^0(X, \mathcal{L}_j \otimes L(a)) \qquad (h\text{-orthogonal sum})$$

Define a map $\tau^1 : H^0(X_A, \mathcal{L}) \longrightarrow \mathbb{C}^{n+1}$ by the composition of $\{\tau_j\}$, which identifies $H^0(X, \mathcal{L})$ with $\mathbb{C}^{n+1}$, and the map ev_1. We thus have the following diagram:

$$\begin{array}{ccc} H^0(X_A, \mathcal{L}) & \xrightarrow{ev_1} H^0(X, \mathcal{L}) \xrightarrow{\{\tau_j\}} \mathbb{C}^{n+1} & \\ \iota_a \uparrow & & \\ H^0(X_A, \mathcal{L} \otimes L(a)) & \supset H^0(X, \mathcal{L} \otimes L(a)) = \bigoplus_{j=0}^{m} H^0(X, \mathcal{L}_j \otimes L(a)). & \end{array}$$

Define line subbundles l_j of the trivial bundle $\mathbb{R}^2 \times \mathbb{C}^{n+1}$ by

$$l_j = \tau^1 \circ \iota_a(H^0(X, \mathcal{L}_j \otimes L(a)) \qquad \text{for} \quad \mathrm{j = 0, 1, \cdots, m.}$$

Then it follows that $\mathbb{R}^2 \times \mathbb{C}^{n+1} = \bigoplus_{j=0}^{m} l_j$, which is an orthogonal direct sum with respect to the inner product $\langle \ , \ \rangle$ on $\mathbb{C}^{n+1}$. To see this, it suffices to prove the following

Lemma 6. *For $z \in \mathbb{R}^2$ and $j = 0, 1, \cdots, m$, let $\sigma_j \in H^0(\mathcal{L}_j \otimes L(a))$. Set $s_j = \iota_a(\sigma_j)$. Then $h(s_j, s_k) = 0$ for $j \neq k$.*

Proof. From Lemma 4 we know that $h(s_j, s_k)$ is constant. Therefore, it suffices to show that when $j \neq k$, $h(s_j, s_k)$ is zero at some point of $\mathbb{P}^1$. Setting $f_j = \sigma_j \theta_a^{-1}$, we see that f_j is a holomorphic function over X_I and $s_j = e^a(\sigma_j \theta_a^{-1})\theta = e^a f_j \theta$. Since σ_j

is a global holomorphic section of $\mathcal{L}\otimes L(a)$, which has a divisor $(m-j)P_0+jQ_0+\sum P_i$ for $j=0,\cdots,m-1$ or a divisor mQ_0 for $j=m$, it follows that f_j has a divisor

$$\begin{cases} (m-j)P_0+jQ_0+\sum P_i & \text{for} \quad \mathrm{j=0,1,\cdots,m-1}, \\ mQ_0 & \text{for} \quad \mathrm{j=m}. \end{cases}$$

Set $r_{jk}=f\cdot\theta\otimes\overline{\rho_{X*}\theta}f_j\overline{\rho_{X*}f_k}$. Then we have $h(s_j,s_k)=\mathrm{Tr(r_{jk})}$. Recall that $X_I=X_0\cup X_\infty$. Denote by U_0 (resp. U_∞) a connected component of X_0 (resp. X_∞) which contains P_0 (resp. Q_0). Recall that there are no branch points on X_0 and X_∞ except P_0 and Q_0. Since $f\cdot\theta\otimes\overline{\rho_{X*}\theta}$ is a meromorphic function with a divisor $(-R)$, it then follows that

$$f\cdot\theta\otimes\overline{\rho_{X*}\theta}=\begin{cases} \zeta^{-m} & \text{in} \quad \mathrm{U_0}, \\ \zeta^m & \text{in} \quad \mathrm{U_\infty}, \\ 1 & \text{elsewhere in} \quad X_I. \end{cases}$$

Therefore, r_{jk} has a divisor

(3.3)
$$\begin{cases} (k-j)P_0+(j-k)Q_0+\sum P_i+\sum Q_i & \text{for} \quad \mathrm{j,k=0,1,\cdots,m-1}, \\ (m-j)P_0+(j-m)Q_0+\sum P_i & \text{for} \quad \mathrm{k=m;j=0,1,\cdots,m-1}. \end{cases}$$

Note that $\pi^{-1}(0)=\{(m+1)P_0,P_1,\cdots,P_{n-m}\}$ and $\pi^{-1}(\infty)=\{(m+1)Q_0,Q_1,\cdots,$ $Q_{n-m}\}$, where $(m+1)P_0$ (resp. $(m+1)Q_0$) stands for the point P_0 (resp. Q_0) with multiplicity $(m+1)$. This, together with (3.3), yields that if $j,k<m$, then

$$\mathrm{Tr(r_{jk})}=\sum_{\pi^{-1}(0)} r_{jk}=0 \qquad \text{when} \quad \mathrm{k>j},$$

$$\mathrm{Tr(r_{jk})}=\sum_{\pi^{-1}(\infty)} r_{jk}=0 \qquad \text{when} \quad \mathrm{j>k},$$

and if $j<k=m$, then

$$\mathrm{Tr(r_{jm})}=\sum_{\pi^{-1}(0)} \mathrm{r_{jm}}=0,$$

proving our assertion. □

Lemma 7. *Let $\sigma : \mathbb{R}^2 \longrightarrow H^0(X)$ be a smooth section for which $\sigma(z, \overline{z})$ is a globally holomorphic section of $\mathcal{F} \otimes L(a)$ for some ideal sheaf $\mathcal{F}$ of $\mathcal{L}$. Let D be the covariant differentiation on $H^0(X)$ induced by the parallel transport of the vector bundle $H^0(X) \to \mathbb{R}^2$. Then, $D_{\partial/\partial z}\sigma$ (resp. $D_{\partial/\partial \overline{z}}\sigma$) is a globally defined holomorphic section of $\mathcal{F}(P_0) \otimes L(a)$ (resp. $\mathcal{F}(Q_0) \otimes L(a)$).*

Remark. Each $\mathcal{L}_j$ is an ideal sheaf of $\mathcal{L}$.

Proof. We can define a connection D on the bundle $H^0(X)$ by

$$D_Z\sigma = \iota_a^{-1}(Z\iota_a(\sigma)),$$

where σ is a section of the bundle $H^0(X) \to \mathbb{R}^2$ and Z is an arbitrary vector field on $\mathbb{R}^2$. Setting $s = \iota_a(\sigma)$ and $f = \sigma\theta_a^{-1}$, we see that $s = e^a f\theta : \mathbb{R}^2 \longrightarrow H^0(X_A, \mathcal{L})$ and $f : \mathbb{R}^2 \longrightarrow H^0(X_I, \mathcal{F} \otimes \mathcal{L}^{-1})$. Recall that $a = z\zeta^{-1} - \overline{z}\zeta$. Then we obtain

$$\begin{cases} \dfrac{\partial s}{\partial z} = \zeta^{-1}e^a f\theta + \dfrac{\partial f}{\partial z}e^a\theta = \left(\zeta^{-1}f + \dfrac{\partial f}{\partial z}\right)e^a\theta \in H^0(X_A, \mathcal{F}(P_0)), \\ \dfrac{\partial s}{\partial \overline{z}} = -\zeta e^a f\theta + \dfrac{\partial f}{\partial \overline{z}}e^a\theta = \left(-\zeta f + \dfrac{\partial f}{\partial \overline{z}}\right)e^a\theta \in H^0(X_A, \mathcal{F}(Q_0)). \end{cases}$$

Thus, it follows from the definition of D that

$$\begin{cases} D_{\partial/\partial z}\sigma = \left(\zeta^{-1}f + \dfrac{\partial f}{\partial z}\right)\theta_a \in H^0(X_I, \mathcal{F}(P_0) \otimes L(a)), \\ D_{\partial/\partial \overline{z}}\sigma = \left(-\zeta f + \dfrac{\partial f}{\partial \overline{z}}\right)\theta_a \in H^0(X_I, \mathcal{F}(Q_0) \otimes L(a)). \end{cases}$$

Since $\partial s/\partial z$ is holomorphic over X_A, so is $D_{\partial/\partial z}\sigma$ over X_A. In consequence, $D_{\partial/\partial z}\sigma$ is holomorphic over $X = X_A \cup X_I$ and defines a global holomorphic section of $\mathcal{F}(P_0) \otimes L(a)$.

The proof for the case of $D_{\partial/\partial \overline{z}}\sigma$ is similar. □

Let $\sigma_0, \sigma_1, \cdots, \sigma_n$ be global holomolphic sections of the bundle $H^0(X) \to \mathbb{R}^2$, for which $H^0(X, \mathcal{L}_j \otimes L(a)) = \mathrm{Span}\{\sigma_\mathrm{j}\}$ for $j = 0, 1, \cdots, m-1$ and $H^0(X, \mathcal{L}_m \otimes L(a)) = \mathrm{Span}\{\sigma_\mathrm{m}, \cdots, \sigma_\mathrm{n}\}$. Set $s_j = \iota_a(\sigma_j)$ for $j = 0, 1, \cdots, n$. Then, $\{s_0, \cdots, s_n\}$ defines a free system of generators for $H^0(X_A, \mathcal{L})$. Recall that B denotes the algebra of holomorphic maps $A \to \mathbb{C}$. Let V_j and V_m be B-modules generated, respectively, by s_j and $s_m, \cdots, s_n$, where $j = 0, 1, \cdots, m-1$. We then have

$$H^0(X_A, \mathcal{L}) = \sum_{j=0}^{m} V_j,$$

which is a h-orthogonal direct sum by Lemma 6. We denote by $\Pi_j : H^0(X_A, \mathcal{L}) \longrightarrow V_j$ the h-orthogonal projection onto V_j.

Lemma 8. *Each map $s_j : \mathbb{R}^2 \longrightarrow H^0(X_A, \mathcal{L})$ satisfies*

$$\begin{cases} \dfrac{\partial s_j}{\partial z} \in V_j \bigoplus V_{j+1} & \text{for} \quad \mathrm{j = 0, 1, \cdots, m-1,} \\ \dfrac{\partial s_k}{\partial z} \in V_m \bigoplus V_0 & \text{for} \quad \mathrm{k = m, \cdots, n,} \end{cases}$$

and

$$\begin{cases} \Pi_{j+1}\left(\dfrac{\partial s_j}{\partial z}\right) \neq 0 & \text{for} \quad \mathrm{j = 0, 1, \cdots, m-1,} \\ \Pi_0\left(\dfrac{\partial^2 s_{m-1}}{\partial z^2}\right) \neq 0. & \end{cases}$$

Proof. As in the proof of Lemma 7, write $s_j = e^a f_j \theta$ with $f_j = \sigma_j \theta_a^{-1}$, where $\sigma_j \in H^0(X, \mathcal{L}_j \otimes L(a))$.

[Case $1: j = 0, 1, \cdots, m-2$] By Lemma 7 we have $D_{\partial/\partial z} \sigma_j \in H^0(X, \mathcal{L}_j(P_0) \otimes L(a))$. Recall that if L is non-special, then so is $L(P)$ for any point $P \in X$. Therefore we see that $\mathcal{L}_j(P_0) \otimes L(a)$ is non-special by Corollary 1. Then it follows from the Riemann-Roch formula that $h^0(X, \mathcal{L}_j(P_0) \otimes L(a)) = 2$.

Now, obviously, $H^0(X, \mathcal{L}_j(P_0) \otimes L(a))$ is generated by σ_j and σ_{j+1}, since $\mathcal{L}_j = \mathcal{L}_j(P_0) \otimes \mathcal{O}_X(-P_0)$ and $\mathcal{L}_{j+1} = \mathcal{L}_j(P_0) \otimes \mathcal{O}_X(-Q_0)$, which show that $\mathcal{L}_j$ and $\mathcal{L}_{j+1}$ are subsheaves of $\mathcal{L}_j(P_0)$. Therefore we obtain

$$\frac{\partial s_j}{\partial z} \in V_j \bigoplus V_{j+1}.$$

Moreover, since $\iota_a^{-1}(\partial s_j / \partial z) = (\zeta^{-1} f_j + \partial f_j / \partial z)\theta_a$ and $\zeta^{-1} f_j \theta_a = \zeta^{-1} \sigma_j$ cannot be an element of $H^0(X_I, \mathcal{L}_j \otimes L(a))$, we must have $\Pi_{j+1}(\partial s_j / \partial z) \neq 0$.

[Case 2 : $j = m-1$] As in Case 1, we have $\partial s_{m-1}/\partial z \in V_{m-1} \bigoplus V_m$, $\iota_a^{-1}(\partial s_{m-1}/\partial z) \in H^0(X, \mathcal{L}_{m-1}(P_0) \otimes L(a))$ and $\Pi_m(\partial s_{m-1}/\partial z) \neq 0$. In this case, although $\mathcal{L}_m$ is not a subsheaf of $\mathcal{L}_{m-1}(P_0)$, it is enough to consider $\mathcal{L}_m(-\sum P_i)$, which is a subsheaf of $\mathcal{L}_m$.

Next, we show $\Pi_0(\partial^2 s_{m-1}/\partial z^2) \neq 0$. We have

$$\begin{aligned}\iota_a^{-1}\left(\frac{\partial^2 s_{m-1}}{\partial z^2}\right) &= \left(\zeta^{-2} f_{m-1} + 2\zeta^{-1}\frac{\partial f_{m-1}}{\partial z} + \frac{\partial^2 f_{m-1}}{\partial z^2}\right)\theta_a \\ &\in H^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a)).\end{aligned}$$

Notice that $\mathcal{L}_{m-1}(2P_0) = \mathcal{L}(P_0 - (m-1)Q_0 - \sum P_i)$ and $h^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a)) = 3$ by the Riemann-Roch formula. Hence, $H^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a))$ has a section induced from a meromorphic section of $\mathcal{L} \otimes L(a)$, which has a pole of order 1 at P_0. Indeed, $\zeta^{-2} f_{m-1}\theta_a$ gives rise to such a section. We observe that $\mathcal{L}_{m-1}, \mathcal{L}_m(-\sum P_i)$ and $\mathcal{L}(P_0 - (m+1)Q_0 - \sum P_i)$ are subsheaves of $\mathcal{L}_{m-1}(2P_0)$. Since $\lambda^{-1}\sigma_0$ is a section of $\mathcal{L}(P_0 - (m+1)Q_0 - \sum P_i)$ (note that $(\lambda) = (m+1)P_0 - (m+1)Q_0$ on $U_0 \cup U_\infty$), it follows that $\mathcal{L}_{m-1}(2P_0)$ is generated by $\sigma_{m-1}, \sigma_m, \cdots \sigma_n$ and $\lambda^{-1}\sigma_0$. Among them, $\lambda^{-1}\sigma_0$ is the only one which has a pole of order 1 at P_0. This implies that $\Pi_0(\partial^2 s_{m-1}/\partial z^2) \neq 0$.

[Case 3 : $k = m, \cdots, n$] Similarly, we have $\iota_a^{-1}(\partial s_k/\partial z) \in H^0(X, \mathcal{L}_m(P_0) \otimes L(a))$ and $h^0(X, \mathcal{L}_m(P_0) \otimes L(a)) = n - m + 2$. In this case, $\mathcal{L}_m$ and $\mathcal{L}(P_0 - (m+1)Q_0 - \sum P_i)$ are subsheaves of $\mathcal{L}_m(P_0)$, and $\mathcal{L}_m(P_0)$ is generated by $\sigma_m, \cdots, \sigma_n$ and $\lambda^{-1}\sigma_0$. Thus we have $\partial s_k/\partial z \in V_m \bigoplus V_0$. □

Now, we are in a position to prove the following theorem.

Theorem 3. *Let $l_0, \cdots, l_m (m \geq 2)$ be the subbundles of $\mathbb{R}^2 \times \mathbb{C}^{n+1}$ constructed above. Then l_0 determines a harmonic map $\psi_0 : \mathbb{R}^2 \longrightarrow \mathbb{C}P^n$ of isotropy order m, where the isotropy order m is defined by*

$$m = \max\{\mathrm{j:\ all\ line\ bundles\ V_0, \ldots, V_j\ are\ mutually\ orthogonal}\}.$$

Here $V_0 = l_0$ and for $i \geqq 1$, under the orthogonal projection $\pi_{i-1}^\perp : \mathbb{C}^{n+1} \longrightarrow V_{i-1}^\perp$ onto the orthogonal bundle $V_{i-1}^\perp$ of V_{i-1}, V_i is a line bundle defined by $V_i = \pi_{i-1}^\perp(\partial V_{i-1}/\partial z)$.

Note that this theorem implies the harmonicity of the map associated with $(X, \pi, \mathcal{L})$, which is constructed in the previous section, since it is equal to ψ_0 by definition.

Proof of Theorem 3. Recall the map $\tau^1 : H^0(X_A, \mathcal{L}) \longrightarrow \mathbb{C}^{n+1}$. We see that $\tau^1(V_j) = l_j$. Obviously, the map τ^1 and the differentiation $\partial/\partial z$ commute. Denote by $\pi_j : \mathbb{C}^{n+1} \longrightarrow l_j$ the orthogonal projection onto l_j. Then we observe that $\tau^1 \circ \Pi_j = \pi_j \circ \tau^1$.

It then follows from Lemma 8 that

$$\begin{cases} \dfrac{\partial}{\partial z} l_j \subset l_j \oplus l_{j+1} & \text{for} \quad \mathrm{j = 0, 1, \cdots, m,} \\ \pi_{j+1}\left(\dfrac{\partial l_j}{\partial z}\right) \neq 0 & \text{for} \quad \mathrm{j = 0, 1, \cdots, m-1,} \\ \pi_0\left(\dfrac{\partial^2 l_{m-1}}{\partial z^2}\right) \neq 0, \end{cases} \tag{3.4}$$

where we use the convention that $l_{m+1} = l_0$. Thus, a map $\psi = (l_0, l_1, \cdots, l_m) : \mathbb{R}^2 \longrightarrow F^r(\mathbb{C}P^n)$ is a primitive map. In fact, the complexification of the tangent bundle of $F^r(\mathbb{C}P^n)$ is given by $T^{\mathbb{C}}(F^r(\mathbb{C}P^n)) = \bigoplus_{i \neq j} \mathrm{Hom(l_i, l_j)}$. On the other hand, $(\psi^*\beta)(\partial/\partial z)$ takes values in $\bigoplus_{i=0}^m \mathrm{Hom(l_i, l_{i+1})}$ with $l_{m+1} = l_0$ by (3.4). Recall that $F^r(\mathbb{C}P^n)$ has the structure of $(m+1)$-symmetric space such that $(l_j)_x$ is a ω^j-eigenspace of the automorphism τ_x of order $(m+1)$, where $\omega = \exp(2\pi\sqrt{-1}/(m+1))$. Then we have $[\mathcal{G}_1] = \bigoplus_{i=0}^m \mathrm{Hom(l_i, l_{i+1})}$ with $l_{m+1} = l_0$. In consequence, we see that ψ is a primitive map.

Now, if $m \geq 2$, then $\varphi = \tilde{\pi} \circ \psi : \mathbb{R}^2 \longrightarrow \mathbb{C}P^n$ is a harmonic map, where $\tilde{\pi} : F^r(\mathbb{C}P^n) \longrightarrow \mathbb{C}P^n$ is the homogeneous projection.

When $m = 1$, we have a map $\psi : \mathbb{R}^2 \longrightarrow F^1(\mathbb{C}P^n) = \mathbb{C}P^n$. Since the condition of the primitivity of ψ is meaningless in this case, the above argument is not applicable. However, we can show that ψ is also harmonic by calculating a holomorphic section of l_0 and investigating the divisor of this section. $\square$

4. Reconstruction of spectral curves from harmonic tori

In this section, we shall recover the spectral data $(X, \pi, \mathcal{L})$ from a given non-isotropic harmonic torus $\psi_0 \colon T^2 \to \mathbb{C}P^n$. Throughout this section, it is convenient to work with $G = U_{n+1}$, $H = \underbrace{U_1 \times \cdots \times U_1}_{r+2 \text{ times}} \times U_{n-r-1}$ and $\mathfrak{g} = \mathfrak{u}_{n+1}$. Here $0 \leqq r \leqq n-1$.

4.1. **Extended frames and loop groups.** First we recall the following

Theorem 4 ([1])**.** *Every non-isotropic weakly conformal harmonic map ψ_0 of a Riemann surface M into $\mathbb{C}P^n$ of isotropy order $r+1$ is covered by a unique bprimitive map $M \to F^r(\mathbb{C}P^n) = G/H$.*

Thus there exists a unique primitive lift $\psi \colon T^2 \to G/H$ for ψ_0 (if ψ_0 is non-conformal, we set $\psi = \psi_0$; 'primitive' will simply mean 'harmonic' for $m = 1$). We may frame this by $\Phi \colon \mathbb{R}^2 \to G$ over the universal cover $\mathbb{R}^2 \to T^2$ of T^2 and normalize the frame so that $\Phi(0) = \mathrm{Id}$.

Set $\alpha = \Phi^{-1} d\Phi$, which is the pull back of the Maurer-Cartan form by Φ, and write

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{p}},$$

according to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. If we define

$$\alpha_\zeta = \zeta^{-1}\alpha_{\mathfrak{p}}' + \alpha_{\mathfrak{h}} + \zeta\alpha_{\mathfrak{p}}'', \quad \zeta \in \mathbb{C}^*,$$

then we get

$$d\alpha_\zeta + \frac{1}{2}[\alpha_\zeta \wedge \alpha_\zeta] = 0.$$

Moreover, since $\alpha_{\mathfrak{p}}'$ takes values in $\mathfrak{g}_{-1}$, this map $\alpha_\zeta \colon \mathbb{C}^* \times \mathbb{R}^2 \to \mathfrak{g}^{\mathbb{C}}$ is ν-equivariant in ζ, i.e., $\nu(\alpha_\zeta) = \alpha_{\omega\zeta}$ (here ω is a primitive $(m+1)$-st root of unity). It follows that we can integrate α_ζ to find an extended frame $\Phi_\zeta \colon \mathbb{C}^* \times \mathbb{R}^2 \to G^{\mathbb{C}}$ which satisfies

$$\alpha_\zeta = \Phi_\zeta^{-1} d\Phi_\zeta.$$

Moreover, we may always choose the integration constants so that Φ_ζ is ν-equivariant and $\Phi_{1/\bar{\zeta}} = \Phi_\zeta^\dagger$, where $\dagger$ denotes Hermitian transpose. We are going to view Φ_ζ as a map from $\mathbb{R}^2$ into a certain loop group, defined as follows.

Let $C = C_\epsilon$ be the union of circles C_1 and C_2 in the ζ-plane of radii ϵ and ϵ^{-1} respectively, where $0 < \epsilon < 1$. Define

$$\Lambda_C^\epsilon(G^{\mathbb{C}}, \nu) = \{g\colon C \to G^{\mathbb{C}} \mid g \text{ is real-analytic, } \nu(g(\zeta)) = g(\omega\zeta)\}$$

and write $g_i = g|_{C_i}$ for $i = 1, 2$. We also denote the Lie algebra of $\Lambda_C^\epsilon(G^{\mathbb{C}}, \nu)$ by $\Lambda_C^\epsilon(\mathfrak{g}^{\mathbb{C}}, \nu)$. For a subgroup K of $G^{\mathbb{C}}$ and the Lie algebra $\mathfrak{k}$ of K, we define $\Lambda_C^\epsilon(K, \nu)$ and $\Lambda_C^\epsilon(k, \nu)$ in a similar way. On this group $\Lambda_C^\epsilon(G^{\mathbb{C}}, \nu)$ there is an anti-holomorphic involution given by

$$g \mapsto \overline{g} = g(\overline{\zeta}^{-1})^{\dagger -1}.$$

We will denote the fixed point subgroup of this involution by $\Lambda^\epsilon(G, \nu)$, which is the group (for any choice of ϵ) that Φ_ζ takes values in. We also write $\Lambda(G, \nu)$ for $\Lambda^\epsilon(G, \nu)$ if there is no confusion.

A crucial fact about this subgroup is that it admits an 'Iwasawa decomposition' in the following sense. First recall that $H^{\mathbb{C}}$ has an Iwasawa decomposition, which we will write as $H^{\mathbb{C}} = HB_0$, inherited from $G^{\mathbb{C}}$. Here B_0 is a subgroup of upper triangular matrices with positive real diagonal entries and $B_0 = \exp(\mathfrak{b}_0)$, where $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \oplus \mathfrak{b}_0$ is the corresponding Lie algebra decomposition. Next, we view C as a pair of circles on the ζ-sphere $\mathbb{P}_\zeta$ so that C is the common boundary for a closed annulus E and a union I of closed discs $I_\epsilon = \{\zeta \in \mathbb{P}_\zeta \mid |\zeta| < \epsilon\}$ and $I_{1/\epsilon} = \{\zeta \in \mathbb{P}_\zeta \mid |\zeta| > 1/\epsilon\}$. Then we can define the following three subgroups of $\Lambda(G, \nu)$:

$$\Lambda_E^\epsilon = \{g \in \Lambda^\epsilon(G, \nu) \mid \text{boundary of a holomorphic map } g\colon E \to G^{\mathbb{C}}\},$$
$$\Lambda_I^\epsilon = \{g \in \Lambda^\epsilon(G, \nu) \mid \text{boundary of a holomorphic map } g\colon I \to G^{\mathbb{C}} \text{ with } g(0) = \mathrm{Id}\},$$
$$B = \{(b, \bar{b}) | b \in B_0\}.$$

For simplicity, we also denote Λ_E^ϵ and Λ_I^ϵ by Λ_E and Λ_I, respectively. Then it follows from a result in [19], [20] that every element $g \in \Lambda(G,\nu)$ has a unique factorization $g = ubn$ where $u \in \Lambda_E$, $b \in B$ and $n \in \Lambda_I$. We will refer to this as the Iwasawa decomposition for $\Lambda(G,\nu)$.

The corresponding Lie algebra decomposition is denoted by

$$\Lambda^\epsilon(\mathfrak{g},\nu) = \Lambda_E^\epsilon(\mathfrak{g},\nu) \oplus \mathfrak{b} \oplus \Lambda_I^\epsilon(\mathfrak{g},\nu).$$

When there is no confusion, we shall often drop superscript ϵ.

4.2. **The dressing orbit of a vacuum solution.** One of the results we will need from [5] is that, up to isometries, every non-isotropic harmonic torus possesses an extended frame belonging to a particular class which we will now describe. In the terminology of [5], these are the extended frames which lie in the dressing orbit of a vacuum solution.

For any positive integer k, let $\Lambda_k \subset \Lambda_E(\mathfrak{g},\nu)$ denote the subspace of Laurent polynomials in ζ of degree $\leqq k$. We define

$$\Delta^\epsilon = \Lambda_1 \oplus \mathfrak{b} \oplus \Lambda_I(\mathfrak{g},\nu) \subset \Lambda(\mathfrak{g},\nu).$$

We also write Δ for Δ^ϵ. Let $\Lambda_{I,B} = \Lambda_{I,B}^\epsilon$ denote the subgroup of $\Lambda^\epsilon(G,\nu)$ generated by the subgroup B and Λ_I. Then $\Lambda_{I,B}$ acts on the subspace Δ as an adjoint action. For a given $(\xi,\bar{\xi}) \in \Delta$, define

$$e(\xi) = \exp[(z\xi, \bar{z}\bar{\xi})],$$

which gives rise to a 2-parameter subgroup of $\Lambda(G,\nu)$. Using the Iwasawa decomposition in $\Lambda(G,\nu)$, we write

$$e(\xi) = \Phi(\xi)b(\xi)n(\xi)$$

and observe that $\Phi(\xi)$ equals the identity at $z = 0$.

It can be shown quite readily that $\Phi(\xi)$ is the extended frame for some primitive harmonic map of $\mathbb{R}^2$ into G/H. In fact, if we use $\mathcal{F}$ to denote the set of all normalized extended frames (i.e., those with $\Phi_\zeta(0) = \mathrm{Id}$) for primitive harmonic maps $\mathbb{R}^2 \to G/H$, then we have defined a map $\Phi\colon \Delta \to \mathcal{F}$. Now, following [5], we observe that, for any $g \in \Lambda_{I,B}$,

$$\Phi(\mathrm{Adg}\xi) = \mathrm{g}\sharp\Phi(\xi),$$

where $g\sharp\Phi(\xi)$ is the Λ_E-factor in the Iwasawa decomposition of $g\Phi(\xi)$. It is not hard to show that the latter defines an action of $\Lambda_{I,B}$ on $\mathcal{F}$, called the dressing action, and the equation shows that Φ intertwines the two actions.

Now let $\mathcal{H}$ denote the space of all primitive maps $\mathbb{R}^2 \to G/H$ based by $\psi(0) = H$. Then $\mathcal{H} \cong \mathcal{F}/C^\infty(\mathbb{R}^2, H)$, i.e., the space of based primitive maps is the quotient of $\mathcal{F}$ by the group of gauge transformations. From [5] we know that the dressing action descends to $\mathcal{H}$; we will denote the gauge equivalence class of $\Phi(\xi)$ by $[\Phi(\xi)]$ and then $g\sharp[\Phi(\xi)] = [g\sharp\Phi(\xi)]$. The orbit $O_\xi = \{g\sharp[\Phi(\xi)] \mid g \in \Lambda_{I,B}\} \subset \mathcal{H}$ is called the dressing orbit of $[\Phi(\xi)]$. One of the principal results of [5] is the following

Theorem 5 ([5])**.** *Let $(\xi, \bar{\xi}) \in \Delta$ and write $\xi = \zeta^{-1}\xi_{-1} + \xi_0 + \zeta\xi_1 + \zeta^2\xi_2 + \cdots$. When ξ_{-1} is semisimple, we can find (possibly after shrinking ϵ) an element $g \in \Lambda_{I,B}$ for which $[\Phi(\xi)] = g\sharp[\Phi(\zeta^{-1}\xi_{-1})]$.*

The proof of this theorem needs the following lemmas. Concerning the adjoint and dressing actions of Λ_I^ϵ, we have

Lemma 9 ([5])**.** *For $g \in \Lambda_I^\epsilon$ and $\eta \in \Delta^\epsilon$,*

$$[\Phi(\mathrm{Adg}\eta)] = \mathrm{g}\sharp[\Phi(\eta)].$$

Lemma 10 ([5])**.** *Let $\mu, \eta \in \Delta^\epsilon$. Then $[\Phi(\mu)] = [\Phi(\eta)]$ if and only if $(\zeta\mu)(0) = (\zeta\eta)(0)$ and*

$$(\mathrm{ad}\eta)^{\mathrm{n}}\mu \in \Lambda_{\mathrm{I}}^\epsilon(\mathfrak{g}, \nu), \tag{4.1}$$

for all $n \geqq 1$.

Proof. $[\Phi(\mu)] = [\Phi(\eta)]$ if and only if $\Phi(\mu) = \Phi(\eta)k$ for some $k \in C^\infty(\mathbb{R}^2, H)$. Using the definitions of $\Phi(\mu)$ and $\Phi(\eta)$, it is straightforward to see that this is the case precisely when

$$e(z) := \exp(-z\mu)\exp(z\eta) \in \Lambda_I^\epsilon$$

for $z \in \mathbb{R}^2$. This, in turn, is the same as demanding that

$$e^{-1}de = (-\mathrm{Ad}\exp(-\mathrm{z}\eta)\mu + \eta)\mathrm{dz}$$

be $\Lambda_I^\epsilon(\mathfrak{g}, \nu)$-valued, that is,

$$e^{-\mathrm{adz}\eta}\mu - \eta \in \Lambda_I^\epsilon(\mathfrak{g}, \nu)$$

for all $z \in \mathbb{R}^2$. Expanding this last relation in powers of z and comparing coefficients proves the lemma. □

Let A be an element of $\mathfrak{g}_{-1}$ such thah $[A, \bar{A}] = 0$. Set $\eta_A = \zeta^{-1}A$. Then applying this to the case where $\eta = \eta_A$, we have the following proposition.

Proposition 1 ([5])**.** *$[\Phi(\mu)] = [\Phi(\eta_A)]$ if and only if $(\zeta\mu)(0) = A$ and $[\mu, A] = 0$.*

Proof. Write $\mu = \sum_{n \geqq -1} \zeta^n \mu_n$ on C_ϵ. Comparing coefficients of ζ in (4.1) gives

$$(\mathrm{adA})^\mathrm{n}\mu_{\mathrm{n}-1} = 0$$

for all $n \geqq 1$. However, since A is semisimple, $\ker(\mathrm{adA})^\mathrm{n} = \ker(\mathrm{adA})$. Hence $[\mu, A] = 0$ as required. □

Moreover we have

Proposition 2 ([5])**.** *For $\mu \in \Delta^\epsilon$ and $g \in \Lambda_{I,B}^\epsilon$, $[\Phi(\mu)] = g\sharp[\Phi(\eta_A)] \in \mathcal{O}_A$ if and only if $(\zeta\mu)(0) = \mathrm{Adg(0)A}$ and*

$$[\mu, \mathrm{AdgA}] = 0.$$

Proof. By Lemma 9, $[\Phi(\mu)] = g\sharp[\Phi(\eta_A)]$ if and only if $[\Phi(\mathrm{Adg}^{-1}\mu)] = [\Phi(\eta_{\mathrm{A}})]$. From Proposition 1, we see that this is the case precisely when

$$(\zeta \mathrm{Adg}^{-1}\mu)(0) = \mathrm{A}$$

and

$$[\mathrm{Adg}^{-1}\mu, \mathrm{A}] = 0.$$

Hence the result follows. ☐

Now we are in a position to prove Theorem 5.

Proof of Theorem 5. First we can find $A \in \mathrm{AdB}\eta_{-1}$ such that $[A, \bar{A}] = 0$, and, after dressing by an element of B, we may assume that $\eta_{-1} = A$. By Proposition 4.3, it now suffices to find $g \in \Lambda^{\epsilon}_{I,B}$, for some $0 < \epsilon \leqq \epsilon^{'}$, such that

$$\mathrm{Adg}(0)\mathrm{A} = \mathrm{A}, \quad [\mathrm{A}, \mathrm{Adg}\eta] = 0.$$

We shall construct g via the inverse function thorem.

Since A is semisimple,

$$\mathfrak{g}^{\mathbb{C}} = \ker \mathrm{adA} \oplus [\mathrm{A}, \mathfrak{g}^{\mathbb{C}}],$$

and we define $\phi \colon \ker \mathrm{adA} \oplus [\mathrm{A}, \mathfrak{g}^{\mathbb{C}}] \to \mathfrak{g}^{\mathbb{C}}$ by

$$\phi(x, y) = \mathrm{Ad}\exp(\mathrm{y})\mathrm{x}.$$

Observe that ϕ is equivariant in the following sense:

$$\omega\nu\phi(x, y) = \phi(\omega\nu x, \nu y) \tag{4.2}$$

for all $(x, y) \in \ker \mathrm{adA} \oplus [\mathrm{A}, \mathfrak{g}^{\mathbb{C}}]$.

Differentiating ϕ at $(A, 0)$ gives

$$\mathrm{d}_{(\mathrm{A},0)}\phi(\mathrm{v}, \mathrm{w}) = \mathrm{v} + [\mathrm{w}, \mathrm{A}]$$

for $(v, w) \in \ker \mathrm{adA} \oplus [\mathrm{A}, \mathfrak{g}^{\mathbb{C}}]$, so that $\mathrm{d}_{(\mathrm{A},0)}\phi$ is an isomorphism. By the holomorphic inverse function theorem there are open neighbourhoods Ω_1 of $(A, 0)$ and Ω_2 of A such that $\phi\colon \Omega_1 \to \Omega_2$ is a biholomorphism. Moreover, since $(A, 0)$ is fixed by the linear automorphism $T\colon (x, y) \mapsto (\omega\nu x, \nu y)$ of order $(r+2)$, we may assume, shringking Ω_1 if necessary, that Ω_1 is T-stable.

Let $(\psi_1, \psi_2) = \phi^{-1}\colon \Omega_2 \to \Omega_1$ so that, for $\chi \in \Omega_2$,

$$\chi = \mathrm{Ad}(\exp(\psi_2(\chi)))\psi_1(\chi),$$

or, equivalently,

$$\mathrm{Ad}\exp(-\psi_2(\chi))\chi = \psi_1 \in \ker \mathrm{ad}\mathrm{A}. \tag{4.3}$$

From (4.2) and the T-stability of Ω_1, we observe that ψ_2 has the following equivariant property:

$$\psi_2(\omega\nu\chi) = \nu\psi_2(\chi)$$

for all $\chi \in \Omega_2$.

Since $\eta \in \Delta^{\epsilon'}$, $\zeta\eta$ is holomorphic on $I_{\epsilon'}$ with $(\zeta\eta)(0) = A$. Hence we can find $0 < \epsilon \leqq \epsilon'$ such that $C_\epsilon \cup I_\epsilon \subset (\zeta\eta)^{-1}(\Omega_2)$. We may therefore define $g\colon C_1 \cup I_\epsilon \to G^{\mathbb{C}}$ by

$$g(\zeta) = \exp(-\psi_2(\zeta\eta(\zeta))).$$

By construction, g is holomorphic on I_ϵ and $g(0) = \exp(-\psi_2(A)) = -1 \in B$ so that

$$\mathrm{Adg}(0)\mathrm{A} = \mathrm{A}.$$

Moreover, from (4.3), for $\zeta \in C_1$ we have

$$\mathrm{Adg}(\zeta)\eta(\zeta) = \zeta^{-1}\mathrm{Ad}\exp(-\psi_2(\zeta\eta(\zeta)))\zeta\eta(\zeta) = \zeta^{-1}\psi_1(\zeta\eta(\zeta)) \in \ker \mathrm{ad}\mathrm{A}$$

so that

$$[A, \mathrm{Adg}\eta] = 0$$

on C_1. Hence g will define our desired element of $\Lambda^\epsilon_{I,B}$ so long as it satisfies the equivariant condition $g(\omega\zeta) = \nu g(\zeta)$. For this, recall that $\eta(\omega\zeta) = \nu\eta(\zeta)$ so that, using (4.4),

$$g(\omega\zeta) = \exp(-\psi_2(\omega\zeta\eta(\omega\zeta))) = \exp(-\psi_2(\psi\nu\zeta\eta(\zeta)))$$
$$= \exp(-\nu\psi_2(\zeta\eta(\zeta))) = \nu g(\zeta)$$

as required. This completes the proof. □

Our aim for the rest of this subsection is to prove:

Theorem 6. *Each primitive lift $\psi\colon T^2 \to G/H$ admits, possibly up to an isometry, an extended frame Φ_ζ given by $\Phi_\zeta = g\sharp\Phi(\zeta^{-1}\Lambda)$, where $g \in \Lambda_{I,B}$ and $\Lambda \in \mathfrak{g}_{-1}$ is a non-zero semisimple element fixed a priori.*

This is actually a fact about primitive maps of finite type, in the sense of [1], [4], which includes all tori worked out by Burstall [1]. The proof relies on the following results.

Lemma 11 ([4])**.** *Each primitive map ψ of finite type admits an element $(\xi, \bar{\xi}) \in \Delta$ for which $\Phi(\xi)$ is an extended frame.*

Lemma 12 ([29])**.** *G/H is a rank one $m+1$-symmetric space, that is, every semisimple element of $\mathfrak{g}_{-1}$ is $\mathrm{Ad}(\mathrm{H}^{\mathbb{C}})$-conjugate to some scalar multiple of a fixed non-zero semisimple $\Lambda \in \mathfrak{g}_{-1}$.*

Lemma 13 ([5])**.** *Let $g \in \Lambda(G,\nu)$ be extended holomorphically into I and define $g \sharp \Phi(\xi)$ to be the Λ_E-component of $g\Phi(\xi)$ in its Iwasawa decomposition. Then $\Phi(\mathrm{Adg}\xi) = \mathrm{g} \sharp \Phi(\xi)\widetilde{\mathrm{k}}$ for some $\widetilde{k}\colon \mathbb{R}^2 \to H$.*

Proof. This lemma follows immediately from Lemma 9. □

Now we can prove Theorem 6. Fix a non-zero seimisimple $\Lambda \in \mathfrak{g}_{-1}$. By Lemma 11, ψ has an extended frame $\Phi(\xi)$, which is, by Theorem 5, gauge equivalent to $\widetilde{g} \sharp \Phi(\zeta^{-1}\xi_{-1})$ for some $\widetilde{g} \in \Lambda_{I,B}$. By Lemma 12 there is some $h \in H^{\mathbb{C}}$ for which $\xi_{-1} = \mathrm{Adh}\Lambda$, so that $\Phi(\mathrm{Ad}(\widetilde{\mathrm{g}}\mathrm{h})\zeta^{-1}\Lambda)$ is an extended frame for ψ. By Lemma 13 this is gauge equivalent to the frame $(\widetilde{g}h)\sharp\Phi(\zeta^{-1}\Lambda)$. Finally, we may write $\widetilde{g}h = kg$ for some $k \in H$, $g \in \Lambda_{I,B}$, so that

$$(kg)\sharp\Phi(\zeta^{-1}\Lambda) = k(g\sharp\Phi(\zeta^{-1}\Lambda))$$

yields an extended frame for ψ. Thus $g\sharp\Phi(\zeta^{-1}\Lambda)$ is an extended frame for the map obtained, up to an isometry of G/H determined by k, from ψ (indeed this isometry preserves the base point $\psi(0)$).

4.2.1. Λ *and its centralizer.* Now we fix an element $\Lambda \in \mathfrak{g}_{-1}$ as follows. Let δ_j denote the column vector ${}^t(0,\dots,1,\dots,0)$, where the '1' lies in the j-th column with $0 \leqq j \leqq n$. Take $\mathbb{C}P^n$ to be the G-orbit of the line $\langle\delta_0\rangle$. Then $F^r(\mathbb{C}P^n)$ is the G-orbit of the flag

$$\langle\delta_0\rangle \subset \langle\delta_0,\delta_1\rangle \subset \langle\delta_0,\dots,\delta_r\rangle \subset \mathbb{C}^{n+1},$$

which is isomorphic to G/H, where H is the fixed point subgroup of the automorphism $\nu\colon g \mapsto \mathrm{Ad}\sigma\mathrm{g}$ with σ the diagonal matrix $\mathrm{diag}(1,\omega^{-1},\dots,\omega^{-m},\omega^{-m},\dots,\omega^{-m})$ for $m = r+1$ and $\omega = \exp(2\pi i/(m+1))$. Thus we may fix Λ to be the following matrix of rank $m+1$

$$\Lambda := \begin{pmatrix} 0 & & & 1 & & \\ 1 & \ddots & & & & \\ & \ddots & 0 & & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Later on we will need to work with the centralizer $\mathfrak{z}$ of Λ in $\mathfrak{g}^{\mathbb{C}}$ and its center $\mathfrak{c}$. It is not hard to see that $\mathfrak{c} = \mathfrak{c}_* \oplus \langle\mathrm{Id}\rangle$, where

$$\mathfrak{c}_* = \langle\Lambda,\Lambda^2,\dots,\Lambda^{m+1}\rangle.$$

It is worth noting that $\mathfrak{c}$ is a reductive Lie algebra, whose single semisimple component is a subalgebra of $\mathfrak{g}_0$. It will be useful to observe that each element of $\mathfrak{z}$ has a block decomposition according to the splitting $\mathfrak{z} = \mathfrak{c}_* \oplus \mathfrak{z}_*$, where $\mathfrak{z}_* = \mathfrak{gl}_{\mathfrak{n}-\mathfrak{m}}$ is the subalgebra of matrices of rank $n - m$, whose non-zero entries occupy the bottom right-hand corner.

4.2.2. *Higher flows in* O_Λ. For simplicity (and consistency with the previous work) we will write Φ^0 for $\Phi(\zeta^{-1}\Lambda)$. As a result of Theorem 6, we see that every primitive map of finite type $\mathbb{R}^2 \to G/H$ is, possibly up to an isometry, found in the dressing orbit of the vacuum solution $[\Phi^{(0)}]$. From Theorem 6 we know that this orbit O_Λ is isomorphic to $\Lambda_{I,B}/\Gamma_{I,B}$, where $\Gamma_{I,B}$ is the stabilizer of $\zeta^{-1}\Lambda$ for the adjoint action of $\Lambda_{I,B}$. Let $[g]$ denote the coset $g\Gamma_{I,B}$. Then this isomorphism is given by $[g] \mapsto g \sharp [\Phi^{(0)}]$.

We will now describe the action of an abelian Lie subgroup of $\Lambda(G, \nu)$ on this orbit, whose 1-parameter subgroups generate the so-called 'higher flows' (the terminology comes from soliton theory, from which the idea of dressing actions originated). These matters will be of use later on.

Observe that $\Lambda(\mathfrak{c}, \nu)$ is the center of the centralizer for $\zeta^{-1}\Lambda$ in $\Lambda(\mathfrak{g}, \nu)$. Let $\mathfrak{C}_R$ denote the subalgebra of finite order elements of $\Lambda(\mathfrak{c}, \nu)$ (that is, those whose projections to $\Lambda_E(\mathfrak{g}, \nu)$ are Laurent polynomials). The abelian Lie group $\exp(\mathfrak{C}_R)$ has a right action on $\Lambda_{I,B}/\Gamma_{I,B}$, which is defined by $\exp a[g] = [(g \exp(a))_I]$, where $(\cdot)_I$ denotes the $\Lambda_{I,B}$ factor in the Iwasawa decomposition. Note that, in particular, the subgroup $\exp(\mathfrak{C}_R) \cap \Gamma_{I,B}$ acts trivially.

We now examine the action of the 2-parameter subgroup $\exp(\mathfrak{m})$, where

$$\mathfrak{m} = \{w\zeta^{-1}\Lambda + \bar{w}\zeta\overline{\Lambda} \mid w \in \mathbb{C}\}.$$

Clearly, the $\exp(\mathfrak{m})$-orbit of $[g]$ can be written as $\{[(g\Phi^{(0)}(w))_I] \mid w \in \mathbb{C}\}$. The corresponding points in O_Λ have extended frames of the form

$$(4.4) \qquad \begin{aligned} (g\Phi^{(0)}(w))_I \sharp \Phi^{(0)}(z) &= (g\Phi^{(0)}(w))_E^{-1} \sharp \Phi^{(0)}(z + w) \\ &= \Phi_\zeta(w)^{-1}\Phi_\zeta(z + w) \end{aligned}$$

for $\Phi_\zeta(z) = g \sharp \Phi^{(0)}(z)$. The right-hand side is clearly an extended frame for the primitive map ψ^w defined by $\psi^w(z) = \Phi_1(w)^{-1}\psi(z+w)$. Thus the corresponding $\exp(\mathfrak{m})$-orbit in O_Λ is nothing but the set of all primitive maps obtained from ψ by the translation $z \mapsto z + w$.

4.3. **Affine schemes corresponding to rings of polynomial Killing fields.** The key to recovering the spectral curve is to understand the Lie algebras of formal and polynomial Killing fields. These give us those local deformations of an extended frame, which correspond to moving along the Jacobi variety of the spectral curve. Hence we will see that we can recover the tangent space $H^1(X, \mathcal{O}_X)$ to the Jacobi variety from the formal Killing fields and that polynomial Killing fields will give us the the coordinate ring $H^0(X_A, \mathcal{O})$ of an affine open subset $X_A \subset X$. (For more details, we refer the reader to the appendix of [21]. However, since in general the Lie algebra of polynomial Killing fields is not abelian, the theory requires more care.)

By Theorem 6 our primitive lift possesses an extended frame Φ_ζ of the form $g \sharp \Phi^{(0)}$ for some $g \in \Lambda_{I,B}$. This means that there is a map $\chi\colon \mathbb{R}^2 \to \Lambda_{I,B}$ for which

$$g\Phi^{(0)} = \Phi_\zeta \, \chi. \tag{4.5}$$

Observe that $\chi(0) = g$. From now on, for convenience, let us drop the subscript 'ζ' from α_ζ, etc. By a real Killing field for $\alpha = \Phi^{-1}d\Phi$ we mean a map $\eta\colon \mathbb{R}^2 \to \Lambda(\mathfrak{g}, \nu)$ such that

(1) $d\eta = [\eta, \alpha]$, and
(2) η has finite order.

Recall that an element η of $\Lambda(\mathfrak{g}, \nu)$ is said to have finite order when its projection η_E to $\Lambda_E(\mathfrak{g}, \nu)$ is a Laurent polynomial in ζ (and more generally, an element of $\Lambda_C(\mathfrak{g}^{\mathbb{C}}, \nu)$ will be said to have finite order if it extends meromorphically into I, where it has at worst poles). A real polynomial Killing field is a real formal Killing field η for which $\eta = \eta_E$. To define the space of all formal or polynomial Killing fields we take the complexification of the space of real ones; this gives us maps with values in $\Lambda_C(\mathfrak{g}, \nu)$.

Let $\mathfrak{Z} = \{\text{finite order } a \in \Lambda_C(\mathfrak{g}^{\mathbb{C}}, \nu) \mid [a, \Lambda] = 0\}$ and observe that this is identical with the subset of $\Lambda_C(\mathfrak{z}, \nu)$ consisting of only the finite order elements. Denote $\mathfrak{Z} \cap \Lambda(\mathfrak{g}, \nu)$ by $\mathfrak{Z}_R$. We are interested in the subalgebra $\mathfrak{Z}_R^{pol} = \{a \in \mathfrak{Z}_R \mid (\mathrm{Adg\,a})_{\mathrm{E}} = \mathrm{Adg\,a}\}$ and its complexification $\mathfrak{Z}^{pol} \subset \mathfrak{Z}$. The following lemma is derived from a lemma in the appendix of [21].

Lemma 14. *The map* $\mathrm{Ad}\chi\colon \mathfrak{Z} \to \{$*formal Killing fields*$\}$ *is an isomorphism of Lie algebras. It identifies* $\mathfrak{Z}^{pol}$ *with* $\{$*polynomial Killing fields*$\}$*.*

In the general case, however, $\mathfrak{Z}$ is not abelian and, as we have seen above, the Lie group acting on the dressing orbit O_Λ of $\Phi^{(0)}$ is the center $\exp(\mathfrak{C}_R)$ of $\exp(\mathfrak{Z}_R)$. We will regard this as an action of $\mathfrak{C}_R$ and let $\mathfrak{C}_R^{[g]}$ denote the stabilizer of $[g]$ for this action; it is not hard to see that this coincides with $(\mathfrak{C}_R \cap \mathfrak{Z}_R) \oplus \mathfrak{Z}_R^{pol}$. Then the following result is proved in [5].

Lemma 15 ([5])**.** $g \sharp \Phi^{(0)}$ *is an extended frame for a primitive map of finite type if and only if* $\mathfrak{C}_R/\mathfrak{C}_R^{[g]}$ *is finite dimensional.*

It follows that ψ is of finite type precisely when $\mathfrak{C}/\mathfrak{C}^{[g]}$ is finite dimensional. Owing to this we are interested in the vector space $\mathfrak{S}(z)$ of polynomial Killing fields, which is given by the complexification of

$$\mathfrak{S}_R(z) = \{(\mathrm{Ad}\chi(\mathrm{z})\,\mathrm{c})_{\mathrm{E}} \mid \mathrm{c} \in \mathfrak{C}_{\mathrm{R}}^{[\mathrm{g}]}\}.$$

Lemma 16. *For each* z*,* $\mathfrak{S}(z)$ *is a space of commuting elements of* $\Lambda_E(\mathfrak{g}^{\mathbb{C}}, \nu)$*, each of which is semisimple-valued on the unit circle.*

Proof. Let c_1, $c_2 \in \mathfrak{C}_R^{[g]}$. Then by the definition $\mathrm{Ad}\chi^{-1}(\mathrm{Ad}\chi\,\mathrm{c_i})_{\mathrm{E}} \in \mathfrak{z}_{\mathrm{R}}$. So

$$[\mathrm{Ad}\chi\,\mathrm{c_i},\, (\mathrm{Ad}\chi\,\mathrm{c_j})_{\mathrm{E}}] = 0 = [\mathrm{Ad}\chi\,\mathrm{c_i},\, (\mathrm{Ad}\chi\,\mathrm{c_j})_{\mathrm{I}}],$$

since each c_i is central. From this we obtain

$$\begin{aligned}[](\mathrm{Ad}\chi\,\mathrm{c_1})_{\mathrm{E}},\, (\mathrm{Ad}\chi\,\mathrm{c_2})_{\mathrm{E}}] &= -[\mathrm{Ad}\chi\,\mathrm{c_{1E}},\, (\mathrm{Ad}\chi\,\mathrm{c_2})_{\mathrm{I}}] \\ &= [\mathrm{Ad}\chi\,\mathrm{c_{1I}},\, (\mathrm{Ad}\chi\,\mathrm{c_2})_{\mathrm{I}}],\end{aligned}$$

which implies that each side is identically zero. Therefore the elements of $\mathfrak{S}_R(z)$ (and hence of $\mathfrak{S}(z)$) commute. By definition, if $\xi \in \mathfrak{S}(z)$, so does its conjugate $\bar{\xi} = -\xi(\bar{\zeta}^{-1})^{\dagger}$. Hence $[\xi, \bar{\xi}] = 0$. Conversely, on $|\zeta| = 1$ we see that ξ is normal and therefore semisimple. □

Now, let $\mathfrak{R}(z)$ denote the $\mathbb{C}$-algebra generated by $\mathfrak{S}(z)$, and let $\mathfrak{R} = \mathfrak{R}(0)$. Observe that $\mathfrak{S}$ contains the $\mathbb{C}$-algebra $\mathfrak{B} = \mathbb{C}[\lambda^{-1}\mathrm{Id}, \lambda\,\mathrm{Id}]$, where $\lambda = \zeta^{m+1}$. Hence $\mathfrak{R}$ is a commutative unital $\mathbb{C}$-algebra without nilpotents, and also is a $\mathfrak{B}$-module.

Lemma 17. *$\mathfrak{R}$ is a torsion free, finitely generated $\mathfrak{B}$-module.*

Proof. It is obvious that $\mathfrak{R}$ is torsion free. To see that $\mathfrak{R}$ is finitely generated as a $\mathfrak{B}$-module, we observe that

$$M = \{v\colon C \to \mathbb{C}^{n+1} \mid v \text{ extends to a Laurent polynomial on } \mathbb{C}^* \text{ and } v(\omega\zeta) = \sigma v(\zeta)\}$$

is a $\mathfrak{B}$-module of rank $(n+1)$ and also is a faithful $\mathfrak{R}$-module, where the multiplication is defined by that of matrices on column vectors. □

Let us define $\mathfrak{U} = \mathrm{Ad}g^{-1}\,\mathfrak{R}$, which is an abelian subalgebra of $\Lambda(\mathfrak{z}, \nu)$. Using the splitting $\mathfrak{z} = \mathfrak{c}_* \oplus \mathfrak{z}_*$, we may write each $a \in \mathfrak{U}$ as $a = a_0 + a_\bullet$, where a_0 takes values in $\mathfrak{c}_*$ and $a_\bullet$ takes values in $\mathfrak{z}_*$. Note that $a_\bullet$ is a function of ζ^{m+1}, since $\mathfrak{z}_* \subset \mathfrak{g}_0$. For a, $b \in \mathfrak{U}$ we have

$$ab = a_0 b_0 + a_\bullet b_\bullet.$$

We will also define $W(z) = \chi^{-1} M$ so that $W(z)$ is a faithful $\mathfrak{U}$-module (of boundaries of meromorphic maps from I to $\mathbb{C}^{n+1}$, each component of which has finite order).

Finally, let us observe that for $\xi(z) \in \mathfrak{R}(z)$ there exists $a \in \mathfrak{U}$ such that $\xi(z) = \mathrm{Ad}\chi(\mathrm{z})\,\mathrm{a} = \mathrm{Ad}\Phi(\mathrm{z})^{-1}\,\xi(0)$, using $g = \chi(0)$. Therefore $\mathfrak{R}(z) = \mathrm{Ad}\Phi^{-1}(\mathrm{z})\,\mathfrak{R}$, which is a property we will find useful later.

4.4. **Construction of Valuations.** By the previous remarks we have a commutative unital $\mathbb{C}$-algebra $\mathfrak{R}$ and therefore an affine variety $\mathrm{Spec}(\mathfrak{R})$. Our aim is to show that, when ψ_0 is linearly full (i.e., its image does not lie in a hyperplane of $\mathbb{C}P^n$), this is an affine algebraic curve whose completion X by non-singular points has the properties required to be the spectral curve. That is, we shall show that X admits a rational function π of the right type and a line bundle (or rank 1 torsion free coherent sheaf) $\mathcal{L}$ from which we recover ψ_0. In this subsection, we will discuss the construction over the affine curve and show that if ψ_0 is full, then the curve must be connected. This will allow us to complete the construction over X in the following subsection.

As a corollary of Lemma 17, we see that $\mathfrak{R}$ is an integral extension of $\mathfrak{B}$, therefore $\mathrm{Spec}(\mathfrak{R})$ is an affine curve and the inclusion $\mathfrak{B} \to \mathfrak{R}$ is dual to a finite morphism

$$\pi\colon \mathrm{Spec}(\mathfrak{R}) \to \mathrm{Spec}(\mathfrak{B}) \cong \mathbb{C}^*.$$

Since $\mathfrak{B}$ is a principal ideal domain, $\mathfrak{R}$ is actually a free $\mathfrak{B}$-module and its rank k equals the degree of π. Since M has rank $n+1$ over $\mathfrak{B}$, we see that $k \leqq n+1$. Now let X denote the completion of $X_A = \mathrm{Spec}(\mathfrak{R})$ by smooth points.

Proposition 3. *The curve X is real and admits a rational function $\pi\colon X \to \mathbb{P}^1$ with a zero P_0 of degree $m+1$. Thus π has $k \geqq m+1$.*

Proof. The algebra $\mathfrak{R}$ possesses the involution $\xi \mapsto \bar{\xi}$, which is dual to a real involution on $\mathrm{Spec}(\mathfrak{R})$ and extends to X. Now we must show that the fiber of the morphism π over $\lambda = 0$ contains a point P_0 with ramification index m.

We take the point of view that each smooth point of X corresponds to a valuation on some subfield of the ring of fractions $\mathfrak{F} = S^{-1}\mathfrak{A}$, where S is the set of all non-zero divisors. Because $\mathfrak{A}$ need not be an integral domain, $\mathfrak{F}$ itself need not be a field. However, each smooth point corresponds to a surjection $\nu\colon \mathfrak{F}^* \to \mathbb{Z}$, which is a multiplicative homomorphism (i.e. $\nu(ab) = \nu(a) + \nu(b)$) and has $\nu(a+b) \geqq \min(\nu(\mathrm{a}), \nu(\mathrm{b}))$. The subring $I = \{a \in \mathfrak{F} \mid \nu(a) \geqq 0\} \cup \{0\}$ is easily seen to be a discrete valuation ring of its field of fractions. We will describe one of the valuations covering the point $\lambda = 0$.

For each $a = (a_1, a_2)$ in $\mathfrak{A}$, the component a_1 has a block decomposition into the sum

$$a_{10}(\zeta) + a_{1\bullet}(\zeta^{m+1})$$

derived from the decomposition described earlier. This provides us with a grading $o_0 \colon \mathfrak{A} \to \mathbb{Z}$ for the ring $\mathfrak{A}$, defined by taking $o_0(a)$ to be the order of a_{10} in ζ^{-1}. With this we define a multiplicative homomorphism

$$\nu_0 \colon \mathfrak{F}^* \to \mathbb{Z}, \quad \nu_0(r/s) = o_0(s) - o_0(r).$$

If we can show that this is surjective, then we are done, for in that case $I_0 = \{a \in \mathfrak{F} \mid \nu_0 \geqq 0\} \cup \{0\}$ is isomorphic to the regular local ring $\mathcal{O}_\zeta$. Around the corresponding smooth point P_0 the map π behaves like $\zeta \mapsto \zeta^{m+1}$.

To show that ν_0 is onto, we use Lemma 15. First observe that $\mathfrak{A}$ contains, by definition, $\mathrm{Ad}\chi^{-1}\,\mathfrak{S}(\mathrm{z})$ and that for any $c \in \mathfrak{C}_R^{[g]}$

$$\mathrm{Ad}\chi^{-1}\,(\mathrm{Ad}\chi\,\mathrm{c})_{\mathrm{E}} \in \mathrm{c}_{\mathrm{E}} + \mathfrak{Z}_{\mathrm{R}}^{\mathrm{I}}.$$

Now observe that $o_0 \colon \mathfrak{C}_R \to \mathbb{Z}$ is onto. It follows from Lemma 15 that $o_0(\mathfrak{C}_R^{[g]})$ contains all but a finite number of positive integers and therefore so does $\mathrm{Ad}\chi^{-1}\,\mathfrak{S}(\mathrm{z}) \subset \mathfrak{A}$. So, for every integer k, there exist r, $s \in \mathfrak{A}$ for which $\nu_0(r/s) = k$. $\square$

Remark. We see from this proof that, with respect to the isomorphism $\mathbb{C}[X_A] \cong \mathfrak{A}$, a regular function on X_A vanishes on the irreducible component barring P_0 precisely when the corresponding component a_{10} is identically zero.

It is not hard to see that π intertwines the real involution on X with the map $\zeta \mapsto \bar{\zeta}^{-1}$ and so it has a pole P_∞ of order $m + 1$. Now we are going to show that the irreducible component X_0 of X, which carries the points P_0 and P_∞, is the completion of Spec($\mathfrak{A}_0$), where

$$\mathfrak{A}_0 = \{a_0 \mid a \in \mathfrak{A}\}.$$

Thus $\mathfrak{A}_0$ is a subalgebra of $\Lambda_C(\mathfrak{c}_*, \nu)$ and also is a quotient algebra of $\mathfrak{A}$.

Lemma 18. $\mathfrak{A}_0$ *is an integral domain.*

Proof. From now on, for an open subset U of X, we denote by $\mathrm{Hol(U, \mathbb{C}^k)}$ the space of $\mathbb{C}^k$-valued holomorphic functions on U. Observe that each element of $\Lambda_C(\mathfrak{c}_*, \nu)$ can be written as a Fourier series in $\zeta^{-1}\Lambda$, and we obtain a $\mathbb{C}$-algebra morphism of $\mathfrak{A}_0$ into $\mathrm{Hol(I^*, \mathbb{C})}$ (where $I^* = I \cap A$) given by $a_0(\zeta^{-1})$. Its image consists solely of holomorphic maps which, unless they are identically zero, do not vanish on either of the connected components of I^*. Therefore, $a_0 b_0 = 0$ if and only if a_0 or b_0 is the zero element. □

It follows from the previous remark that Spec($\mathfrak{A}_0$) is the irreducible component of Spec($\mathfrak{A}$) $\cong \mathrm{X_A}$ carrying the point P_0 and its conjugate P_∞. Although X_A need not be irreducible, we will shortly see that when ψ_0 is full, it must be connected. First we must introduce a sheaf over X_A whose sections provide the harmonic map.

Recall the $\mathfrak{R}$-module M, which is torsion free as a $\mathfrak{B}$-module. Since $\mathfrak{B}$ is an integral domain, it is elementary to show that M must also be torision free over $\mathfrak{R}$, so that it determines a torsion free coherent sheaf $\mathcal{L}_A$ over X_A. We want to show that its restriction to X_{0A} = Spec($\mathfrak{A}_0$) is rank one. This is easily seen by looking at the equivalent picture of $W(0) = \chi(0)^{-1}M$ as an $\mathfrak{A}$-module. For any $f = {}^t(f_0, \ldots, f_n) \in W(0)$, let us write

$$f = f^0 + f_\bullet = {}^t(f_0, \ldots, f_m, 0, \ldots, 0) + {}^t(0, \ldots, 0, f_{m+1}, \ldots, f_n).$$

So, for $a \in \mathfrak{A}$, we clearly have $(af)^0 = a_0 f^0$. Therefore the vector space $W^0(0) = \{f^0 \mid f \in W(0)\}$ is an $\mathfrak{A}_0$-module, which is clearly torsion free. Now observe that the injection

$$\Sigma\colon W^0(0) \to \mathrm{Hol}(\mathrm{I}^*, \mathbb{C}); \quad \mathrm{f}^0 \mapsto \mathrm{f}_0 + \cdots + \mathrm{f}_\mathrm{m},$$

is an $\mathfrak{A}$-module morphism (one readily verifies that $\Sigma(a_0 f^0) = a_0(\zeta^{-1})\Sigma(f^0)$, using the representation $a_0(\zeta^{-1}\Lambda) \mapsto a_0(\zeta^{-1})$). Therefore, in a smooth neighbourhood of P_0, each stalk of $\mathcal{L}_A$ is a module of regular $\mathbb{C}$-valued functions of ζ with respect to the appropriate regular local ring, so that $\mathcal{L}_A$ has rank one in this neighbourhood. Thus we have shown:

Proposition 4. *The restriction of $\mathcal{L}_A$ to X_{0A} (and therefore to the connected component of X_A containing it) is a rank one torsion free coherent sheaf. In particular, when X_A is connected, $\mathcal{L}_A$ has rank one.*

We can easily repeat the previous results for each z, replacing $\mathfrak{R}$ by $\mathfrak{R}(z)$ and $W(0)$ by $W(z)$. This gives us, for each z, a sheaf $\mathcal{L}_A(z)$ over X_A whose restriction to X_{0A} has rank one and whose direct image under π is the vector bundle $\mathcal{E}_A$ of rank $n+1$, corresponding to the $\mathfrak{B}$-module M. This vector bundle comes equipped with a trivialization determined by the isomorphism

$$\begin{aligned}\Gamma(A, \mathcal{E}_A) \cong M \to \{f : A \to \mathbb{C}^{n+1} \mid \text{Laurent polynomial}\} \\ f(\zeta) \mapsto (\kappa f)(\lambda),\end{aligned}$$

where $A = \mathbb{C}^*_\lambda$ and $\kappa = \mathrm{diag}(1, \zeta, \ldots, \zeta^\mathrm{m}, \ldots, \zeta^\mathrm{m})$, so that $\kappa(\omega\zeta) = \kappa(\zeta)\sigma^{-1}$ and therefore κf is a function of λ.

The effect of this isomorphism is to remove the ν-equivalence, which has permeated into the construction so far. Because of this we will find it most convenient to remove the effects of ν-equivalence from all the objects we are dealing with. To this end we redefine, throughout this subsection and the next,

$$\begin{aligned} M &= \{\text{Laurent polynomial } f\colon A \to \mathbb{C}^{n+1}\} \\ \mathfrak{A} &= \mathrm{Ad}\kappa\, \mathfrak{A} \\ \Phi(z) &= \mathrm{Ad}\kappa\, \Phi(\mathrm{z}) \\ \chi(z) &= \mathrm{Ad}\kappa\, \chi(\mathrm{z}) \end{aligned}$$

and so forth. Observe that, in particular, Φ and χ are still holomorphic in their vector bundle $\mathcal{E}(z)$ over $\mathbb{P}^1$ characterized by the transition relations

$$\chi(z)\, \hat{\tau}_z = \tau_z \quad \text{on } A \cap I,$$

where τ_z is a trivialization over A and $\hat{\tau}_z$ a trivialization over I. Furthermore, we can always choose I so that it contains no branch points of π other than 0 and ∞ and that $\pi^{-1}(I)$ contains only smooth points.

Eventually, we will be able to show that when ψ_0 is full, $\mathcal{E}(z)$ is the direct image of a rank one torsion free coherent sheaf $\mathcal{L}(z)$ over X obtained by moving $\mathcal{L}$ (the extension of $\mathcal{L}_A$ to X) linearly around the Picard variety. Before we can do this we must establish that, for ψ_0 to be full, the curve X must be connected, so that $\mathcal{L}_A$ must be rank one by the previous proposition.

Proposition 5. *X is disconnected if and only if ψ_0 is not full.*

Proof. First we will show that if X is disconnected, then $\Phi_1\, \delta_0$ (i.e., $\Phi\, \delta_0$ evaluated at $\lambda = 1$) takes values in a proper subspace of $\mathbb{C}^{n+1}$. When X is disconnected, we may write $X = Y + Z$, where Y is the connected component carrying the irreducible component X_0. For each z, there must be $\varepsilon_Y(z), \varepsilon_Z \in \mathfrak{R}(z)$ representing the globally regular characteristic functions 1_Y, 1_Z on X (i.e., 1_Y is identically one on Y and identically zero on Z). Each of these is clearly independent of λ. Thus we obtain a direct sum decomposition

$$M = \varepsilon_Y(z)\, M \oplus \varepsilon_Z\, M$$

corresponding to a global decomposition of $\mathcal{E}(z)$ into a sum of trivial bundles.

Consider now the global section $\sigma_0(z)$ of $\mathcal{E}(z)$ defined by $\tau_z(\sigma_0(z)) = \delta_0$. Since $\mathfrak{R}(z) = \mathrm{Ad}\Phi(\mathrm{z})^{-1}\,\mathfrak{R}$, we deduce $\Phi\tau_z = \tau_0$ and therefore $\Phi\,\delta_0 = \tau_0(\sigma_0(z))$. The proof that ψ_0 is not full will be finished if we can show

$$\varepsilon_Z(z)\tau_z(\sigma_0(z)) = 0,$$

since then $\varepsilon_Z(0)\Phi\,\delta_0 = 0$ and therefore $\Phi_1\,\delta_0$, where $a_Z = \mathrm{Ad}\chi^{-1}\,\varepsilon_{\mathrm{Z}}$ represents 1_Z in $\mathfrak{A}$. Since 1_Z vanishes identically on X_0, we know $(a_Z)_0 = 0$ (recall an earlier remark). From the Fourier series for χ^{-1} on the circle C_1 we see that $\chi^{-1}\delta_0$ has the form

$$\chi^{-1}\delta_0 = (\alpha_0, 0, \dots, 0) + O(\lambda)$$

about $\lambda = 0$. Therefore $a_Z\,\chi^{-1}\delta_0$ vanishes at $\lambda = 0$, whence it vanishes everywhere, since it represents a global section of $\mathcal{E}(z)$.

Now let us show that when ψ_0 is not full, the algebra $\mathfrak{R}$ possesses idempotent ε different from the identity. For then ε, $\mathrm{Id} - \varepsilon$ are a pair of 'orthogonal idempotent', whence Spec $(\mathfrak{R})$ is disconnected from the next Lemma (see, for example, [11]):

Lemma 19 ([11])**.** *Let A be a ring. Then the following conditions are equivalent:*

(1) Spec A *is disconnected.*

(2) *There exist nonzero elements $e_1, e_2 \in A$ such that $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called orthogonal idempotents).*

(3) *A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.*

We may assume without loss of generality that ψ_0 is full in the projective k-plane of points whose last $n-k$ coordinates vanish. From the preceding argument ψ_0 determines a subalgebra $\mathfrak{R}_k \subset \Lambda_C(\mathfrak{gl}_{\mathfrak{k}+1}, \nu)$ which must contain the identity matrix in $\mathfrak{gl}_{\mathfrak{k}+1}$. We take this for ε. Now observe that $\mathfrak{R}$ is a unital subalgbra of $\Lambda_C(\mathfrak{gl}_{\mathfrak{n}+1}, \nu)$ containing $\mathfrak{R}_k$, and therefore $\varepsilon \in \mathfrak{R}$ satisfies the conditions required. $\square$

4.5. **Completion of the affine curve and the line bundle.** From now on we will assume that ψ_0 is full and thus, by the previous result, X is connected. It follows that $\mathcal{L}_A(z)$ has rank one and therefore π has degree $n+1$. Using these facts, we will describe the extension $\mathcal{L}(z)$ of $\mathcal{L}_A(z)$ to all X and produce from it global sections which reconstruct the map ψ. We aim to prove:

Theorem 7. *$\mathcal{L}(z)$ has degree $p+n$, is real (i.e., satisfies Condition (2.1)) and moves linearly with $z, \bar{z}$ around the Picard variety of X. The primitive lift ψ determines (and is determined by), up to scaling, global sections $\sigma_0(z), \dots, \sigma_{m-1}(z)$, where each $\sigma_j(z)$ has a divisor of zeros at least $(m-j)P_0 + jP_\infty + E_0$ (while $(m+1)P_0 + E_0$ is the divisor of zeros of π).*

Remark. In the statement of this theorem, by 'Picard variety' we mean the moduli space of maximal rank 1 torsion free coherent sheaves, when this is relevant (we will see later on that $\mathcal{L}$ must be maximal, after Proposition 7).

Before proving this theorem, we need to describe $\mathcal{L}(z)$. First, recall that $I \subset \mathbb{P}^1$ has chosen so that $X_I = \pi^{-1}(I)$ consists of smooth points and has ramification points only over $\lambda = 0, \infty$. Using the direct image isomorphism $\mathrm{Hol}(\mathrm{X_I}, \mathbb{C}) \to \mathrm{Hol}(\mathrm{I}, \mathbb{C}^{\mathrm{n+1}})$ (see, for example, [8]), it is easy to see that $\mathrm{Hol}(\mathrm{X_I}, \mathbb{C})$ can be represented, as an algebra of endomorphisms on $\mathrm{Hol}(\mathrm{I}, \mathbb{C}^{\mathrm{n+1}})$, by $\mathfrak{g}^{\mathbb{C}}$-valued functions on I which are diagonalisable at each value of λ (indeed, for $\lambda \neq 0, \infty$, the eigenvalues give the $n+1$ values of the function on X_I).

We deduce from these remarks that for each $\lambda \in I \setminus \{0, \infty\}$ the commutative Lie algebra $\mathfrak{A}_\lambda \subset \mathfrak{z}$ obtained by evaluating elements of $\mathfrak{A}$ at λ, consists of semisimple elements. Moreover, $\mathfrak{A}$ has rank $n+1$ as a $\mathfrak{B}$-module and therefore $\mathfrak{A}_\lambda$ is a maximal torus subalgebra of semisimple elements — a Cartan subalgebra for $\mathfrak{g}^{\mathbb{C}}$. We also deduce that we can complete this holomorphic family at $\lambda = 0, \infty$. Since all Cartan subalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate and ours lie in $\mathfrak{z}$, there is a holomorphic map $\gamma\colon I \to Z$ (where the Lie subgroup $Z \subset G^{\mathbb{C}}$ is the stabilizer of Λ), for which $\mathrm{Ad}\gamma_\lambda^{-1} \mathfrak{A}_\lambda = \mathfrak{a}$, where $\mathfrak{a}$ can be any fixed Cartan subalgebra in $\mathfrak{z}$. In particular, we will fix $\mathfrak{a} = \mathfrak{c}_* + \mathfrak{d}$, where $\mathfrak{d}$ is the torus of diagonal matrices in $\mathfrak{z}_*$. Therefore, we define $\widetilde{\mathfrak{A}} = \mathrm{Ad}\gamma^{-1}\,\mathfrak{A}$. Then each element of $\widetilde{\mathfrak{A}}$ takes values in $\mathfrak{a}$, so that it has a block decomposition

$$a = a_0 + a_1 + \cdots + a_{n-m},$$

where a_0 takes values in $\mathfrak{c}_*$, and $a_j \neq 0$ is a diagonal matrix function whose non-zero entry lies only in the $m+j$-th place with $0 \leqq j \leqq n$. In the proof of Proposition 3, we saw how to use the components a_0 to obtain a discrete valuation ring in the ring of fraction $\mathfrak{F}$, and this provided us with the ramification point P_0. To get all the points lying over $\lambda = 0$, it is not hard to show that one defines gradings $o_j \colon \widetilde{\mathfrak{A}} \to \mathbb{Z}$, for which $o_j(a)$ is the order of a_{1j} in λ^{-1} (for $a = (a_1, a_2)$ we define $(a_{1j}, a_{2j}) = a_j$, using the block decomposition above). We use these to define valuations of the ring of fractions $\widetilde{\mathfrak{A}}$ of $\widetilde{A}$ in a manner described earlier.

Next, we will describe explicitly the extension $\mathcal{L}(z)$ of $\mathcal{L}_A(z)$ to all of X. In what follows, let us suppose we are dealing with the generic case in which $\pi^{-1}(0)$ has $n+1-m$ distinct points. First, define $\widetilde{W}(z) = \gamma^{-1}W(z)$ so that this represents the module of sections of $\mathcal{L}_A(z)$ over $X_A \cong \mathrm{Spec}(\widetilde{\mathfrak{A}})$. Then define $\mathcal{W}$ to be the $\widetilde{\mathcal{F}}$-module of fractions $\widetilde{S}\widetilde{W}(z)$ (where $\widetilde{S}$ is the set of all non-zero divisors of $\widetilde{\mathfrak{A}} \setminus \{0\}$). For each point P in $X \setminus X_A$, we can construct an $\mathcal{O}_p$-module $\mathcal{W}_P \subset \mathcal{W}$ which is torsion free (and therefore free, since $\mathcal{O}_P$ is a discrete valuation ring). This will be the stalk for $\mathcal{L}(z)$ over P. For a point over λ, for example, we get each $\mathcal{W}_P$ in the following manner.

On $\widetilde{W}(z)$, define a map $o_P \colon \widetilde{W}(z) \setminus \{0\} \to \mathbb{Z}$ such that: (i) $o_P(w_1 + w_2) \geqq \min(o_P(w_1), o_P(w_2))$ and, (ii) $o_P(aw) = \nu_P(a) + o_P(w)$ whenever $a \in \widetilde{\mathfrak{A}}$, where ν_P is the valuation for $\mathcal{O}_P$. This extends to give $o_P \colon \mathcal{W} \setminus \{0\} \to \mathbb{Z}$ by setting $o_P(w/a) = o_P - \nu_P$. So we define

$$\mathcal{W}_P = \{w \in \mathcal{W} \setminus \{0\} \mid o_P(w) \geqq 0\} \cup \{0\}.$$

This is clearly a torsion free $\mathcal{O}_P$-module, since $\widetilde{W}(z)$ is torsion free. Now we need to describe precisely what the map o_P is.

Let $w \in \widetilde{W}(z)$ and write its restriction w_1 to C_1 as the column vector

$$w_1 = {}^t(f_0(\lambda), \dots, f_n(\lambda)).$$

For $j = 1, \dots, n-m$, define

$$o_j(w) = -(\text{order of pole of } f_{m+j} \text{ at } \lambda = 0).$$

For any $a \in \widetilde{\mathfrak{A}}$ it is easy to see that the correspondence $w \mapsto aw$ multiplies each f_{m+j} by a function of λ whose order is $\nu_j(a)$. It follows that o_j satisfies both properties (i) and (ii) above. For $j = 0$ define

$$f(\zeta) = \zeta^{m+1} \sum_{j=0}^{m} \zeta^{-j} f_j(\zeta^{m+1})$$

and

$$o_0(w) = -(\text{order of pole of } f(\zeta) \text{ at } \zeta = 0).$$

This clearly satisfies property (ii), so we only have to check (i). Recall that the ν-equivalent representation of a, that is $\alpha = \mathrm{Ad}\kappa^{-1}\,\mathrm{a}$, has a block decomposition $\alpha_0 + \alpha_1 + \dots + \alpha_{n-m}$ for which the restriction α_{10} of α_0 to C_1 has a Fourier expansion of the form

$$\alpha_{10} = \sum \beta_j \zeta^{-j} \Lambda^j$$

with only finitely many negative powers of ζ. Let us define

$$\beta(\zeta) = \sum \beta_j \zeta^{-j}.$$

Then we have

$$o_0(aw) = -(\text{order of } \beta f \text{ at } \zeta = 0) = \nu_0(a) + o_0(w).$$

In fact, set $\mathbf{1} = {}^t(1, \dots, 1, 0, \dots, 0)$ with $m+1$ entries. Then $f(\zeta) = \zeta^m \, {}^t\mathbf{1}\kappa^{-1}w_1$. Using the fact ${}^t\mathbf{1}\Lambda^j = {}^t\mathbf{1}$, we compute

$$\begin{aligned} {}^t\mathbf{1}\kappa^{-1}a_1 w_1 &= {}^t\mathbf{1}\alpha_{10}\kappa^{-1}w_1 \\ &= {}^t\mathbf{1}(\sum \beta_j \zeta^{-j}\Lambda^j)\kappa^{-1}w_1 \\ &= \beta f, \end{aligned}$$

from which the result easily follows.

By its construction, the direct image $\pi_*\mathcal{L}(z)$ over $\mathbb{P}^1$ has transition relations

$$\widetilde{\chi}\,\widetilde{\tau}_z = \tau_z, \tag{4.6}$$

where $\widetilde{\chi} = \chi\gamma$ and $\widetilde{\tau}_z = \gamma\widehat{\tau}_z$. The advantage of these transition relations over I is that $\widetilde{\tau}_z$ now occurs as a direct image from a trivialization of θ_z for $\mathcal{L}(z)$ over X_I (we will write $\widetilde{\tau}_z = \pi_*\theta_z$). To see this, let s be a section of $\mathcal{L}(z)$ over $\pi^{-1}(I)$ (and identify it with the corresponding section of $\mathcal{E}(z)$). Then

$$\widetilde{W}(z) \ni \widetilde{\tau}_z(s) = \begin{cases} {}^t(f_0, \dots, f_n) & \text{about } \lambda = 0, \\ {}^t(h_0, \dots, h_n) & \text{about } \lambda^{-1} = 0. \end{cases}$$

In the construction of $\mathcal{L}(z)$ over X_I we used the one-to-one correspondence

$$\begin{aligned} (f_j) &\leftrightarrow (f, f_{m+1}, \dots, f_n), \quad \text{where } f(\zeta) = \zeta^{m+1}\sum_0^m f_j(\zeta^{m+1})\zeta^{-j}, \\ (h_j) &\leftrightarrow (h, h_{m+1}, \dots, h_n), \quad \text{where } h(\zeta) = \sum_0^m h_j(\zeta^{m+1})\zeta^{-j}. \end{aligned} \tag{4.7}$$

The right-hand side above gives us a trivialization over X_I, since this is a union of $n+1-m$ distinct pairs of discs. Moreover, the definitions of $f(\zeta)$ and $h(\zeta)$ describe the property of direct image about a point of ramification (see, for example, [8]).

Lemma 20. *Let $[e^a]$ denote the line bundle of degree zero over X determined by the 1-cocycle (e^a, X_A, X_I). Then $\mathcal{L}(z) \otimes \mathcal{L} \cong [e^a]$ for $a = z\zeta^{-1} - \bar{z}\zeta$.*

Proof. We have already observed that $\Phi\tau_z = \tau_0$, from which it follows (using (4.6) and (4.5)) that $\Phi^{(0)}\pi_*\theta_z = \pi_*\theta_0$. It suffices to show that this implies $\theta_z = e^a\theta_0$, where we regard θ_z as a non-vanishing local section of $\mathcal{L}(z)$. This identity follows if we see

$$\pi_*\theta_z(\zeta^{-1}s) = \Lambda\pi_*\theta_z(s) \tag{4.8}$$

for any local section s, where ζ is defined to be zero except in the discs about P_0 and P_∞. Indeed, then we see that $\pi_*\theta_z(e^a s) = \Phi^{(0)}\pi_*\theta_z(s) = \pi_*\theta_0(s)$, whence $e^a s/\theta_z = s/\theta_0$. So let us prove (4.8) about $\lambda = 0$ — the argument about $\lambda^{-1} = 0$ is much the same.

From (4.7) we know that if $\pi_*\theta_z(s) = {}^t(f_0, \dots, f_n)$ about $\lambda = 0$, then $s/\theta_z = \zeta^{m+1}\sum_0^m \zeta^{-j} f_j(\zeta^{m+1})$ about P_0. Now a simple calculation shows that

$$\pi_*\theta_z(\zeta^{-1}s) = {}^t(\lambda^{-1}f_m, f_0, \dots, f_{m-1}, 0, \dots, 0) = \Lambda\pi_*\theta_z(s)$$

as required. □

Now we can prove Theorem 7.

Proof of Theorem 7. Since $\pi_*\mathcal{L}(z)$ is trivial, $\mathcal{L}(z)$ must have degree $p+n$ and the transition relations (4.6) show that $\mathcal{L}(z)$ is real. Also, by the previous lemma, $\mathcal{L}(z) \cong \mathcal{L} \otimes [e^a]$ moves linearly around the Picard variety.

Now define a global section $\sigma_j(z)$ of $\mathcal{L}(z)$ by $\delta_j = \tau_z(\sigma_j(z))$. Thus $\tau_0(\sigma_j(z)) = \Phi\,\delta_j$ arises from the map ψ. We must show that the global section $\sigma_j(z)$ has a divisor of zeros $(m-j)P_0 + jP_\infty + E_0$. Near $\lambda = 0$, we have

$$\pi_*\theta_z(\sigma_j(z)) = \widetilde{\chi}(z)^{-1}\delta_j.$$

Examining the leading order terms in the two Fourier series' for $\widetilde{\chi}$ (one on C_1 and the other on C_2), we see that

$$\widetilde{\chi}(z)^{-1}\delta_j = \begin{cases} {}^t(a_0, \dots, a_j, 0, \dots, 0) + O(\lambda) & \text{about } \lambda = 0, \\ {}^t(0, \dots, 0, b_j, \dots, b_n) + O(\lambda^{-1}) & \text{about } \lambda^{-1} = 0. \end{cases}$$

It follows from (4.7) that

$$\sigma_j(z)/\theta_z^0 = \begin{cases} \zeta^{m+1}\sum_0^j a_i\zeta^{-i} + O(\zeta^{m+1}) & \text{about } P_0, \\ 0 + O(\lambda) & \text{about } P_0 \in X_0, \quad k \neq 0, \\ \sum_j^m b_i\zeta^{-i} + O(\zeta^{-m-1}) & \text{about } P_\infty. \end{cases}$$

This shows that $\sigma_j(z)$ has a divisor of zeros at least $(m-j)P_0 + jP_\infty + E_0$. $\square$

4.6. **Computation of arithmetic genus of spectral curves.** Recall that the dressing orbit O_Λ is isomorphic to $\Lambda_{I,B}/\Gamma_{I,B}$ and therefore every primitive map in this orbit corresponds to a coset $g\Gamma_{I,B}$. An examination of the definition of the ring $\mathfrak{R}$ (via $\mathfrak{S}$) shows that it depends only upon this coset and not the particular choice of g. Given $\mathfrak{R}$, the module M fixes $\mathcal{L}$ and we see that the results above provide a bijective corresponding between (i) 'full' primitive maps $\mathbb{R}^2 \to G/H$ of finite type, based by $\psi(0) = H$, up to base point preserving isometries, and (ii) triplets $(X, \pi, \mathcal{L})$ satisfying the conditions described at the beginning of Subsection 2.1 (including the possibility that X is singular or reducible).

Recall also that the group of higher flows $\exp(\mathfrak{C}_R)$ acts on O_Λ and, in particular, its two parameter subgroup $\exp(\mathfrak{m})$ induces the translation flow $\psi \mapsto \psi^w$ corresponding to the translation $z \mapsto z + w$ in $\mathbb{R}^2 \cong \mathbb{C}$. One readily sees (from the previous subsection) that in terms of the triplet $(X, \pi, \mathcal{L})$ this action fixes X, π and maps $\mathcal{L}$ to $\mathcal{L}(w)$. In particular, whenever $\psi(z)$ is doubly periodic (and therefore of finite type) with periods z_1, z_2, we must have $\psi^{z_j} = \psi$ and therefore $\mathcal{L}(z_j) = \mathcal{L}$. We obtain

Proposition 6. *A necessary condition for $(X, \pi, \mathcal{L})$ to correspond to a harmonic 2-torus is that there exist linearly independent $z_1, z_2 \in \mathbb{C}$ for which $\mathcal{L}(z_j) \cong \mathcal{L}$.*

Remark. This is not a sufficient condition. An examination of (4.4) shows that $\psi^w = \psi$ is not sufficient to imply $\psi(z + w) = \psi(z)$; they will in general differ by a factor depending upon the extended frame $\Phi_\zeta(w)$.

It should be possible (with some extra work) to exhibit an analytic isomorphism between the $\exp(\mathfrak{C}_R)$-orbit of a map of finite type and the $J_R(X)$-orbit of the corresponding $\mathcal{L}$. Since the $J_R(X)$-orbit of (a maximal sheaf) $\mathcal{L}$ is isomorphic to $J_R(X)$ itself, this would identify the arithmetic genus p of X with the dimension of the $\exp(\mathfrak{C}_R)$-orbit. But we can get this useful result more quickly from:

Proposition 7. $H^1(X, \mathcal{O}_X) \cong \mathfrak{C}/\mathfrak{C}^{[g]}$.

Proof. For convenience, let U_0, U_∞ be the open discs about P_0, P_∞ obtained from X_I. Let U denote their union and set $U^* = U \setminus \{P_0, P_\infty\}$, $X^* = X \setminus \{P_0, P_\infty\}$. Since X is connected, X^* is a Stein manifold and therefore the sequence

$$0 \to \mathrm{Hol(U, \mathbb{C})} + \mathrm{Hol(X^*, \mathbb{C})^{alg}} \to \mathrm{Hol(U^*, \mathbb{C})^{alg}} \to \mathrm{H^1(X, \mathcal{O}_X)} \to 0$$

is exact, where the subscript 'alg' denotes functions with only finite order poles at P_0, P_∞. Now observe that $\mathcal{C}_* = \Lambda_C(\mathfrak{c}_*, \nu)^{\mathrm{alg}}$ is isomorphic to $\mathrm{Hol(U^*, \mathbb{C})}$. The isomorphism is given, as we have already seen, by $c(\zeta^{-1}\Lambda) \mapsto c(\zeta^{-1})$. Moreover, we observe that $\mathfrak{C} = \widehat{\mathfrak{B}} \oplus \mathfrak{C}_*$, where $\widehat{\mathfrak{B}}$ is the subspace of all multiples of the identity matrix, and that $\widehat{\mathfrak{B}}$ is clearly contained in $\mathfrak{C}^{[g]}$. Therefore $\mathfrak{C}/\mathfrak{C}^{[g]} \cong \mathfrak{C}_*/\mathfrak{C}_*^{[g]}$, where $\mathfrak{C}_*^{[g]} = \mathfrak{C}_* \cap \mathfrak{C}^{[g]}$. So it remains to show that $\mathfrak{C}_*^{[g]}$ corresponds to the kernel of the exact sequence above.

Clearly, $\mathrm{Hol(X^*, \mathbb{C})^{alg}}$ is isomorphic to $\mathbb{C}[X^*]$, which can be realized as the subalgebra $\widetilde{\mathfrak{A}^0}$ of $\widetilde{\mathfrak{A}}$ consisting of those elements which extend holomorphically to every point except P_0, P_∞. That is, in the decomposition $a = a_0 + a_1 + \cdots + a_{n-m}$, each a_j for $j \neq 0$ extends holomorphically into I. Let $\mathfrak{C}_*^0$ denote the image of this subspace under the projection of $\Lambda_C(\mathfrak{a}, \nu)^{\mathrm{alg}}$ onto $\mathfrak{C}_*$. Note that this identifies these two spaces. First, we see that the kernel is $\{a \in \widetilde{\mathfrak{A}^0} \mid a_0 = 0\}$. Moreover each element of the kernel represents a regular function on X^* which vanishes on the irreducible component $X_0 \cap X^*$, where X_0 carries P_0. But such function must be identically zero, since its restriction to other irreducible components of X^* (which are complete subvarieties) must be globally regular. Now take $\mathfrak{C}_*^I$ to be the complexification of $\Lambda(\mathfrak{c}_*, \nu) \cap (\mathfrak{b} + \Lambda_I(\mathfrak{g}, \nu))$. Then it is not hard to see that $\mathfrak{C}_*^I$ is identified with $\mathrm{Hol(U, \mathbb{C})}$ under the isomorphism $\mathfrak{C}_* \cong \mathrm{Hol(U^*, \mathbb{C})}$. So it remains to show that $\mathfrak{c} \in \mathfrak{C}_*^{[g]}$ if and only if $c = a_0 + b_0$, where $a_0 \in \mathfrak{C}_*^0$ and $b_0 \in \mathfrak{C}_*^I$. In fact, it suffices to prove this when c is real.

First, observe that $c = c_0$. When $c = a_0 + b_0$, we can define $b = b_0 - a_1 - \cdots - a_{n-m}$ so that $c = a + b$. This exhibits c as an element of $(\mathfrak{C}_*^{[g]})_R = \mathfrak{C}_* \cap \mathfrak{Z}_R^{\mathrm{pol}} \oplus \mathfrak{Z}_R^I$. Conversely, if $c \in (\mathfrak{C}_*^{[g]})_R$, then $c = a + b$, where $a \in \widetilde{\mathfrak{A}}_R$ and we see that $b \in \mathfrak{Z}_R^I$ commutes with every element of $\widetilde{\mathfrak{A}}_R$ by the proof of Lemma 16. Since X is connected, $\widetilde{\mathfrak{A}}$ is a $\mathfrak{B}$-module of rank $n+1$, so elements of $\widetilde{\mathfrak{A}}$ have maximal rank almost everywhere. Thus b must take values in $\mathfrak{a}$ so that $c = a_0 + b_0$. $\square$

5. Properties of spectral data with a compact connected Riemann surface

This section is devoted to the proof of Theorem 1. First, properties of smooth real curves are described, from which we choose spectral curves. Second, properties of meromorphic functions on the above spectral curves which satisfy Conditions (2) and (4) in Definition 2.1 are determined (Proposition 8). Finally, after preparing a tool (Proposition 9) useful to select line bundles satisfying Condition (3) in Definition 2.1, we prove Theorem 1.

5.1. **Properties of smooth real curves.** First, we define subsets in the rational curve $\mathbb{P}^1$. Let S^+ (resp. S^-) be the northern (resp. southern) hemisphere defined by $S^+ = \{\lambda \in \mathbb{P}^1 \mid |\lambda| > 1\}$ (resp. $S^- = \{\lambda \in \mathbb{P}^1 \mid |\lambda| < 1\}$). Let X be a compact connected Riemann surface. Let ρ_X be an anti-holomorphic involution on X and X^ρ a subset of X formed by the fixed points for ρ_X.

It should be remarked that it is not suitable for our purpose to choose a Riemann surface with an anti-holomorphic involution ρ_X such that $X^\rho = \emptyset$, since ρ_X has no fixed points on X and hence violates Condition (4) in Definition 2.1.

Theorem 8 ([6])**.** *Let (X, ρ_X) be as above and $X^\rho \neq \emptyset$. Then $X \setminus X^\rho$ consists of* (F0) *two connected components or* (F1) *one connected component.*

If X is a Riemann surface of type (F0), then X^ρ consists of $\nu(X)$ circles $S^1_1, \dots, S^1_{\nu(X)}$.

Proposition 8. *Let π be a non-constant holomorphic map from X to $\mathbb{P}^1$ satisfying the following conditions:*

(1) $\pi \circ \rho_X = \rho \circ \pi$,
(2) *ρ_X fixes every point of $\pi^{-1}(S^1)$,*
(3) *π has no branch points on S^1.*

Then X is a Riemann surface of type (F0)*. Moreover, π is a meromorphic function on X of degree $N = n + 1$ satisfying all poles are contained in X^N and all zeros are* *contained in X^S, or all poles are contained in X^S and all zeros are contained in X^N. Moreover π has a zero P_0 of order $\geqq 2$ and has a point $x \in X^\rho$ such that $|\pi(x)| = 1$ and the set of poles is the image of the set of zeros by ρ_X.*

Proof of Proposition 8. The proof is divided into several lemmas.

Lemma 21. *There exist no non-constant holomorphic maps from a connected compact Riemann surface X of type* (F1) *to $\mathbb{P}^1$ satisfying Condition* (2) *in Proposition 8.*

Proof. Suppose that such a map exists. Let $X^* = X \setminus X^\rho$, $X^+ = \{x \in X^* \mid \pi(x) \in S^+\}$, and $X^- = \{x \in X^* \mid \pi(x) \in S^-\}$. Then X^+ and X^- are open and $X^* = X^+ \cup X^-$. Since X^* is connected, X^* coincides with either X^+ or X^-. In particular, π is not surjective, which is a contradiction. □

On account of Lemma 21, we may assume that X is a compact connected Riemann surface of type (F0).

Lemma 22. *The map π satisfies Condition* (1) *in Proposition 8 if and only if π is a meromorphic function on X of degree $N = n+1$ satisfying all poles are contained in X^N and all zeros are contained in X^S, or all poles are contained in X^S and all zeros are contained in X^N. Moreover π has a point $x \in X^\rho$ such that $|\pi(x)| = 1$.*

Proof. The map π intertwines the involution ρ_X on X and ρ on $\mathbb{P}^1$ if and only if

$$\pi(u)\overline{\pi(\rho_X(u))} = 1. \tag{5.1}$$

From this it follows that if π has a pole (resp. zero) of order k at p, then $\rho_X(p)$ is the zero (resp. pole) of π of order k. Since ρ_X fixes every point of X^ρ, there exist no zeros and poles on X^ρ. Suppose that $\pi \colon X \to \mathbb{P}^1$ satisfies Condition (1) in Proposition 8. Then the divisor of π must be of the following form

$$(\pi) = (\alpha_1) + \cdots + (\alpha_k) - (\beta_1) - \cdots - (\beta_k), \tag{5.2}$$

where α_i, β_i are points on $X \setminus X^\rho$ which satisfy $\beta_i = \rho_X(\alpha_i)$. Take a point P on X^ρ. Using (5.1), we get $\pi(P)\overline{\pi(P)} = 1$, that is, $|\pi(P)| = 1$.

Conversely, let π be the map which satisies (5.2) and has a point $p \in X^\rho$ with $|\pi(p)| = 1$. Then, clearly π satisfies the equation (5.1). $\square$

Lemma 23. *Let π be a map as in Lemma 22. Then π satisfies Condition (2) in Proposition 8 if and only if π is either* (A) χ *or* (B) $1/\chi$*, where χ is the meromorphic function as in Proposition 8.*

Proof. Since X is a compact connected Riemann surface of type (F0), $X^* = X \setminus X^\rho$ consists of two connected components. More precisely, $X^* = X^N \cup X^S$. Let $X^{N,+}$ and $X^{N,-}$ be the subsets of X defined by $X^{N,\pm} = \{x \in X^N \mid \pi(x) \in S^\pm\}$, respectively. Similarly, define $X^{S,\pm} = \{x \in X^S \mid \pi(x) \in S^\pm\}$.

Suppose that π satisfies Condition (2) in Proposition 8. Then we see that $\pi(X^*) \cap S^1 = \emptyset$. It then follows that $X^N = X^{N,+} \cup X^{N,-}$ and $X^S = X^{S,+} \cup X^{S,-}$. Since X^N and X^S are connected, we see that (a) $X^N = X^{N,+}$, $X^S = X^{S,-}$ or (b) $X^N = X^{N,-}$, $X^S = X^{S,+}$. In the case (a) (resp. (b)), π must be a function of type (A) (resp. (B)) as in Proposition 8.

Conversely, if π is either (A) χ or (B) $1/\chi$, then it is easy to see that π maps $S^1_1, \dots, S^1_{\nu(X)}$ into S^1. Let π_i denotes the restriction of π to S^1_i and d_i be the degree of the map $\pi_i \colon S^1_i \to S^1$ for $1 \leqq i \leqq \nu(X)$. Since $|d_i| + \cdots + |d_{\nu(X)}|$ coincides with the degree of π by the residue theorem, we see that for any point $p \in S^1$, $\pi^{-1}(p)$ is contained in $X^\rho = S^1_1 \cup \cdots \cup S^1_{\nu(X)}$. This implies that π satisfies Condition (2) in Proposition 8. $\square$

Lemma 24. *Let π be a map as in Proposition 8. Then the ramification divisor does not intersect $X^\rho = S^1_1 \cup \cdots \cup S^1_{\nu(X)}$.*

Proof. Let π be a meromorphic function of type (A) as in Proposition 8. Note that the number of zeros of π on X^S is given by the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial X^S} \frac{1}{\pi(u)} d\pi(u),$$

which is equal to $N = n+1$ from Proposition 12. Since π maps $S^1_1, \dots, S^1_{\nu(X)}$ into S^1, for every point $p \in S^1$ we have

$$\sharp\{\pi^{-1}(p)\} = N. \tag{5.3}$$

Suppose that there exists a point x such that $x \in R \cap (S^1_1 \cup \cdots \cup S^1_{\nu(X)})$, where R is the ramification divisor of π. Setting $q = \pi(x)$, we see that $\sharp\{\pi^{-1}(q)\} = N$ by the identity (5.3).

Let $\pi^{-1}(q) = \{P_1, \dots, P_N\}$ and U_i a neighbourhood of P_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Let $V(q)$ be the neighbourhood of q defined by $V(q) = \bigcap_i \pi(U_i)$. Denote by e the degree of π at x. It then follows from the assumption $e \geqq 2$ that there exists a neighbourhood $W(x)$ of x such that $\pi(W(x)) \subset V(q)$ and the degree of $\pi|_{W(x)\setminus\{x\}}$, the restriction of π to $W(x) \setminus \{x\}$, is e. Take a point $y \in \pi(W(x)) \setminus \{q\}$. Then, there exist a point $Y_i \in U_i$ for each $i \neq 1$ and points $Z_1, \dots, Z_e \in U_1$ such that π maps all of these points to y. Also, we see that $\sharp\{\pi^{-1}(y)\} \geqq N - 1 + e \geqq N + 1$. This contradicts that the degree of π is N. Hence R does not intersect $S^1_1 \cup \cdots \cup S^1_{\nu(X)}$.

The proof for a meromorphic function of type (B) as in Proposition 8 proceeds in a similar manner. $\square$

By Lemma 22, Lemma 23 and Lemma 24, Proposition 8 has been proved.

5.2. **Δ-invariants of divisors on Riemann surfaces.**

Proposition 9. *Let (X, ρ_X) be a compact connected Riemann surface of type* (F0). *Let E and F be divisors on X*

$$E + \rho_X(E) \cong F + \rho_X(F), \tag{5.4}$$

where $\cong$ means linearly equivalence. Let f be a non-constant meromorphic function such that

$$(f) = E + \rho_X(E) - (F + \rho_X(F)), \quad \overline{\rho_X^* f} = f, \tag{5.5}$$

where (f) is the divisor of f. Then f^ρ, the restriction of f to $S_1^1 \cup \cdots \cup S_{\nu(X)}^1$, is a non-negative or non-positive real function if and only if $\Delta(E-F) = 0$ where $\Delta(E-F)$ is a number defined as follows:

$$\Delta(E-F) = \nu(X) - |\sharp\{s_i \in \Lambda \mid f(s_i)/f(s_1) > 0\} - \sharp\{s_i \in \Lambda \mid f(s_i)/f(s_1) < 0\}|.$$

Here Λ is a set consisting of points $s_1, s_2, \ldots, s_{\nu(X)}$ such that $s_i \in S_i^1$, $f(s_i) \neq 0, \infty$.

Proof. Let S_{zp} be a intersection of $S_1^1 \cup \cdots \cup S_{\nu(X)}^1$ with the set of zeros and poles of f^ρ. Restricting f^ρ to $(S_1^1 \cup \cdots \cup S_{\nu(X)}^1) \setminus S_{zp}$, we get a real function f^*. Considering the restriction of $(E + \rho_X(E) - F - \rho_{X*}(F))$ to $S_1^1 \cup \cdots \cup S_{\nu(X)}^1$, we see that f^ρ has only zeros and poles with even order. So the sign of f^* does not change at each point of S_{zp}. Thus f^ρ is non-negative or non-positive on each connected component of $S_1^1 \cup \cdots \cup S_{\nu(X)}^1$. Hence f^ρ is a non-negative or non-positive real function on $S_1^1 \cup \cdots \cup S_{\nu(X)}^1$ if and only if there exist points $p_1 \in S_1^1 \setminus S_{zp}, \ldots, p_{\nu(X)} \in S_{\nu(X)}^1 \setminus S_{zp}$ such that $f(p_i)/f(p_1) > 0$ for $1 \leqq i \leqq \nu(X)$, that is, $\Delta(E-F) = 0$. □

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Conditions (2) and (4) in Definition 2.1 are equivalent to the following assertions:

(1) π is a meromorphic function as in Proposition 8.

(2) π has a zero P_0 of order $m+1 \geqq 2$.

This means that Conditions (2) and (4) in Definition 2.1 are satisfied precisely when π satisfies Condition (2) in Theorem 1. It is clear that $R = R_+ + \rho_{X*}(R_+)$. Applying Proposition 9 to $E = D$ and $F = R_+$, we see that $\delta(\mathcal{L}) = \Delta(D - R_+)$. Thus Condition (3) in Definition 2.1 is equivalent to Condition (3) in Theorem 1. Hence Theorem 1 is proved. □

6. Rational or elliptic spectral curves

6.1. **Jacobi's theta functions and Weierstrass' zeta functions.** C. G. J. Jacobi introduced four functions θ_1, θ_2, θ_3 and θ_4 of variables $p(u) = \exp(\pi\sqrt{-1}u)$ and $q = \exp(\pi\sqrt{-1}\tau)$, where u is the usual covering coordinate of an elliptic curve $X = \mathbb{C}/\mathbb{L}$ and τ stands for its period ratio with familiar standardization that the imaginary part $\mathrm{Im}\tau$ of τ is positive. If we take $\mathbb{L}$ to be $\mathbb{Z}\oplus\tau\mathbb{Z}$ for simplicity, then these Jacobi's theta functions are given as follows:

$$\begin{aligned}
\theta_1(u) &= \theta_1(u|\tau) = \sqrt{-1}\sum(-1)^n p^{2n-1}q^{(n-1/2)^2},\\
\theta_2(u) &= \theta_2(u|\tau) = \sum p^{2n-1}q^{(n-1/2)^2},\\
\theta_3(u) &= \theta_3(u|\tau) = \sum p^{2n}q^{n^2},\\
\theta_4(u) &= \theta_4(u|\tau) = \sum(-1)^n p^{2n}q^{n^2}.
\end{aligned}$$

Here the sums are taken over $n \in \mathbb{Z}$. Under the addition of half-periods, these functions transform according to the following table.

	$u+1/2$	$u+\tau/2$	$u+1/2+\tau/2$	$u+1$	$u+\tau$	$u+1+\tau$
θ_1	θ_2	$-\sqrt{-1}a\theta_4$	$-a\theta_3$	$-\theta_1$	$-b\theta_1$	$b\theta_1$
θ_2	$-\theta_1$	$-a\theta_3$	$\sqrt{-1}a\theta_4$	$-\theta_2$	$b\theta_2$	$-b\theta_2$
θ_3	θ_4	$a\theta_2$	$\sqrt{-1}a\theta_1$	θ_3	$b\theta_3$	$b\theta_3$
θ_4	θ_3	$\sqrt{-1}a\theta_1$	$a\theta_2$	θ_4	$-b\theta_4$	$-b\theta_4$

For example, we have the transformation rules

$$\theta_1(u+\tau) = -b(u)\theta_1(u), \tag{6.1}$$

$$\theta_1(u+1/2) = \theta_2(u), \tag{6.2}$$

$$\theta_1(u+\tau/2) = -\sqrt{-1}a(u)\theta_4(u), \tag{6.3}$$

$$\theta_3(u+\tau/2) = a(u)\theta_2(u), \tag{6.4}$$

$$\theta_4(u+1/2) = \theta_3(u), \tag{6.5}$$

where $a(u) = p(u)^{-1}q^{-1/4}$ and $b(u) = p(u)^{-2}q^{-1}$. Special values of these functions are obtained as follows:

$$\begin{aligned}
&\lim_{t\to\infty} q^{-1/4}\frac{\partial\theta_1}{\partial u}(0|\sqrt{-1}t) = 2\pi, \quad \lim_{t\to\infty} q^{-1/4}\theta_2(0|\sqrt{-1}t) = 2,\\
&\lim_{t\to\infty}\theta_3(0|\sqrt{-1}t) = 1, \quad \lim_{t\to\infty}\theta_4(0|\sqrt{-1}t) = 1.
\end{aligned} \tag{6.6}$$

On the other hand, Weierstrass' zeta function ζ_w is defined by

$$\zeta_w(u) = \zeta_{w,\tau}(u) = \frac{1}{u} + \sum_{\omega \in \mathbb{L} \setminus (0,0)} \left\{ \frac{1}{(u-\omega)} + \frac{u}{\omega^2} + \frac{1}{\omega} \right\}. \tag{6.7}$$

Note that these functions have the following properties. θ_1 is a odd function. θ_2, θ_3 and θ_4 are even functions. Concerning ζ_w, there exist complex numbers $A = A_\tau$ and $B = B_\tau$ depending only on τ such that

(6.8)
$$\zeta_w(u+1) - \zeta_w(u) = A, \quad \zeta_w(u+\tau) - \zeta_w(u) = B, \quad A\tau - B = 2\pi\sqrt{-1}.$$

Moreover, if τ is pure imaginary, we have $\overline{\theta_1(u)} = \theta_1(\overline{u})$, $\overline{\zeta_w(u)} = \zeta_w(\overline{u})$, $\overline{A} = A$ and $\overline{B} = -B$.

For further details and formulas regarding these functions, we refer the reader to McKean and Moll [22, Chapter 3].

6.2. **Main results.** Our main theorems which refine the correspondence proved by McIntosh may be stated as follows. (See Section 2.2 for the detail of this correspondence.)

Theorem 9. *Let X be the smooth rational curve. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if the following conditions are satisfied:*

(1) *(X, ρ_X) is real isomorphic to $(\mathbb{P}^1, \rho)$. By the affine coordinate λ, π is expressed as*

$$\pi(\lambda) = \alpha_0 \lambda^{m+1} \frac{\prod_{j=1}^{n-m}(\lambda - P_j)}{\prod_{j=1}^{n-m}(\lambda - Q_j)}, \quad P_0 = 0, \quad \alpha_0 = \frac{\prod_{j=1}^{n-m}(1 - Q_j)}{\prod_{j=1}^{n-m}(1 - P_j)}$$

for some m and n with $1 \leqq m \leqq n-1$. Here $P_j \in X^S = \{\lambda \in X \mid 0 < |\lambda| < 1\}$ and $Q_j = 1/\overline{P_j}$ for any $1 \leqq j \leqq n-m$.

(2) *$\mathcal{L}$ is a line bundle of degree n.*

Theorem 10. *Choosing a complex coordinate on the source suitably, the harmonic map $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ corresponding to the spectral data $(X,\, \pi,\, \mathcal{L} = \mathcal{O}_X(D))$ in Theo-*

rem 9 is given by

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i(z)$ is a function defined by

$$\Psi_i(z) = \exp\left(\eta_i^{-1} z - \eta_i \overline{z}\right) \cdot \frac{\prod_{j=1}^{n-m} (\eta_i - P_j)}{\prod_{j=1}^{n-m} (\eta_i - R_j)}. \tag{6.9}$$

Here $\{\eta_0, \ldots, \eta_n\}$ is the inverse image $\pi^{-1}(1)$ of 1 by π and $R_+ = \sum_{j=1}^{n} R_j$ is a divisor given by the intersection of X^S with R, that is, $R_+ = X^S \cap R$.

Furthermore we obtain the following

Theorem 11. *Ψ is doubly periodic with periods v_1, $v_2 \in \mathbb{C}$ if and only if the set*

$$V = \bigcap_{1 \leqq i \leqq n} \frac{\pi}{\beta_i} (\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}) \tag{6.10}$$

contains the 2-dimensional lattice $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, where $\beta_1, \ldots, \beta_n$ are complex numbers defined by $\beta_i = \eta_i^{-1} - \eta_0^{-1}$.

Proof. Let $T^n = \{(w_1, \ldots, w_n) \in \mathbb{C}^n \mid |w_i| = 1\ (1 \leqq i \leqq n)\}$ be a real n-dimensional torus group defined by the rule

$$(a_1, \ldots, a_n) \times (b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n).$$

We define a group homomorphism $\Phi = (\Phi_1, \ldots, \Phi_n)$ from the additive group $\mathbb{R}^2$ to T^n by $z = x + \sqrt{-1}y \mapsto (\Psi_1/\Psi_0, \ldots, \Psi_n/\Psi_0)$.

Note that Ψ has two periods v_1, v_2 if and only if so is Φ. If Φ has two periods v_1, v_2, then the set $\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ is contained in V, since V is the set of all points on which the value of Φ is equal to the initial value $\Phi(0) = (1, \ldots, 1) \in T^n$. Conversely, if V contains a 2-dimensional lattice $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, then clearly v_1 and v_2 are periods of Φ, since Φ is a homomorphism. Hence condition (6.10) is a necessary and sufficient condition for Ψ to be doubly periodic with periods v_1, v_2. □

Corollary 2. *Let* $(X, \pi, \mathcal{L})$ *be a spectral data in Theorem* 9 *such that the degree of* π *is* 3*. Then the corresponding harmonic map* $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^2$ *in Theorem* 10 *is always doubly periodic with periods* v_1*,* v_2*, where* v_1 *and* v_2 *are complex numbers in the set*

$$\mathbb{Z}v_+ \oplus \mathbb{Z}v_- = \mathbb{Z}\,\pi\left(\beta_1\mathrm{Im}(\beta_2/\beta_1)\right)^{-1} \oplus \mathbb{Z}\,\pi\left(\beta_2\mathrm{Im}(\beta_1/\beta_2)\right)^{-1}.$$

Proof. In this case, the set V in Theorem 11 reduces to $\mathbb{Z}v_+ \oplus \mathbb{Z}v_-$. Hence Corollary 2 follows from Theorem 11. □

Now we turn to the case of a smooth elliptic spectral curve X. Let us denote by $\mathrm{Pic^d(X)}$ and $J(X)$ the set of line bundles on X of degree d and the Jacobian of X, respectively. Note that $J(X)$ can be identified with $X = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$. We then define a biholomorphic map $J\colon \mathrm{Pic^0(X)} \to \mathrm{J(X)}$ by $J(L) = \sum_{j=1}^{k}(P_j - Q_j) \pmod{\mathbb{Z}\oplus\mathbb{Z}\tau}$, provided that $L \in \mathrm{Pic^0(X)}$ is expressed as a divisor line bundle $\mathcal{O}_X(\sum_{j=1}^{k}(P_j - Q_j))$.

Theorem 12. *Let* X *be a smooth elliptic curve. Then* $(X, \pi, \mathcal{L})$ *is a spectral data if and only if the following conditions are satisfied:*

(1) X *is an elliptic curve* $X_\tau = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$*, where* τ *is a pure imaginary number* $\sqrt{-1}t$ *with* $t > 0$*.* ρ_X *is an anti-holomorphic involution induced by the usual conjugation of* $\mathbb{C}$*. Regarded as a doubly periodic meromorphic function on* $\mathbb{C}$*,* π *is expressed as*

$$\pi(u) = C\frac{\theta_1(u-P_0)^{m+1}\prod_{j=1}^{n-m-1}\theta_1(u-P_j)\cdot\theta_1(u-P_{n-m}+W)}{\theta_1(u-Q_0)^{m+1}\prod_{j=1}^{n-m}\theta_1(u-Q_j)}$$

for some m *and* n *with* $1 \leqq m \leqq n-1$*. Here* $P_i \in X^S = \{x \in X \mid 0 < \mathrm{Im\,x} < \mathrm{Im}\,\tau/2 \pmod{\mathrm{Im}\,\tau\mathbb{Z}}\}$ *and* $Q_i = \overline{P_i} \pmod{\mathbb{Z}\oplus\mathbb{Z}\tau}$ *for any* $0 \leqq i \leqq n-m$*;* $W = (m+1)P_0 + \sum_{i=1}^{n-m}P_i - (m+1)Q_0 - \sum_{i=1}^{n-m}Q_i$*;* $P_0 \neq P_i$ *for* $i \neq 0$*;* W *belongs to* $\mathbb{Z}\oplus\mathbb{Z}\tau$*; and* C *is the unique constant such that* $\pi(0) = 1$*.*

(2) *Let* $r : \mathrm{Pic^{n+1}(X)} \to \mathrm{Pic^0(X)}$ *be a map defined by* $\mathcal{F} \mapsto \mathcal{F}\otimes\mathcal{O}_X(-R_+)$*, where* $R_+ = \sum_{j=0}^{n}R_j$ *is a divisor of degree* $n+1$ *given by the intersection of* X^S *with* R*, that is,* $R_+ = X^S \cap R$*. Then,* $\mathcal{L}$ *is an element of the inverse image of* $\left(\mathbb{Z}\oplus\sqrt{-1}\mathbb{R}\right)/(\mathbb{Z}\oplus\tau\mathbb{Z})$ *by the composition* $J\circ r$*.*

Theorem 13. *Choosing a complex coordinate on the source suitably, the harmonic map* $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ *corresponding to the spectral data* $(X_\tau,\, \pi,\, \mathcal{L} = \mathcal{O}_X(D))$ *in Theorem* 12 *is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i(z)$ *is a function defined by*

$$\Psi_i(z) = \mu_i^{-1} \exp\left(z[\zeta_w(\eta_i - P_0) - A\,\eta_i] - \overline{z}[\zeta_w(\eta_i - Q_0) - A\,\eta_i]\right)$$

(6.11)

$$\cdot \frac{\theta_1(\eta_i - P_0)^m \prod_{j=1}^{n-m} \theta_1(\eta_i - P_j)\theta_1(\eta_i + mP_0 + \sum_{j=1}^{n-m} P_j - D - z + \bar{z})}{\prod_{j=0}^{n} \theta_1(\eta_i - R_j)}.$$

Here $\{\eta_0, \ldots, \eta_n\}$ *is the inverse image* $\pi^{-1}(1)$ *of* 1 *by* π*,* μ_i *is a constant given by* $\mu_i = \exp\left(2\pi\sqrt{-1}(D - R_+)\mathrm{Im}\,\eta_\mathrm{i}/\mathrm{t}\right)$*, and* A *is a constant given in the equation* (6.8)*.*

Moreover we prove the following

Theorem 14. *The harmonic map* $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ *in Theorem* 13 *is doubly periodic with periods* $v_1,\, v_2 \in \mathbb{C}$ *if and only if the set* $V = \bigcap_{0 \leqq i \leqq n} V_i$ *contains the* 2*-dimensional lattice* $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$*, where* $V_0, \ldots, V_n$ *are the sets defined by*

$$V_i = \begin{cases} \pi\beta_i^{-1}\left(\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}\right), & \text{if } \beta_i \neq 0, \\ \mathbb{C}, & \text{otherwise.} \end{cases}$$

Here $\beta_0, \beta_1, \ldots, \beta_n$ *are complex numbers defined by*

$$\beta_0 = -2\pi/t, \quad \beta_i = [\zeta_w(\eta_0 - P_0) - \zeta_w(\eta_i - P_0) - B\,(\eta_0 - \eta_i)\tau^{-1}] \quad (1 \leqq i \leqq n).$$

Corollary 3. *Let* $(X,\, \pi,\, \mathcal{L})$ *be a spectral data in Theorem* 12 *such that the degree of* π *is* 2 *and* $\mathrm{Im}\,\beta_1 \neq 0$*. Then the corresponding harmonic map* $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^1$ *in Theorem* 13 *is always doubly periodic with periods* $v_1,\, v_2$*, where* v_1 *and* v_2 *are complex numbers in the set*

$$\mathbb{Z}v_+ \oplus \mathbb{Z}v_- = \mathbb{Z}\,\pi\,(\mathrm{Im}\beta_1)^{-1} \oplus \mathbb{Z}\,\overline{\beta_1}\,(\mathrm{Im}\beta_1)^{-1}\,t/2.$$

Proof. In this case, the set V in Theorem 14 reduces to $\mathbb{Z}v_+ \oplus \mathbb{Z}v_-$. Hence Corollary 3 follows from Theorem 14. □

We now give some explicit examples of harmonic maps by applying the theorems above.

Example 1. *Let* $(X = \mathbb{P}^1, \pi, \mathcal{L})$ *be a spectral data defined as follows. The map* $\pi\colon \mathbb{P}^1 \to \mathbb{P}^1$ *is given by* $\lambda \mapsto \lambda^{n+1}$. $\mathcal{L}$ *is the divisor line bundle*

$$\mathcal{L} = O_X(n0),$$

and $P_0 = 0$*, a point as in Condition (2) of Definition 1. Then we choose the constant function* $f = 1$ *as a meromorphic function in Condition (3) of Definition 1. Setting* $\omega = \exp(2\pi\sqrt{-1}/(n+1))$*, we see that* $\pi^{-1}(1)$ *is given by* $\{1, \omega, \omega^2, \ldots, \omega^n\}$*. Then the corresponding harmonic map* $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ *is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i = \exp(\omega^{-j}z - \omega^j\bar{z})$*. Note that* Ψ *is a superconformal map. Moreover, if n = 1, 2, 3 or 5, then* ψ *is doubly periodic.*

Example 2. *Let* $(X = \mathbb{P}^1, \pi, \mathcal{L})$ *be a spectral data defined as follows. The map* $\pi\colon \mathbb{P}^1 \to \mathbb{P}^1$ *is now given by*

$$\lambda \mapsto \frac{1-\beta}{1-\alpha}\lambda^2\left(\frac{\lambda-\alpha}{\lambda-\beta}\right),$$

where α *is a real number such that* $0 < |\alpha| < 1$ *and* $\beta = 1/\alpha$*. The ramification divisor* R *of* π *is given by* $R = (R_1) + (0) + (\rho_X(R_1)) + (\infty)$*, where* $R_1 = (\alpha^2 + 3 - \sqrt{\alpha^4 - 10\alpha^2 + 9})/4\alpha$. $\mathcal{L}$ *is the divisor line bundle given by*

$$\mathcal{L} = O_X(R_1 + \infty),$$

and $P_0 = 0$*. Moreover,* $\pi^{-1}(1) = \{\eta_0, \eta_1, \eta_2\}$ *is given by*

$$\eta_0 = 1, \quad \eta_1 = \frac{\alpha - 1 + \sqrt{4-(\alpha-1)^2}\sqrt{-1}}{2}, \quad \eta_2 = \frac{\alpha - 1 - \sqrt{4-(\alpha-1)^2}\sqrt{-1}}{2}.$$

Then the corresponding harmonic map $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^2$ *is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \Psi_2(z)],$$

where

$$\Psi_i(z) = \exp\left(\eta_i^{-1} z - \eta_i \bar{z}\right) \cdot \frac{(\eta_i - \alpha)}{(\eta_i - R_1)}.$$

Note that Ψ is a harmonic map of isotropy order 1 and is nowhere conformal. Moreover, by Corollary 2, Ψ has two complex periods v_1 and v_2, which are in the lattice $\mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ defined by

$$v_+ = \left(-\frac{1}{\sqrt{4-(\alpha-1)^2}} + \frac{\sqrt{-1}}{\alpha - 3} \right)\pi, \quad v_- = \left(\frac{1}{\sqrt{4-(\alpha-1)^2}} + \frac{\sqrt{-1}}{\alpha - 3} \right)\pi.$$

Example 3. *Let $(X_\tau = X_{\sqrt{-1}}, \pi, \mathcal{L})$ be a spectral data defined as follows. We define the map $\pi \colon X_\tau \to \mathbb{P}^1$ by $u \mapsto \lambda = g(u)/g(1/2)$, where $g(u)$ is a meromorphic function on X given by*

$$g(u) = \frac{\theta_1(u-R_0)^2 \theta_1(u - R_0 - 2\sqrt{-1})}{\theta_1(u-R_3)^3}$$

with $R_0 = 1/2 + \sqrt{-1}/6$ and $R_3 = 1/2 + 5\sqrt{-1}/6$. In this case, there exists a point $R_2 \in X^S$ such that the ramification divisor R is expressed as $2R_0 + R_2 + \rho_X(2R_0 + R_2)$. We define the divisor line bundle $\mathcal{L}$ by

$$\mathcal{L} = O_X(2R_0 + R_2).$$

Set $P_0 = R_0$ as a distinguished zero of π as in Condition (2) of Definition 1. We choose the constant function $f = 1$ as a meromorphic function in Condition (3) of Definition 1. In this case, $\zeta_w(\sqrt{-1}r) = -\sqrt{-1}\zeta_w(r)$ for $r \in \mathbb{R}$. From this, together with (6.8), we get $A = \pi$. Since $\pi^{-1}(1)$ is $\left\{0, 1/2, \sqrt{-1}/2\right\}$, the corresponding harmonic map $\Psi \colon \mathbb{R}^2 \to \mathbb{C}P^2$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi(0,z) : \psi(1/2, z) : \psi(\sqrt{-1}/2, z)],$$

where

$$\psi(u,z) = \exp\left[z\left\{\zeta_{w,\tau}(u - R_0) - \pi u\right\} - \bar{z}\left\{\zeta_{w,\tau}(u - R_3) - \pi u\right\}\right] \frac{\theta_1(u - R_2 - z + \bar{z})}{\theta_1(u - R_2)}.$$

Note that Ψ is a superconformal map into $\mathbb{C}P^2$.

Example 4. *Let* $(X_\tau = X_{\sqrt{-1}}, \pi, \mathcal{L})$ *be a spectral data defined as follows. We now define the map* $\pi\colon X_\tau \to \mathbb{P}^1$ *by* $u \mapsto \lambda = \mathfrak{p}(u - R_2)/\mathfrak{p}(-3\sqrt{-1}/4)$*, where* $R_2 = 3\sqrt{-1}/4$ *and* $\mathfrak{p}$ *is Weierstrass'* $\mathfrak{p}$ *function defined by*

$$\mathfrak{p}(u) = \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{(u - (m + \sqrt{-1}n))^2} - \frac{1}{(m + \sqrt{-1}n)^2} \right\}.$$

The ramification divisor R *of* π *is given by* $R = R_0 + R_1 + R_2 + R_3$*, where* $R_0 = \sqrt{-1}/4$*,* $R_1 = (2 + \sqrt{-1})/4$ *and* $R_3 = (2 + 3\sqrt{-1})/4$*. Define the divisor line bundle* $\mathcal{L}$ *by*

$$\mathcal{L} = O_X(R_0 + R_1).$$

Set $P_0 = R_0$ *as a distinguished zero of* π *as in Condition (2) of Definition 1. The constant function* $f = 1$ *can be taken as a meromorphic function in Condition (3) of Definition 1. Since* $\pi^{-1}(1)$ *is* $\{0, \sqrt{-1}/2\}$*, the corresponding harmonic map* $\Psi\colon \mathbb{R}^2 \to \mathbb{C}P^1$ *is given by*

$$z = x + \sqrt{-1}y \mapsto [\psi(0, z) : \psi(\sqrt{-1}/2, z)],$$

where

$$\psi(u, z) = \exp\left[z\left\{\zeta_{w,\tau}(u - R_0) - \pi u\right\} - \bar{z}\left\{\zeta_{w,\tau}(u - R_2) - \pi u\right\}\right] \frac{\theta_1(u - R_1 - z + \bar{z})}{\theta_1(u - R_1)}.$$

Note that Ψ *is a harmonic map of isotropy order* 1 *and is nowhere conformal.*

Concerning the periodicity of Ψ*, the corresponding set* V *in Theorem 14 then consists of the lattice points in Figure 1.*

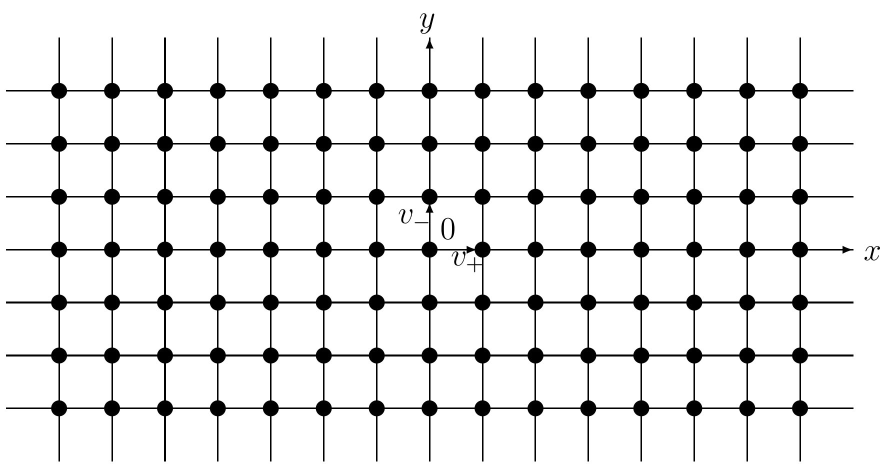

Figure 1.

From Corollary 3, we see that Ψ *has two periods* v_+ *and* v_- *defined by*

$$v_+ = 2\pi/(4\zeta_w(1/4) - \pi) \fallingdotseq 0.4962....., \quad v_- = \sqrt{-1}/2,$$

that is, $\Psi(v_- + z) = \Psi(v_+ + z) = \Psi(z)$*. Moreover,* Ψ *maps the torus* $T = \mathbb{C}/(\mathbb{Z}v_+ \oplus \mathbb{Z}v_-)$ *to an annulus in the Riemann sphere* $\mathbb{C}P^1$*.*

6.3. **Classification of spectral data with the smooth rational spectral curve.** This section is devoted to the proof of Theorem 9. First, we shall describe the real structures of the smooth rational curve $\mathbb{P}^1$.

We first note that there are two real structures on $\mathbb{P}^1$ (cf. §2.1 in [6]). One is $(\mathbb{P}^1, \rho)$. The other is $(\mathbb{P}^1, \sigma)$, where σ is the anti-holomorphic involution defined by

$$\lambda \mapsto -1/\bar{\lambda}.$$

However, it is not suitable to choose the latter as the involution of the spectral curve $X = \mathbb{P}^1$, since it has no fixed points on $\mathbb{P}^1$ and does not satisfy Condition (4) in Definition 1.

Throughout this section, we shall always assume that $X = \mathbb{P}^1$ and $\rho_X = \rho$.

Proposition 10. *Let* π *be a non-constant holomorphic map from* X *to* $\mathbb{P}^1$ *satisfying Conditions* (1) *and* (2) *in Theorem 1.*

Then π *is either* (A) χ *or* (B) $1/\chi$*, where* χ *is a meromorphic function defined by*

$$\chi(\lambda) = \alpha_0 \lambda^k \frac{\prod_{j=1}^{l}(\lambda - \alpha_j)}{\prod_{j=1}^{l}(\lambda - \beta_j)}.$$

Here k *and* l *are some non-negative integers with* $k + l \neq 0$*;* $\alpha_0 \in \mathbb{C}^* = \mathbb{C} \setminus 0$*;* $\alpha_1, \dots, \alpha_l$ *are non zero complex numbers satisfying* $|\alpha_i| < 1$ *and* $|\alpha_0 \alpha_1 \cdots \alpha_l| = 1$*; and* $\beta_i = 1/\bar{\alpha}_i$*. Moreover* π *has a zero* P_0 *of order* $\geqq 2$*.*

Conversely, any map π *expressed as above satisfies Conditions* (2) *in Theorem 1.*

Proof. Assume that X and π satisfy Condition (1) and (2) in Theorem 1. From Condition (2), π has a zero P_0 of order $\geqq 2$, and the divisor of π must be of the following form

$$(\pi) = (\alpha_1) + \cdots + (\alpha_l) - (\beta_1) - \cdots - (\beta_l), \tag{6.12}$$

where $\{\alpha_1, \ldots, \alpha_l\}$ is a subset of S^+ or S^-, $\beta_i = \rho_X(\alpha_i)$, that is, $\beta_i = 1/\bar{\alpha}_i$.

Thus π is either (A) χ or (B) $1/\chi$, where χ is a meromorphic function defined by

$$\pi(\lambda) = \alpha_0 \lambda^k \frac{\prod_{j=1}^{l}(\lambda - \alpha_j)}{\prod_{j=1}^{l}(\lambda - \beta_j)}, \tag{6.13}$$

where $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l$ are all complex numbers contained in $\mathbb{C}^* \backslash S^1$ with $|\alpha_i| < 1$ and $\alpha_0 \in \mathbb{C}^*$.

The function $\pi \overline{\rho_X^* \pi}$ is a constant function since its divisor vanishes. By the assumption that π has the point $x \in X^\rho$ with $|\pi(x)| = 1$, we see that

$$\pi(0)\overline{\rho_X^* \pi(0)} = \pi(0)\overline{\pi(0)} = \pi(x)\overline{\pi(x)} = 1.$$

Using the above equation, we get

$$|\alpha_0 \alpha_1 \cdots \alpha_l| = 1. \tag{6.14}$$

Moreover, from Condition (2) in Theorem 1, π has a zero P_0 of order $\geqq 2$.

Conversely, let π be the map defined as above. Then, clearly π satisfies Conditions (1) and (2) in Theorem 1. □

Proposition 11. *Let π be a meromorphic function on $X = \mathbb{P}^1$ and $\mathcal{L}$ a line bundle over X. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if it satisfies the following conditions:*

(1) *π is a meromorphic function as in Proposition* 10.
(2) *The degree of $\mathcal{L}$ is $N-1$, where N is the degree of π.*

Proof. Conditions (1) and (2) in Theorem 1 are equivalent to that Condition (1) in Proposition 11 by Proposition 10. Let $\mathcal{L} \cong \mathcal{O}_X(D)$ be a line bundle which satisfy Condition (3) in Theorem 1. Then the degree of D must be equal to $N-1$ since the degree of R is equal to $2N-2$.

Conversely let $\mathcal{L} = \mathcal{O}_X(D)$ be any line bundle of degree $N-1$. We see that $\delta(\mathcal{L})$ is automatically 0. Thus Condition (3) in Theorem 1 is equivalent to Condition (2) in Proposition 11. Hence Proposition 11 is proved. □

Now let us prove Theorem 9.

Proof of Theorem 9. To prove this theorem, it suffices to show that for every spectral data $(X, \pi, \mathcal{L})$ with P_0 as in Proposition 11, there exists a real automorphism ϕ on (X, ρ_X) such that the value of λ at $\phi^{-1}(P_0)$ is equal to 0 and the pull-back of π by ϕ is of a form in Condition (1) of Theorem 9. But this is quite straightforward. □

6.4. **Classification of spectral data with smooth elliptic spectral curves.** This section is devoted to the proof of Theorem 12. First, we describe all smooth real elliptic curves which can be spectral curves. Second, meromorphic functions on these spectral curves, which satisfy Condition (2) in Theorem 1, are determined (Proposition 12). Finally, after preparing a device (Proposition 13) useful to select line bundles satisfying Condition (3) in Theorem 1, we prove Theorem 12

Let $X = X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ be an elliptic curve, where τ belongs to the upper half plane $\mathfrak{H} := \{\mathrm{Im}\tau > 0\}$. Let ρ_X be an anti-holomorphic involution of X and X^ρ the fixed point set of ρ_X.

It should be remarked that a real elliptic curve (X, ρ_X) with $X^\rho = \emptyset$ is not suitable for our purpose, since ρ_X has no fixed points on X and hence violates Condition (4) in Definition 1.

Theorem 15 ([6])**.** *Let (X, ρ_X) be as above and $X^\rho \neq \emptyset$. Then (X, ρ_X) is isomorphic to $(\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}), \sigma)$, where τ belongs to* (F0) $\{\sqrt{-1}t \mid t \in \mathbb{R},\ t > 0\}$ *or* (F1) $\{1/2 + \sqrt{-1}t \mid t \in \mathbb{R},\ t > 0\}$*, and σ is the anti-holomorphic involution on $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ induced by the usual conjugation of $\mathbb{C}$.*

If X is an elliptic curve of type (F0), then X^ρ consists of two circles S^1_A and S^1_B defined by

$$S^1_A = (\mathbb{R} \oplus \tau\mathbb{Z})/(\mathbb{Z} \oplus \mathbb{Z}\tau), \quad S^1_B = (\mathbb{R} \oplus \tau(1/2 + \mathbb{Z}))/(\mathbb{Z} \oplus \mathbb{Z}\tau),$$

and $X \setminus X^\rho$ consists of two tubes X^N and X^S defined by

$$X^N = (\{x \in \mathbb{C} \mid \mathrm{Im}\,\tau/2 < \mathrm{Im}\,\mathrm{x} < \mathrm{Im}\tau\} \oplus \mathbb{Z}\tau)\,/(\mathbb{Z} \oplus \mathbb{Z}\tau),$$
$$X^S = (\{x \in \mathbb{C} \mid 0 < \mathrm{Im}\,\mathrm{x} < \mathrm{Im}\tau/2\} \oplus \mathbb{Z}\tau)\,/(\mathbb{Z} \oplus \mathbb{Z}\tau).$$

Proposition 12. *Let X_τ be an elliptic curve and ρ_X an anti-holomorphic involution on X_τ with $X^\rho \neq \emptyset$. Let π be a non-constant holomorphic map from X_τ to $\mathbb{P}^1$ satisfying Conditions* (1) *and* (2) *in Theorem 1.*

Then X_τ is an elliptic curve of type (F0). *Moreover, regarded as a doubly periodic meromorphic function on $\mathbb{C}$, π is either* (A) χ *or* (B) $1/\chi$*, where χ is a meromorphic function defined by*

$$\chi(u) = C\exp(-2\pi\sqrt{-1}qu)\prod_{i=1}^{n+1}\frac{\theta_1(u-\alpha_i)}{\theta_1(u-\beta_i)}.$$

Here θ_1 is Jacobi's theta function as in Section 6.1; n is a positive integer; q, $\alpha_1,\dots,\alpha_{n+1}$, $\beta_1,\dots,\beta_{n+1}$, and C are constants satisfying the following conditions:

(1) *$\alpha_i \in X^S$ and $\sum_i(\alpha_i-\beta_i)$ is expressed as $p+q\tau \in \mathbb{Z}\oplus\mathbb{Z}\tau$.*
(2) *$\beta_i = \rho_X(\alpha_i)$, that is, $\alpha_i+\beta_i$ is expressed as $r_i+s_i\tau \in \mathbb{R}\oplus\mathbb{Z}\tau$.*
(3) $|C| = \exp\left(\pi\sqrt{-1}\sum_i s_i(\alpha_i-\beta_i)\right)$.

Moreover π has a zero P_0 of order $\geqq 2$. Conversely any map π expressed as above satisfies Conditions (2) *and* (3) *in Theorem 1.*

Proof. Assume that X and π Condition (1) and (2) in Theorem 1. From Condition (1) in Theorem 1, the number of connected components of $X \setminus X^\rho$ is 2, and hence X is an elliptic curve of type (F0).

From Condition (2) in Theorem 1, the divisor of π must be of the following form

(6.15)
$$(\pi) = (\alpha_1)+(\alpha_2)+\cdots+(\alpha_{n+1})-(\beta_1)-(\beta_1)-\cdots-(\beta_{n+1}),$$

where $\{\alpha_1,\dots,\alpha_{n+1}\}$ is a subset of X^N or X^S, $\beta_i = \rho_X(\alpha_i)$, that is, $\alpha_i+\beta_i$ is expressed as $r_i+s_i\tau \in \mathbb{R}\oplus\mathbb{Z}\tau$ ($0 \leqq \mathrm{i} \leqq \mathrm{n}-\mathrm{m}$).

By Abel's theorem, $\sum_{i=1}^{n+1}(\alpha_i-\beta_i)$ belongs to $\mathbb{Z}\oplus\tau\mathbb{Z}$, and hence there exist integers p and q such that $\sum_{i=1}^{n+1}(\alpha_i-\beta_i) = p+q\tau$. Thus$\pi$ is either (A) χ or (B) $1/\chi$, where χ is a meromorphic function defined by

$$\chi(u) = C\exp(-2\pi\sqrt{-1}qu)\prod_{i=1}^{n+1}\frac{\theta_1(u-\alpha_i)}{\theta_1(u-\beta_i)}. \tag{6.16}$$

Here θ_1 is Jacobi's theta function as in Section 6.1; n is a positive integer; q, $\alpha_1,\dots,\alpha_{n+1}$, $\beta_1,\dots,\beta_{n+1}$, and C are constants satisfying Conditions (1) and (2) in Proposition 12.

The function $\pi\overline{\rho_X^*\pi}$ is a constant function since its divisor vanishes. By the assumption that π has the point $x \in X^\rho$ with $|\pi(x)| = 1$, we see that

$$\pi(0)\overline{\rho_X^*\pi(0)} = \pi(0)\overline{\pi(0)} = \pi(x)\overline{\pi(x)} = 1.$$

Using the above equation, we get

$$|C| = \exp\left(\pi\sqrt{-1}\sum_i s_i(\alpha_i - \beta_i)\right) \tag{6.17}$$

Moreover, from Condition (2) in Theorem 1, π has a zero P_0 of order $\geqq 2$.

Conversely, let π be the map defined as above. Then, clearly π satisfies Conditions (1) and (2) in Theorem 1. □

Proposition 13. *Let* $(X = \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}t\mathbb{Z}), \rho_X)$ *be a real curve of type* (F0)*, which is identified with its Jacobian* $J(X)$*. Let* E *and* F *be divisors on* X *satisfying*

$$E + \rho_X(E) \cong F + \rho_X(F), \tag{6.18}$$

where $\cong$ *means linear equivalence. Let* f *be a non-constant meromorphic function such that*

$$(f) = E + \rho_X(E) - (F + \rho_X(F)), \quad \overline{\rho_X^* f} = f, \tag{6.19}$$

where (f) *is the divisor of* f*. Then* f^ρ*, the restriction of* f *to* $X^\rho = S_A^1 \cup S_B^1$*, is a non-negative or a non-positive real function if and only if*

$$J(E - F) \in \left(\mathbb{Z} \oplus \sqrt{-1}\mathbb{R}\right) / \left(\mathbb{Z} \oplus \sqrt{-1}t\mathbb{Z}\right), \tag{6.20}$$

where $J(E - F)$ *is defined by*

$$\sum_i (P_i - Q_i) \mod \mathbb{Z} \oplus \mathbb{Z}\sqrt{-1}t,$$

provided $E - F$ *is expressed as* $E - F = \sum_i (P_i - Q_i)$*.*

Proof. Let S_{zp} be the intersection of $S_A^1 \cup S_B^1$ with the set of zeros and poles of f^ρ. Restricting f^ρ to $(S_A^1 \cup S_B^1) \setminus S_{zp}$, we get a real function f^*. Considering the restriction of $(E + \rho_X(E) - F - \rho_X(F))$ to $S_A^1 \cup S_B^1$, we see that f^ρ has only zeros and poles with even order. So the sign of f^* remains invariant at each point of S_{zp}, and hence f^ρ is non-negative or non-positive on each connected component of $S^1_A \cup S^1_B$. Consequently, f^ρ is a non-negative or a non-positive real function on $S^1_A \cup S^1_B$ if and only if there exist points $\alpha \in S^1_A \setminus S_{zp}$ and $\beta \in S^1_B \setminus S_{zp}$ such that $f(\beta)/f(\alpha) > 0$.

Note that the divisors E and F satisfy the equivalence (6.18) precisely when $J(E-F)$ belongs to $L(0)$ or $L(1/2)$, where $L(s)$ $(0 \leqq s < 1)$ is defined by $L(s) = ((\mathbb{Z}+s) \oplus \sqrt{-1}\mathbb{R})/(\mathbb{Z} \oplus \sqrt{-1}t\mathbb{Z})$. Then the following lemma completes the proof of Proposition 13. □

Lemma 25. *In the case $J(E-F) \in L(0)$, there exist $\alpha \in S^1_A$ and $\beta \in S^1_B$ such that $f(\beta)/f(\alpha) > 0$. In the case $J(E-F) \in L(1/2)$, there exist $\alpha \in S^1_A$ and $\beta \in S^1_B$ such that $f(\beta)/f(\alpha) < 0$.*

Proof. The divisor $E + \rho_X(E) - (F + \rho_X(F))$ is expressed as $\sum_{i=1}^{2k}(P_i - Q_i)$ with $P_i \neq Q_j$ $(1 \leqq i, j \leqq 2k)$. By Abel's theorem, there exist integers p and q such that

$$p + q\tau = \sum_{i=1}^{2k} (P_i - Q_i)\,. \tag{6.21}$$

Then the meromorphic function g having this divisor is determined up to a non-zero constant and is expressed as follows:

$$g(u) = \gamma \exp(-2\pi\sqrt{-1}qu)\frac{\theta_1(u-P_1)\cdots\theta_1(u-P_{2k})}{\theta_1(u-Q_1)\cdots\theta_1(u-Q_{2k})}, \tag{6.22}$$

where γ is a non-zero complex number and q is the integer given in (6.21).

It is not hand to see by moving the points $P_1, \dots, P_{2k}$, $Q_1, \dots, Q_{2k}$ appropriately that we can construct a 1-parameter family g_s of meromorphic functions on X which satisfies the following conditions:

(1) If $J(E-F) \in L(0)$, then $g_0 = g$ and $g_1 = \begin{cases} \gamma G_k^{(0)} & \text{for } k \geqq 2, \\ \gamma G_k^{(0)} \text{ or } \gamma/G_k^{(0)} & \text{for } k = 1. \end{cases}$
If $J(E-F) \in L(1/2)$, then $g_0 = g$ and $g_1 = \gamma G_k^{(1/2)}$. Here $G_k^{(0)}$ and $G_k^{(1/2)}$ are meromorphic functions on X_τ defined by

$$G_k^{(0)}(u) = \exp(-2\pi\sqrt{-1}ku)\left(\frac{\theta_1(u-1/2-\tau/2)}{\theta_1(u-1/2)}\right)^{2k},$$

$$G_k^{(1/2)}(u) = \left(\frac{\theta_1(u-1/2-\tau/2)}{\theta_1(u-\tau/2)}\right)^{2} G_{k-1}^{(0)}(u).$$

(2) g_s depends smoothly on the parameter s for $0 \leqq s \leqq 1$. If we denote the divisors consisting of poles and zeros of g_s by $\sum_i P_i^s$ and $\sum_i Q_i^s$ respectively, then they are invariant under ρ_X and $P_i^s \neq Q_j^s$ for $1 \leqq i, j \leqq 2k$.

Also, we can construct 1-parameter families of points $\alpha_s \in S_A^1$ and $\beta_s \in S_B^1$ satisfying the following conditions:

(1) For each $0 \leqq s \leqq 1$, α_s and β_s do not belong to $\{P_1^s, \dots, P_{2k}^s, Q_1^s, \dots, Q_{2k}^s\}$.

(2) $\alpha_1 = \epsilon + 1/2$ and $\beta_1 = \epsilon + 1/2 + \tau/2 = \epsilon + 1/2 + \sqrt{-1}t/2$, where ϵ is a small positive constant.

We see that the sign of $g_s(\beta_s)/g_s(\alpha_s)$ does not depend on the choice of s, and hence $f(\beta_0)/f(\alpha_0) = g_0(\beta_0)/g_0(\alpha_0)$ and $g_1(\beta_1)/g_1(\alpha_1)$ have the same sign.

Assume that $J(E-F) \in L(0)$ and $k \geqq 2$. Let us determine the sign of $g_1(\beta_1)/g_1(\alpha_1) = G_k^{(0)}(\epsilon + 1/2 + \tau/2)/G_k^{(0)}(\epsilon + 1/2)$. Using the identities (6.1) and (6.3), we see that

$$\begin{aligned}
&G_k^{(0)}(\epsilon + 1/2 + \tau/2)/G_k^{(0)}(\epsilon + 1/2) \\
&= \exp(-2\pi\sqrt{-1}k(\tau/2)) \left(\frac{\theta_1(\epsilon)^2}{\theta_1(\epsilon - \tau/2)\theta_1(\epsilon + \tau/2)} \right)^{2k} \\
&= \exp(-2\pi\sqrt{-1}k(\tau/2)) \left(\frac{\theta_1(\epsilon)^2}{\left(\sqrt{-1}a(-\epsilon)\theta_4(\epsilon)\right)\left(-\sqrt{-1}a(\epsilon)\theta_4(\epsilon)\right)} \right)^{2k} \\
&= \left(\frac{\theta_1(\epsilon)}{\theta_4(\epsilon)} \right)^{4k} = \left(\frac{\theta_1(\epsilon|\sqrt{-1}t)}{\theta_4(\epsilon|\sqrt{-1}t)} \right)^{4k}.
\end{aligned}$$

If we fix ϵ, we get a nowhere vanishing real function ϕ defined by

$$\phi(t) = \left(\frac{\theta_1(\epsilon|\sqrt{-1}t)}{\theta_4(\epsilon|\sqrt{-1}t)} \right)^{4k} \quad (t > 0).$$

By (6.6), we get the following Taylor expansion:

$$\lim_{t\to\infty} q^{-k}\phi(t) = (2\pi)^{4k}\epsilon^{4k} + O(\epsilon^{4k+1}), \tag{6.23}$$

from which we see that for a small positive ϵ, this is positive. If $k = 1$, then we can see that the sign of $f(\beta_0)/f(\alpha_0)$ is positive in a similar fashion. Thus Lemma 25 is verified in the case that $J(E-F) \in L(0)$.

In the case $J(E-F) \in L(1/2)$, the sign of $g_1(\beta_1)/g_1(\alpha_1) = G_k^{(1/2)}(\epsilon + 1/2 + \tau/2)/G_k^{(1/2)}(\epsilon + 1/2)$ is similarly determined as follows. Using the identities (6.1), (6.2), (6.3), (6.4) and (6.5), we obtain

$$
\begin{aligned}
&G_k^{(1/2)}(\epsilon+1/2+\tau/2)/G_k^{(1/2)}(\epsilon+1/2)\\
&=\left(\frac{\theta_1(\epsilon)}{\theta_1(\epsilon+1/2)}\right)^2\left(\frac{\theta_1(\epsilon-\tau/2)}{\theta_1(\epsilon+1/2-\tau/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}\\
&=\left(\frac{\theta_1(\epsilon)}{\theta_2(\epsilon)}\right)^2\left(\frac{\theta_1(\epsilon+\tau/2)/b(\epsilon-\tau/2)}{\theta_1(\epsilon+1/2+\tau/2)/b(\epsilon+1/2-\tau/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}\\
&=\left(\frac{b(\epsilon+1/2-\tau/2)}{b(\epsilon-\tau/2)}\right)^{-2}\left(\frac{\theta_1(\epsilon)}{\theta_2(\epsilon)}\right)^2\left(\frac{\theta_1(\epsilon+\tau/2)}{\theta_1(\epsilon+1/2+\tau/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}\\
&=\left(\frac{b(\epsilon+1/2-\tau/2)}{b(\epsilon-\tau/2)}\right)^{-2}\left(\frac{\theta_1(\epsilon)}{\theta_2(\epsilon)}\right)^2\left(\frac{\sqrt{-1}a(\epsilon)\theta_4(\epsilon)}{\sqrt{-1}a(\epsilon+1/2)\theta_4(\epsilon+1/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}\\
&=\left(\frac{b(\epsilon+1/2-\tau/2)a(\epsilon)}{b(\epsilon-\tau/2)a(\epsilon+1/2)}\right)^{-2}\left(\frac{\theta_1(\epsilon)}{\theta_2(\epsilon)}\right)^2\left(\frac{\theta_4(\epsilon)}{\theta_4(\epsilon+1/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}\\
&=\left(\frac{b(\epsilon+1/2-\tau/2)a(\epsilon)}{b(\epsilon-\tau/2)a(\epsilon+1/2)}\right)^{-2}\left(\frac{\theta_1(\epsilon)}{\theta_2(\epsilon)}\right)^2\left(\frac{\theta_4(\epsilon)}{\theta_3(\epsilon)}\right)^{-2}\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}\\
&=-\left(\frac{\theta_3(\epsilon)}{\theta_2(\epsilon)\theta_4(\epsilon)}\right)^2\theta_1(\epsilon)^2\frac{G_{k-1}^{(0)}(\epsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\epsilon+1/2)}.
\end{aligned}
$$

From (6.23), together with (6.6), we get the following Taylor expansion:

$$\lim_{t\to\infty} q^{-(k-1)}G_k^{(1/2)}(\epsilon+1/2+\tau/2)/G_k^{(1/2)}(\epsilon+1/2)=-2^{4(k-1)}\pi^{4k-2}\epsilon^{4k-2}+O(\epsilon^{4k-1}).$$

If we take a small positive ϵ, this is negative. Thus Lemma 25 also holds in the case $J(E-F)\in L(1/2)$. □

Now we are in a position to prove Theorem 12.

Proof of Theorem 12. Conditions (1) and (2) in Theorem 1 are equivalent to the following assertion: π is a meromorphic function as in Proposition 12.

It is clear that $R = R_+ + \rho_{X*}(R_+)$. Applying Proposition 13 to $E = D$ and $F = R_+$, we see that Condition (3) in Theorem 1 is equivalent to Condition (2) in Theorem 12.

Take any spectral data, that is, a triple $(X, \pi, \mathcal{L})$ with P_0, which satisfies the above assertions and Condition (2) in Theorem 12. Consider the following real automorphism ϕ_a on (X, ρ_X) defined by $u \mapsto u + a$, where a is a real number. Then, by using ϕ_a and ρ_X, we can construct a real automorphism ϕ on (X, ρ_X) such that $(X, \phi^*\pi, \phi^*\mathcal{L})$ is a triplet in Theorem 12, where $\phi^*\pi$ and $\phi^*\mathcal{L}$ denote the pull-backs by ϕ of π and $\mathcal{L}$, respectively. Hence Theorem 12 follows. $\square$

6.5. **Construction of harmonic maps in terms of the rational spectral curve.** Using the results in Section 2.2, let us now construct harmonic maps corresponding to spectral data whose spectral curves are smooth rational curves, and prove Theorem 10.

Let $(X, \pi, \mathcal{L})$ be a spectral data as in Theorem 9. We may assume that π, R and $\mathcal{L}$ are of the following form:

$$\pi(\lambda) = \alpha_0 \lambda^{m+1} \frac{\prod_{j=1}^{n-m}(\lambda - P_j)}{\prod_{j=1}^{n-m}(\lambda - Q_j)}, \quad P_0 = 0, \quad R = D + \rho_X(D), \quad \mathcal{L} = \mathcal{O}_X(D),$$

where α_0 is a constant as in Theorem 9 and D is a divisor defined by $D = mP_0 + \sum_{i=1}^{n-m} R_i$. First we prove the following

Lemma 26. *Let $(X, \pi, \mathcal{L})$ be a spectral data as above. Define a function $\psi(z,\, \bar{z},\, \lambda)$ on X with parameter z by*

$$\psi(z,\, \bar{z},\, \lambda) = \exp\left(\frac{z}{\kappa}\lambda^{-1} - \overline{\left(\frac{z}{\kappa}\right)}\lambda\right) \cdot \frac{\prod_{j=1}^{n-m}(\lambda - P_j)}{\prod_{j=1}^{n-m}(\lambda - R_j)}. \tag{6.24}$$

Here $\kappa = (\partial\zeta/\partial\lambda)|_{\lambda=P_0}$ is the value of the differential of the meromorphic function ζ as in (2.4) at $\lambda = P_0$. Then $\psi(z,\, \bar{z},\, u)\theta_A(z)$ is an element of $H^0(X,\, \mathcal{L}_0 \otimes L(z))$ for any $z \in \mathbb{C}$.

Proof. Denote by $D|_{P_0 \cup Q_0}$ the restriction of the divisor $D = mP_0 + \sum_{i=1}^{n-m} R_i$ to $P_0 \cup Q_0$. Then, applying Lemma 2 to $M = D - D|_{P_0\cup Q_0} - E_0$, $N = D|_{P_0 \cup Q_0} - mP_0$, and $\phi = \psi$, we get the assertion. $\square$

Next we construct a special orthonormal basis of global sections of $\mathcal{L} = \mathcal{O}_X(mP_0 + \sum_{i=1}^{n-m} R_i)$ following the method explained above. Here we choose $f = 1$ as a meromorphic function on X in Condition (3) of Definition 1. For $0 \leqq i \leqq n$, let us denote by σ_i the following element

$$\sigma_i = \frac{\eta_i^m \prod_{j=1}^{i-1}(\eta_i - R_j)}{\prod_{j=0}^{i-1}(\eta_i - \eta_j) \cdot \prod_{j=i+1}^{n}(\eta_i - \eta_j)} \frac{\prod_{j=0}^{i-1}(\lambda - \eta_j) \cdot \prod_{j=i+1}^{n}(\lambda - \eta_j)}{\lambda^m \prod_{j=1}^{i-1}(\lambda - R_j)}.$$

Then we see that $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$ and $h(\sigma_i, \sigma_i) = 1$ for $0 \leqq i \leqq n$. Thus we get an orthonormal basis $\{\sigma_i\}_{0 \leqq i \leqq n}$ of $H^0(X,\, \mathcal{L})$, that is, $h(\sigma_i,\, \sigma_j) = \delta_{ij}$.

Owing to (2.11), the corresponding harmonic map : $\mathbb{R}^2 \to \mathbb{C}P^n$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi_0^z(1) : \psi_1^z(1) : \cdots : \psi_n^z(1)],$$

where each $\psi_i^z(1)$ is a function defined by

$$\psi_i^z(1) = \exp\left(\frac{z}{\kappa}\eta_i^{-1} - \overline{\left(\frac{z}{\kappa}\right)}\eta_i\right) \cdot \frac{\prod_{j=1}^{n-m}(\eta_i - P_j)}{\prod_{j=1}^{n-m}(\eta_i - R_1)}. \tag{6.25}$$

Define a map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $z = x + \sqrt{-1}y \mapsto \kappa z$. Then the composition $\psi \circ F$ gives rise to the harmonic map given in (6.9). This completes the proof of Theorem 10

6.6. **Construction of harmonic maps in terms of elliptic spectral curves.** By an argument similar to that in section 6.2, we now construct harmonic maps corresponding to spectral data whose spectral curves are smooth elliptic curves, and prove Theorem 13.

Lemma 27. *Let* $\left(X = X_{\sqrt{-1}t}, \pi, \mathcal{L} = \mathcal{O}_X(\sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i)\right)$ *be a spectral data as in Theorem* 12. *Define a function* $\psi(z,\, \bar{z},\, u)$ *on* X *with parameter* z *by*

$$\psi(z,\, \bar{z},\, u) = \exp\left(\frac{z}{\kappa}[\zeta_w(u - P_0) - A\,u] - \overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(u - Q_0) - A\,u]\right)$$
$$\cdot \frac{\prod_{j=1}^{k} \theta_1(u - F_j) \cdot \theta_1(u - P_0)^m \cdot \prod_{j=1}^{n-m} \theta_1(u - P_j) \cdot \theta_1(u - G - H)}{\prod_{j=1}^{k+n+1} \theta_1(u - E_j)}. \tag{6.26}$$

Here ζ_w *is Weierstrass' zeta function as in* (6.7),

$$G = \sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i - mP_0 - \sum_{i=1}^{n-m} P_i, \quad H = H(z,\, \bar{z}) = \frac{z}{\kappa} - \overline{\left(\frac{z}{\kappa}\right)},$$

A is the constant as in (6.8), *and* $\kappa = (\partial\, \zeta/\partial u)|_{u=P_0}$ *is the value of the differential of the meromorphic function* ζ *in* (2.4) *at* $u = P_0$. *Then* $\psi(z,\, \bar{z},\, u)\theta_A(z)$ *is an element of* $H^0(X,\, \mathcal{L}_0 \otimes L(z))$ *for any* $z \in \mathbb{C}$.

Proof. The proof of this lemma is similar to that of Lemma 26 $\square$

Next we construct a special orthonormal basis of global sections of $\mathcal{L} = \mathcal{O}_X(\sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i)$ following the method used in Section 2.2. Here we choose

$$f = \frac{\prod_{j=1}^{k+n+1} \theta_1(u - E_j)}{\prod_{j=1}^{k} \theta_1(u - F_j) \prod_{j=0}^{n} \theta_1(u - R_j)} \cdot \frac{\prod_{j=1}^{k+n+1} \theta_1(u - \overline{E_j})}{\prod_{j=1}^{k} \theta_1(u - \overline{F_j}) \prod_{j=0}^{n} \theta_1(u - \overline{R_j})}$$

as a meromorphic function on X in Condition (3) of Definition 1. Let μ_i be the constant in Theorem 13 and set $\widehat{\eta_i} = \sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i - (\eta_0 + \cdots + \eta_{i-1} + \eta_{i+1} + \cdots + \eta_n)$. Denoting by σ_i the element

$$\mu_i^{-1} \frac{\prod_{j=0}^{n} \theta_1(\eta_i - R_j) \cdot \prod_{j=1}^{k} \theta_1(u - F_j) \cdot \prod_{j=0}^{i-1} \theta_1(u - \eta_j) \cdot \theta_1(u - \widehat{\eta_i}) \cdot \prod_{j=i+1}^{n} \theta_1(u - \eta_j)}{\prod_{j=1}^{i-1} \theta_1(\eta_i - \eta_j) \cdot \theta_1(\eta_i - \widehat{\eta_i}) \cdot \prod_{j=i+1}^{n} \theta_1(\eta_i - \eta_j) \cdot \prod_{j=1}^{k+n+1} \theta_1(u - E_j)},$$

we see that $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$ and $h(\sigma_i, \sigma_i) = 1$ for $0 \leqq i \leqq n$. Thus we get an orthonormal basis $\{\sigma_i\}_{0 \leqq i \leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$. These are well-defined by the following lemma.

Lemma 28. *The above constants $\widehat{\eta_i}$ are not equal to η_i* (mod $\mathbb{Z} \oplus \mathbb{Z}\tau$).

Proof. If $\widehat{\eta_i} = \eta_i$ mod $\mathbb{Z} \oplus \mathbb{Z}\tau$, then $h(\sigma_i, \sigma_i) = 0$, which is a contradiction because h is positive definite. $\square$

On account of (2.11), the corresponding harmonic map : $\mathbb{R}^2 \to \mathbb{C}P^n$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi_0^z(1) : \psi_1^z(1) : \cdots : \psi_n^z(1)],$$

where each $\psi_i^z(1)$ is a function defined by

$$\psi_i^z(1) = \mu_i \exp\left(\frac{z}{\kappa}[\zeta_w(\eta_i - P_0) - A\,\eta_i] - \overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(\eta_i - Q_0) - A\,\eta_i]\right)$$

$$\cdot \frac{\theta_1(\eta_i - P_0)^m \prod_{j=1}^{n-m} \theta_1(\eta_i - P_j) \cdot \theta_1(\eta_i - G - H(z, \bar{z}))}{\prod_{j=0}^{n} \theta_1(\eta_i - R_j)}. \tag{6.27}$$

Define a map $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $z = x + \sqrt{-1}y \mapsto \kappa z$. Then the composition $\psi \circ F$ gives rise to the harmonic map given in (6.11). This completes the proof of Theorem 13.

6.7. **Periodicity conditions of harmonic maps in terms of generalized Jacobians.** McIntosh studied periodicity conditions of the corresponding harmonic maps by introducing certain homomorphisms into generalized Jacobians. In this section, when X is a smooth elliptic curve, we reformulate McIntosh's periodicity conditions by introducing certain families of lines on the complex plane $\mathbb{C}$, and prove Theorem 14.

Let $(X, \pi, \mathcal{L})$ be a spectral data as in Definition 1. Let $L(z)$ be the line bundle as in Section 2.2 and $\theta_A(z)$ the local trivialization of $L(z)$ over X_A as in (2.4). Let $J(X_\mathfrak{o})$ be a generalized Jacobian defined by

$$J(X_\mathfrak{o}) = \bigcup_{L \in J(X)} \{(\mathrm{Hom}\,(\mathrm{L}|_{\eta_1}, \mathrm{L}|_{\eta_0}) \setminus \{0\}) \times \cdots \times (\mathrm{Hom}\,(\mathrm{L}|_{\eta_n}, \mathrm{L}|_{\eta_0}) \setminus \{0\})\}\,.$$

We define a map $\widehat{L}\colon \mathbb{R}^2 \to J(X_\mathfrak{o})$ by $z = x + \sqrt{-1} \mapsto (L(z), h_1(z), \ldots, h_n(z))$, where $h_i(z)$ is an element of $\mathrm{Hom}\,(\mathrm{L(z)}|_{\eta_i}, \mathrm{L(z)}|_{\eta_0}) \setminus \{0\}$ ($\cong \mathbb{C}^*$) defined by the condition that $h_i(z)$ maps $\theta_A(z)|_{\eta_i}$ to $\theta_A(z)|_{\eta_0}$. Then McIntosh proved the following

Theorem 16 ([18]). *The harmonic map $\psi\colon \mathbb{R}^2 \to \mathbb{C}P^n$ corresponding to the above spectral data is doubly periodic if and only if $\widehat{L}\colon \mathbb{R}^2 \to J(X_\mathfrak{o})$ is doubly periodic.*

In the case of the smooth rational curve X, the maps Φ in the proof of Theorem 11 and $\widehat{L}$ are essentially the same.

Let us determine the map $\widehat{L}$ when $(X, \pi, \mathcal{L})$ is a spectral data with a smooth elliptic curve as its spectral curve. First, we compute the map $L\colon \mathbb{R}^2 \to J(X)$ defined by $z = x + \sqrt{-1}y \mapsto L(z)$. Let T_z be a divisor defined by

$$T_z = (D) - m(P_0) - (S) - E_0, \tag{6.28}$$

where S is a point on X defined by $S = G + H$ and E_0 is the divisor given in Section 2.2. Then $\psi(z, \bar{z}, u) \otimes \theta_A(z)$ belongs to $H^0(X, \mathcal{O}_X(T_z) \otimes L(z))$ ($\cong H^0(X, \mathcal{L}_0(-S) \otimes L(z))$) by Lemma 27. Moreover, we see that $\psi(z, \bar{z}, u) \otimes \theta_A(z)$ is a non-vanishing global holomorphic section of $\mathcal{O}_X(T_z) \otimes L(z)$. In particular, the line bundle $L(z) \otimes \mathcal{O}_X(T_z)$ is trivial, that is, $L(z) \otimes \mathcal{O}_X(T_z) \cong \mathcal{O}_X$, and hence $L(z) \cong \mathcal{O}_X(-T_z)$. Using (6.28) and identifying Jacobian $J(X)$ with $X \cong \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}t\mathbb{Z})$, we see that $L\colon \mathbb{R}^2 \to J(X)$ is given by

$$z = x + \sqrt{-1}y \mapsto -D + m\,P_0 + S + E_0 = H(z, \bar{z}) = z/\kappa - \overline{(z/\kappa)} \mod \mathbb{Z} \oplus \mathbb{Z}\sqrt{-1}t,$$

where κ is the complex number in Lemma 27.

Second, we determine $\theta_A(z)$. Let Θ be a meromorphic function on $\mathbb{C}^2$ defined by

$$\Theta(w,u)=\frac{\prod_{j=1}^{k+n+1}\theta_1(u-E_j)}{\prod_{j=1}^{k}\theta_1(u-F_j)\cdot\theta_1(u-P_0)^m\prod_{j=1}^{n-m}\theta_1(u-P_j)\cdot\theta_1(u-G-w)}.$$

Using $\psi(z,\,\bar z,\,u)\otimes\theta_A(z)\in H^0(X,\,L(z)\otimes\mathcal{O}_X(T_z))=H^0(X,\,\mathcal{O}_X)\cong\mathbb{C}$, we see that

$$\theta_A(z)=C\exp\left(-\frac{z}{\kappa}[\zeta_w(u-P_0)-A\,u]+\overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(u-Q_0)-A\,u]\right)\Theta\left(H(z,\bar z),u\right),$$

where C is a non-zero constant.

Now we give an explicit description of $\widehat{L}$. Let $v\colon S^1_J=\{e^{\sqrt{-1}\theta}\mid 0\leqq\theta<2\pi\}\to J(X)$ be a map defined by $e^{\sqrt{-1}\theta}\mapsto\sqrt{-1}t\theta/2\pi \mod \mathbb{Z}\oplus\mathbb{Z}\sqrt{-1}t$. Let $J_S\to S^1_J$ be the pull-back of $J(X_{\mathfrak{o}})$ by v. For $0\leqq i\leqq n$, we define $B_i\colon e^{\sqrt{-1}\theta}\in S^1_J\mapsto B_i(e^{\sqrt{-1}\theta})\in \mathrm{Hom}\left(\mathrm{v}(\mathrm{e}^{\sqrt{-1}\theta})|_{\eta_\mathrm{i}},\,\mathrm{v}(\mathrm{e}^{\sqrt{-1}\theta})|_{\eta_0}\right)$, sections of $J_S\to S^1_J$, by the condition that each $B_i(e^{\sqrt{-1}\theta})$ maps the element $\exp(\sqrt{-1}\eta_i\theta)\Theta(\sqrt{-1}t\theta/(2\pi),\eta_i)$ of $\mathcal{O}_X(-T_z)|_{\eta_i}$ to the element
$\exp(\sqrt{-1}\eta_0\theta)\Theta(\sqrt{-1}t\theta/(2\pi),\eta_0)$ of $\mathcal{O}_X(-T_z)|_{\eta_0}$. Since the image of $\mathbb{R}^2$ by L is contained in $\mathbb{Z}\oplus\mathbb{R}\tau \mod \mathbb{Z}\oplus\mathbb{Z}\tau$ ($\subset J(X)$), we can regard $\widehat{L}\colon\mathbb{R}^2\to J(X_{\mathfrak{o}})$ as a map $\mathbb{R}^2\to J_S$. Using this identification, the map $\widehat{L}\colon\mathbb{R}^2\to J_S$ is given by

$$z=x+\sqrt{-1}y\mapsto\left(\exp(2\pi H(z,\,\bar z)/t)\in S^1_J,\,h_1(z,\,\bar z),\,h_2(z,\,\bar z),\,\cdots,\,h_n(z,\,\bar z)\right),$$

where $h_i(z,\,\bar z)$ is an element of $\mathrm{Hom}\left(\mathrm{v}(\exp(2\pi\mathrm{H}(z,\,\bar z)/t))|_{\eta_\mathrm{i}},\,\mathrm{v}(\exp(2\pi\mathrm{H}(z,\,\bar z)/t))|_{\eta_0}\right)$ being defined by $h_i(z,\,\bar z)=\exp(b_i(z,\bar z))\,B_i(\exp(2\pi H(z,\,\bar z)/t))$ with

$$\begin{aligned}b_i(z,\,\bar z)=&\frac{z}{\kappa}[\zeta_w(\eta_0-P_0)-\zeta_w(\eta_i-P_0)-\frac{B}{\tau}(\eta_0-\eta_i)]\\&-\overline{(\frac{z}{\kappa})}[\zeta_w(\eta_0-Q_0)-\zeta_w(\eta_i-Q_0)-\frac{B}{\tau}(\eta_0-\eta_i)].\end{aligned}$$

Lemma 29. *For $1\leqq i\leqq n$, each $b_i(z,\bar z)$ is pure imaginary.*

Proof. We may assume that $0\leqq\mathrm{Im}\,\mathrm{P}_0,\,\mathrm{Im}\,\mathrm{Q}_0,\,\mathrm{Im}\,\eta_0,\,\ldots,\,\mathrm{Im}\,\eta_\mathrm{n}<\mathrm{Im}\,\tau$. On this assumption, $Q_0=\overline{P_0}+\tau$. Using $\overline{\zeta_w(u)}=\zeta_w(\bar u)$ and $\bar B=-B$, we then get

$$\begin{aligned}&\overline{[\zeta_w(\eta_0-P_0)-\zeta_w(\eta_i-P_0)-B\tau^{-1}(\eta_0-\eta_i)]}\\&=[\zeta_w(\overline{\eta_0-P_0})-\zeta_w(\overline{\eta_i-P_0})]-B\tau^{-1}\overline{(\eta_0-\eta_i)}\\&=[\zeta_w(\overline{\eta_0}-Q_0+\tau)-\zeta_w(\overline{\eta_i}-Q_0+\tau)]-B\tau^{-1}\overline{(\eta_0-\eta_i)}.\end{aligned}\tag{6.29}$$

In the case that $\eta_0 \in S^1_A$ and $\eta_i \in S^1_B$, it follows from $\zeta_w(u+\tau) = \zeta_w(u) + B$ that the right hand side of (6.29) is equal to

$$\begin{aligned}&[\zeta_w(\eta_0 - Q_0 + \tau) - \zeta_w(\eta_i - \tau - Q_0 + \tau)] - B\tau^{-1}(\eta_0 - \eta_i + \tau)\\&\quad = [\zeta_w(\eta_0 - Q_0) - \zeta_w(\eta_i - Q_0)] - B\tau^{-1}(\eta_0 - \eta_i),\end{aligned}$$

which implies see that b_i is pure imaginary. Similarly, we can also see that b_i is pure imaginary in other cases. □

Thus we can consider $\widehat{L} \colon \mathbb{R}^2 \to J_S$ to be a map $L_T \colon \mathbb{R}^2 \to T^{n+1} = S^1_J \times S^1 \times \cdots \times S^1$ defined by

$$z = x + \sqrt{-1}y \mapsto (\exp(2\pi H(z,\, \bar{z})/t),\, \exp(b_1(z,\, \bar{z})),\, \ldots\,,\, \exp(b_n(z,\, \bar{z})))\,.$$

Evidently, $\widehat{L}$ is doubly periodic if and only if L_T is doubly periodic. Then we have the following

Proposition 14. *The harmonic map $\psi \colon \mathbb{R}^2 \to \mathbb{C}P^n$, defined by* (6.27)*, corresponding to a spectral data $(X,\, \pi,\, \mathcal{L})$ is doubly periodic with periods $v_1,\, v_2 \in \mathbb{C}$ if and only if the set $V = \bigcap_{0\leqq i\leqq n} V_i$ contains the 2-dimensional lattice $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, where $V_0, \ldots, V_n$ are the sets defined by*

$$V_i = \begin{cases} \pi\beta_i^{-1}\left(\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}\right), & \text{if } \beta_i \neq 0,\\ \mathbb{C}, & \text{otherwise.}\end{cases} \tag{6.30}$$

Here $\beta_0,\, \beta_1,\, \ldots\,,\, \beta_n$ are complex constants defined by

$$\beta_0 = 2\pi/(\kappa\, t), t, \quad \beta_i = [\zeta_w(\eta_0 - P_0) - \zeta_w(\eta_i - P_0) - B\,(\eta_0 - \eta_i)\tau^{-1}]/\kappa \quad (1 \leqq i \leqq n).$$

Proof. Recall that ψ has two periods $v_1,\, v_2$ if and only if L_T has two periods $v_1,\, v_2$ by Theorem 16. If L_T has two periods $v_1,\, v_2$, then the set $\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ is contained in V, since V is the set of all points on which the value of L_T is equal to the initial value $L_T(0) = (1,\, \ldots\,,\, 1) \in T^{n+1}$.

Conversely, if V contains a 2-dimensional lattice $M = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, then clearly v_1 and v_2 are periods of L_T, since L_T is a homomorphism from the additive group $\mathbb{R}^2$ to T^{n+1}. Hence Condition (6.30) is a necessary and sufficient condition for L_T to be doubly periodic with periods $v_1,\, v_2$. □

Now let us prove Theorem 14.

Proof of Theorem 14. From the argument in the proof of Theorem 13, we see that the map given in Theorem 14 is a composition $\psi \circ F$, where ψ is the map in Proposition 14 and F is a map defined by $\mathbb{R}^2 \to \mathbb{R}^2$, $z = x + \sqrt{-1}y \mapsto \kappa z$. Thus Theorem 14 follows immediately from Proposition 14. $\square$

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