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Hypoellipticity of second order differential operators with sign-changing principal symbols

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Hypoellipticity of second order differential operators with sign-changing principal symbols

A thesis presented

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Contents

Introduction

Let P be a partial differential operator with coefficients of class $\boldsymbol{C}^\infty$ defined in an open set Ω of $\boldsymbol{R}^d$. P is said to be hypoelliptic in Ω if

$$\begin{cases} \text{for any } u \in \mathcal{D}'(\Omega) \text{ and for any open subset } \Omega' \text{ of } \Omega, \\ Pu \in \boldsymbol{C}^\infty(\Omega') \text{ implies } u \in \boldsymbol{C}^\infty(\Omega'). \end{cases}$$

It is one of basic problems in the theory of partial differential equations to analyze to what extent a distribution solution u to the equation $Pu = f$ is smooth according to the smoothness of an arbitrarily given function f. Hypoellipticity of P is considered as a part of this problem. The study of hypoellipticity of operators whose principal symbols have real-valued coefficients and constant signs has progressed by the following methods. The first one is based on an a priori estimate of solutions in Sobolev spaces. Once we find such an inequality, we can prove the hypoellipticity with the aid of interpolation inequalities or the theory of pseudo-differential operators. The second is based on the study of elementary solution. It is not easy to construct an elementary solution in general. In some cases we apply the theory of stochastic differential equations, and in other cases we construct a parametrix instead of an elementary solution.

In this thesis, we study the hypoellipticity of second order partial differential operators whose principal symbols are complex-valued or change sign. For this, we must apply both of the above two methods except for some particular cases. Indeed we will divide the dual space into two microlocal domains in one of which we use an estimation of norm and in the other of which we construct a parametrix. Let us introduce the development of the study of hypoellipticity of such operators from the viewpoint of norm estimate.

As is well-known, elliptic operators are hypoelliptic. This suggests that the principal symbol of hypoelliptic operators has a kind of positivity. In 1967, Hörmander [10] proved "If a second order differential operator with real-valued coefficients is hypoelliptic, then the principal symbol does not change sign when the dual variables vary." Let us refer to it as the Hörmander principle. A natural question arises. "Is it necessary for hypoellipticity that the principal symbol does not change sign when the space variables vary?" In 1971, Kannai [12] proved that $L_1 = \partial_t + t\partial_x^2$, an operator of two variables, is hypoelliptic, while $L_2 = \partial_t - t\partial_x^2$ is not. L_1 and L_2 are typical examples of operators with sign-changing principal symbols. This illustrates that the semi-definiteness of the principal symbol is not necessary for hypoellipticity and that the type of changing sign is important. The sign of the principal symbol of L_2 above changes from minus to plus as t increases. This is a condition similar to Nirenberg–Treves criterion for local solvability of differential equations of principal type. (See [26] and [27].) In 1976, Beals and Fefferman [5] generalized Kannai's result to higher dimensional cases. Their result was obtained by getting a suitable a priori estimate with weight and using the general theory of pseudo-differential operators due to Beals (See [3]). An advantage of this method is that the hypoellipticity follows from a single a priori estimate with weight. However, there are many restrictions on the weight. Therefore, the class of functions which control the sign of the principal symbol are strictly limited. Beals and Fefferman's result was extended by many authors. (Kumano-go and Taniguchi [15], Taniguchi [32], Akamatsu [1], Zuily [35], Amano [2] and Lanconelli [17]) However, these works treat operators neither with complex-valued coefficients nor with degeneracy of infinite order.

On the other hand, in 1987, Morimoto [23] presented a new idea for the study of hypoellipticity. This is described simply as follows. First, suppose that the following inequality without weight holds. Given any positive number N,

$$\|u\| \le C\Big(\|Pu\| + \|u\|_{-N}\Big) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(\Omega), \tag{1}$$

where $\|\cdot\|_s$ denotes the norm of the Sobolev space H_s of order s (s is a real number) and $\|\cdot\|$ stands for $\|\cdot\|_0$. Let u be a distribution on Ω. If $Pu \in \boldsymbol{C}^\infty(\Omega)$, then we expect that $u \in L^2(\Omega)$ from (1). But the hypoellipticity of P does not follow simply from this observation. Now, let (s, ϕ, ψ) be an arbitrary element of $\boldsymbol{R} \times \boldsymbol{C}_0^\infty(\Omega) \times \boldsymbol{C}_0^\infty(\Omega)$ such that $\psi = 1$ in a neighborhood of the support of ϕ. If u is a distribution, there exists an N such that $\phi u \in H_{-N}(\boldsymbol{R}^d)$. Applying (1) to $\langle D_x\rangle^s \phi u$ and $N+s$ in place of u and N respectively, we have

$$\|\langle D_x\rangle^s \phi u\| \le C\Big(\|P\langle D_x\rangle^s \phi u\| + \|\phi u\|_{-N}\Big), \tag{2}$$

where $\langle D_x\rangle^s = (1+|D_x|^2)^{s/2}$. We admit for the moment that the first term on the right hand side is evaluated as follows:

$$\|P\langle D_x\rangle^s \phi u\| \le C\Big(\|\langle D_x\rangle^s \psi P u\| + \|\phi u\|_{-N}\Big). \tag{3}$$

Combining (2) with (3), we obtain

$$\|\langle D_x\rangle^s \phi u\| \le C\Big(\|\langle D_x\rangle^s \psi P u\| + \|\phi u\|_{-N}\Big). \tag{4}$$

So, (4) holds if (1) and (3) hold. The hypoellipticity of P follows from (4). This is because $\psi P u \in H_s(\boldsymbol{R}^d)$ implies $\langle D_x\rangle^s \phi u \in L^2(\boldsymbol{R}^d)$ from (4), so $\phi u \in H_s(\boldsymbol{R}^d)$. Since (s, ϕ, ψ) is arbitrary, P is hypoelliptic by Sobolev's imbedding theorem. The question is how to prove (3). To do this, it suffices to prove that the norm $\|[\langle D_x\rangle^s \phi, P]\, u\|$ is estimated by the right hand side of (3). Successive derivatives of the symbol of P appear in the expansion of the symbol of $[\langle D_x\rangle^s \phi, P]$. So, Morimoto introduced four kinds of estimates with weight for operators with the differentiated symbols of P and proved that these estimates are sufficient for (3) to hold. This is a method applicable to a very large class of operators because the hypoellipticity follows from a priori estimate like (1) without regularity gain. However, we have to verify many inequalities to hold.

We improve Morimoto's method to apply to a special microlocal domain and obtain the results of hypoellipticity for a large class of second order partial differential operators with complex-valued principal symbols which contains the operators treated in [5].

The operators which we study in this thesis are of the form:

$$P = \partial_t + f(t,x) \sum_{j,k=1}^{n} a_{jk}(t,x)\, L_j(x,\partial_x)\, L_k(x,\partial_x) \quad \text{in } \boldsymbol{R}^{d+1}, \tag{\textbf{A}}$$

where $(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^d$, $f(t,x)$ is a real-valued function of class $\boldsymbol{C}^\infty$, $a_{jk}(t,x)$ $(j,k = 1,\dots,n)$ are complex-valued functions of class $\boldsymbol{C}^\infty$ and $L_j(x,\partial_x)$ $(j = 1,\dots,n)$ are first order differential operators in $\boldsymbol{R}^d_x$ with real-valued coefficients. d and n are positive integers, $d \geq n$ or $d < n$. Operators of the form (**A**) are a sufficiently general object of the study of hypoellipticity of operators with sign-changing principal symbols.

In what follows, we introduce our results in the order of composition of this thesis. There are five results in this thesis, only the forth of them is on non-hypoellipticity, the others are on hypoellipticity. The first result is our main theorem, the second and the third are extensions of the first result. In the first result, we study the case where f is real-valued and f depends on x or not. The second result is devoted to the case where f is complex-valued. In the third result, we restrict ourselves to the case where f is real-valued and independent of x and suppose that orders of degeneracy of vector fields $L_1, L_2, \dots, L_n$ are not the same. In the forth result, we investigate the question of non-hypoellipticity of P of the form (**A**) in the case where f is independent of x. Our fifth result is a criterion for hypoellipticity of operators with $f(t) = t^p + i\,t^q$. This is related to the second result. But the criterion is not a simple corollary to the second result.

[1°] The first result is on hypoellipticity of operators of the form (**A**). Let us enumerate our basic assumptions on $f(t,x)$, $a_{jk}(t,x)$ and L_j.

(1°) Either the following (1°-α) or (1°-β) holds.

- (1°-α) ***f* does not depend on *x*** and there exists no non-empty open interval on which $f(t)$ vanishes identically. Moreover, if $f(t_0) > 0$ at a point $t_0 \in \boldsymbol{R}$, then $f(t) \geq 0$ everywhere on $[t_0, +\infty]$.
- (1°-β) ***f* depends on *x*** and does not change sign. Moreover, for any $x_0 \in \boldsymbol{R}^d$, there exists no non-empty open interval on which $f(t,x_0)$ vanishes identically.

(2°) The matrix $A(t,x) = (a_{jk}(t,x))_{j,k=1}^n$ satisfies the following:

For any compact set K of $\boldsymbol{R}^{d+1}$ there exists a constant $\delta = \delta(K) > 0$ such that

$$A(t,x) + {}^t\bar{A}(t,x) \geq \delta I \quad \text{on } K.$$

(3°) The Lie algebra generated by $\{L_j\}_{j=1}^n$ is of dimension d at every point of $\boldsymbol{R}_x^d$.

Now our first result is the following.

Theorem A *Suppose that P is an operator of the form* (**A**) *satisfying Conditions* (1°), (2°) *and* (3°). *Then P is hypoelliptic in $\boldsymbol{R}^{d+1}$.*

Let us explain briefly about each of the conditions. Condition (1°) restricts the type of changing sign of f and the largeness or smallness of the set of zeros of f. In the manner of changing sign, (1°) admits the case where f changes from minus sign to plus sign as t increases and does not admit the opposite case. On the size of the set of zeros, (1°) means that it is sufficiently small. But this is not so restrictive, because there exists an f satisfying (1°) whose set of zeros is of positive Lebesgue measure. Moreover, (1°) does not restrict the vanishing order of f, so f may vanish in infinite order. Condition (2°) implies that the real part of the principal symbol of P is non-negative when ξ runs over $\boldsymbol{R}^d$, which is consistent to the Hörmander principle. Condition (3°) is the assumption that the degeneracy of P with respect to x is of finite order. We hope to relax this assumption so that we can treat infinitely degenerate operators with respect to x.

We will mention the proof of this theorem in the last part of Introduction. The method of the proof of this theorem is applicable to show the hypoellipticity of various operators. In fact, the results [**2°**], [**3°**], [**5°**] below are proved in a similar way to the proof of Theorem **A**.

[2°] Next, we consider the case where f in (**A**) is complex-valued. Let P be a second order differential operator with coefficients of class $\boldsymbol{C}^\infty$ of the form:

$$(\mathbf{B}) \qquad P = \partial_t + \big(f(t,x) + i\,g(t,x)\big) \sum_{j,k=1}^{n} a_{jk}(t,x) L_j L_k \quad \text{in } \boldsymbol{R}^{d+1},$$

where $f(t,x)$ and $g(t,x)$ are real-valued functions of class $\boldsymbol{C}^\infty$ and L_j ($j = 1, \dots, n$) are the same as in (**A**) and satisfy Condition (3°) above. Assumptions on $f(t,x)$, $g(t,x)$, a_{jk} are the following:

(1$^\sharp$) $f(t,x)$ satisfies (1°) in Theorem **A**.

(2$^\sharp$) For any compact set K of $\boldsymbol{R}^{d+1}$, either the following (2$^\sharp$-1) or (2$^\sharp$-2) holds.

(2$^\sharp$-1) ***f* changes sign** and there exist positive constants $\rho = \rho(K) > 3/4$ and $C = C(K, \rho)$ such that

$$|\, g(t,x)| \le C\,|f(t,x)| \ \text{ and } \sum_{k=1}^{d} |\,\partial_{x_k} g(t,x)| \le C\,|f(t,x)|^{\rho} \quad \text{on } K,$$

(2$^\sharp$-2) ***f* does not change sign** and there exists a positive constant $C = C(K)$ such that

$$|\, g(t,x)| + \sum_{k=1}^{d} |\,\partial_{x_k} g(t,x)|^2 \le C\,|f(t,x)| \quad \text{on } K.$$

(3$^\sharp$) $a_{jk}(t,x)$ are complex-valued functions of class $\boldsymbol{C}^\infty$ and the matrix $A(t,x) = (a_{jk}(t,x))_{j,k=1}^{n}$ satisfies the following:
For any compact set K of $\boldsymbol{R}^{d+1}$, either the following (3$^\sharp$-1) or (3$^\sharp$-2) holds.

(3$^\sharp$-1) ***f* changes sign** and there exists a positive constant $\delta = \delta(K)$ such that

$$\mathrm{Re}\Big(\big(f(t,x) + i\, g(t,x)\big)A(t,x)\eta,\ f(t,x)\eta\Big) \ge \delta\,|f(t,x)|^2\,|\eta|^2$$

for all $((t,x),\, \eta) \in K \times \boldsymbol{C}^n$.

(3$^\sharp$-2) ***f* does not change sign** and there exists a positive constant $\delta = \delta(K)$ such that

$$\Big|\mathrm{Re}\Big((f(t,x) + i\, g(t,x))A(t,x)\eta,\ \eta\Big)\Big| \ge \delta\,|f(t,x)|\,|\eta|^2$$

for all $((t,x),\, \eta) \in K \times \boldsymbol{C}^n$.

Here $(\cdot\,,\cdot)$ stands for the Hermitian scalar product on $\boldsymbol{C}^n$.

If the inequality in (3$^\sharp$-1) holds, then the inequality in (3$^\sharp$-2) holds even if f changes sign. Now our second result is stated as follows.

Theorem B *Suppose that P is an operator of the form* (**B**) *satisfying Conditions* (1$^\sharp$), (2$^\sharp$) *and* (3$^\sharp$). *Then P is hypoelliptic in $\boldsymbol{R}^{d+1}$.*

Briefly speaking, this theorem says that P of the form (**B**) is hypoelliptic if P is hypoelliptic in the case where g is equal to 0 and if g is small in comparison with f. Condition (2$^\sharp$) is not necessary in general for P to be hypoelliptic. For example, let us consider the following operator of two variables:

$$L_{p,q} \;=\; \partial_t + \big(t^p + i\, t^q\big)\partial_x^2,$$

where p, q are non-negative integers. This is one of operators of the form (**B**) satisfying ($1^\sharp$) and ($3^\sharp$). As will be seen in [**5°**] below, $L_{p,q}$ is hypoelliptic if and only if $p \le 2q$. In the case where $p/2 < q < p$, $L_{p,q}$ does not satisfy ($2^\sharp$) but it is hypoelliptic.

[3°] In the third place, we extend Theorem **A** to a different direction from Theorem **B**. That is to say, we study the case where orders of degeneracy of vector fields $L_1, L_2, \ldots, L_n$ in (**A**) are not the same. Let Q be a second order differential operator with coefficients of class $\boldsymbol{C}^\infty$ and of the form:

$$(\mathbf{C}) \qquad Q = \partial_t + f_0(t) \sum_{j,k=1}^{n} a_{jk}(t,x) f_j(t) L_j f_k(t) L_k \quad \text{in } \boldsymbol{R}^{d+1}.$$

This is a generalization of P in Theorem **A** in the case where f is independent of x (See (1°-α) above). Let $\{f_j(t)\}_{j=0}^n$ be real-valued functions of class $\boldsymbol{C}^\infty$ defined in $\boldsymbol{R}$, and let Z_j be the set of zeros of f_j ($j = 0, \ldots, n$). For every compact set I of $\boldsymbol{R}$, we define the set $N(I)$ to be

$$N(I) = \Big\{ j \in \{1, \ldots, n\} \,;\, Z_j \cap I \neq \emptyset \Big\}.$$

Assume the following conditions on $\{f_j(t)\}_{j=0}^n$.

($1^\flat$-1) For every $j \in \{0, \ldots, n\}$, Z_j does not contain any non-empty open interval.

($1^\flat$-2) If $f_0(t_0) > 0$ at a point $t_0 \in \boldsymbol{R}$, then $f_0(t) \ge 0$ everywhere on $[t_0, +\infty]$.

($1^\flat$-3) Given a compact set $I \subset \boldsymbol{R}$, suppose that $N(I)$ is not empty. Then the following statement holds.
“ For any $j \in N(I)$, there exist positive constants $C = C(I,j)$, $\lambda = \lambda(I,j)$ such that

$$|f_0(t)| \le C\,|f_j(t)|^\lambda \quad \text{on } I. \text{ ”}$$

Our third result is the following.

Theorem C *Suppose that Q satisfies* ($1^\flat$-1), ($1^\flat$-2), ($1^\flat$-3), (2°) *and* (3°). *Then Q is hypoelliptic in $\boldsymbol{R}^{d+1}$.*

Roughly speaking, Q is hypoelliptic if the functions $\{f_j\}_{j=1}^n$, which stand for the degeneracy of vector fields with respect to t, are controlled by a single function f_0.

[4°] Now, we study a condition of non-hypoellipticity of an operator P of the form (**A**) under (2°) and (3°) in Theorem **A**. We restrict ourselves to the case where f is independent of x. Our result is the following:

Theorem D *Suppose that P is an operator of the form* (**A**) *satisfying Conditions* $(2°)$ *and* $(3°)$*. Then P is not hypoelliptic in $\boldsymbol{R}^{d+1}$ if there exist an $s \in \boldsymbol{R}$ and an open interval I containing s such that*

$$(\mathbf{D}) \qquad \begin{gathered} f(t) \geq 0 \text{ for every } t \in I \text{ satisfying } t < s \\ \text{and } f(t) \leq 0 \text{ for every } t \in I \text{ satisfying } t > s. \end{gathered}$$

This is a refinement of the result on non-hypoellipticity in [5]. Theorem **D** admits the case where the vanishing order of f is infinite. However, since there exist f's such that neither Condition (**D**) nor Condition $(1°)$ holds, we do not know from Theorem **D** whether P with such an f is hypoelliptic or not. However, assuming that Conditions $(2°)$ and $(3°)$ hold and that f is real-analytic, $(1°)$ is necessary and sufficient for P to be hypoelliptic owing to Theorem **D**.

[5°] Finally, we investigate the hypoellipticity of the following operator of two variables:

$$(\mathbf{E}) \qquad L_{p,q} = \partial_t + \left(t^p + i\, t^q\right)\partial_x^2,$$

where p, q are non-negative integers. Our result is the following.

Theorem E *$L_{p,q}$ is hypoelliptic in $\boldsymbol{R}^2$ if and only if $p \leq 2q$.*

Theorem **E** indicates that Condition $(2^\sharp)$ in Theorem **B** is not necessary for hypoellipticity.

Sketch of the proof of Theorem A

Theorems **B** and **C** on hypoellipticity are proved in a similar way. To simplify the explanation, we restrict ourselves to the case where ***f* changes sign and it does not depend on *x***. Let (τ, ξ) be the dual variables of (t, x) and Ω an open subset of $\boldsymbol{R}^{d+1}$. Also, let l be positive number. For the proof, we divide the space $\boldsymbol{R}^{d+1}_{\tau,\xi}$ into two microlocal domains **(i)** $|\tau| \geq l|\xi|^2/2$ and **(ii)** $|\tau| \leq l|\xi|^2$.

Definition We say that P is hypoelliptic in the microlocal domain D of $\boldsymbol{R}^{d+1}_{\tau,\xi}$ if $Pu \in \boldsymbol{C}^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$ implies $\chi(D_t, D_x)u \in \boldsymbol{C}^\infty(\Omega)$ for every real-valued function $\chi(\tau, \xi)$ such that χ is equal to 1 identically on D and the support of χ is contained in a neighborhood of D.

It suffices for the proof to prove that P is hypoelliptic in each of **(i)** and **(ii)**.

(i) P satisfies Hörmander's condition in the microlocal domain $|\tau| \geq l|\xi|^2/2$. So there exists a left parametrix of P in this domain belonging to an appropriate class of pseudo-differential operators. Consequently, P is hypoelliptic in this domain. This will be proved in §1. (P is Γ-elliptic, where $\Gamma = \left(-\infty \ \ \overset{1/2}{\underset{\sqrt{l/2}}{}} \ \ \phi \ \ \phi \ \ 1 \right)$ (see §1 of [31]).)

(ii) In the domain $|\tau| \leq l|\xi|^2$, P is microlocally a weakly elliptic operator. Here, we say that P is weakly elliptic if the inequality (4) holds for P.

Let us explain how the proof of **(ii)** proceeds. As is mentioned above, Morimoto gave a sufficient condition for a general partial differential operator to be weakly elliptic. We rewrite this condition in the microlocal domain $|\tau| \leq l|\xi|^2$. The sufficient condition obtained in this way is composed of one a priori estimate without weight analogous to (1) and four kinds of estimates for operators whose symbols are derivatives of the total symbol of P. These are introduced in §2. (See Conditions (I)–(V).) Thus, there are five kinds of estimates to be verified. Let us sketch how to do it.

First, since the principal symbol of P in question is a quadratic form of the vector fields $\{L_j\}_{j=1}^n$ multiplied by f, an operator with a differentiated symbol of P is roughly of the form:

$$\sum_{j=1}^n M_j f L_j + \sum_{j=1}^n \sum_{k=1}^d N_{j,k} f_{x_k} L_j + M_0,$$

where M_j, $N_{j,k}$ are pseudo-differential operators. From this representation, we see that verifying four kinds of estimates with weight is equivalent to studying inequalities for operators $\{L_j\}_{j=1}^n$. From this, we have not so many inequalities to be verified. Furthermore, we can investigate the hypoellipticity of an operator of the form (**A**) even if we generalize the class of f to a certain extent. This is the reason why we have set the form of the operators as in (**A**). Therefore, the idea of the proof of Theorem **A** can also be applied to the proofs of Theorem **B** and of Theorem **C**.

In view of the fact explained above, the four kinds of inequalities with weight for P will follow from the following two estimates for L_j. Without loss of generality, we may assume that f changes sign at $t = 0$.

Lemma 3.1 *For any $K \subset\subset \Omega$ and any $\rho > 3/4$, there exists a constant C depending only on (K, ρ) such that*

$$\sum_{j=1}^n \| |f|^\rho L_j u \|^2 \leq C \left\{ \left| \mathrm{Re} \left(Pu, (\operatorname{sgn} t) |f|^{2\rho-1} u \right) \right| + \|u\|^2 \right\} \tag{3.1}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$.

Proposition 4.1 *For any $K \subset\subset \Omega$, any $N > 0$ and any $\chi \in S_\Psi$, there exist positive constants $\kappa = \kappa(K), C = C(K, N, \chi)$ such that*

$$
\begin{aligned}
(4.1) \qquad & \sum_{j=1}^{n} \left\| \langle D_x \rangle^\kappa f L_j \chi u \right\|^2 + \left\| \langle D_x \rangle^\kappa f \chi u \right\|^2 + \sum_{k=1}^{d} \left\| \langle D_x \rangle^\kappa (\partial_{x_k} f) \chi u \right\|^2 \\
& \leq C \Big(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2 \Big) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K),
\end{aligned}
$$

where S_Ψ is a class of pseudo-differential operators defined in §1.

We explain simply each of these two inequalities. Let us begin with (3.1). The smaller ρ, the better (3.1). For example, if ρ were equal to 0, (3.1) would be an estimate which is not affected by the degeneracy from f. The function $(\operatorname{sgn} t)|f|^{2\rho-1}u$ appearing on the right hand side is smooth if $\rho = 1$. On the other hand, this function is not smooth in general if $\rho < 1$. An estimate like (3.1) involving a non-smooth function can not be treated in the usual method based on the theory of pseudo-differential operators, because we can not choose such a non-smooth function as a weight. One of advantages of our method is that we can use such an estimate. The proof of **(ii)** starts from (3.1). We can deduce from (3.1) many estimates involving only smooth functions. (See §4, §5, §6.) Lemma 3.1 is proved in §3.

Next, let us explain (4.1). The number κ is smaller than 1 in general. The loss of derivatives of (4.1) with respect to x is equal to $1 - \kappa$, which is smaller than 1 if f does never vanish. So, (4.1) would be regarded as a subelliptic estimate and a good estimate if f did not vanish. If we take the vanishing of f into account, (4.1) is not really so good. Our proof goes well without obtaining a good estimate, which is another advantage of our method. To prove (4.1), we divide the microlocal domain $|\tau| \leq l|\xi|^2$ into two subdomains $|f\langle\xi\rangle^\kappa| \leq 1$, $|f\langle\xi\rangle^\kappa| \geq 1$ and evaluate the left hand side of (4.1) in each domain, where κ is a constant depending only on $(K, \{L_j\}_{j=1}^n)$. (4.1) in the former domain follows from (3.1), and (4.1) in the latter is obtained by using Condition (3°) and applying Oleinik–Radkevich's method [28]. (See also [14].) Proposition 4.1 will be proved in §4.

The remaining problem is to show an inequality analogous to (1) which we need. This follows from the next proposition.

Proposition 5.1 *For any open set $K \subset\subset \boldsymbol{R}^{d+1}$, any $N > 0$, any $\chi \in S_\Psi$ and any $\mu > 0$, there exists a constant $C = C(K, N, \chi, \mu)$ such that*

$$
(5.1) \qquad \|\chi u\| \leq \mu \|P\chi u\| + C\|u\|_{-N} \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).
$$

(5.1) is an improvement of (1), because we can take any positive number μ in advance. In application of (5.1) to our problem, it is important that μ can be chosen arbitrarily small (See Conditions (II) and (V) in §2). The proof of Proposition 5.1 is done by using

(4.1) and by a partition of unity according to a given small μ. The construction of such a partition of unity is the key to the proof of Proposition 5.1 and it is our new idea. We make use only of the smallness of the set of zeros of f and the way of changing sign of f in this construction, so our result is also available in the case where the order of vanishing of f is infinite and the set of zeros of f is complicated.

This is the sketch of the proof.

We believe, by this thesis, that the condition for operators with sign-changing symbols to be hypoelliptic becomes clear to some extent. In this thesis, we can not treat, unfortunately, the case where f which controls a sign of the principal symbol depends on x if f changes sign. This will be our future problem.

1 Preliminary

In this section, we introduce some notation used for the proof of Theorem **A**. Moreover, we prove **(i)** and prepare for the proof of **(ii)**, where **(i)** and **(ii)** are stated in the Introduction. Before going into the detail, we give some examples of Theorem **A**.

Examples of Theorem A

1) The following operator P of 3 variables is hypoelliptic:

$$P = \partial_t + t^{2k+1}\left(\partial_x^2 + \partial_y^2\right),$$

where k is a non-negative integer. On the contrary,

$$Q = \partial_t - t^{2k+1}\left(\partial_x^2 + \partial_y^2\right)$$

is not hypoelliptic. (See [15], [5] ($k = 0$)).

2) The following operator P of 3 variables is hypoelliptic:

$$P = \partial_t + f(t)\left(\partial_x^2 + x^2\partial_y^2\right),$$

$$\text{where } f(t) = \begin{cases} (\operatorname{sgn} t)\left(1 + \sin\dfrac{1}{t}\right)e^{-\frac{1}{|t|}} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

$f(t)$ has a countably infinite number of zeros which accumulate at the origin and $f(t)$ changes sign there. On the contrary,

$$Q = \partial_t - f(t)\left(\partial_x^2 + x^2\partial_y^2\right)$$

is not hypoelliptic. (See §10).

3) The following operator P of 3 variables is hypoelliptic:

$$P = \partial_t + f(t,x,y)\left(\partial_x^2 + x^2\partial_y^2\right),$$

$$\text{where } f(t,x,y) = \begin{cases} \left(1 + \sin\dfrac{1}{t^2+x^2+y^2}\right)\exp\left(-\dfrac{1}{t^2+x^2+y^2}\right) \\ \qquad \text{for } (t,x,y) \neq (0,0,0), \\ 0 \quad \text{for } (t,x,y) = (0,0,0). \end{cases}$$

We shall mention further examples in §7.

In what follows we shall use systematically the notation in Chapter 2 of Kumano-go [14]. We say that a function $a(t,x,\tau,\xi)$ of class $\boldsymbol{C}^\infty$ defined on $\boldsymbol{R}^{2d+2} = \boldsymbol{R}^1_t \times \boldsymbol{R}^d_x \times \boldsymbol{R}^1_\tau \times \boldsymbol{R}^d_\xi$ belongs to a symbol of class $S^m_{\rho,\delta}$ if for any multi-index α, β there exists a constant $C_{\alpha,\beta}$ such that

$$\left| a_{(\beta)}{}^{(\alpha)}(t,x,\tau,\xi) \right| \le C_{\alpha,\beta} \langle \tau;\xi \rangle^{m+\delta|\beta|-\rho|\alpha|} \quad \text{in } \boldsymbol{R}^{2d+2},$$

where $\langle \tau;\xi\rangle = \sqrt{|\tau|^2+|\xi|^2+1}$, $a_{(\beta)}{}^{(\alpha)}(t,x,\tau,\xi) = \partial^\alpha_{\tau,\xi} D^\beta_{t,x} a(t,x,\tau,\xi)$ and $D_{t,x} = -i\partial_{t,x}$. Let $\mathcal{S}(\boldsymbol{R}^{d+1})$ be the space of rapidly decreasing functions. We say that a linear operator A from $\mathcal{S}(\boldsymbol{R}^{d+1})$ to $\mathcal{S}(\boldsymbol{R}^{d+1})$ is a pseudo-differential operator with symbol $a(t,x,\tau,\xi)$ of class $S^m_{\rho,\delta}$ if $a(t,x,\tau,\xi) \in S^m_{\rho,\delta}$ and if Au can be defined to be

$$Au(t,x) = (2\pi)^{-d-1} \int e^{it\tau + ix\cdot\xi}\, a(t,x,\tau,\xi)\, \mathcal{F}[u](\tau,\xi)\, d\tau\, d\xi \quad \text{for } u \in \mathcal{S}(\boldsymbol{R}^{d+1}),$$

where $\mathcal{F}[u]$ is the Fourier transform:

$$\mathcal{F}[u](\tau,\xi) = \int e^{-is\tau - iy\cdot\xi}\, u(s,y)\, ds\, dy.$$

We write $A = a(t,x,D_t,D_x) \in S^m_{\rho,\delta}$ and denote the symbol $a(t,x,\tau,\xi)$ of A by $\sigma(A)(t,x,\tau,\xi)$. (See §1 of Chapter 2 in [14].)

Since the hypoellipticity is a local property, we may assume, without loss of generality, that the coefficients of P are bounded as well as their derivatives of any order. We introduce some notation. Let (τ,ξ) be the dual variables of (t,x) and l, l', m any positive numbers such that $l < l'$. We denote by $\boldsymbol{Z}_+$ the set of non-negative integers. We say that a smooth function $\chi(\tau,\xi)$ belongs to the family $\Psi_{l,l',m}$ if

$$\chi \in S^0_{1/2,0}\,, \quad 0 \le \chi \le 1\,,$$
$$\text{supp}\chi \subset \{|\tau| \le l'|\xi|^2\} \cap \{|\tau| + |\xi| \ge m\}\,, \quad \chi \equiv 1 \text{ on } \{|\tau| \le l|\xi|^2\}$$

and $\langle \tau\rangle^\alpha \langle \xi \rangle^{|\beta|} \partial^\alpha_\tau \partial^\beta_\xi \chi$ is bounded in $\boldsymbol{R}^{d+1}_{\tau,\xi}$ for every $(\alpha,\beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}^d_+$,
where $\langle\tau\rangle = \sqrt{1+\tau^2}$ and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

Set

$$\Psi = \bigcup_{\substack{l,l',m>0\\ l<l'}} \Psi_{l,l',m}\ .$$

For χ, χ' belonging to $\boldsymbol{C}^\infty(\boldsymbol{R}_\tau \times \boldsymbol{R}^d_\xi)$, we define the notation $\chi \subset\subset \chi'$ if $\chi' \equiv 1$ on suppχ ($\chi \subseteq \chi'$ means that $\chi \subset\subset \chi'$ or $\chi = \chi'$). We see immediately that if $\chi \in \Psi_{l,l',m}$, $\chi' \in \Psi_{L,L',M}$ $(l' < L, M < m)$ then $\chi \subset\subset \chi'$. Therefore, for any element χ of Ψ, we can

find a $\chi' \in \Psi$ such that $\chi \subset\subset \chi'$. Also if $\chi \subset\subset \chi'$, then $1-\chi' \subset\subset 1-\chi$.
We define the class of operators $\mathcal{S}_\Psi$ to be

$$\mathcal{S}_\Psi = \Big\{ \chi(D_t, D_x) \ ; \ \chi \in \Psi \Big\}.$$

Obviously, $\mathcal{S}_\Psi \subset \mathcal{S}^0_{1/2,0}$.

Theorem **A** follows from the following two propositions. Propositions 1.1 and 1.2 correspond to **(i)** and **(ii)** in the Introduction respectively.

Proposition 1.1 *Suppose that P is an operator of the form* (**A**). *Then P is hypoelliptic in the microlocal domain* $\operatorname{supp}(1-\chi)$ *for every $\chi \in \Psi$, that is to say, for any $\Omega \subset \boldsymbol{R}^{d+1}$ fixed, if $u \in \mathcal{D}'(\boldsymbol{R}^{d+1})$ and $Pu \in \boldsymbol{C}^\infty(\Omega)$, then $\big(1-\chi(D_t, D_x)\big)u \in \boldsymbol{C}^\infty(\Omega)$.*

Proposition 1.2 *Suppose that P is an operator of the form* (**A**) *satisfying* (1°), (2°) *and* (3°). *For any $\Omega \subset\subset \boldsymbol{R}^{d+1}$, if $u \in \mathcal{D}'(\boldsymbol{R}^{d+1})$ and $Pu \in \boldsymbol{C}^\infty(\Omega)$, then $\chi(D_t, D_x)u \in \boldsymbol{C}^\infty(\Omega)$ for every $\chi \in \Psi$.*

Indeed, since $u = \big(1-\chi(D_t, D_x)\big)u + \chi(D_t, D_x)u$ for every $\chi \in \Psi$, the hypoellipticity of P follows from these propositions. First, let us prove Proposition 1.1.

Proof of Proposition 1.1. Since P is hypoelliptic in a domain of $\boldsymbol{R}^{d+1}$ on which f does not vanish, we may assume that $\sup_{(t,x)\in\Omega} |f(t,x)|$ is sufficiently small in Ω according to a given $\chi \in \Psi$. First, P satisfies (H)-condition in $\operatorname{supp}(1-\chi)$, that is to say, there exist constants C, C' and for any $\alpha, \beta \in \boldsymbol{Z}^{d+1}_+$ there exists a constant $C_{\alpha\beta}$ independent of (t, x, τ, ξ) such that

$$(1.1) \qquad \begin{cases} |p(t,x,\tau,\xi)| \;\geq\; C\langle\tau;\xi\rangle \text{ on } \{|\tau|+|\xi| \geq C'\} \cap \operatorname{supp}(1-\chi), \\[2ex] \Big|{p_{(\beta)}}^{(\alpha)}(t,x,\tau,\xi)\Big| \;\leq\; C_{\alpha\beta}\langle\tau;\xi\rangle^{-|\alpha|/2}\,|p(t,x,\tau,\xi)| \\ \qquad\qquad\qquad \text{on } \{|\tau|+|\xi| \geq C'\} \cap \operatorname{supp}(1-\chi), \end{cases}$$

where $p(t,x,\tau,\xi)$ is the symbol of P i.e., $p(t,x,\tau,\xi) = e^{-it\tau - ix\cdot\xi}Pe^{it\tau+ix\cdot\xi}$.
These inequalities allow us to define a formal left parametrix of P as a sum of pseudo-differential operators. First, we choose a function $\psi(\tau,\xi) \in \boldsymbol{C}^\infty(\boldsymbol{R}_\tau \times \boldsymbol{R}^d_\xi)$ satisfying

$$0 \leq \psi(\tau,\xi) \leq 1\,, \quad \psi(\tau,\xi) = 0 \ (\,|\tau|+|\xi| \leq C'\,)\,,\ = 1 \ (\,|\tau|+|\xi| \geq 2C'\,)\,,$$

and set

$$(1.2) \qquad \begin{cases} q_0(t,x,\tau,\xi) \;=\; p(t,x,\tau,\xi)^{-1}\psi(\tau,\xi), \\[2ex] q_k(t,x,\tau,\xi) \;=\; -\left(\displaystyle\sum_{\substack{|\gamma|+j=k \\ j<k}} \frac{1}{\gamma!}\, q_j^{(\gamma)} p_{(\gamma)}\right) q_0 \quad (\,k \geq 1\,). \end{cases}$$

Since q_k is not globally smooth, we define new functions q'_k by setting $q'_k = q_k(1-\chi)$ ($k \geq 0$). Then from (1.1) and (1.2) we get

$$\begin{aligned} \left| {q'_0}_{(\beta)}^{(\alpha)} \right| &\leq C_{0,\alpha,\beta} \langle \tau; \xi \rangle^{-1-|\alpha|/2}, \\ \left| {q'_k}_{(\beta)}^{(\alpha)} \right| &\leq C_{k,\alpha,\beta} \langle \tau; \xi \rangle^{-1-k/2-|\alpha|/2}, \end{aligned}$$

which implies $q'_k \in S^{-1-k/2}_{1/2,0}$ ($k \geq 0$). Next we can find a symbol $q(t,x,\tau,\xi) \in S^{-1}_{1/2,0}$ such that $q \sim \sum_{k=0}^{\infty} q'_k$ by the general theory of pseudo-differential operators. (See Chapter 2 in [14].) Then for any N we have

$$\sigma(QP) - \sum_{|\gamma|<N} \frac{1}{\gamma!} \left(\sum_{k=0}^{N-1} q'_k \right) p_{(\gamma)} \in S^{1-N/2}_{1/2,0}, \tag{1.3}$$

where $Q = q(t,x,D_t,D_x)$ and $\sigma(QP)$ is the symbol of QP. On the other hand, we write

$$\begin{aligned} &\sum_{|\gamma|<N} \frac{1}{\gamma!} \left(\sum_{k=0}^{N-1} q'_k \right) p_{(\gamma)} \\ &\quad = \sum_{|\gamma|<N} \frac{1}{\gamma!} \left(\sum_{k=0}^{N-1} q_k \right) p_{(\gamma)}(1-\chi) + \sum_{|\gamma|<N} \frac{1}{\gamma!} \sum_{k=0}^{N-1} \sum_{\substack{\beta<\gamma \\ \beta\neq 0}} \binom{\gamma}{\beta} q_k^{(\gamma-\beta)} (1-\chi)^{(\beta)} p_{(\gamma)} \\ &\quad \left(\text{set } K_N = \sum_{|\gamma|<N} \frac{1}{\gamma!} \sum_{k=0}^{N-1} \sum_{\substack{\beta<\gamma \\ \beta\neq 0}} \binom{\gamma}{\beta} q_k^{(\gamma-\beta)} (1-\chi)^{(\beta)} p_{(\gamma)} \right) \\ &\quad = \sum_{|\gamma|<N} \frac{1}{\gamma!} \left(\sum_{k=0}^{N-1} q_k \right) p_{(\gamma)}(1-\chi) + K_N \\ &\quad = q_0 p(1-\chi) + \sum_{k=1}^{N-1} \left\{ q_k p + \sum_{|\gamma|+j=k} \frac{1}{\gamma!} q_j^{(\gamma)} p_{(\gamma)} \right\} (1-\chi) \\ &\qquad + \sum_{\substack{|\gamma|+j\geq N \\ j<N, |\gamma|<N}} \frac{1}{\gamma!} q_j^{(\gamma)} p_{(\gamma)}(1-\chi) + K_N. \end{aligned}$$

Then we have

$$\sum_{|\gamma|<N} \frac{1}{\gamma!} \left(\sum_{k=0}^{N-1} q'_k \right) p_{(\gamma)} - (1-\chi) - K_N \in S^{1-N/2}_{1/2,0},$$

which implies, together with (1.3),

$$\sigma\Big(QP - \big(1-\chi(D_t,D_x)\big) - K_N(t,x,D_t,D_x) \Big) \in S^{1-N/2}_{1/2,0}.$$

We take $\chi' \in \Psi$ such that $\chi \subset\subset \chi'$. Since $\big(1-\chi'(D_t,D_x)\big) K_N(t,x,D_t,D_x) \in \mathcal{S}^{-\infty}$ due to $(1-\chi') \subset\subset (1-\chi)$, we obtain

$$\sigma\Big(\big(1-\chi'(D_t,D_x)\big) QP - \big(1-\chi'(D_t,D_x)\big) \Big) \in S^{1-N/2}_{1/2,0}.$$

Since N is arbitrary, we have for any $\chi, \chi' \in \Psi$ ($\chi \subset\subset \chi'$)

$$(1.4) \qquad \Big(1 - \chi'(D_t, D_x)\Big) = \Big(1 - \chi(D_t, D_x)\Big)QP + R,$$

where $\Big(1 - \chi(D_t, D_x)\Big)Q \in \mathcal{S}^{-1}_{1/2,0}$ and $R \in \mathcal{S}^{-\infty}$. Let x_0 be any fixed point in Ω and $\varphi, \psi \in \boldsymbol{C}_0^\infty(\Omega)$ be such that $\varphi \subset\subset \psi$ and φ is identically equal to 1 on some neighborhood of x_0. For the proof of Proposition 1.1, it suffices to show that $\varphi\Big(1-\chi(D_t, D_x)\Big)u \in \boldsymbol{C}_0^\infty(\Omega)$ provided that $u \in \mathcal{D}'(\boldsymbol{R}^{d+1})$ and that $Pu \in \boldsymbol{C}^\infty(\Omega)$. From (1.4), we have

$$\Big(1 - \chi(D_t, D_x)\Big)\psi u = Q'P\psi u + R\psi u,$$

where $Q' \in \mathcal{S}^{-1}_{1/2,0}$. Multiplying φ from the left to the above equality, we obtain

$$\varphi\Big(1 - \chi(D_t, D_x)\Big)u = \varphi Q'Pu + \varphi\Big[Q'P, \psi\Big]u - \varphi\Big[1 - \chi(D_t, D_x), \psi\Big]u.$$

Since $\varphi\Big[Q'P, \psi\Big]$, $\varphi\Big[1 - \chi(D_t, D_x), \psi\Big] \in \mathcal{S}^{-\infty}$, we have $\varphi\Big(1 - \chi(D_t, D_x)\Big)u \in \boldsymbol{C}_0^\infty(\Omega)$. □

Next, we prepare the proof of Proposition 1.2. By Sobolev's imbedding theorem, $\chi(D_t, D_x)u \in \boldsymbol{C}^\infty(\Omega)$ is equivalent to the following:

$$(1.5) \qquad \langle D_t; D_x\rangle^s \psi\chi(D_t, D_x)u \in L^2(\boldsymbol{R}^{d+1}) \text{ for any real } s \text{ and for any } \psi \in \boldsymbol{C}_0^\infty(\Omega).$$

Since $\langle\xi\rangle \leq \langle\tau; \xi\rangle \leq C\langle\xi\rangle^2$ on suppχ, where C depends only on $\chi \in \Psi$, (1.5) is equivalent to the following:

$$(1.6) \qquad \langle D_x\rangle^s \psi\chi(D_t, D_x)u \in L^2(\boldsymbol{R}^{d+1}) \text{ for any real } s \text{ and for any } \psi \in \boldsymbol{C}_0^\infty(\Omega).$$

And again, (1.6) holds if

$$(1.7) \qquad \langle D_x\rangle^s \chi(D_t, D_x)\psi u \in L^2(\boldsymbol{R}^{d+1}) \text{ for any real } s \text{ and for any } \psi \in \boldsymbol{C}_0^\infty(\Omega).$$

Indeed, suppose (1.7) holds. Then for any $\psi \in \boldsymbol{C}_0^\infty(\Omega)$, we can take $\psi' \in \boldsymbol{C}_0^\infty(\Omega)$ such that $\psi \subset\subset \psi'$. From (1.7), $\langle D_x\rangle^s\chi(D_t, D_x)\psi'u \in L^2(\boldsymbol{R}^{d+1})$, so $\psi\langle D_x\rangle^s\chi(D_t, D_x)\psi'u \in L^2(\boldsymbol{R}^{d+1})$. Since $[\psi, \langle D_x\rangle^s]\chi(D_t, D_x)\psi'u \in L^2(\boldsymbol{R}^{d+1})$ from (1.7), we have $\langle D_x\rangle^s\psi\chi(D_t, D_x)$ $\psi'u \in L^2(\boldsymbol{R}^{d+1})$. Finally (1.6) holds because $\langle D_x\rangle^s\psi[\chi(D_t, D_x), \psi'] \in \mathcal{S}^{-\infty}$. Now, (1.7) implies $u \in \boldsymbol{C}^\infty(\Omega)$. So, for the proof of Proposition 1.2, it suffices to see that $u \in \mathcal{D}'(\boldsymbol{R}^{d+1})$ and $Pu \in \boldsymbol{C}^\infty(\Omega)$ implies (1.7).

On the other hand, for any $\Omega \subset\subset \boldsymbol{R}^{d+1}$ fixed, if $Pu \in \boldsymbol{C}^\infty(\Omega)$, then $P\chi(D_t, D_x)u \in \boldsymbol{C}^\infty(\Omega)$ for every $\chi \in \Psi$, because $P\chi(D_t, D_x)u = -P\Big(1 - \chi(D_t, D_x)\Big)u + Pu$ and $\Big(1 - \chi(D_t, D_x)\Big)u \in \boldsymbol{C}^\infty(\Omega)$ from Proposition 1.1. Thus in order to show (1.7) for $u \in \mathcal{D}'(\boldsymbol{R}^{d+1})$ such that $Pu \in \boldsymbol{C}^\infty(\Omega)$, it suffices to show the following proposition.

Proposition 1.3 *Suppose that P satisfies* (1°), (2°) *and* (3°). *For each real s, each $\chi, \chi' \in \Psi$ satisfying $\chi \subset\subset \chi'$, the following statement holds.*
" Suppose that $\langle D_x\rangle^s \psi P \chi''(D_t, D_x)u \in L^2(\boldsymbol{R}^{d+1})$ for any $\psi \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^{d+1})$ and any $\chi'' \in \Psi$ satisfying $\chi \subseteq \chi'' \subseteq \chi'$. Then $\langle D_x\rangle^s \chi(D_t, D_x)\psi u \in L^2(\boldsymbol{R}^{d+1})$ for any $\psi \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^{d+1})$. "

This means that P is weakly elliptic in microlocal domain suppχ. (see §2 in [22].) We will prove Proposition 1.3 in §6 by making use of a result in the next section. If we admit this for the moment, Proposition 1.2 is proved, and the proof of Theorem **A** is completed.

2 Criterion for hypoellipticity in microlocal domain

In this section we shall give a refinement of Theorem 2.1 in [23]. We use this to prove Proposition 1.3. The statement of Proposition 1.3 is almost the same as Proposition 2.1 below. Let Ω be an open set of $\boldsymbol{R}^{d+1}$ and $P(t,x,D_t,D_x)$ a differential operator of order m with coefficients in $\boldsymbol{C}^\infty(\Omega)$. P in this section is not necessarily the same as P in §1. $\|\cdot\|_s$ denotes the norm of the Sobolev space H_s for real number s and $\|\cdot\|$ stands for $\|\cdot\|_0$. If there is no confusion, we identify a function $\chi \in \Psi$ with an operator $\chi(D_t,D_x) \in \mathcal{S}_\Psi$. For $\chi, \chi' \in \mathcal{S}_\Psi$, the notation $\chi \subset\subset \chi'$ (resp. $\chi \subseteq \chi'$) means that $\sigma(\chi) \subset\subset \sigma(\chi')$ (resp. $\sigma(\chi) \subseteq \sigma(\chi')$). We assume five conditions for P as follows:

(I) For any $K \subset\subset \Omega$, any $N > 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C_1 = C_1(K,N,\chi)$ such that

$$\|\chi u\| \le C_1 \left(\|P\chi u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{2.1}$$

(II) For any $K \subset\subset \Omega$, any $\beta = (0,\beta') \in \{0\} \times \boldsymbol{Z}_+^d$ ($|\beta| \neq 0$), any $\mu > 0$, any $N > 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C_2 = C_2(K,\beta,\mu,N,\chi)$ such that

$$\left\|\langle D_x\rangle^{-|\beta|}(P\chi)_{(\beta)}u\right\| \le \mu \|P\chi u\| + C_2 \|u\|_{-N} \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K), \tag{2.2}$$

where $p_{(\beta)}(t,x,\tau,\xi) = D^\beta_{t,x}p(t,x,\tau,\xi)$ and $D_{t,x} = -i\partial_{t,x}$.

(III) For any $K \subset\subset \Omega$, any $\alpha \in \boldsymbol{Z}_+^{d+1}$, any $N > 0$ and any $\chi, \chi' \in \mathcal{S}_\Psi$ satisfying $\chi \subset\subset \chi'$, there exists a constant $C_3 = C_3(K,\alpha,N,\chi,\chi')$ such that

$$\left\|(P\chi)^{(\alpha)}u\right\| \le C_3 \left(\|P\chi' u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K), \tag{2.3}$$

where $p^{(\alpha)}(t,x,\tau,\xi) = \partial^\alpha_{\tau,\xi}p(t,x,\tau,\xi)$.

(IV) For any $(t_0,x_0) \in \Omega$ and any neighborhood U of (t_0,x_0), there exist $\phi, \psi \in \boldsymbol{C}_0^\infty(U)$ such that

$$\begin{cases} \phi(t,x) = 1 & \text{in some neighborhood of } (t_0,x_0), \\ \phi \subset\subset \psi & (\text{ that is, } \quad \psi = 1 \text{ in a neighborhood of supp } \phi \text{ }), \end{cases}$$

and the inequality

$$\begin{aligned} &\|\langle D_x\rangle^\kappa P\chi\phi u\| \\ &\quad \le C_4 \left(\|\langle D_x\rangle^\kappa \psi P\chi u\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right) \text{ for all } u \in \boldsymbol{C}_0^\infty(K) \end{aligned} \tag{2.4}$$

holds for any open set $K \subset\subset \Omega$, any $N > 0$ and any $\chi, \chi' \in \mathcal{S}_\Psi$ ($\chi \subset\subset \chi'$), where $C_4 = C_4(K,N,\chi,\chi',\phi,\psi)$ is a constant depending on $(K,N,\chi,\chi',\phi,\psi)$ and κ is a positive number smaller than 1 depending only on K.

(V) For any $K \subset\subset \Omega$, any $\beta = (0, \beta') \in \{0\} \times \boldsymbol{Z}_+^d$ ($|\beta| \neq 0$), any $\mu > 0$, any $N > 0$, any $\chi, \chi' \in \mathcal{S}_\Psi$ ($\chi \subset\subset \chi'$), there exists a constant $C_5 = C_5(K, \beta, \mu, N, \chi, \chi', \psi)$ such that

$$(2.5)\ \left\| \langle D_x \rangle^{\kappa - |\beta|} (\psi P \chi)_{(\beta)} u \right\| \\ \leq \mu \left\| \langle D_x \rangle^{\kappa} \psi P \chi u \right\| + C_5 \left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N} \right) \text{ for all } u \in \boldsymbol{C}_0^\infty(K),$$

where ψ is the function introduced in (IV) and κ is the number introduced in (IV).

Proposition 2.1 *Suppose that a differential operator $P = p(t, x, D_t, D_x)$ satisfies Conditions* (I)–(V). *For each real s, each $\chi, \chi' \in \mathcal{S}_\Psi$ satisfying $\chi \subset\subset \chi'$, the following statement holds.*

" Suppose that $u \in \mathcal{D}'(\Omega)$ and $\langle D_x \rangle^s \psi P \chi'' u \in L^2(\boldsymbol{R}^{d+1})$ for any $\psi \in \boldsymbol{C}_0^\infty(\Omega)$ and any $\chi'' \in \mathcal{S}_\Psi$ satisfying $\chi \subseteq \chi'' \subseteq \chi'$. Then $\langle D_x \rangle^s \chi \psi u \in L^2(\boldsymbol{R}^{d+1})$ for any $\psi \in \boldsymbol{C}_0^\infty(\Omega)$. "
Therefore P is hypoelliptic in the microlocal domain $\mathrm{supp}\chi \times \Omega$.

Remark. Proposition 2.1 holds even if we omit the term $\|P\chi' u\|$ from the right hand side of (2.4) and (2.5) in Conditions (IV) and (V) respectively. We need this term in the application of Proposition 2.1 to P specified in Theorem **A**. This is because we have to investigate operators with differentiated symbols of $P\chi$. The derivatives of the symbol $P\chi$ involve derivatives of χ. The term $\|P\chi u\|$ alone does not suffice to evaluate operators with such a symbol applied to u. So the term $\|P\chi' u\|$ is needed for applying Proposition 2.1 to P in question.

Before proving this proposition, let us sketch the roles of Conditions (I)–(V) in the proof. Let χ, χ' be as above and u an element of a Sobolev space. Given any point $(t_0, x_0) \in \Omega$, let $\phi, \psi \in \boldsymbol{C}_0^\infty(\Omega)$ be as in Condition (IV). The proposition follows from the inequality

$$\left\| \langle D_x \rangle^s \chi \phi u \right\| \leq C \left(\left\| \langle D_x \rangle^s \psi P \chi' u \right\| + \|\psi u\|_{-N} \right). \tag{2.6}$$

Obviously, (2.1) in Condition (I) is a version of (2.6) in the case $s = 0$. To obtain (2.6) for $s > 0$, we apply (2.1) to $\langle D_x \rangle^s \chi \phi u$ in place of u to have

$$\left\| \langle D_x \rangle^s \chi \phi u \right\| \leq C_1 \left(\left\| P \langle D_x \rangle^s \chi \phi u \right\| + \|\psi u\|_{-N} \right). \tag{2.7}$$

Rewriting $P \langle D_x \rangle^s \chi \phi u = [P\chi, \langle D_x \rangle^s]\, \phi u + \langle D_x \rangle^s P \chi \phi u$, we have to show two inequalities

$$\left\| \left[P\chi, \langle D_x \rangle^s \right] \phi u \right\| \leq C \left(\left\| \langle D_x \rangle^s P \chi \phi u \right\| + \|\psi u\|_{-N} \right), \tag{2.8}$$

$$\left\| \langle D_x \rangle^s P \chi \phi u \right\| \leq C \left(\left\| \langle D_x \rangle^s \psi P \chi u \right\| + \|\psi u\|_{-N} \right). \tag{2.9}$$

Let us begin with (2.8). By an asymptotic expansion of the symbol, we have

$$\left\| \left[P\chi, \langle D_x \rangle^s \right] \phi u \right\| \leq C \left(\sum_{0 < |\beta| < 2(s+m+N)} \left\| \langle D_x \rangle^{s - |\beta|} (P\chi)_{(\beta)} \phi u \right\| + \|\phi u\|_{-N} \right),$$

where β runs over indices of type $(0, \beta')$. Condition (II) guarantees that each term of the sum on the right hand side does not exceed $C\left(\|\langle D_x\rangle^s P\chi\phi u\| + \|\psi u\|_{-N}\right)$ (see Lemma 2.2). (2.8) is verified in this way (Arbitrariness of μ in Condition (II) will be necessary only to prove Lemma 2.2).

Next, the proof of (2.9) is divided into four steps.

First step : We rewrite $\langle D_x\rangle^s P\chi\phi u$ as

$$\langle D_x\rangle^s P\chi\phi u = \langle D_x\rangle^\kappa \left[P\chi, \langle D_x\rangle^{s-\kappa}\right]\phi u + \langle D_x\rangle^\kappa P\chi\langle D_x\rangle^{s-\kappa}\phi u,$$

and deal with the first term on the right hand side, where κ is the number introduced in Condition (IV). Again by Lemma 2.2, we have

$$\left\|\langle D_x\rangle^\kappa \left[P\chi, \langle D_x\rangle^{s-\kappa}\right]\phi u\right\| \leq \mu \left\|\langle D_x\rangle^s P\chi\phi u\right\| + C(\mu)\left\|\psi u\right\|_{-N}.$$

Choosing a μ small enough, we have

$$\left\|\langle D_x\rangle^s P\chi\phi u\right\| \leq C\left(\left\|\langle D_x\rangle^\kappa P\chi\langle D_x\rangle^{s-\kappa}\phi u\right\| + \|\psi u\|_{-N}\right). \tag{2.10}$$

Second step : We rewrite $\langle D_x\rangle^\kappa P\chi\langle D_x\rangle^{s-\kappa}\phi u$ as

$$\langle D_x\rangle^\kappa P\chi\langle D_x\rangle^{s-\kappa}\phi u = \langle D_x\rangle^\kappa P\chi\left[\langle D_x\rangle^{s-\kappa}, \phi\right]u + \langle D_x\rangle^\kappa P\chi\phi\langle D_x\rangle^{s-\kappa}u.$$

We expand the symbol of $\left[\langle D_x\rangle^{s-\kappa}, \phi\right]$ and evaluate the commutators between each term in the expansion and $P\chi$ applied to u. Then, we have

$$\left\|\langle D_x\rangle^\kappa P\chi\left[\langle D_x\rangle^{s-\kappa}, \phi\right]u\right\| \leq C\left(\left\|\langle D_x\rangle^{s-1}P\chi u\right\| + \left\|\langle D_x\rangle^{s-1}P\chi' u\right\| + \|\psi u\|_{-N}\right).$$

Here we make use of Condition (III) to evaluate commutators between $P\chi$ and multiplication by functions (see Lemma 2.3).

Third step : Condition (IV) yields

$$\begin{aligned}&\left\|\langle D_x\rangle^\kappa P\chi\phi\langle D_x\rangle^{s-\kappa}u\right\| \\ &\quad\leq C\left(\left\|\langle D_x\rangle^\kappa\psi P\chi\langle D_x\rangle^{s-\kappa}u\right\| + \left\|P\chi\langle D_x\rangle^{s-\kappa}u\right\| + \left\|P\chi'\langle D_x\rangle^{s-\kappa}u\right\| + \|\psi u\|_{-N}\right).\end{aligned}$$

By Condition (II), the second and third terms on the right hand side are smaller than $C\left(\left\|\langle D_x\rangle^{s-\kappa}P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi' u\right\| + \|\psi u\|_{-N}\right)$. Hence we obtain

$$\begin{aligned}&\left\|\langle D_x\rangle^\kappa P\chi\phi\langle D_x\rangle^{s-\kappa}u\right\| \\ &\quad\leq C\left(\left\|\langle D_x\rangle^\kappa\psi P\chi\langle D_x\rangle^{s-\kappa}u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi' u\right\| + \|\psi u\|_{-N}\right).\end{aligned}$$

Here, notice that not $\langle D_x\rangle^s$ but $\langle D_x\rangle^{s-\kappa}$ appears in the second and third term on the right hand side. Thus, we need Condition (IV) to lower the order of $\langle D_x\rangle^s P\chi$.

Forth step : We rewrite $\langle D_x\rangle^{\kappa}\psi P\chi\langle D_x\rangle^{s-\kappa}u$ as

$$\langle D_x\rangle^{\kappa}\psi P\chi\langle D_x\rangle^{s-\kappa}u = \langle D_x\rangle^{\kappa}\left[\psi P\chi, \langle D_x\rangle^{s-\kappa}\right]u + \langle D_x\rangle^{s}\psi P\chi u.$$

We expand the symbols $\left[\psi P\chi, \langle D_x\rangle^{s-\kappa}\right]$ and evaluate the composition of $\langle D_x\rangle^{\kappa}$ and each term of the expansion applied to u. By Condition (V), we have

$$\begin{aligned}&\left\|\langle D_x\rangle^{\kappa}\left[\psi P\chi, \langle D_x\rangle^{s-\kappa}\right]u\right\| \\ &\quad\leq\; C\left(\left\|\langle D_x\rangle^{s}\psi P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi' u\right\| + \|\psi u\|_{-N}\right).\end{aligned}$$

Condition (V) plays the same role as (II) for $\psi P\chi$ instead of $P\chi$. (See Lemma 2.4 and Corollary 2.5.)

Combining all these steps, we have

$$\begin{aligned}&\left\|\langle D_x\rangle^{s}P\chi\phi u\right\| \\ &\quad\leq C\left(\left\|\langle D_x\rangle^{s}\psi P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi' u\right\| + \|\psi u\|_{-N}\right).\end{aligned} \tag{2.11}$$

(see Lemma 2.6) The remaining problem is to estimate the second and third terms on the right hand side of (2.11). We take functions $\phi', \psi' \in \boldsymbol{C}_0^{\infty}(\Omega)$ such that $\psi \subset\subset \phi' \subset\subset \psi'$ and (2.4) holds for (ϕ', ψ') in place of (ϕ, ψ). Substituting $\phi' u$ for u in (2.11), we have

$$\begin{aligned}&\left\|\langle D_x\rangle^{s}P\chi\phi u\right\| \\ &\quad\leq C\left(\left\|\langle D_x\rangle^{s}\psi P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi\phi' u\right\| + \left\|\langle D_x\rangle^{s-\kappa}P\chi'\phi' u\right\| + \|\phi' u\|_{-N}\right),\end{aligned} \tag{2.12}$$

because $\psi\,[P\chi, \phi']$ is a smoothing operator. The second and third terms on the right hand side of (2.12) are of the same type as in the left hand side of (2.11). Then we have easily

$$\begin{aligned}&\left\|\langle D_x\rangle^{s}P\chi\phi u\right\| \\ &\quad\leq C\Big(\left\|\langle D_x\rangle^{s}\psi P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}\psi' P\chi u\right\| + \left\|\langle D_x\rangle^{s-\kappa}\psi' P\chi' u\right\| \\ &\qquad + \left\|\langle D_x\rangle^{s-2\kappa}P\chi u\right\| + \left\|\langle D_x\rangle^{s-2\kappa}P\chi' u\right\| + \left\|\langle D_x\rangle^{s-2\kappa}P\chi'' u\right\| + \|\psi' u\|_{-N}\Big),\end{aligned}$$

where $\chi'' \in S_\Psi$ satisfying $\chi' \subset\subset \chi''$. Repeating a finite number of times of this argument, the second and third terms will be estimated by $C\,\|\psi' u\|_{-N}$. Therefore we have an inequality analogous to (2.6).

Now, we mention lemmas which are used above, and prove them. As in §2 of [22] we employ a pseudo-differential operator $\Lambda_{s,k,\varepsilon} = \langle D_x\rangle^{s}(1+\varepsilon\langle D_x\rangle)^{-k}$ for real s, $\varepsilon > 0$ and $k \geq 0$. The first lemma is used to control $\|[\langle D_x\rangle^{s}, P\chi]u\|$.

Lemma 2.2 (cf. Lemma 2.1 of [23]) *Suppose that P satisfies Condition* (II). *Then for any $K \subset\subset \Omega$, any $\beta = (0, \beta') \in \{0\} \times \boldsymbol{Z}_+^d$ ($|\beta| \neq 0$), any real s, any $\mu > 0$, $N > 0$, $0 < \varepsilon < 1$, $k \geq 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C = C(K, \beta, s, \mu, N, k, \chi)$ independent of ε such that*

$$\left\|\Lambda_{s-|\beta|,k,\varepsilon}(P\chi)_{(\beta)}u\right\| \leq \mu \left\|\Lambda_{s,k,\varepsilon}P\chi u\right\| + C\left\|u\right\|_{-N} \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{2.13}$$

Furthermore, for any $K \subset\subset \Omega$, any real s, s', any $\mu > 0$, $N > 0$, $0 < \varepsilon < 1$, $k \geq 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C' = C'(K, s, s', \mu, N, k, \chi)$ independent of ε such that

$$\left\|\langle D_x\rangle^{s'}\left[P\chi, \Lambda_{s,k,\varepsilon}\right]u\right\| \leq \mu \left\|\Lambda_{s+s',k,\varepsilon}P\chi u\right\| + C'\left\|u\right\|_{-N} \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{2.14}$$

Proof of Lemma 2.2. We take $\phi, \psi \in \boldsymbol{C}_0^\infty(\Omega)$ such that $\phi \subset\subset \psi$, $\phi \equiv 1$ on K. Applying (2.2) to $K = \mathrm{supp}\psi$, $\psi\Lambda_{s,k,\varepsilon}\phi u \in \boldsymbol{C}_0^\infty((\mathrm{supp}\psi)^\circ)$ and $\psi\Lambda_{s,k,\varepsilon}\phi u$ for u, we obtain

$$\left\|\langle D_x\rangle^{-|\beta|}(P\chi)_{(\beta)}\psi\Lambda_{s,k,\varepsilon}\phi u\right\| \leq \mu\left\|P\chi\psi\Lambda_{s,k,\varepsilon}\phi u\right\| + C\left\|u\right\|_{-N}.$$

Then we have easily

$$\begin{aligned}\left\|\Lambda_{s-|\beta|,k,\varepsilon}(P\chi)_{(\beta)}u\right\| \;\leq\; & \left\|\langle D_x\rangle^{-|\beta|}\left[(P\chi)_{(\beta)}\psi, \Lambda_{s,k,\varepsilon}\right]\phi u\right\| \\ & + \mu\left\|\Lambda_{s,k,\varepsilon}P\chi u\right\| + \mu\left\|\left[P\chi\psi, \Lambda_{s,k,\varepsilon}\right]\phi u\right\| + C\left\|u\right\|_{-N+s},\end{aligned}$$

where C is independent of ε. Since we can regard $\Lambda_{s,k,\varepsilon}$ as an element of $\mathcal{S}^s_{1/2,0}$ on $\mathrm{supp}\chi$, the expansion formula yields

$$\begin{aligned}\left[(P\chi)_{(\beta)}\psi, \Lambda_{s,k,\varepsilon}\right] \;\equiv\; & \sum_{0<|\alpha|<2(s+m+N)} \frac{(-1)^{|\alpha|}}{\alpha!}\Lambda_{s,k,\varepsilon}^{(\alpha)}\left\{(P\chi)_{(\beta)}\psi\right\}_{(\alpha)} \mod \mathcal{S}^{-N}_{1/2,0} \\ \left[(P\chi)\psi, \Lambda_{s,k,\varepsilon}\right] \;\equiv\; & \sum_{0<|\alpha|<2(s+m+N)} \frac{(-1)^{|\alpha|}}{\alpha!}\Lambda_{s,k,\varepsilon}^{(\alpha)}(P\chi\psi)_{(\alpha)} \mod \mathcal{S}^{-N}_{1/2,0} \\ & \text{(where } \alpha \text{ runs over indices of type } (0,\alpha')\text{)},\end{aligned}$$

and since N is arbitrary and $\left\{(P\chi)_{(\beta)}\psi\right\}_{(\alpha)}\phi - (P\chi)_{(\alpha+\beta)}\phi \in \mathcal{S}^{-\infty}$, we see that

$$\begin{aligned}&\left\|\Lambda_{s-|\beta|,k,\varepsilon}(P\chi)_{(\beta)}u\right\| \\ &\quad\leq\; \mu\left\|\Lambda_{s,k,\varepsilon}P\chi u\right\| + C'\mu\sum_{0<|\alpha|<2(s+m+N)}\left\|\Lambda_{s-|\alpha|,k,\varepsilon}(P\chi)_{(\alpha)}u\right\| \\ &\qquad + C''\left(\sum_{0<|\alpha|<2(s+m+N)}\left\|\Lambda_{s-|\alpha|-|\beta|,k,\varepsilon}(P\chi)_{(\alpha+\beta)}u\right\| + \left\|u\right\|_{-N}\right),\end{aligned} \tag{2.15}$$

where C' is independent of (ε, μ) and C'' is independent of ε. It is obvious that (2.13) holds for $|\beta| \geq 2(s + m + N)$, so we may suppose that $|\beta| \leq 2(s + m + N) - 1$. Set

$a_0 = \|\Lambda_{s,k,\varepsilon} P\chi u\|$, $a_j = \sum_{|\beta|=j} \left\|\Lambda_{s-|\beta|,k,\varepsilon}(P\chi)_{(\beta)} u\right\|$ $(j = 1, \ldots, 2(s+m+N)\ -1)$ and $a_{2(s+m+N)} = \|u\|_{-N}$. (2.15) yields

$$a_j \ \leq \ \mu \sum_{k=0}^{2(s+m+N)-1} a_k + C_{\mu,j} \sum_{k=j+1}^{2(s+m+N)} a_k \qquad (j = 1, \ldots, 2(s+m+N)-1\,). \tag{2.16}$$

Applying Lemma 2.9 of [22] to (2.16), we have

$$a_j \leq \mu' a_0 + C'_{\mu',j} a_{2(s+m+N)}$$

for any $\mu' > 0$. This inequality is equivalent to (2.13). (2.14) follows from (2.13) and the above expansion formula. □

Next lemma is used to evaluate the commutators between $P\chi$ and multiplication by functions.

Lemma 2.3 (cf. Lemma 2.2 of [23]) *Suppose that $\phi(t,x)$ belongs to $\boldsymbol{C}_0^\infty(\Omega)$ and that P satisfies Conditions* (II) *and* (III). *Then for any $K \subset\subset \Omega$, any real s, any $N > 0$, $0 < \varepsilon < 1$, $k \geq 0$ and any $\chi, \chi' \in \mathcal{S}_\Psi$ $(\chi \subset\subset \chi')$, there exists a constant $C = C(K, s, N, k, \chi, \chi', \phi)$ independent of ε such that*

$$\|\Lambda_{s,k,\varepsilon} P\chi\phi u\| \leq C\left(\|\Lambda_{s,k,\varepsilon} P\chi u\| + \|\Lambda_{s,k,\varepsilon} P\chi'\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{2.17}$$

Proof of Lemma 2.3. By means of (2.14), we have

$$\begin{aligned}\left\|\Lambda_{s,k,\varepsilon} P\chi\phi u\right\| &\leq \left\|P\chi\Lambda_{s,k,\varepsilon}\phi u\right\| + \left\|\left[\Lambda_{s,k,\varepsilon}, P\chi\right]\phi u\right\| \\ &\leq \left\|P\chi\Lambda_{s,k,\varepsilon}\phi u\right\| + \mu\left\|\Lambda_{s,k,\varepsilon} P\chi\phi u\right\| + C_\mu \|\phi u\|_{-N}\,.\end{aligned}$$

Then we obtain

$$\left\|\Lambda_{s,k,\varepsilon} P\chi\phi u\right\| \leq C\left(\left\|P\chi\Lambda_{s,k,\varepsilon}\phi u\right\| + \|u\|_{-N}\right).$$

Here and in what follows we denote different constants independent of ε by the same notation C. Using the expansion formula

$$\Lambda_{s,k,\varepsilon}\phi \ \equiv \sum_{0\leq|\alpha|<2(s+m+N)} \phi_{(\alpha)}\Lambda_{s,k,\varepsilon}^{(\alpha)}/\alpha! \quad \text{mod } S_{1/2,0}^{-N-m} \text{ on supp}\chi, \tag{2.18}$$

we have

$$\left\|P\chi\Lambda_{s,k,\varepsilon}\phi u\right\| \leq C\left(\sum_{|\alpha|<2(s+m+N)} \left\|P\chi\phi_{(\alpha)}\Lambda_{s,k,\varepsilon}^{(\alpha)} u\right\| + \|u\|_{-N}\right). \tag{2.19}$$

And each of the sum satisfies

$$
\begin{aligned}
(2.20)\quad & \left\| P\chi\phi_{(\alpha)}\Lambda^{(\alpha)}_{s,k,\varepsilon}u \right\| \\
& \le \left\| \phi_{(\alpha)} P\chi\Lambda^{(\alpha)}_{s,k,\varepsilon}u + \sum_{0<|\beta|\le 2(s+m+N)} \phi_{(\alpha+\beta)}(P\chi)^{(\beta)}\Lambda^{(\alpha)}_{s,k,\varepsilon}u/\beta! \right\| + C\|u\|_{-N} \\
& \le C\left(\left\| P\chi\Lambda^{(\alpha)}_{s,k,\varepsilon}u \right\| + \left\| P\chi'\Lambda^{(\alpha)}_{s,k,\varepsilon}u \right\| + \|u\|_{-N} \right) \text{ (by Condition (III)) .}
\end{aligned}
$$

If χ'' is one of χ and χ', we have

$$
\begin{aligned}
(2.21)\quad & \left\| P\chi''\Lambda^{(\alpha)}_{s,k,\varepsilon}u \right\| \\
& \le \left\| \Lambda^{(\alpha)}_{s,k,\varepsilon} P\chi'' u + \sum_{0<|\beta|<2(s+m+N-|\alpha|)} (-1)^{|\beta|}\Lambda^{(\alpha+\beta)}_{s,k,\varepsilon}(P\chi'')_{(\beta)}u/\beta! \right\| + C\|u\|_{-N} \\
& \le C\left(\left\| \Lambda_{s-|\alpha|,k,\varepsilon} P\chi'' u \right\| + \|u\|_{-N} \right) \text{ (by (2.13)) .}
\end{aligned}
$$

Combining three inequalities (2.19), (2.20) and (2.21), we have (2.17). □

Next lemma is used to control $\|[\psi P\chi, \langle D_x\rangle^s]u\|$.

Lemma 2.4 (cf. Lemma 2.3 of [23]) *Suppose that P satisfies Conditions* (II) *and* (V)*. Then for any $K \subset\subset \Omega$, any $\beta = (0,\beta') \in \{0\} \times \boldsymbol{Z}^d_+$ ($|\beta| \neq 0$), any real s, any $\mu > 0$, $0 < \varepsilon < 1$, $N > 0$, $k \ge 0$ and any $\chi, \chi' \in \mathcal{S}_\Psi$ ($\chi \subset\subset \chi'$), there exists a constant $C = C(K, \beta, s, \mu, k, N, \chi, \chi', \psi)$ independent of ε such that*

$$
\begin{aligned}
(2.22)\quad & \left\| \Lambda_{s-|\beta|,k,\varepsilon}(\psi P\chi)_{(\beta)}u \right\| \\
& \le \mu\left\| \Lambda_{s,k,\varepsilon}(\psi P\chi)u \right\| + C\left(\left\| \Lambda_{s-\kappa,k,\varepsilon}P\chi u \right\| + \left\| \Lambda_{s-\kappa,k,\varepsilon}P\chi' u \right\| + \|u\|_{-N} \right)
\end{aligned}
$$

for all $u \in \boldsymbol{C}^\infty_0(K)$, where $\kappa > 0$ and ψ are the same as in Condition (V)*.*

Proof of Lemma 2.4. As in the beginning of the proof of Lemma 2.2, we take $\phi, \phi' \in \boldsymbol{C}^\infty_0(\Omega)$ such that $\phi \subset\subset \phi'$, $\phi \equiv 1$ on K. Applying (2.5) to $K = \mathrm{supp}\phi'$, we obtain

$$
\begin{aligned}
(2.23)\quad & \left\| \langle D_x\rangle^{\kappa-|\beta|}(\psi P\chi)_{(\beta)}\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u \right\| \\
& \le \mu\left\| \langle D_x\rangle^{\kappa}(\psi P\chi)\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u \right\| \\
& \quad + C\left(\left\| P\chi\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u \right\| + \left\| P\chi'\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u \right\| + \|\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u\|_{-N} \right) .
\end{aligned}
$$

The left hand side of (2.23) is estimated from below as follows.

$$
\begin{aligned}
& \left\| \langle D_x\rangle^{\kappa-|\beta|}(\psi P\chi)_{(\beta)}\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u \right\| \\
& = \left\| \Lambda_{s-\beta,k,\varepsilon}(\psi P\chi)_{(\beta)}u + \langle D_x\rangle^{\kappa-|\beta|}\left[(\psi P\chi)_{(\beta)}\phi', \Lambda_{s-\kappa,k,\varepsilon} \right]\phi u \right\| \\
& \ge \left\| \Lambda_{s-\beta,k,\varepsilon}(\psi P\chi)_{(\beta)}u \right\| - \left\| \langle D_x\rangle^{\kappa-|\beta|}\left[(\psi P\chi)_{(\beta)}\phi', \Lambda_{s-\kappa,k,\varepsilon} \right]\phi u \right\| .
\end{aligned}
$$

Using the expansion formula

$$\Big[(\psi P\chi)_{(\beta)}\phi', \Lambda_{s-\kappa,k,\varepsilon}\Big]\,\phi \equiv \sum_{0<|\alpha|<2(s+m+N)} (-1)^{|\alpha|}\Lambda^{(\alpha)}_{s-\kappa,k,\varepsilon}\left\{(\psi P\chi)_{(\beta)}\right\}_{(\alpha)}\phi/\alpha! \ \mathrm{mod}\ \mathcal{S}^{-N}_{1/2,0}$$

(where α runs over indices of type $(0,\alpha')$), we have

$$(2.24)\qquad \begin{aligned} &\left\|\langle D_x\rangle^{\kappa-|\beta|}(\psi P\chi)_{(\beta)}\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u\right\| \\ &\quad\ge \left\|\Lambda_{s-\beta,k,\varepsilon}(\psi P\chi)_{(\beta)}u\right\| \\ &\qquad - C\left(\sum_{0<|\alpha|<2(s+m+N)}\left\|\Lambda_{s-|\alpha+\beta|,k,\varepsilon}(\psi P\chi)_{(\alpha+\beta)}u\right\| + \|u\|_{-N}\right). \end{aligned}$$

Next, the first term on the right hand side of (2.23) is written as follows.

$$(2.25)\qquad \begin{aligned} &\mu\left\|\langle D_x\rangle^{\kappa}(\psi P\chi)\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u\right\| \\ &= \ \mu\left\|\Lambda_{s,k,\varepsilon}(\psi P\chi)u + \langle D_x\rangle^{\kappa}\Big[(\psi P\chi\phi'), \Lambda_{s-\kappa,k,\varepsilon}\Big]\phi u\right\| \\ &\le \ \mu\left\|\Lambda_{s,k,\varepsilon}(\psi P\chi)u\right\| + C\mu\sum_{0<|\alpha|<2(s+m+N)}\left\|\Lambda_{s-|\alpha|,k,\varepsilon}(\psi P\chi)_{(\alpha)}u\right\| + C'\|u\|_{-N}, \end{aligned}$$

where C is independent of (μ,ε) and C' is independent of ε. Also the second (third) terms on the right hand side of (2.23) are estimated as follows.

$$(2.26)\qquad \begin{aligned} &\left\|P\chi\phi'\Lambda_{s-\kappa,k,\varepsilon}\phi u\right\| \\ &= \ \left\|\Lambda_{s-\kappa,k,\varepsilon}P\chi u + \Big[P\chi\phi', \Lambda_{s-\kappa,k,\varepsilon}\Big]\phi u\right\| \\ &\le \ \left\|\Lambda_{s-\kappa,k,\varepsilon}P\chi u\right\| + C\|u\|_{-N} \ \ (\text{ by (2.14) when } s'=0\ ), \end{aligned}$$

where C is a constant independent of μ,ε.
Combining above three inequalities (2.24),(2.25) and (2.26), we obtain

$$\begin{aligned} &\left\|\Lambda_{s-\beta,k,\varepsilon}(\psi P\chi)_{(\beta)}u\right\| \\ &\le \ \mu\left\|\Lambda_{s,k,\varepsilon}(\psi P\chi)u\right\| + \mu C\sum_{0<|\alpha|<2(s+m+N)}\left\|\Lambda_{s-|\alpha|,k,\varepsilon}(\psi P\chi)_{(\alpha)}u\right\| \\ &\quad + C'\sum_{0<|\alpha|<2(s+m+N)}\left\|\Lambda_{s-|\alpha+\beta|,k,\varepsilon}(\psi P\chi)_{(\alpha+\beta)}u\right\| \\ &\quad + C''\left(\left\|\Lambda_{s-\kappa,k,\varepsilon}P\chi u\right\| + \left\|\Lambda_{s-\kappa,k,\varepsilon}P\chi' u\right\| + \|u\|_{-N}\right), \end{aligned}$$

where C, C' are constants independent of (μ,ε).
By the same way as in the proof of Lemma 2.2, we obtain (2.22). □

We have a corollary by the same way as in the proof of (2.14) from (2.13).

Corollary 2.5 (cf. Corollary 2.4 of [23]) *Suppose that P satisfies Conditions* (II) *and* (V). *Then for any $K \subset\subset \Omega$, any real s, s', any $N > 0$, $0 < \varepsilon < 1$, $k \ge 0$ and any*

$\chi, \chi' \in S_\Psi$ $(\chi \subset\subset \chi')$, there exists a constant $C = C(K, s, s', N, k, \chi, \chi', \psi)$ independent of ε such that

$$\text{(2.27)} \quad \left\| \langle D_x \rangle^{s'} [\psi P \chi, \Lambda_{s,k,\varepsilon}]\, u \right\| \leq C \left(\left\| \Lambda_{s+s',k,\varepsilon} \psi P \chi u \right\| + \left\| \Lambda_{s+s-\kappa,k,\varepsilon} P \chi u \right\| + \left\| \Lambda_{s+s'-\kappa,k,\varepsilon} P \chi' u \right\| + \|u\|_{-N} \right)$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where (κ, ψ) are the same as in Condition (V).

The next lemma plays the most important role in the proof of Proposition 2.1. (see (2.11).) If P satisfies Conditions (I),(II) and the inequality in the next lemma holds, then Conditions (III)–(V) are not necessary to assume.

Lemma 2.6 (cf. Lemma 2.5 of [23]) *Suppose that P satisfies Conditions* (II)–(V). *Then for any $K \subset\subset \Omega$, any real s, any $N > 0$, $0 < \varepsilon < 1$, $k \geq 0$ and any $\chi, \chi' \in S_\Psi$ $(\chi \subset\subset \chi')$, there exists a constant $C = C(K, s, N, k, \chi, \chi')$ independent of ε such that*

$$\text{(2.28)} \quad \left\| \Lambda_{s+\kappa,k,\varepsilon} P \chi \phi u \right\| \leq C \left(\left\| \Lambda_{s+\kappa,k,\varepsilon} \psi P \chi u \right\| + \left\| \Lambda_{s,k,\varepsilon} P \chi u \right\| + \left\| \Lambda_{s,k,\varepsilon} P \chi' u \right\| + \|u\|_{-N} \right)$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where κ and $\phi, \psi \in \boldsymbol{C}_0^\infty(\Omega)$ are the same as in Condition (IV).

Proof of Lemma 2.6. The left hand side of (2.28) is written as follows.

$$\begin{aligned} \left\| \Lambda_{s+\kappa,k,\varepsilon} P \chi \phi u \right\| &= \left\| \langle D_x \rangle^\kappa P \chi \Lambda_{s,k,\varepsilon} \phi u + \langle D_x \rangle^\kappa \left[\Lambda_{s,k,\varepsilon}, P\chi \right] \phi u \right\| \\ &\leq \left\| \langle D_x \rangle^\kappa P \chi \Lambda_{s,k,\varepsilon} \phi u \right\| + \mu \left\| \Lambda_{s+\kappa,k,\varepsilon} P \chi \phi u \right\| + C \|u\|_{-N}, \end{aligned}$$

(by Lemma 2.2 (2.14) with $s' = \kappa$). Taking μ small enough, we have

$$\left\| \Lambda_{s+\kappa,k,\varepsilon} P \chi \phi u \right\| \leq C \left(\left\| \langle D_x \rangle^\kappa P \chi \Lambda_{s,k,\varepsilon} \phi u \right\| + \|u\|_{-N} \right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).$$

In view of the expansion formula (2.18) we have

$$\begin{aligned} &\left\| \langle D_x \rangle^\kappa P \chi \Lambda_{s,k,\varepsilon} \phi u \right\| \\ &\leq \left\| \langle D_x \rangle^\kappa P \chi \phi \Lambda_{s,k,\varepsilon} u \right\| + C \left(\sum_{0 < |\alpha| < 2(s+m+N+\kappa)} \left\| \langle D_x \rangle^\kappa P \chi \phi_{(\alpha)} \Lambda_{s,k,\varepsilon}^{(\alpha)} u \right\| + \|u\|_{-N} \right) \\ &\qquad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \end{aligned}$$

By means of (2.17) with $s = \kappa, k = 0$ we have for $|\alpha| \neq 0$

$$\begin{aligned} &\left\| \langle D_x \rangle^\kappa P \chi \phi_{(\alpha)} \Lambda_{s,k,\varepsilon}^{(\alpha)} u \right\| \\ &\leq C \left(\left\| \langle D_x \rangle^\kappa P \chi \Lambda_{s,k,\varepsilon}^{(\alpha)} u \right\| + \left\| \langle D_x \rangle^\kappa P \chi' \Lambda_{s,k,\varepsilon}^{(\alpha)} u \right\| + \|u\|_{-N} \right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K) \end{aligned}$$

If χ'' is one of χ and χ', we have by (2.13)

$$\begin{aligned}
&\left\|\langle D_x\rangle^\kappa P\chi''\Lambda^{(\alpha)}_{s,k,\varepsilon}u\right\| \\
&\quad = \left\|\langle D_x\rangle^\kappa \Lambda^{(\alpha)}_{s,k,\varepsilon}P\chi''u + \langle D_x\rangle^\kappa\left[P\chi'',\Lambda^{(\alpha)}_{s,k,\varepsilon}\right]u\right\| \\
&\quad \le C\left(\left\|\Lambda_{s+\kappa-|\alpha|,k,\varepsilon}P\chi''u\right\| + \sum_{0<|\beta|<2(s+m+N+\kappa-|\alpha|)}\left\|\Lambda_{s+\kappa-|\alpha+\beta|,k,\varepsilon}(P\chi'')_{(\beta)}u\right\| + \|u\|_{-N}\right) \\
&\quad \le C\left(\left\|\Lambda_{s+\kappa-|\alpha|,k,\varepsilon}P\chi''u\right\| + \|u\|_{-N}\right) \quad \text{for all } u\in \boldsymbol{C}_0^\infty(K).
\end{aligned}$$

Combining these three inequalities we have

$$\begin{aligned}
(2.29)\qquad &\left\|\Lambda_{s+\kappa,k,\varepsilon}P\chi\phi u\right\| \\
&\quad \le C\left(\left\|\langle D_x\rangle^\kappa P\chi\phi\Lambda_{s,k,\varepsilon}u\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi u\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi' u\right\| + \|u\|_{-N}\right).
\end{aligned}$$

Next, as in the beginning of the proof of Lemma 2.2, we take $\varphi,\varphi'\in \boldsymbol{C}_0^\infty(\Omega)$ such that $\varphi\subset\subset\varphi'$, $\varphi\equiv 1$ on K and $\varphi'\equiv 1$ on suppψ. Then we have that for $u\in\boldsymbol{C}_0^\infty(K)$

$$\begin{aligned}
\left\|\langle D_x\rangle^\kappa P\chi\phi\Lambda_{s,k,\varepsilon}u\right\| &= \left\|\langle D_x\rangle^\kappa P\chi\phi\Lambda_{s,k,\varepsilon}\varphi'\varphi u\right\| \\
&\le \left\|\langle D_x\rangle^\kappa P\chi\phi\varphi'\Lambda_{s,k,\varepsilon}\varphi u\right\| + \left\|\langle D_x\rangle^\kappa P\chi\phi\left[\Lambda_{s,k,\varepsilon},\varphi'\right]\varphi u\right\| \\
&\le \left\|\langle D_x\rangle^\kappa P\chi\phi\varphi'\Lambda_{s,k,\varepsilon}\varphi u\right\| + C\|u\|_{-N},
\end{aligned}$$

because $\langle D_x\rangle^\kappa P\chi\phi\left[\Lambda_{s,k,\varepsilon},\varphi'\right]\varphi$ is smoothing. Substituting $\varphi'\Lambda_{s,k,\varepsilon}\varphi u$ for u in (2.4), we obtain

$$\begin{aligned}
&\left\|\langle D_x\rangle^\kappa P\chi\phi\varphi'\Lambda_{s,k,\varepsilon}\varphi u\right\| \\
&\quad \le C\left(\left\|\langle D_x\rangle^\kappa \psi P\chi\varphi'\Lambda_{s,k,\varepsilon}\varphi u\right\| + \left\|P\chi\varphi'\Lambda_{s,k,\varepsilon}\varphi u\right\| + \left\|P\chi'\varphi'\Lambda_{s,k,\varepsilon}\varphi u\right\| + \|u\|_{-N}\right).
\end{aligned}$$

By means of $\psi P\chi\varphi' - \psi P\chi, [P\chi,\varphi']\Lambda_{s,k,\varepsilon}\varphi, [P\chi',\varphi']\Lambda_{s,k,\varepsilon}\varphi \in S^{-\infty}$, the right hand side is smaller than the following:

$$\begin{aligned}
&\le C\left(\left\|\langle D_x\rangle^\kappa\psi P\chi\Lambda_{s,k,\varepsilon}u\right\| + \left\|P\chi\Lambda_{s,k,\varepsilon}u\right\| + \left\|P\chi'\Lambda_{s,k,\varepsilon}u\right\| + \|u\|_{-N}\right) \\
&\le C\Big(\left\|\Lambda_{s+\kappa,k,\varepsilon}\psi P\chi u\right\| + \left\|\langle D_x\rangle^\kappa\left[\psi P\chi,\Lambda_{s,k,\varepsilon}\right]u\right\| \\
&\qquad + \left\|\Lambda_{s,k,\varepsilon}P\chi u\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi' u\right\| + \|u\|_{-N}\Big) \text{ (by (2.14) with } s'=0\text{)} \\
&\le C\left(\left\|\Lambda_{s+\kappa,k,\varepsilon}\psi P\chi u\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi u\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi' u\right\| + \|u\|_{-N}\right) \\
&\quad \text{(by Corollary 2.5 (2.27) with } s'=\kappa\text{).}
\end{aligned}$$

Combining above inequality and (2.29), we have (2.28). □

Remark. Set $k=2(s_0+m+N+\kappa)$ for $s_0>0$. Then, for any $v\in H_{-N}(\boldsymbol{R}^{d+1})\cap\mathcal{E}'(K)$, we have

$$\left\|\Lambda_{s+\kappa,k,\varepsilon}P\chi\phi v\right\| \le C\left(\left\|\Lambda_{s+\kappa,k,\varepsilon}\psi P\chi v\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi v\right\| + \left\|\Lambda_{s,k,\varepsilon}P\chi' v\right\| + \|u\|_{-N}\right),$$

where $s \le s_0$ and C is a constant independent of ε. This can be verified from (2.28) in the same way as in the remark to Lemma 2.5 in [23].

Proof of Proposition 2.1. Let (t_0, x_0) be any fixed point in Ω and let $\psi(t,x) \in \boldsymbol{C}_0^\infty(\Omega)$ be such that $\psi \equiv 1$ in a neighborhood $U(t_0, x_0)$ of (t_0, x_0). Then, for any positive integer l and any $\chi, \chi' \in \mathcal{S}_\Psi(\chi \subset\subset \chi')$, we can find finite sequences $\{\phi_j\}_{j=1}^l, \{\psi_j\}_{j=1}^l \subset \boldsymbol{C}_0^\infty(\Omega), \{\chi_j\}_{j=1}^l \subset \mathcal{S}_\Psi$ satisfying the following two conditions.

condition (a) :

$$\phi_1 \subset\subset \psi_1 \subset\subset \phi_2 \subset\subset \psi_2 \subset\subset \cdots \subset\subset \phi_l \subset\subset \psi_l \subset\subset \psi$$

$$\chi \subset\subset \chi_1 \subset\subset \chi_2 \subset\subset \cdots \subset\subset \chi_l \subset\subset \chi'(= \chi_{l+1})$$

condition (b) : For any $K \subset\subset \Omega$ and any $N > 0$, there exists a constant $C = C(K, N, \chi_j, \chi_{j+1}, \phi_{j'}, \psi_{j'})$ such that

$$\left\| \langle D_x \rangle^\kappa P \chi_j \phi_{j'} u \right\| \le C \left(\left\| \langle D_x \rangle^\kappa \psi_{j'} P \chi_j u \right\| + \|P\chi_j u\| + \|P\chi_{j+1} u\| + \|u\|_{-N} \right) \tag{2.30}$$

$$\text{for all } u \in \boldsymbol{C}_0^\infty(K) \; (\, j, j' = 1, \ldots, l\,),$$

where κ is a positive number depending only on K.

Indeed, we can find these sequences as follows.
First, we take a sequence $\{\chi_j\}_{j=1}^l$ satisfying the condition (a). Next, from Condition (IV), we can take $\widetilde{\phi}_1, \widetilde{\psi}_1 \in \boldsymbol{C}_0^\infty(U(t_0, x_0))$ such that $\widetilde{\phi}_1 \subset\subset \widetilde{\psi}_1, \widetilde{\phi}_1 \equiv 1$ in some neighborhood $V(t_0, x_0)$ of (t_0, x_0) and $(\widetilde{\phi}_1, \widetilde{\psi}_1)$ satisfies (2.4) in place of (ϕ, ψ). Similarly we can take again $\widetilde{\phi}_2, \widetilde{\psi}_2 \in \boldsymbol{C}_0^\infty(V(t_0, x_0))$ such that $\widetilde{\phi}_2 \subset\subset \widetilde{\psi}_2, \widetilde{\phi}_2 \equiv 1$ in some neighborhood $W(t_0, x_0)$ of (t_0, x_0) and $(\widetilde{\phi}_2, \widetilde{\psi}_2)$ satisfies (2.4) in place of (ϕ, ψ). Here we used the arbitrariness of U in Condition (IV). Since $W(t_0, x_0) \subset\subset V(t_0, x_0) \subset\subset U(t_0, x_0)$, $\widetilde{\phi}_2 \subset\subset \widetilde{\psi}_2 \subset\subset \widetilde{\phi}_1 \subset\subset \widetilde{\psi}_1 \subset\subset \psi$. Repeating these steps l times, we have sequences $\{\widetilde{\phi}_j\}_{j=1}^l$ and $\{\widetilde{\psi}_j\}_{j=1}^l \subset \boldsymbol{C}_0^\infty(\Omega)$. Set $\phi_j = \widetilde{\phi}_{l-j+1}, \psi_j = \widetilde{\psi}_{l-j+1} (j = 1, \ldots, l)$. Then, $\{\phi_j\}_{j=1}^l$, $\{\psi_j\}_{j=1}^l$ are sequences we want.

As is well-known, for $u \in \mathcal{D}'(\Omega)$, there exists an $N > 0$ such that $\psi u \in H_{-N}(\boldsymbol{R}^{d+1})$. Let us choose l larger than $2(s+m+N)/\kappa$. By means of Lemma 2.11 of [22], its remark and (I), for any $\phi_1 u = \phi_1 \psi v \in H_{-N}(\boldsymbol{R}^{d+1}) \cap \mathcal{E}'(K)$ (, where K is some neighborhood of supp ψ), we have

$$\left\| \Lambda_{s,k,\varepsilon} \chi_1 \phi_1 u \right\| \le C \left(\left\| \Lambda_{s,k,\varepsilon} P \chi_1 \phi_1 u \right\| + \|\psi u\|_{-N} \right) \tag{2.31}$$

for a constant C independent of ε and $k = 2(s+m+N)$. From (2.30) and the remark to Lemma 2.6 with $k = 2(s+m+N)$, we have for any $s' \le s$,

$$
\begin{aligned}
&\left\|\Lambda_{s',k,\varepsilon}P\chi_j\phi_{j'}u\right\| \\
&\quad = \left\|\Lambda_{s',k,\varepsilon}P\chi_j\phi_{j'}\phi_{j'+1}u\right\| \\
&\quad \leq C\Big(\left\|\Lambda_{s',k,\varepsilon}\psi_{j'}P\chi_j\phi_{j'+1}u\right\| \\
&\qquad\qquad + \left\|\Lambda_{s'-\kappa,k,\varepsilon}P\chi_j\phi_{j'+1}u\right\| + \left\|\Lambda_{s'-\kappa,k,\varepsilon}P\chi_{j+1}\phi_{j'+1}u\right\| + \|\psi u\|_{-N}\Big) \\
&\quad \leq C'\left(\left\|\Lambda_{s',k,\varepsilon}\psi_{j'}P\chi_j u\right\| + \left\|\Lambda_{s'-\kappa,k,\varepsilon}P\chi_j\phi_{j'+1}u\right\| + \left\|\Lambda_{s'-\kappa,k,\varepsilon}P\chi_{j+1}\phi_{j'+1}u\right\| + \|\psi u\|_{-N}\right).
\end{aligned}
$$

Because $\psi_{j'}P\chi_j\phi_{j'+1} - \psi_{j'}P\chi_j \in \mathcal{S}^{-\infty}$ and $\phi_{j'}u = \phi_{j'}\phi_{j'+1}u$. From (2.31) and above estimate, we have

$$
\begin{aligned}
&\left\|\Lambda_{s,k,\varepsilon}\chi_1\phi_1 u\right\| \\
&\quad \leq C\left(\sum_{j,j'=1}^{l}\left\|\Lambda_{s-\kappa(j'-1),k,\varepsilon}\psi_{j'}P\chi_j u\right\| + \sum_{j=1}^{l+1}\left\|\Lambda_{s-\kappa l,k,\varepsilon}P\chi_j\psi u\right\| + \|\psi u\|_{-N}\right),
\end{aligned}
$$

where C is a constant independent of ε.
Since $\|\Lambda_{s-\kappa(j'-1),k,\varepsilon}\psi_{j'}P\chi_j u\| \leq \|\langle D_x\rangle^{s-\kappa(j'-1)}\psi_{j'}P\chi_j u\|$ $(j, j' = 1, \dots, l)$ and the family $\{\Lambda_{s-\kappa l,k,\varepsilon}P\chi_j\psi\}_{0<\varepsilon<1}$ is bounded in $\mathcal{S}^{-N}_{1/2,0}$ for all j $(j = 1, \dots, l)$, we obtain

$$
\left\|\Lambda_{s,k,\varepsilon}\chi_1\phi_1 u\right\| \leq C\left(\sum_{j,j'=1}^{l}\left\|\langle D_x\rangle^{s-\kappa(j'-1)}\psi_{j'}P\chi_j u\right\| + \|\psi u\|_{-N}\right).
$$

Since the right hand side is bounded uniformly with respect to ε from the hypothesis of the proposition, we finally obtain by letting ε tend to 0

$$
\left\|\langle D_x\rangle^s\chi_1\phi_1 u\right\| \leq C\left(\sum_{j,j'=1}^{l}\left\|\langle D_x\rangle^s\psi_{j'}P\chi_j u\right\| + \|\psi u\|_{-N}\right).
$$

This shows that $\langle D_x\rangle^s\chi_1\phi_1 u \in L^2(\boldsymbol{R}^{d+1})$. Let ψ be any fixed element in $\boldsymbol{C}_0^\infty(\Omega)$. Since (t_0, x_0) is arbitrary point in Ω, we see that $\langle D_x\rangle^s\chi_1\psi u \in L^2(\boldsymbol{R}^{d+1})$. This completes the proof of Proposition 2.1. □

3 The basic inequalities

From now on, we investigate the operator

$$P = \partial_t + f(t,x) \sum_{j,k=1}^{n} a_{jk}(t,x) L_j L_k$$

specified in Theorem **A**. In view of Proposition 2.1, for the proof of Proposition 1.3 (see §6), it suffices to show that P satisfies Conditions (I)–(V) in §2. We need many preliminary inequalities. Let us sketch how they are used in the following sections.

1. We use inequalities (3.1), (3.2) and (3.3) in Lemma 3.1 below to show that P satisfies Conditions (I), (II), (IV) and (V). More precisely, they are used to evaluate $fL_j\chi u$ $(j = 1, \ldots, n)$ which are principal terms of $\langle D_x\rangle^{-1} (P\chi)_{(\beta)} u$ $(\beta = (0, \beta'), |\beta'| = 1)$.

2. Inequalities (3.12) and (3.13) in Lemma 3.3 are used to have the inequality

$$\|\chi u\| \leq \mu \|P\chi u\| + C_{\mu,N}\|u\|_{-N} \quad \text{for every } \mu, N > 0 \text{ and every } \chi \in \mathscr{S}_\Psi$$

   (see Proposition 5.1 in §5). P satisfies Condition (I) due to this inequality.

3. We use the inequality (3.14) in Lemma 3.4 only for the proof of Lemma 3.5.

4. Inequality (3.16) in Lemma 3.5 is an a priori estimate with weight. Combining (3.12) and (3.13) with (3.16), we can show that P satisfies Conditions (I), (IV) and (V). More precisely, these inequalities are used to prove Proposition 4.1. (see Lemma 4.5 in §4.)

Let Ω be a bounded domain in $\boldsymbol{R}^{d+1}$ and let π_t be the projection from Ω to $\boldsymbol{R}_t$ $(\pi_t((t,x)) = t)$. **If $\boldsymbol{f}$ does not depend on $\boldsymbol{x}$,** let $f(t) = f(t,x)$. (We set always $f(t) = f(t,x)$ in this case.) Then, from the hypothesis (1°-α), there exists at most one point where $f(t)$ changes sign. Without loss of generality, we may suppose that the point is $t = 0$.

Definition
We say that (f, Ω) is of **type ($\boldsymbol{\alpha}$-1)** if $f(t,x)$ does not depend on x and if $\pi_t(\Omega)$ contains $t = 0$.
We say that (f, Ω) is of **type ($\boldsymbol{\alpha}$-2)** if $f(t,x)$ does not depend on x and if $\pi_t(\Omega)$ does not contain $t = 0$.
We say that (f, Ω) is of **type ($\boldsymbol{\beta}$) if $\boldsymbol{f}$ depends on $\boldsymbol{x}$.**

We denote by $(\,\cdot\,,\,\cdot\,)$ the ordinary scalar product on $L^2(\boldsymbol{R}^{d+1})$.

Lemma 3.1 (i) *If (f,Ω) is of type (α-1), for any $K \subset\subset \Omega$ and any $\rho > 3/4$, there exists a constant C depending only on (K,ρ) such that*

$$\sum_{j=1}^{n} \||f|^{\rho} L_j u\|^2 \leq C\left\{\left|\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\right| + \|u\|^2\right\} \tag{3.1}$$

for all $u \in \boldsymbol{C}_0^{\infty}(K)$.

(ii) *If (f,Ω) is of type (α-2) or of type (β), for any $K \subset\subset \Omega$, there exists a constant C depending only on K such that*

$$\sum_{j=1}^{n} \left\||f|^{1/2} L_j u\right\|^2 \leq C\left\{|\mathrm{Re}\,(Pu,u)| + \|u\|^2\right\}, \tag{3.2}$$

for all $u \in \boldsymbol{C}_0^{\infty}(K)$.

(iii) *If (f,Ω) is of type (α-2) or of type (β), for any $K \subset\subset \Omega$, there exists a constant C depending only on K such that*

$$\sum_{j=1}^{n} \|fL_j u\|^2 \leq C\left\{|\mathrm{Re}\,(Pu, fu)| + \sum_{k=1}^{d}\|(\partial_{x_k} f)u\|^2 + \|fu\|^2 + \left|\left((\partial_t f)u, u\right)\right|\right\} \tag{3.3}$$

for all $u \in \boldsymbol{C}_0^{\infty}(K)$.

Remark. In (i), the lower bound 3/4 of the exponent ρ of $|f|$ can not be replaced by smaller one in general. This is because we need for the proof of Lemma 3.1 the estimate $\mathrm{Re}(\partial_t u, (\operatorname{sgn} t)|f|^{2\rho-1}u) \leq C\|u\|^2$. If $2\rho - 1 \leq 1/2$, then $(\operatorname{sgn} t)|f|^{2\rho-1}$ is not of class $\boldsymbol{C}^1(\boldsymbol{R})$ in general even if $(df/dt)(0) = 0$. For example, if we take

$$f(t) = \begin{cases} (\operatorname{sgn} t)\left(1 + \sin\dfrac{\pi}{2t}\right)e^{-\frac{1}{|t|}}, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

then $(\operatorname{sgn} t)|f(t)|^{1/2} \notin \boldsymbol{C}^1(\boldsymbol{R})$, because $|f(t)|^{1/2}$ is not differentiable at $t = 1/(3+4l)$ for every $l \in \boldsymbol{Z}$. Since the set $\{1/(3+4l)\}_{l\in\boldsymbol{Z}}$ accumulates at the origin, $(\operatorname{sgn} t)|f(t)|^{1/2} \notin \boldsymbol{C}^1(\pi_t(\Omega))$ for any Ω satisfying $0 \in \pi_t(\Omega)$.

On the other hand, the exponent 1/2 in (ii) can not be replaced by smaller one. In (iii), the second, the third and the forth terms on the right hand side should not be replaced by $\|u\|^2$. The reason will be seen in §4. In the following sections, we study the case where the exponent ρ is smaller than 1. This is because, as is mentioned in the Introduction, the smaller ρ, the better (3.1).

Proof of Lemma 3.1. Let us begin with (i). We define $E(u)$ to be

$$E(u) = -\int (\operatorname{sgn} t)\,|f(t)|^{2\rho-1}\,\partial_t\left(|u|^2\right)dt\,dx. \tag{3.4}$$

Then we can rewrite $\mathrm{Re}(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u)$ as follows.

$$
\begin{aligned}
&\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\\
&\quad = \mathrm{Re}\left(\partial_t u, (\operatorname{sgn} t)|f|^{2\rho-1}u\right) + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}a_{jk}L_jL_ku, u\right)\\
&\quad = \frac{1}{2}\int (\operatorname{sgn} t)|f|^{2\rho-1}\partial_t\left(|u|^2\right)dt\,dx\\
&\qquad + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}a_{jk}L_ku, L_j^*u\right) + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}\left[a_{jk}, L_j\right]L_ku, u\right),
\end{aligned}
$$

where L_j^* is the formal adjoint of L_j. Furthermore, we have

$$
\begin{aligned}
&\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\\
&\quad = -\mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}a_{jk}L_ku, L_ju\right) + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}a_{jk}L_ku, (L_j+L_j^*)u\right)\\
&\qquad + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}\left[a_{jk}, L_j\right]L_ku, u\right) - \left(-\frac{1}{2}\int (\operatorname{sgn} t)|f|^{2\rho-1}\partial_t|u|^2dtdx\right)\\
&\quad = -\frac{1}{2}\sum_{j,k=1}^{n}\left(|f|^{2\rho}\left\{a_{jk}+\overline{a_{kj}}\right\}L_ku, L_ju\right) + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}a_{jk}L_ku, (L_j+L_j^*)u\right)\\
&\qquad + \mathrm{Re}\sum_{j,k=1}^{n}\left(|f|^{2\rho}\left[a_{jk}, L_j\right]L_ku, u\right) - \frac{1}{2}E(u).
\end{aligned}
$$

Since $L_j+L_j^*$ reduces to a multiplication by a smooth function, we have by using Condition (2°) and Schwarz' inequality

$$
\begin{aligned}
&\frac{1}{2}\delta(K)\sum_{j=1}^{n}\||f|^{\rho}L_ju\|^2 + \frac{1}{2}E(u)\\
&\quad \le \left|\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\right| + \varepsilon\sum_{j=1}^{n}\||f|^{\rho}L_ju\|^2 + C_{\varepsilon,K,\rho}\|u\|^2,
\end{aligned}
$$

where ε is any positive constant, $C_{\varepsilon,K,\rho}$ is a constant depending only on (ε, K, ρ) and $\delta(K)$ is the number introduced in Condition (2°). Taking $\varepsilon = \delta(K)/4$, we have

$$
\frac{1}{4}\delta(K)\sum_{j=1}^{n}\||f|^{\rho}L_ju\|^2 + \frac{1}{2}E(u) \le \left|\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\right| + C_{K,\rho}\|u\|^2. \tag{3.5}
$$

Next, we introduce the following condition for $E(u)$.

- **(F)** $E(u)$ can be represented as $E(u) = E_1(u) + E_2(u)$, where $E_1(u) \ge 0$ and $|E_2(u)| \le C'_K\|u\|^2$ for any $u \in \boldsymbol{C}_0^\infty(K)$. ($C'_K$ depends only on K)

If (F) is satisfied, it follows from (3.5) that

$$
\frac{1}{4}\delta(K)\sum_{j=1}^{n}\||f|^{\rho}L_ju\|^2 \le \frac{1}{4}\delta(K)\sum_{j=1}^{n}\||f|^{\rho}L_ju\|^2 + \frac{1}{2}\left(E(u) - E_2(u)\right)
$$

$$\leq \left|\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\right| + C_{K,\rho}\left\|u\right\|^2 - \frac{1}{2}E_2(u)$$
$$\leq \left|\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\rho-1}u\right)\right| + (C_{K,\rho} + C'_K/2)\left\|u\right\|^2.$$

This implies (3.1).

Now let us show that (F) is satisfied. The proof is divided into two cases where $f'(0)$ is equal to zero or not.

Case 1 $f'(0) = 0$

If $(\operatorname{sgn} t)|f(t)|^{2\rho-1} \in \boldsymbol{C}^1(\pi_t(\Omega))$, then $|E(u)| \leq C'_K\|u\|^2$ for all $u \in \boldsymbol{C}_0^\infty(K)$ by integration by parts. So (F) is satisfied if we set $E_1(u) \equiv 0$ and $E_2(u) = E(u)$. Therefore we have only to show that $(\operatorname{sgn} t)|f(t)|^{2\rho-1} \in \boldsymbol{C}^1(\pi_t(\Omega))$. In this case, $f(0) = f'(0) = f''(0) = 0$ by the Taylor expansion. Since $f(t)$ does not change sign except at $t = 0$, $|f(t)|$ is smooth except at $t = 0$. Hence $|f(t)|$ belongs to $\boldsymbol{C}^2(\boldsymbol{R})$. The next lemma allows us to prove that $(\operatorname{sgn} t)|f(t)|^{2\rho-1} \in \boldsymbol{C}^1(\pi_t(\Omega))$.

Lemma 3.2 *Let $g(t) \in \boldsymbol{C}^2(\boldsymbol{R})$ be a non-negative function. Then, $g(t)^\lambda$ belongs to $\boldsymbol{C}^1(\boldsymbol{R})$ for every $\lambda > 1/2$.*

Proof of Lemma 3.2. Let Z_g be the set of zeros of $g(t)$. $g(t)^\lambda$ is differentiable at every point of $\boldsymbol{R}$ and

$$\frac{d}{dt}\left\{g(t)^\lambda\right\} = \begin{cases} 0 & \text{if } t \in Z_g, \\ \lambda g(t)^{\lambda-1}g'(t) & \text{otherwise.} \end{cases} \tag{3.6}$$

Therefore, we shall show that the derivative of $g(t)^\lambda$ is continuous at every point t_0 of Z_g. As is well-known, for non-negative $\boldsymbol{C}^2$-function $g(t)$, there exist a neighborhood U of t_0 and a constant C such that

$$\left|\frac{d}{dt}g(t)\right| \leq C\sqrt{g(t)} \quad \text{in } U. \tag{3.7}$$

(cf. Lemma 1.7.1 in §7 of [28]) Consequently, we have

$$\left|\frac{d}{dt}g(t)^\lambda\right| \leq \lambda\left|g(t)^{\lambda-1}g'(t)\right| \leq C\lambda\left|g(t)\right|^{\lambda-\frac{1}{2}} \text{ in } U,$$

which implies, together with (3.6), $g(t)^\lambda \in \boldsymbol{C}^1(\boldsymbol{R})$. □

Now we apply this to $g(t) = |f(t)|$ and $\lambda = 2\rho - 1 > 1/2$. Then we have $|f(t)|^{2\rho-1} \in \boldsymbol{C}^1(\boldsymbol{R})$ and $\frac{d}{dt}\{|f(t)|^{2\rho-1}\} \to 0$ as $t \to 0$. Therefore, $(\operatorname{sgn} t)|f(t)|^{2\rho-1} \in \boldsymbol{C}^1(\boldsymbol{R})$.

Case 2 $f'(0) \neq 0$

In this case, set $h(t) = f(t)/t$ for $t \neq 0$ and $h(0) = f'(0)$. Then $h(t)$ is non-negative and

$|h(t)|^{2\rho-1} \in \boldsymbol{C}^1(\boldsymbol{R})$ by Lemma 3.2. So we have $(\operatorname{sgn} t)|f(t)|^{2\rho-1} \in \boldsymbol{C}^1(\boldsymbol{R} \setminus \{0\}) \cap \boldsymbol{C}(\boldsymbol{R})$. We rewrite $E(u)$ as follows.

$$E(u) = -\lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} \left\{ (\operatorname{sgn} t)|f|^{2\rho-1} \right\} \partial_t \left(|u|^2 \right) dtdx$$

by Lebesgue's convergence theorem. Then we have

$$\begin{aligned} E(u) &= \lim_{\varepsilon \to 0} \Big\{ \int_{|t|>\varepsilon} |u|^2 \partial_t \left((\operatorname{sgn} t)|t|^{2\rho-1}|h(t)|^{2\rho-1} \right) dtdx \\ &\qquad + \int_{\boldsymbol{R}^d_x} \left(|f(\varepsilon)|^{2\rho-1}|u(\varepsilon,x)|^2 + |f(-\varepsilon)|^{2\rho-1}|u(-\varepsilon,x)|^2 \right) dx \Big\} \\ &= \lim_{\varepsilon \to 0} \Big\{ (2\rho-1) \int_{|t|>\varepsilon} |t|^{2\rho-2}|h(t)|^{2\rho-1}|u|^2 \, dtdx \\ &\qquad + \int_{|t|>\varepsilon} (\operatorname{sgn} t)|t|^{2\rho-1}|u|^2 \partial_t \left(|h(t)|^{2\rho-1} \right) dtdx \Big\}, \end{aligned}$$

because $|f(\pm\varepsilon)| = O(\varepsilon^{2\rho-1})$ and $2\rho - 1 > 1/2$. Furthermore, since $|t|^{2\rho-2}$ is locally integrable,

$$\begin{aligned} E(u) &= (2\rho-1) \int |t|^{2\rho-2}|h(t)|^{2\rho-1}|u|^2 dtdx + \int (\operatorname{sgn} t)|t|^{2\rho-1}|u|^2 \partial_t \left\{ |h(t)|^{2\rho-1} \right\} dtdx \\ &= H_1(u) + H_2(u). \end{aligned}$$

$(\operatorname{sgn} t)|t|^{2\rho-1}\partial_t \{|h(t)|^{2\rho-1}\}$ is bounded in $\pi_t(K)$, so there exists a constant C'_K such that $|H_2(u)| \le C'_K \|u\|^2$ for all $u \in \boldsymbol{C}_0^\infty(K)$. Moreover $H_1(u)$ is non-negative because the integrand is non-negative and $2\rho - 1 > 1/2$. So (F) is satisfied if we set $E_1(u) = H_1(u)$, $E_2(u) = H_2(u)$. This completes the proof of (i).

Next we shall prove (ii). (The proof of (iii) is done in a similar way, so we omit it.) Without loss of generality, we may suppose that $f(t,x)$ is non-negative. We rewrite $\operatorname{Re}(Pu,u)$ as follows:

$$\begin{aligned} &\operatorname{Re}(Pu,u) \\ &= \operatorname{Re}(\partial_t u, u) + \operatorname{Re} \sum_{j,k=1}^n \left(f(t,x)a_{jk}L_jL_ku, u \right) \\ &= \operatorname{Re} \sum_{j,k=1}^n \left(f(t,x)a_{jk}L_ku, L_j^*u \right) + \operatorname{Re} \sum_{j,k=1}^n \left([f(t,x)a_{jk}, L_j]\, L_ku, u \right) \\ &= -\operatorname{Re} \sum_{j,k=1}^n \left(f(t,x)a_{jk}L_ku, L_ju \right) + \operatorname{Re} \sum_{j,k=1}^n \left(f(t,x)a_{jk}L_ku, (L_j + L_j^*)u \right) \\ &\qquad + \operatorname{Re} \sum_{j,k=1}^n \left([f(t,x)a_{jk}, L_j]\, L_ku, u \right) \end{aligned}$$

and rewrite $[f(t,x)a_{jk}, L_j]$ as

$$\left[f(t,x)a_{jk}, L_j \right] = f(t,x)b_{jk0} + \sum_{l=1}^d b_{jkl}(\partial_{x_l} f(t,x)), \tag{3.8}$$

where b_{jkl} is a smooth function. Then we have

(3.9) $\mathrm{Re}\,(Pu,u)$

$$= -\frac{1}{2}\sum_{j,k=1}^{n}\Big(f(t,x)\left\{a_{jk}+\overline{a_{kj}}\right\}L_k u, L_j u\Big) + \mathrm{Re}\sum_{j,k=1}^{n}\Big(f(t,x)a_{jk}L_k u, (L_j+L_j^*)u\Big)$$
$$+\,\mathrm{Re}\sum_{j,k=1}^{n}\Big(f(t,x)L_k u, \overline{b_{jk0}}u\Big) + \mathrm{Re}\sum_{j,k=1}^{n}\sum_{l=1}^{d}\Big((\partial_{x_l}f(t,x))L_k u, \overline{b_{jkl}}u\Big).$$

Note that the forth term on the right hand side does not appear in the case where (f,Ω) is of type (α-2). Since $f(t,x)$ does not change sign in K, we have

$$\sum_{l=1}^{d}|\partial_{x_l}f(t,x)|^2 \le C(K)f(t,x) \quad \text{on } K. \tag{3.10}$$

(This is a generalization of (3.7) to $(d+1)$-dimensional case. (see also [28].)) Then we have

$$\sum_{j=1}^{n}\sum_{l=1}^{d}\|(\partial_{x_l}f(t,x))L_j u\|^2 \le C(K)\sum_{j=1}^{n}\left\|(f(t,x))^{1/2}L_j u\right\|^2. \tag{3.11}$$

Since $L_j + L_j^*$ reduces to a multiplication by a smooth function, combining (3.9) and (3.10), using Condition (2°) and Schwarz' inequality, we have

$$\frac{1}{2}\,\delta(K)\sum_{j=1}^{n}\left\||f|^{1/2}L_j u\right\|^2 \le |\mathrm{Re}\,(Pu,u)| + \varepsilon\sum_{j=1}^{n}\left\||f|^{1/2}L_j u\right\|^2 + C(K,\varepsilon)\,\|u\|^2,$$

where ε is any positive constant, $C(K,\varepsilon)$ is a constant depending only on (K,ε) and $\delta(K)$ is the number introduced in Condition (2°). Taking $\varepsilon = \delta(K)/4$, we obtain (3.2). The proof is completed. □

Lemma 3.3 (i) *If (f,Ω) is of type (α-1), for any $K \subset\subset \Omega$, there exists a constant C depending only on K such that*

$$\|u\|^2 \le C\left(|\mathrm{Re}\,(Pu,tu)| + \|tu\|^2\right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{3.12}$$

(ii) *If (f,Ω) is of type (α-2) or (β), for any $K \subset\subset \Omega$ and any $a \in \boldsymbol{R}$, there exists a constant C such that*

$$\|u\|^2 \le C\left(\left|\mathrm{Re}\Big(Pu,(t-a)u\Big)\right| + \sum_{j=1}^{n}\left\||(t-a)f(t,x)|^{1/2}L_j u\right\|^2\right) \tag{3.13}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where C depends only on K and the diameter of $\{a\}\cup\pi_t(K)$.

Proof of Lemma 3.3. First, by means of (3.8), we rewrite $\mathrm{Re}\,(Pu,(t-a)u)$ as

$$
\begin{aligned}
&\mathrm{Re}\,(Pu,(t-a)u)\\
= & -\frac{1}{2}\|u\|^2 - \mathrm{Re}\sum_{j,k=1}^{n}\Big((t-a)f(t,x)a_{jk}L_k u, L_j u\Big)\\
&+\mathrm{Re}\sum_{j,k=1}^{n}\Big((t-a)f(t,x)a_{jk}L_k u, \Big(L_j+L_j^*\Big)u\Big)\\
&+\mathrm{Re}\sum_{j,k=1}^{n}\Big(\left[f(t,x)a_{jk},L_j\right]L_k u,(t-a)u\Big)\\
= & -\frac{1}{2}\|u\|^2 - \mathrm{Re}\sum_{j,k=1}^{n}\Big((t-a)f(t,x)a_{jk}L_k u, L_j u\Big)\\
&+\mathrm{Re}\sum_{j,k=1}^{n}\left((t-a)f(t,x)L_k u, \overline{a_{jk}}\Big(L_j+L_j^*\Big)u\right)\\
&+\mathrm{Re}\sum_{j,k=1}^{n}\Big((t-a)f(t,x)L_k u,\overline{b_{jk0}}u\Big)+\mathrm{Re}\sum_{j,k=1}^{n}\sum_{l=1}^{d}\Big((t-a)(\partial_{x_l}f(t,x))L_k u,\overline{b_{jkl}}u\Big).
\end{aligned}
$$

In the case (i), we have by setting $a=0$

$$
\frac{1}{2}\|u\|^2+\frac{\delta(K)}{2}\sum_{j=1}^{n}\left\||tf|^{1/2}L_j u\right\|^2 \le |\mathrm{Re}\,(Pu,tu)|+\varepsilon\sum_{j=1}^{n}\left\||tf|^{1/2}L_j u\right\|^2+C_{\varepsilon,K}\left\||tf|^{1/2}u\right\|^2
$$

due to $tf(t)\ge 0$. $|tf(t)|^{1/2}/t$ is bounded in $\pi_t(K)$, so we obtain (3.12) if we set $\varepsilon=\delta(K)/2$.

In the case (ii), we have from (3.10)

$$
\sum_{j=1}^{n}\sum_{l=1}^{d}\left\|(t-a)(\partial_{x_l}f(t,x))L_j u\right\|^2 \le C(K,a)\sum_{j=1}^{n}\left\||(t-a)f(t,x)|^{1/2}L_j u\right\|^2,
$$

where $C(K,a)$ is a constant depending only on K and the diameter of $\{a\}\cup\pi_t(K)$. Combining this inequality with the above equality, we have by Schwarz' inequality

$$
\begin{aligned}
\|u\|^2 \;\le\; & 2\left|\mathrm{Re}\Big(Pu,(t-a)u\Big)\right|+\varepsilon\|u\|^2\\
&+C(K,a,\varepsilon)\sum_{j=1}^{n}\left\||(t-a)f(t,x)|^{1/2}L_j u\right\|^2.
\end{aligned}
$$

We finally obtain (3.13) by taking $\varepsilon=1/2$. □

We prepare a notation for the next lemma. Given a multi-index $J=(j_1,\dots,j_l)$ with $l\ge 1$, set $\widetilde{J}=(\,j_1,\dots,j_{l-1}\,)$ and define vector fields R_J inductively by

$$
R_J=L_{j_1}\quad(l=1)\,,\quad R_J=[R_{\tilde{J}},L_{j_l}]\quad(l\ge 2).
$$

For example,

$$
\begin{aligned}
&R_1=L_1\\
R_{12}&=[R_1,L_2]=[L_1,L_2]\\
R_{123}&=[R_{12},L_3]=\left[[L_1,L_2],L_3\right].
\end{aligned}
$$

The number l is said to be the length of J and denoted by $\|J\|$.

Lemma 3.4 (cf. Lemma 3.2 in Chapter 4 of [14]) *For any $K \subset\subset \Omega$, any multi-index J, there exists a constant C depending only on (K, J) such that*

$$\left\| \langle D_x \rangle^{2^{1-\|J\|}-1} f^{\|J\|} R_J u \right\|^2 \leq C \left(\|Pu\|^2 + \|u\|^2 \right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{3.14}$$

Proof of Lemma 3.4. We proceed by induction with respect to the length l of J. When $l = 1$, we get (3.14) from (3.1) and (3.2). Now we assume that $l \geq 2$ and write $J = (j_1, \ldots, j_l)$, $\widetilde{J} = (j_1, \ldots, j_{l-1})$. If we prove

$$\begin{aligned} &\left\| \langle D_x \rangle^{2^{1-\|J\|}-1} f^{\|J\|} R_J u \right\|^2 \\ &\qquad \leq C \left(\sum_{j=1}^{n} \|f L_j u\|^2 + \left\| \langle D_x \rangle^{2^{1-\|\tilde{J}\|}-1} f^{\|\tilde{J}\|} R_{\tilde{J}} u \right\|^2 + \|u\|^2 \right) \end{aligned} \tag{3.15}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$ and apply the induction hypothesis to $\widetilde{J}$, then (3.14) will follow from (3.1) and (3.2). Set $T_J = \langle D_x \rangle^{2^{2-\|J\|}-2} f^{\|J\|} R_J$ and we write

$$\begin{aligned} \left\| \langle D_x \rangle^{2^{1-\|J\|}-1} f^{\|J\|} R_J u \right\|^2 &= \left(f^{\|J\|} \left[L_{j_l}, R_{\tilde{J}} \right] u, \langle D_x \rangle^{2^{2-\|J\|}-2} f^{\|J\|} R_J u \right) \\ &= \left(L_{j_l} f^{\|J\|} R_{\tilde{J}} u, T_J u \right) + \left(\left[f^{\|J\|}, L_{j_l} \right] R_{\tilde{J}} u, T_J u \right) \\ &\qquad - \left(R_{\tilde{J}} f^{\|J\|} L_{j_l} u, T_J u \right) - \left(\left[f^{\|J\|}, R_{\tilde{J}} \right] L_{j_l} u, T_J u \right). \end{aligned}$$

Since $\left[f^{\|J\|}, L_{j_l} \right]$, $\left[f^{\|J\|}, R_{\tilde{J}} \right]$ are rewritten as

$$\left[f^{\|J\|}, L_{j_l} \right] = U_1 f^{\|\tilde{J}\|} \quad \text{and} \quad \left[f^{\|J\|}, R_{\tilde{J}} \right] = U_2 f,$$

where U_1 and U_2 are multiplications by functions of class $\boldsymbol{C}^\infty$. Thus, we have

$$\begin{aligned} &\left\| \langle D_x \rangle^{2^{1-\|J\|}-1} f^{\|J\|} R_J u \right\|^2 \\ &= \left(f(t,x)^{\|\tilde{J}\|} R_{\tilde{J}} u, T_J f L_{j_l}^* u \right) + \left(f^{\|\tilde{J}\|} R_{\tilde{J}} u, \left[f L_{j_l}^*, T_J \right] u \right) + \left(f^{\|\tilde{J}\|} R_{\tilde{J}} u, U_1 T_J u \right) \\ &\qquad - \left(f L_{j_l} u, T_J f^{\|\tilde{J}\|} R_{\tilde{J}}^* u \right) - \left(f L_{j_l} u, \left[f^{\|\tilde{J}\|} R_{\tilde{J}}^*, T_J \right] u \right) - \left(f L_{j_l} u, U_2 T_J u \right). \end{aligned}$$

$\left[f L_{j_l}^*, T_J \right]$, $\left[f^{\|\tilde{J}\|} R_{\tilde{J}}^*, T_J \right]$ and T_J belong to the symbol class $S^{2^{1-\|\tilde{J}\|}-1}(\boldsymbol{R}_x^d)$ with parameter t, and they are bounded in $S^{2^{1-\|\tilde{J}\|}-1}(\boldsymbol{R}_x^d)$ when t runs over a compact set of $\boldsymbol{R}$. Then, using $L_{j_l} + L_{j_l}^* \equiv 0$ and $R_{\tilde{J}} + R_{\tilde{J}}^* \equiv 0 \mod S_{1,0}^0$, we have (3.15) by Schwarz' inequality. □

Remark. If (f, Ω) is of type (α-1) (resp. (α-2)), Lemma 3.4 holds even if we replace $f(t,x)^{\|J\|}$ by $|f(t)|^{\rho\|J\|}$ (resp. $|f(t)|^{\|J\|/2}$), where ρ is a constant satisfying $\rho > 3/4$.

Lemma 3.5 (cf. Theorem 3.4 in Chapter 4 of [14]) *For any $K \subset\subset \Omega$, there exist a positive integer $\widetilde{k}$ depending only on K and a constant C depending only on K such that*

$$\left\| \langle D_x \rangle^{2^{1-\widetilde{k}}} f^{\widetilde{k}} u \right\|^2 \le C \left(\|Pu\|^2 + \|u\|^2 \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{3.16}$$

Proof of Lemma 3.5. From Condition (3°), we have the following:
For any $K \subset\subset \Omega$, there exist a positive integer $\widetilde{k}$ and functions $b_{jJ}(x), c_j(x) \subset \boldsymbol{C}^\infty(K)$ such that

$$\partial_{x_l} = \sum_{\|J\| \le \widetilde{k}} b_{lJ}(x) R_J(x, D_x) + c_l(x) \quad (\, l = 1, \dots, d\,).$$

Then we write

$$\begin{aligned} &\left\| \langle D_x \rangle^{2^{1-\widetilde{k}}} f^{\widetilde{k}} u \right\|^2 \\ &= \left\| \langle D_x \rangle^{2^{1-\widetilde{k}}-1} \langle D_x \rangle f^{\widetilde{k}} u \right\|^2 \le C \left(\sum_{l=1}^d \left\| \langle D_x \rangle^{2^{1-\widetilde{k}}-1} f^{\widetilde{k}} \partial_{x_l} u \right\|^2 + \left\| \langle D_x \rangle^{2^{1-\widetilde{k}}-1} u \right\|^2 \right) \\ &\le C \left(\sum_{l=1}^d \sum_{\|J\| \le \widetilde{k}} \left\| \langle D_x \rangle^{2^{1-\widetilde{k}}-1} f^{\widetilde{k}} \left(b_{lJ} R_J + c_l \right) u \right\|^2 + \|u\|^2 \right) \ (\text{ since } 2^{1-\widetilde{k}} - 1 \le 0 \ ) \\ &\le C \left(\sum_{l=1}^d \sum_{\|J\| \le \widetilde{k}} \left\| \langle D_x \rangle^{2^{1-\widetilde{k}}-1} f^{\widetilde{k}} R_J u \right\|^2 + \|u\|^2 \right) \le C \left(\|Pu\|^2 + \|u\|^2 \right) \ (\text{ by (3.14) }) . \end{aligned}$$

The proof is finished. □

Remark. If (f, Ω) is of type (α-1) (resp. (α-2)), Lemma 3.5 holds even if we replace $f(t,x)^{\widetilde{k}}$ to $|f(t)|^{\rho \widetilde{k}}$ (resp. $|f(t)|^{\widetilde{k}/2}$), where ρ is a constant satisfying $\rho > 3/4$.

4 The basic a priori estimate with weight

As is mentioned in the Introduction, we need an estimate for L_j with weight for the proof of Theorem **A**. In this section, we shall prove Proposition 4.1 below. Inequality (4.1) there is an estimate for L_j with the weight $\langle\xi\rangle^\kappa f$. By making use of this proposition, we will verify that P in question satisfies Conditions (IV) and (V) in §2. Furthermore, we will prove Proposition 5.1 in the next section by using the corollary to Proposition 4.1. Proposition 5.1 will guarantees Condition (I) for P to hold. As is stated above, Proposition 5.1 plays an important role in the proof of Theorem **A**.

Proposition 4.1 *For any $K \subset\subset \Omega$, any $N > 0$ and any $\chi \in S_\Psi$, there exist a positive number $\kappa = \kappa(K)$ and a positive constant $C = C(K, N, \chi)$ such that*

$$\begin{aligned}(4.1)\qquad & \sum_{j=1}^{n}\left\|\langle D_x\rangle^\kappa f L_j \chi u\right\|^2 + \left\|\langle D_x\rangle^\kappa f \chi u\right\|^2 + \sum_{k=1}^{d}\left\|\langle D_x\rangle^\kappa (\partial_{x_k} f)\chi u\right\|^2 \\ & \leq\ C\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).\end{aligned}$$

We have immediately the following corollary.

Corollary 4.2 *For any $K \subset\subset \Omega$, any $N > 0$, any $\chi \in S_\Psi$ and any $\mu > 0$, there exists a constant $C = C(K, N, \chi, \mu)$ such that*

$$(4.2)\ \sum_{j=1}^{n}\|fL_j\chi u\|^2 + \|f\chi u\|^2 \leq \mu\left(\|P\chi u\|^2 + \|\chi u\|^2\right) + C\,\|u\|_{-N}^2 \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).$$

Proof of Corollary 4.2. By interpolation inequality, we have for any $\lambda, M > 0$

$$(4.3)\qquad \|v\|^2 \leq \lambda\,\|\langle D_x\rangle^\kappa v\|^2 + C(\lambda, M)\left\|\langle D_x\rangle^{-M} v\right\|^2 \quad \text{for all } v \in \mathcal{S}(\boldsymbol{R}^{d+1}),$$

where κ is the number specified in Proposition 4.1. Applying (4.3) to $v = fL_j\chi u$ and $M = 2(N+1)$, we have

$$\|fL_j\chi u\|^2 \leq \lambda\,\|\langle D_x\rangle^\kappa f L_j\chi u\|^2 + C(K, N, \chi, \lambda)\,\|u\|_{-N}^2\,.$$

Moreover, applying (4.3) to $v = f\chi u$ and $M = 2N$, we have

$$\|f\chi u\|^2 \leq \lambda\,\|\langle D_x\rangle^\kappa f\chi u\|^2 + C(K, N, \chi, \lambda)\,\|u\|_{-N}^2\,.$$

Combining above two inequalities and using (4.1), we obtain (4.2). □

The proof of Proposition 4.1 is done by dividing $\boldsymbol{R}^{2d+1}_{t,x,\xi}$ into two domains $\{\,(t,x,\xi)\,;\ |f(t,x)|\langle\xi\rangle^{\kappa'} \leq 1\,\}$ and $\{\,(t,x,\xi)\,;\ |f(t,x)|\langle\xi\rangle^{\kappa'} > 1\,\}$ with a suitable $\kappa' > 0$ and evaluating the left hand side of (4.1) in each domain.

Let $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ be functions of class $\boldsymbol{C}^\infty$ in $\boldsymbol{R}$ such that

$$\begin{cases} \mathrm{supp}\widetilde{\phi}_1 \subset \{|s|<2\}\ , & \widetilde{\phi}_1 \equiv 1 \quad \text{on} \quad \{|s| \le 1\} \\ \mathrm{supp}\widetilde{\phi}_2 \subset \{|s|>1\}\ , & \widetilde{\phi}_2 \equiv 1 \quad \text{on} \quad \{|s| \ge 2\} \end{cases}$$

and

$$\widetilde{\phi}_1 + \widetilde{\phi}_2 \equiv 1 \text{ on } \boldsymbol{R}.$$

And set $\widetilde{\chi}_{j,\kappa'}(t,x,\xi) = \widetilde{\phi}_j\left(f(t,x)\langle\xi\rangle^{\kappa'}\right)$ $(\,j=1,2\,)$, where κ' is a positive constant smaller than 1. By the definition of $\widetilde{\phi}_j$, we have

$$\text{(4.4)} \qquad \langle\xi\rangle^{-\kappa'} \le |f(t,x)| \le 2\langle\xi\rangle^{-\kappa'} \quad \text{on supp}\left(\partial^\alpha_{t,x,\xi}\widetilde{\chi}_{j,\kappa'}\right) \quad (\,|\alpha|>0,\ j=1,2\,).$$

Furthermore, in the case where (f,Ω) is of type (β), we have

$$\text{(4.5)} \qquad |\partial_t f(t,x)|^2 + \sum_{k=1}^{d} |\partial_{x_k} f(t,x)|^2 \le C(K) f(t,x) \quad \text{on } K$$

because $f(t,x)$ does not change sign from $(1°\text{-}\beta)$. In the case where (f,Ω) is of type $(\alpha\text{-}1)$ or of type $(\alpha\text{-}2)$, $\widetilde{\chi}_{1,\kappa'}$ and $\widetilde{\chi}_{2,\kappa'}$ belongs to $S^0_{1,0}(\boldsymbol{R}^d_x)$ with parameter t (see (4.4)) and they remain bounded in $S^0_{1,0}(\boldsymbol{R}^d_x)$ for every κ' as t runs over a compact set of $\boldsymbol{R}_t$. On the other hand, in the case where (f,Ω) is of type (β), $\widetilde{\chi}_{1,\kappa'}$ and $\widetilde{\chi}_{2,\kappa'}$ belong to $S^0_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ with parameter t (see (4.4) and (4.5)) and they remain bounded in $S^0_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ for every κ' as t runs over a compact set of $\boldsymbol{R}_t$. In both cases, $\widetilde{\chi}_{1,\kappa'} + \widetilde{\chi}_{2,\kappa'}$ is identically equal to 1 on $\boldsymbol{R}^{2d+1}$. If there is no confusion, we identify functions $\widetilde{\chi}_{1,\kappa'}, \widetilde{\chi}_{2,\kappa'}$ with operators $\widetilde{\chi}_{1,\kappa'}(t,x,D_x), \widetilde{\chi}_{2,\kappa'}(t,x,D_x)$ respectively.

For the proof of Proposition 4.1, it suffices to show the following two lemmas.

Lemma 4.3 *For any $K \subset\subset \Omega$, any $N>0$, any $\chi \in S_\Psi$ and any $0<\kappa'<1$, there exist a positive number $\kappa_1 = \kappa_1(\kappa')$ and a positive constant $C = C(K,N,\chi,\kappa')$ such that*

$$\text{(4.6)} \qquad \begin{aligned} &\sum_{j=1}^{n} \left\| \langle D_x\rangle^{\kappa_1} f L_j \widetilde{\chi}_{1,\kappa'}\chi u \right\|^2 + \left\| \langle D_x\rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'}\chi u \right\|^2 + \sum_{k=1}^{d} \left\| \langle D_x\rangle^{\kappa_1} (\partial_{x_k} f) \widetilde{\chi}_{1,\kappa'}\chi u \right\|^2 \\ &\le\ C\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|^2_{-N} \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K), \end{aligned}$$

where κ_1 depends only on κ' and C depends only on (K,N,χ,κ').

Lemma 4.4 *For any $K \subset\subset \Omega$, any $N>0$ and any $\chi \in S_\Psi$, there exist positive numbers $\kappa' = \kappa'(K) < 1$, $\kappa_2 = \kappa_2(K)$ and a positive constant $C = C(K,N,\chi)$ such that*

$$\text{(4.7)} \qquad \begin{aligned} &\sum_{j=1}^{n} \left\| \langle D_x\rangle^{\kappa_2} f L_j \widetilde{\chi}_{2,\kappa'}\chi u \right\|^2 + \left\| \langle D_x\rangle^{\kappa_2} f \widetilde{\chi}_{2,\kappa'}\chi u \right\|^2 + \sum_{k=1}^{d} \left\| \langle D_x\rangle^{\kappa_2} (\partial_{x_k} f) \widetilde{\chi}_{2,\kappa'}\chi u \right\|^2 \\ &\le\ C\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|^2_{-N} \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K), \end{aligned}$$

where (κ',κ_2) depends only on K and C depends only on (K,N,χ).

Proof of Proposition 4.1. Given (K, N, χ), let κ' be the same as in Lemma 4.4 which depends only on K. Set $\kappa_1 = \kappa_1(\kappa')$ as in Lemma 4.3 and $\kappa_2 = \kappa_2(K)$ as in Lemma 4.4. Now we define $\kappa = \min\{\kappa_1, \kappa_2\}$. Since $\widetilde{\chi}_{1,\kappa'} + \widetilde{\chi}_{2,\kappa'}$ is equal to the identity operator, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa} f L_j \chi u\right\| + \left\|\langle D_x\rangle^{\kappa} f \chi u\right\| + \sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k} f)\chi u\right\| \\
&\le \sum_{j=1}^{n}\Big(\left\|\langle D_x\rangle^{\kappa} f L_j \widetilde{\chi}_{1,\kappa'}\chi u\right\| + \left\|\langle D_x\rangle^{\kappa} f L_j \widetilde{\chi}_{2,\kappa'}\chi u\right\|\Big) + \left\|\langle D_x\rangle^{\kappa} f \widetilde{\chi}_{1,\kappa'}\chi u\right\| \\
&\qquad + \left\|\langle D_x\rangle^{\kappa} f \widetilde{\chi}_{2,\kappa'}\chi u\right\| + \sum_{k=1}^{d}\Big(\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k} f)\widetilde{\chi}_{1,\kappa'}\chi u\right\| + \left\|\langle D_x\rangle^{\kappa}(\partial_{x_k} f)\widetilde{\chi}_{2,\kappa'}\chi u\right\|\Big) \\
&\le C(K,N,\chi)\Big(\|P\chi u\| + \|\chi u\| + \|u\|_{-N}\Big) \qquad (\text{ by Lemma 4.3 and Lemma 4.4 }).
\end{aligned}$$

Proposition 4.1 is proved. We see that κ depends only on K. □

In the following subsections, we shall prove these lemmas. The proofs of them are done by reducing estimates for L_j to inequalities in Lemma 3.1. So we will divide the proofs into some steps according to stages and kinds of the reduction.

4.1 Proof of Lemma 4.3

We prove Lemma 4.3 in two cases where (f, Ω) is of type (α-1) or (β). The proof in the case of type (α-2) is done in a similar way as in the case of type (α-1) (or is done by the proof in the case of type (β), because the case of type (α-2) is a special case of type (β)). Let us begin with the case of type (α-1).

Case of type ($\boldsymbol{\alpha}$-1) :

Since $\partial_{x_k} f(t)$ ($k = 1, \ldots, d$) vanishes identically in this case, it suffices to show that

$$\begin{aligned}
(4.8)\qquad &\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa_1} f L_j \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 + \left\|\langle D_x\rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 \\
&\le C(K,N,\chi,\kappa')\Big(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\Big) \quad \text{for all } u \in \boldsymbol{C}_0^{\infty}(K).
\end{aligned}$$

Since $|f(t)|\langle\xi\rangle^{\kappa'} \le 2$ on supp$\widetilde{\chi}_{1,\kappa'}$, we have

$$|f(t)|^{\rho'}\langle\xi\rangle^{\rho'\kappa'} \le 2^{\rho'} \text{ on supp}\widetilde{\chi}_{1,\kappa'},$$

where ρ' is a positive number smaller than 1/4. Set $\boldsymbol{\kappa_1 = \rho'\kappa'}$, then we have

$$\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'} L_j \chi u\right\|^2 + \left\|\langle D_x\rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 \le 2^{2\rho'}\left(\sum_{j=1}^{n}\left\||f|^{1-\rho'} L_j \chi u\right\|^2 + \left\||f|^{1-\rho'}\chi u\right\|^2\right).$$

Setting $\rho = 1 - \rho'$ $(> 3/4)$ and using Lemma 3.1 (3.1), we have

$$
\begin{aligned}
&\sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'} L_j \chi u \right\|^2 + \left\| \langle D_x \rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'} \chi u \right\|^2 \\
&\qquad \leq \; C(K, \rho, N, \chi) \left(\| P\chi u \|^2 + \| \chi u \|^2 + \| u \|_{-N}^2 \right).
\end{aligned} \tag{4.9}
$$

By using an asymptotic expansion formula, $\langle D_x \rangle^{\kappa_1} f \, [L_j, \widetilde{\chi}_{1,\kappa'}]$ is an element of $S^0_{1,0}(\boldsymbol{R}^d_x)$ with parameter t and remains bounded in $S^0_{1,0}(\boldsymbol{R}^d_x)$ when t runs over a compact set of $\boldsymbol{R}$. Thus, we have

$$
\begin{aligned}
&\sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_1} f L_j \widetilde{\chi}_{1,\kappa'} \chi u \right\| \\
&\quad \leq \; \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'} L_j \chi u \right\| + \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_1} f \, [L_j, \widetilde{\chi}_{1,\kappa'}] \, \chi u \right\| + C(K, N, \chi) \, \| u \|_{-N} \\
&\quad \leq \; \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_1} f \widetilde{\chi}_{1,\kappa'} L_j \chi u \right\| + C(K, \rho, \chi) \, \| \chi u \| + C(K, N, \chi) \, \| u \|_{-N} \, .
\end{aligned}
$$

Combining this inequality with (4.9), we obtain (4.8). And hence, Lemma 4.3 is proved in the case of type (α-1).

Case of type ($\boldsymbol{\beta}$) :

The multiplication by f can be regarded as an element of $S^{-\kappa'}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ on the support of $\widetilde{\chi}_{1,\kappa'}(t, x, \xi)$. Moreover, the multiplication by $\partial_{x_k} f$ or by $\partial_t f$ can be regarded as an element of $S^{-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$. By making use of the fact above, we prove Lemma 4.3 in the case of type (β).

Let κ be a positive number satisfying $\boldsymbol{0 < \kappa < \kappa'/4}$. To prove Lemma 4.3, it suffices to show the following two inequalities:

$$
\sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa} f L_j \widetilde{\chi}_{1,\kappa'} \chi u \right\|^2 \; \leq \; C(K, N, \chi, \kappa, \kappa') \left(\| P\chi u \|^2 + \| \chi u \|^2 + \| u \|_{-N}^2 \right). \tag{4.10}
$$

$$
\begin{aligned}
&\left\| \langle D_x \rangle^{\kappa} f \widetilde{\chi}_{1,\kappa'} \chi u \right\|^2 + \sum_{k=1}^{d} \left\| \langle D_x \rangle^{\kappa} (\partial_{x_k} f) \widetilde{\chi}_{1,\kappa'} \chi u \right\|^2 \\
&\qquad \leq \; C(K, N, \chi, \kappa, \kappa') \left(\| P\chi u \|^2 + \| \chi u \|^2 + \| u \|_{-N}^2 \right).
\end{aligned} \tag{4.11}
$$

Let us begin with (4.11). Since $\langle D_x \rangle^{\kappa} f \widetilde{\chi}_{1,\kappa'} \in S^{\kappa-\kappa'}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ and $\langle D_x \rangle^{\kappa} (\partial_{x_k} f) \widetilde{\chi}_{1,\kappa'} \in S^{\kappa-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ $(k = 1, \ldots, d)$ with parameter t, they remain bounded in $S^0_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ as t runs over a compact set of $\boldsymbol{R}$. So they are bounded in $L^2(K)$. Thus, the left hand side of (4.11) does not exceed $C(K, N, \chi, \kappa, \kappa') \, \| \chi u \|^2$. This yields (4.11).

Next, we shall prove (4.10). The proof is divided into four steps.

First step :

Applying (3.3) to $\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u$ for u, we have

$$
(4.12)\quad \sum_{j=1}^{n}\left\| fL_j\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2
$$
$$
\leq\ C(K,N,\chi)\Big(\Big|\mathrm{Re}\Big(P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\Big| + \sum_{k=1}^{d}\left\|(\partial_{x_k}f)\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2
$$
$$
+\Big|\Big((\partial_t f)\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\Big| + \|u\|_{-N}^2\Big).
$$

Since $[fL_j,\langle D_x\rangle^\kappa]\,\widetilde{\chi}_{1,\kappa'} \in \mathcal{S}^{\kappa-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x) \subset \mathcal{S}^{0}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ due to $\kappa < \kappa'/4$, the left hand side of (4.12) is estimated from below as follows.

$$
\begin{aligned}
(4.13)\qquad & \sum_{j=1}^{n}\left\| fL_j\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 \\
&= \sum_{j=1}^{n}\left\|\langle D_x\rangle^\kappa fL_j \widetilde{\chi}_{1,\kappa'}\chi u + [fL_j,\langle D_x\rangle^\kappa]\,\widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 \\
&\geq \frac{1}{2}\sum_{j=1}^{n}\left\|\langle D_x\rangle^\kappa fL_j \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 - \sum_{j=1}^{n}\left\|[fL_j,\langle D_x\rangle^\kappa]\,\widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 \\
&\geq \frac{1}{2}\sum_{j=1}^{n}\left\|\langle D_x\rangle^\kappa fL_j \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 - C(K,\kappa,\kappa')\,\|\chi u\|^2 .
\end{aligned}
$$

Since $(\widetilde{\chi}_{1,\kappa'})^*\,\langle D_x\rangle^\kappa(\partial_t f)\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'} \in \mathcal{S}^{2\kappa-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ and (4.11) holds, the second and third terms on the right hand side of (4.12) are evaluated as

$$
\begin{aligned}
(4.14)\qquad & \sum_{k=1}^{d}\left\|(\partial_{x_k}f)\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 + \Big|\Big((\partial_t f)\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\Big| \\
&\leq C(K,\kappa,\kappa')\,\|\chi u\|^2 + \Big|\Big((\widetilde{\chi}_{1,\kappa'})^*\,\langle D_x\rangle^\kappa(\partial_t f)\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, \chi u\Big)\Big| \\
&\leq C'(K,\kappa,\kappa')\,\|\chi u\|^2 .
\end{aligned}
$$

Combining (4.12) with (4.13) and (4.14), we obtain

$$
(4.15)\quad \sum_{j=1}^{n}\left\|\langle D_x\rangle^\kappa fL_j \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2
$$
$$
\leq\ C(K,N,\chi,\kappa,\kappa')\left(\Big|\mathrm{Re}\Big(P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\Big| + \|\chi u\|^2 + \|u\|_{-N}^2\right).
$$

Second step :

By means of (4.15), it suffices for the proof of (4.10) to show that

“ For any $K \subset\subset \Omega$, any $N > 0$, any κ' $(0 < \kappa' < 1)$, any κ $(0 < \kappa < \kappa'/4)$ and any $\mu > 0$, there exists a constant $C = C(K,N,\chi,\kappa,\kappa',\mu)$ such that

$$\left| \mathrm{Re}\Big(P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big) \right| \tag{4.16}$$
$$\leq \mu \sum_{j=1}^n \Big\| \langle D_x\rangle^\kappa f L_j \widetilde{\chi}_{1,\kappa'}\chi u\Big\|^2 + C\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right)$$
$$\text{for all } u \in \boldsymbol{C}_0^\infty(K).\text{”}$$

Indeed, choosing a μ sufficiently small, we have (4.10) from (4.16) above. To obtain (4.16), we rewrite $P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}$ as

$$\begin{aligned} P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'} &= \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'} P + \Big[P, \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\Big] \\ &= \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'} P + \Big[P, \langle D_x\rangle^\kappa\Big]\widetilde{\chi}_{1,\kappa'} + \langle D_x\rangle^\kappa \Big[P, \widetilde{\chi}_{1,\kappa'}\Big]. \end{aligned}$$

So the left hand side of (4.16) is evaluated as

$$\begin{aligned} &\left| \mathrm{Re}\Big(P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big) \right| \\ &\leq \left|\Big(P\chi u, (\widetilde{\chi}_{1,\kappa'})^* \langle D_x\rangle^\kappa f \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| \\ &\qquad + \left|\Big(\Big[P, \langle D_x\rangle^\kappa\Big]\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| + \left|\Big(\langle D_x\rangle^\kappa \Big[P, \widetilde{\chi}_{1,\kappa'}\Big]\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right|. \end{aligned}$$

Since $(\widetilde{\chi}_{1,\kappa'})^* \langle D_x\rangle^\kappa f \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'} \in S^{2\kappa-\kappa'}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$, we have

$$\begin{aligned} &\left| \mathrm{Re}\Big(P\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big) \right| \\ &\leq \left|\Big(\Big[P, \langle D_x\rangle^\kappa\Big]\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| + \left|\Big(\langle D_x\rangle^\kappa \Big[P, \widetilde{\chi}_{1,\kappa'}\Big]\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| \\ &\qquad + C(K,\kappa,\kappa')\left(\|P\chi u\|^2 + \|\chi u\|^2\right). \end{aligned}$$

Therefore, it suffices for the proof of (4.16) to show that the first and second terms on the right hand side of the above inequality do not exceed the right hand side of (4.16). So we shall prove the following two inequalities:

$$\begin{aligned} &\left|\Big(\Big[P, \langle D_x\rangle^\kappa\Big]\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| \\ &\leq \mu \sum_{j=1}^n \Big\| \langle D_x\rangle^\kappa f L_j \widetilde{\chi}_{1,\kappa'}\chi u\Big\|^2 \\ &\qquad + C(K,N,\chi,\kappa,\kappa',\mu)\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right), \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\left|\Big(\langle D_x\rangle^\kappa \Big[P, \widetilde{\chi}_{1,\kappa'}\Big]\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| \\ &\leq C(K,N,\chi,\kappa,\kappa')\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right). \end{aligned} \tag{4.18}$$

The form of $\Big[P, \langle D_x\rangle^\kappa\Big]$ is similar to one of $\Big[P, \widetilde{\chi}_{1,\kappa'}\Big]$ (see (4.19) and (4.20) below), but (4.17) and (4.18) are treated in a different way. This is because $N_{j,k}$ in (4.19) and M_{jk}

in (4.20) belong to a different class of operators. (4.17) and (4.18) will be proved in the Final step below and in the next step respectively.

Third step :

In this step, we prove (4.18). First, $[P, \widetilde{\chi}_{1,\kappa'}]$ is represented as follows.

$$\left[P, \widetilde{\chi}_{1,\kappa'}\right] = \sum_{j=1}^{n} N_{j,0} f L_j + \sum_{j=1}^{n}\sum_{k=1}^{d} N_{j,k}(\partial_{x_k} f) L_j + N_{0,0}, \tag{4.19}$$

where $N_{j,k} \in S^{\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ ($j \in \{0, \ldots, n\}$, $k \in \{0, \ldots, d\}$). So the left hand side of (4.18) is evaluated as follows.

$$\begin{aligned}
&\left|\left(\langle D_x\rangle^\kappa \left[P, \widetilde{\chi}_{1,\kappa'}\right]\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right| \\
&\quad \le \sum_{j=1}^{n}\left|\left(\langle D_x\rangle^\kappa N_{j,0} f L_j \chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right| \\
&\qquad + \sum_{j=1}^{n}\sum_{k=1}^{d}\left|\left(\langle D_x\rangle^\kappa N_{j,k}(\partial_{x_k} f) L_j \chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right| \\
&\qquad + \left|\left(\langle D_x\rangle^\kappa N_{0,0}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right|.
\end{aligned}$$

Since $N^*_{j,k}\langle D_x\rangle^\kappa f \langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'} \in S^{2\kappa-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x) \subset S^0_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ due to $\kappa < \kappa'/4$, we have by Schwarz' inequality

$$\begin{aligned}
&\left|\left(\langle D_x\rangle^\kappa \left[P, \widetilde{\chi}_{1,\kappa'}\right]\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right| \\
&\quad \le C(K, N, \chi, \kappa, \kappa')\left(\sum_{j=1}^{n}\|f L_j \chi u\|^2 + \sum_{j=1}^{n}\sum_{k=1}^{d}\|(\partial_{x_k} f) L_j \chi u\|^2 + \|\chi u\|^2 + \|u\|^2_{-N}\right).
\end{aligned}$$

Combining (4.5), (3.2), (3.3) and this inequality, we obtain (4.18).

Final step :

In this step, we shall prove (4.17). We rewrite $[P, \langle D_x\rangle^\kappa]$ as follows.

$$\left[P, \langle D_x\rangle^\kappa\right] = \sum_{j=1}^{n} M_{j,0} f L_j + \sum_{j=1}^{n}\sum_{k=1}^{d} M_{j,k}(\partial_{x_k} f) L_j + M_{0,0}, \tag{4.20}$$

where $M_{j,k} \in S^\kappa_{1,0}(\boldsymbol{R}^d_x)$ ($j \in \{0, \ldots, n\}$, $k \in \{0, \ldots, d\}$). Therefore (4.17) follows from the following two inequalities:

$$\begin{aligned}
&\sum_{j=1}^{n}\left|\left(M_{j,0} f L_j \widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right| + \left|\left(M_{0,0}\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^\kappa \widetilde{\chi}_{1,\kappa'}\chi u\right)\right| \\
&\quad \le \mu\sum_{j=1}^{n}\left\|\langle D_x\rangle^\kappa f L_j \widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 + C_1(K, N, \chi, \kappa, \kappa', \mu)\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|^2_{-N}\right)
\end{aligned} \tag{4.21}$$

$$(4.22) \quad \sum_{j=1}^{n}\sum_{k=1}^{d}\left|\Big(M_{j,k}(\partial_{x_k}f)L_j\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big)\right|$$
$$\le \mu\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa}fL_j\widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 + C_2(K,N,\chi,\kappa,\kappa',\mu)\left(\|P\chi u\|^2+\|\chi u\|^2+\|u\|_{-N}^2\right).$$

First, let us prove (4.21). Since $M_{j,0} \in S^{\kappa}_{1,0}(\boldsymbol{R}^d_x)$ and $f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'} \in S^{\kappa-\kappa'}_{1,\kappa'/2}(\boldsymbol{R}^d_x) \subset S^{0}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$, we have by Schwarz' inequality

$$\sum_{j=1}^{n}\left|\Big(M_{j,0}fL_j\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big)\right| + \left|\Big(M_{0,0}\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big)\right|$$
$$\le \mu\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa}fL_j\widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 + C(K,N,\chi,\kappa,\kappa',\mu)\left(\|\chi u\|^2+\|u\|_{-N}^2\right),$$

which implies (4.21).

Next, we prove (4.22). We rewrite $(M_{j,k}(\partial_{x_k}f)L_j\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u)$ as follows.

$$\Big(M_{j,k}(\partial_{x_k}f)L_j\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big)$$
$$= \Big(fM_{j,k}(\partial_{x_k}f)L_j\widetilde{\chi}_{1,\kappa'}\chi u, \langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big)$$
$$= \Big(\widetilde{\chi}^*_{1,\kappa'}\langle D_x\rangle^{\kappa}M_{j,k}(\partial_{x_k}f)fL_j\widetilde{\chi}_{1,\kappa'}\chi u, \chi u\Big) + \Big([f,M_{j,k}]\,(\partial_{x_k}f)L_j\widetilde{\chi}_{1,\kappa'}\chi u, \langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big).$$

The expansion formula of the symbol yields $L_j^*(\partial_{x_k}f)\,[f,M_{j,k}]^*\,\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'} \in S^{2\kappa-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$. Moreover, we see that $\widetilde{\chi}^*_{1,\kappa'}\langle D_x\rangle^{\kappa}M_{j,k}(\partial_{x_k}f) \in S^{2\kappa-\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x) \subset S^{\kappa}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ and $\widetilde{\chi}_{1,\kappa'} \in S^{0}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$. Thus, by Schwarz' inequality, we obtain

$$\sum_{j=1}^{n}\sum_{k=1}^{d}\left|\Big(M_{j,k}(\partial_{x_k}f)L_j\widetilde{\chi}_{1,\kappa'}\chi u, f\langle D_x\rangle^{\kappa}\widetilde{\chi}_{1,\kappa'}\chi u\Big)\right|$$
$$\le \mu\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa}fL_j\widetilde{\chi}_{1,\kappa'}\chi u\right\|^2 + C(K,N,\chi,\kappa,\kappa',\mu)\left(\|\chi u\|^2+\|u\|_{-N}^2\right),$$

which implies (4.22). So (4.17) holds and hence (4.16) holds. Therefore, (4.10) is proved. Now the proof of Lemma 4.3 is completed. □

4.2 Proof of Lemma 4.4

Since f does not vanish on the support of $\widetilde{\chi}_{2,\kappa'}(t,x,\xi)$, P does not degenerate on the support of $\widetilde{\chi}_{2,\kappa'}(t,x,\xi)$. By making use of the above fact, we get a subelliptic estimate (4.23) in Lemma 4.5 below. We prove Lemma 4.4 by using this estimate.

Lemma 4.5 *For any $K \subset\subset \Omega$, any $N > 0$, any $\chi \in S_\Psi$ and any κ' satisfying $0 < \kappa' < 2^{1-\widetilde{k}}/(\widetilde{k}+1)$, there exists a positive constant $C = C(K,N,\chi,\kappa')$ such that*

$$(4.23) \quad \left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2 \le C\left(\|P\chi u\|^2+\|\chi u\|^2+\|u\|_{-N}^2\right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K),$$

where C depends only on (K,N,χ,κ') and $\widetilde{k}$ is as in Lemma 3.5.

Proof of Lemma 4.5. First, we consider the operator $f^{-\widetilde{k}}\widetilde{\chi}_{2,\kappa'}\langle D_x\rangle^{-2^{1-\widetilde{k}}}$. Since f does not vanish on the support of $\widetilde{\chi}_{2,\kappa'}$, $f^{-\widetilde{k}}\widetilde{\chi}_{2,\kappa'}\langle D_x\rangle^{-2^{1-\widetilde{k}}}$ is a pseudo-differential operator with the symbol $f^{-\widetilde{k}}\langle\xi\rangle^{-2^{1-\widetilde{k}}}\widetilde{\chi}_{2,\kappa'}(t,x,\xi)$ which belongs to $S^{\widetilde{k}\kappa'-2^{1-\widetilde{k}}}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ due to (4.4). Set

$$R_{\kappa'}=\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}-\langle D_x\rangle^{\kappa'}f^{-\widetilde{k}}\widetilde{\chi}_{2,\kappa'}\langle D_x\rangle^{-2^{1-\widetilde{k}}}f^{\widetilde{k}}\langle D_x\rangle^{2^{1-\widetilde{k}}}.$$

By an asymptotic expansion of the symbol, $R_{\kappa'}\in S^{(\widetilde{k}+1)\kappa'-1}_{1,\kappa'/2}(\boldsymbol{R}^d_x)\subset S^0_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ due to $0<\kappa'<2^{1-\widetilde{k}}/(\widetilde{k}+1)$. Thus, $\left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\|$ is evaluated as follows.

$$\begin{aligned}\left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\| &\le \left\|\langle D_x\rangle^{\kappa'}f^{-\widetilde{k}}\widetilde{\chi}_{2,\kappa'}\langle D_x\rangle^{-2^{1-\widetilde{k}}}f^{\widetilde{k}}\langle D_x\rangle^{2^{1-\widetilde{k}}}\chi u\right\|\\ &\quad +C(K,N,\kappa')\Big(\|\chi u\|+\|u\|_{-N}\Big)\\ &\le C(K,N,\chi,\kappa')\left(\left\|f^{\widetilde{k}}\langle D_x\rangle^{2^{1-\widetilde{k}}}\chi u\right\|+\|\chi u\|+\|u\|_{-N}\right)\\ &\le C(K,N,\chi,\kappa')\Bigg(\left\|\langle D_x\rangle^{2^{1-\widetilde{k}}}f^{\widetilde{k}}\chi u\right\|+\left\|\left[f^{\widetilde{k}},\langle D_x\rangle^{2^{1-\widetilde{k}}}\right]\chi u\right\|\\ &\quad +\|\chi u\|+\|u\|_{-N}\Bigg).\end{aligned}$$

Since $\left[f^{\widetilde{k}},\langle D_x\rangle^{2^{1-\widetilde{k}}}\right]\in S^0_{1,0}(\boldsymbol{R}^d_x)$, combining this inequality with (3.16), we obtain (4.23). □

Lemma 4.5 guarantees that second and third terms on the left hand side in (4.7) does not exceed $C(K,N,\chi)\left(\|P\chi u\|^2+\|\chi u\|^2+\|u\|^2_{-N}\right)$ if we choose a κ_2 so small that $\kappa_2\le\kappa'$.

Proof of Lemma 4.4. Given $K\subset\subset\Omega$, let $\widetilde{k}$ be as in Lemma 3.5 and κ' a positive number satisfying $0<\kappa'<2^{1-\widetilde{k}}/(\widetilde{k}+1)$. Set $\boldsymbol{\kappa_2=\kappa'/4}$. We treat the first term on the left hand side in (4.7). It suffices to show the following inequality:

$$\sum_{j=1}^{n}\left\|\langle D_x\rangle^{\kappa_2}fL_j\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2\le C(K,N,\chi)\left(\|P\chi u\|^2+\|\chi u\|^2+\|u\|^2_{-N}\right). \tag{4.24}$$

In what follows, we prove this inequality. The proof is divided into three steps.

First step :
In the case of type (α-1), we apply (3.1) with $\rho=1$ to $\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u$ for u. In the case of type (α-2) or of type (β), we apply (3.3) to (3.3) $\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u$ for u. Then, we have

$$\begin{aligned}&\sum_{j=1}^{n}\left\|fL_j\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2\\ &\le C(K,N,\chi)\left(\left|\mathrm{Re}\Big(P\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u,f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big)\right|+\left\|\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2+\|u\|^2_{-N}\right).\end{aligned}$$

In the same way as in the proof of Lemma 4.3 for type (β), we have

$$
\begin{aligned}
(4.25)\quad \sum_{j=1}^{n}\Big\| fL_j\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2
&\le C(K,N,\chi)\Big(\Big|\mathrm{Re}\Big(P\chi u, \widetilde{\chi}^*_{2,\kappa'} f\langle D_x\rangle^{2\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big)\Big| \\
&\quad + \Big|\Big(\Big[P,\langle D_x\rangle^{\kappa_2}\Big]\widetilde{\chi}_{2,\kappa'}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big)\Big| \\
&\quad + \Big|\Big(\langle D_x\rangle^{\kappa_2}\Big[P,\widetilde{\chi}_{2,\kappa'}\Big]\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big)\Big| + \Big\|\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2 + \|u\|^2_{-N}\Big)
\end{aligned}
$$

Since $[fL_j, \langle D_x\rangle^{\kappa_2}] \in S^{\kappa_2}_{1,0}(\boldsymbol{R}^d_x)$, the left hand side of (4.25) is estimated from below as follows.

$$
\begin{aligned}
(4.26)\qquad \sum_{j=1}^{n}\Big\| fL_j\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2
\ge \frac{1}{2}\sum_{j=1}^{n}\Big\|\langle D_x\rangle^{\kappa_2} fL_j\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2 - C(K)\Big\|\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2.
\end{aligned}
$$

Since $\|\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\|^2 \le \Big\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2$ due to $\kappa_2 < \kappa'$, combining (4.25) with (4.26), we have by Lemma 4.5

$$
\begin{aligned}
(4.27)\quad \sum_{j=1}^{n}\Big\|\langle D_x\rangle^{\kappa_2} fL_j\widetilde{\chi}_{2,\kappa'}\chi u\Big\|^2
&\le C(K,N,\chi)\Big(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|^2_{-N} \\
&\quad + \Big|\Big(\Big[P,\langle D_x\rangle^{\kappa_2}\Big]\widetilde{\chi}_{2,\kappa'}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big)\Big| \\
&\quad + \Big|\Big(\langle D_x\rangle^{\kappa_2}\Big[P,\widetilde{\chi}_{2,\kappa'}\Big]\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\Big)\Big|\Big).
\end{aligned}
$$

In view of (4.19) and (4.20) , $[P, \langle D_x\rangle^{\kappa_2}]\,\widetilde{\chi}_{2,\kappa'}$, $\langle D_x\rangle^{\kappa_2}\,[P, \widetilde{\chi}_{1,\kappa'}]$ are rewritten as follows:

$$
(4.28)\quad \Big[P,\langle D_x\rangle^{\kappa_2}\Big]\widetilde{\chi}_{2,\kappa'} = \sum_{j=1}^{n} M_{j,0} fL_j\widetilde{\chi}_{2,\kappa'} + \sum_{j=1}^{n}\sum_{k=1}^{d} M_{j,k}(\partial_{x_k} f)L_j\widetilde{\chi}_{2,\kappa'} + M_{0,0}\widetilde{\chi}_{2,\kappa'},
$$

where $M_{j,k} \in S^{\kappa_2}_{1,0}(\boldsymbol{R}^d_x)$ ($j \in \{0,\dots,n\},\ k \in \{0,\dots,d\}$).

$$
(4.29)\qquad \Big[P,\widetilde{\chi}_{2,\kappa'}\Big] = \sum_{j=1}^{n} N_{j,0} fL_j + \sum_{j=1}^{n}\sum_{k=1}^{d} N_{j,k}(\partial_{x_k} f)L_j + N_{0,0},
$$

In the case of type (α-1), $N_{j,0} \in S^0_{1,0}(\boldsymbol{R}^d_x)$ for $j \in \{1,\dots,n\}$ and $N_{0,0} \in S^{\kappa'}_{1,0}(\boldsymbol{R}^d_x)$. In the case of type ($\alpha$-2) or of type ($\beta$), $N_{j,k} \in S^{\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ for $(j,k) \in \{0,\dots,n\}\times\{0,\dots,d\}$. More precisely, $N_{0,0}$ in the case of type (α-1) is represented as

$$
(4.30)\qquad N_{0,0} = (\partial_t\widetilde{\chi}_{2,\kappa'})(t, D_x) + \widetilde{N}_{0,0} \quad \text{where } \widetilde{N}_{0,0} \in S^0_{1,0}(\boldsymbol{R}^d_x).
$$

By means of (4.27), (4.28) and (4.29), it suffices to show the following two inequalities for the proof of (4.24):

$$
\begin{aligned}
(4.31) \qquad & \sum_{j=1}^{n} \Big| \Big(M_{j,0} f L_j \widetilde{\chi}_{2,\kappa'} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \\
& + \sum_{j=1}^{n} \sum_{k=1}^{d} \Big| \Big(M_{j,k} (\partial_{x_k} f) L_j \widetilde{\chi}_{2,\kappa'} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \\
& + \Big| \Big(M_{0,0} \widetilde{\chi}_{2,\kappa'} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \\
& \le \mu \sum_{j=1}^{n} \Big\| \langle D_x \rangle^{\kappa_2} f L_j \widetilde{\chi}_{2,\kappa'} \chi u \Big\|^2 \\
& \qquad + C_1(K, N, \chi, \mu) \Big(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2 \Big) .
\end{aligned}
$$

$$
\begin{aligned}
(4.32) \qquad & \sum_{j=1}^{n} \Big| \Big(\langle D_x \rangle^{\kappa_2} N_{j,0} f L_j \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \\
& + \sum_{j=1}^{n} \sum_{k=1}^{d} \Big| \Big(\langle D_x \rangle^{\kappa_2} N_{j,k} (\partial_{x_k} f) L_j \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \\
& + \Big| \Big(\langle D_x \rangle^{\kappa_2} N_{0,0} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \\
& \le C_2(K, N, \chi) \Big(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2 \Big) .
\end{aligned}
$$

Second step :
In this step, we prove (4.32). First, each term in the first and second sums on the left hand side of (4.32) is rewritten as follows.

$$
\begin{aligned}
\Big(\langle D_x \rangle^{\kappa_2} N_{j,0} f L_j \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) &= \Big(f L_j \chi u, N_{j,0}^{*} \langle D_x \rangle^{\kappa_2} f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big), \\
\Big(\langle D_x \rangle^{\kappa_2} N_{j,k} (\partial_{x_k} f) L_j \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) &= \Big((\partial_{x_k} f) L_j \chi u, N_{j,k}^{*} \langle D_x \rangle^{\kappa_2} f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big).
\end{aligned}
$$

Since $N_{j,k}^{*} \langle D_x \rangle^{\kappa_2} f \langle D_x \rangle^{\kappa_2} \in S_{1,\kappa'/2}^{\kappa'}(\boldsymbol{R}_x^d)$ for $(j,k) \neq (0,0)$, by Schwarz' inequality, the left hand side of (4.32) does not exceed

$$
\begin{aligned}
(4.33) \qquad & C(K,N) \Big(\sum_{j=1}^{n} \|f L_j \chi u\|^2 + \sum_{j=1}^{n} \sum_{k=1}^{d} \|(\partial_{x_k} f) L_j \chi u\|^2 \\
& + \Big\| \langle D_x \rangle^{\kappa'} \widetilde{\chi}_{2,\kappa'} \chi u \Big\|^2 + \Big| \Big(\langle D_x \rangle^{\kappa_2} N_{0,0} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \Big).
\end{aligned}
$$

In the case of type (α-1), from (3.1) with $\rho = 1$, (4.33) does not exceed

$$
C(K,N,\chi) \Big(\|P\chi u\|^2 + \|u\|^2 + \|u\|_{-N}^2 + \Big| \Big(\langle D_x \rangle^{\kappa_2} N_{0,0} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \Big) \Big| \Big).
$$

In the case of type (α-2) or of (β), the same estimate holds from (3.2), (3.3) and (3.11).

The remaining problem is to evaluate $|(\langle D_x\rangle^{\kappa_2} N_{0,0}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u)|$. This is divided into two cases. Suppose that (f,Ω) is of type (α-2) or (β). Remembering $N_{0,0}\in \mathcal{S}^{\kappa'/2}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$, we have $N^*_{0,0}\langle D_x\rangle^{\kappa_2} f\langle D_x\rangle^{\kappa_2}\in \mathcal{S}^{\kappa'}_{1,\kappa'/2}(\boldsymbol{R}^d_x)$ by $\kappa_2=\kappa'/4$. Thus, we have

$$\begin{aligned}(4.34)\quad \left|\left(\langle D_x\rangle^{\kappa_2} N_{0,0}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right| &\le \left|\left(\chi u, N^*_{0,0}\langle D_x\rangle^{\kappa_2} f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\\ &\le C(K)\left(\|\chi u\|^2+\left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2\right).\end{aligned}$$

Suppose that (f,Ω) is of type (α-1). Also in this case, (4.34) holds. This is proved in the following way. We have from (4.30)

$$\begin{aligned}\left|\left(\langle D_x\rangle^{\kappa_2} N_{0,0}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right| &\le \left|\left(\langle D_x\rangle^{\kappa_2}\left(\partial_t\widetilde{\chi}_{2,\kappa'}\right)(t,D_x)\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\\ &\quad+\left|\left(\chi u, \widetilde{N}^*_{0,0}\langle D_x\rangle^{\kappa_2} f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\\ &\le \left|\left(f\left(\partial_t\widetilde{\chi}_{2,\kappa'}\right)(t,D_x)\chi u, \langle D_x\rangle^{2\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\\ &\quad+\left|\left(\chi u, \widetilde{N}^*_{0,0}\langle D_x\rangle^{\kappa_2} f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\end{aligned}$$

(The multiplication by f commutes with $\langle D_x\rangle^{\kappa_2}$ in this case.) Since $\widetilde{N}^*_{0,0}\langle D_x\rangle^{\kappa_2} f\langle D_x\rangle^{\kappa_2}\in \mathcal{S}^{2\kappa_2}_{1,0}(\boldsymbol{R}^d_x)\subset \mathcal{S}^{\kappa'}_{1,0}(\boldsymbol{R}^d_x)$, the second term on the right hand side of the above inequality is evaluated as

$$\left|\left(\chi u, \widetilde{N}^*_{0,0}\langle D_x\rangle^{\kappa_2} f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\le C(K)\left(\|\chi u\|^2+\left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2\right).$$

Next, we treat the first term. The symbol of $f\left(\partial_t\widetilde{\chi}_{2,\kappa'}\right)(t,D_x)$ is equal to $(\partial_t f(t))f(t)\langle\xi\rangle^{\kappa'}$ $\widetilde{\phi}'_2\left(f(t)\langle\xi\rangle^{\kappa'}\right)$. Since $|f(t)|\,\langle\xi\rangle^{\kappa'}\le 2$ on the support of $\widetilde{\phi}'_2\left(f(t)\langle\xi\rangle^{\kappa'}\right)$, $f\left(\partial_t\widetilde{\chi}_{2,\kappa'}\right)(t,D_x)$ belongs to the operator class $\mathcal{S}^0_{1,0}(\boldsymbol{R}^d_x)$ and remains bounded when t runs over a compact set of $\boldsymbol{R}$. So the first term is evaluated as

$$\left|\left(f\left(\partial_t\widetilde{\chi}_{2,\kappa'}\right)(t,D_x)\chi u, \langle D_x\rangle^{2\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\le C(K)\left(\|\chi u\|^2+\left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2\right).$$

Therefore, (4.34) holds also in this case.

Applying Lemma 4.5 to (4.34), we obtain (4.32) in both cases.

Final step :

We prove (4.31). First, we evaluate the first and third terms on the left hand side of (4.31). Since $M_{j,0}\in \mathcal{S}^{\kappa_2}_{1,0}(\boldsymbol{R}^d_x)\subset \mathcal{S}^{\kappa'}_{1,0}(\boldsymbol{R}^d_x)$, we have by Schwarz' inequality

$$\begin{aligned}(4.35)\quad &\sum_{j=1}^n\left|\left(M_{j,0} fL_j\widetilde{\chi}_{2,\kappa'}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|+\left|\left(M_{0,0}\widetilde{\chi}_{2,\kappa'}\chi u, f\langle D_x\rangle^{\kappa_2}\widetilde{\chi}_{2,\kappa'}\chi u\right)\right|\\ &\le \mu\sum_{j=1}^n\left\|\langle D_x\rangle^{\kappa_2} fL_j\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2+C(K,\mu)\left\|\langle D_x\rangle^{\kappa'}\widetilde{\chi}_{2,\kappa'}\chi u\right\|^2\end{aligned}$$

$$\leq \mu \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_2} f L_j \widetilde{\chi}_{2,\kappa'} \chi u \right\|^2 + C(K, N, \chi, \mu) \left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2 \right)$$

(by Lemma 4.5).

Next, we evaluate the second term on the left hand side of (4.31).
We rewrite $(M_{j,k}(\partial_{x_k} f) L_j \widetilde{\chi}_{2,\kappa'} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u)$ as follows.

$$\begin{aligned} &\left(M_{j,k}(\partial_{x_k} f) L_j \widetilde{\chi}_{2,\kappa'} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \right) \\ &\quad = \left(M_{j,k}(\partial_{x_k} f) f L_j \widetilde{\chi}_{2,\kappa'} \chi u, \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \right) \\ &\qquad + \left(\widetilde{\chi}_{2,\kappa'} \chi u, L_j^*(\partial_{x_k} f) \, [f, M_{j,k}]^* \, \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \right). \end{aligned}$$

Since $M_{j,k}(\partial_{x_k} f) \in S^{\kappa_2}_{1,0}(\boldsymbol{R}^d_x)$ and $L_j^*(\partial_{x_k} f) \, [f, M_{j,k}]^* \, \langle D_x \rangle^{\kappa_2} \in S^{\kappa'}_{1,0}(\boldsymbol{R}^d_x)$, the second term on the left hand side of (4.31) is evaluated by Schwarz' inequality as

$$\begin{aligned} &\sum_{j=1}^{n} \sum_{k=1}^{d} \left| \left(M_{j,k}(\partial_{x_k} f) L_j \widetilde{\chi}_{2,\kappa'} \chi u, f \langle D_x \rangle^{\kappa_2} \widetilde{\chi}_{2,\kappa'} \chi u \right) \right| \\ &\quad \leq \mu \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa_2} f L_j \widetilde{\chi}_{2,\kappa'} \chi u \right\|^2 + C(K, \mu) \left\| \langle D_x \rangle^{\kappa'} \widetilde{\chi}_{2,\kappa'} \chi u \right\|^2. \end{aligned} \tag{4.36}$$

By using Lemma 4.5, (4.35) and (4.36) yield (4.31). And hence (4.24) holds. The proof of Lemma 4.4 is finished. □

5 A priori estimate without weight

In this section, we shall prove Proposition 5.1 below. Inequality (5.1) there is an improvement of (2.1) in Condition (I). Obviously (5.1) guarantees Condition (I) for P to hold. We need the arbitrariness of μ to verify Conditions (II) and (V). (see Proposition 6.1 in §6.) Moreover, (5.1) allows us to neglect $\|\chi u\|$ in the course of estimation in the following sections.

Proposition 5.1 *For any open set $K \subset\subset \boldsymbol{R}^{d+1}$, any $N > 0$, any $\chi \in S_\Psi$ and any $\mu > 0$, there exists a constant $C = C(K, N, \chi, \mu)$ such that*

$$\|\chi u\| \le \mu \|P\chi u\| + C \|u\|_{-N} \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{5.1}$$

Proof of Proposition 5.1. For every subset M in an Euclidian space, $\overline{M}$ denotes the closure of M. For any $K \subset\subset \boldsymbol{R}^{d+1}$, there exist a bounded open interval I of $\boldsymbol{R}_t$ and a bounded open set U of $\boldsymbol{R}^d_x$ such that $K \subset\subset I \times U$. There are two cases according as I and U can be or cannot chosen so that $(f, I \times U)$ is of type (α-2) or of type (β). We prove Proposition 5.1 by a partition of unity of $I \times U$. First, suppose that $(f, I \times U)$ is of **type ($\boldsymbol{\alpha}$-2)** or of **type ($\boldsymbol{\beta}$)**. Without loss of generality, we may suppose that $f(t, x)$ is non-negative on $I \times U$.

Let us begin with the construction of a partition of I. A partition of U will be constructed later. For any $y \in U$ and any $\varepsilon > 0$, we can find families of open intervals $\{I_q(y, \varepsilon)\}_{q=1}^{N(y,\varepsilon)}, \{J_q(y, \varepsilon)\}_{q=1}^{N(y,\varepsilon)-1} \subset \boldsymbol{R}_t$ satisfying the following conditions:

(B1) $N(y, \varepsilon)$ is finite for every $y \in U$ and every $\varepsilon > 0$.

(B2) $\overline{I} \subset (\bigcup_{q=1}^{N(y,\varepsilon)} I_q(y, \varepsilon)) \cup (\bigcup_{q=1}^{N(y,\varepsilon)-1} J_q(y, \varepsilon))$.

(B3) $I_1(y, \varepsilon)$ contains the left end point of $\overline{I}$ and $I_{N(y,\varepsilon)}(y, \varepsilon)$ contains the right end point of $\overline{I}$.

(B4) $\overline{I_p(y, \varepsilon)} \cap \overline{I_q(y, \varepsilon)} = \emptyset$ and $J_p(y, \varepsilon) \cap J_q(y, \varepsilon) = \emptyset$ if $p \neq q$.

(B5) $I_q(y, \varepsilon) \cap J_q(y, \varepsilon) \neq \emptyset$ and $I_{q+1}(y, \varepsilon) \cap J_q(y, \varepsilon) \neq \emptyset$ for any q.

(B6) $|I_q(y, \varepsilon)| \le \varepsilon$ for any q, where $|I_q(y, \varepsilon)|$ is the length of I_q.

(B7) $|J_q(y, \varepsilon)| \le \varepsilon/2$ for any q.

(B8) $f(t, y) > 0$ in $\overline{J_q(y, \varepsilon)}$ for any q.

Indeed, we construct them in the following way.

Given $y \in U$, let I_1 be any open interval of length ε containing the left end point of $\overline{I}$. If $f(t,y)$ vanishes at the right end point of $\overline{I_1}$, then we take a smaller one such that $f(t,y)$ does not vanish at the right end point of $\overline{I_1}$ and $|I_1| \geq \varepsilon/2$. This is possible because, by the hypothesis (1°), there is no non-empty interval on which $f(t,y)$ vanishes identically, so the set $\{\, t \in \boldsymbol{R}_t \,;\, f(t,y) > 0 \,\}$ is dense. Let t_1 be the right end point of $\overline{I_1}$ and we choose an open interval J_1 such that $t_1 \in J_1$, $f(t,y) > 0$ in J_1 and $|J_1| < \varepsilon/2$. (Note that the set $\{\, t \in \boldsymbol{R}_t \,;\, f(t,y) > 0 \,\}$ is open.) Let s_1 be the right end point of J_1. Obviously, $t_1 < s_1$ and $|s_1 - t_1| < \varepsilon/2$. Next we choose a point $t_2 \in (t_1 + \varepsilon, s_1 + \varepsilon)$ satisfying $f(t_2, y) > 0$. This is also possible by hypothesis (1°). Set $I_2 = (t_2 - \varepsilon, t_2)$, then $|I_2| = \varepsilon$. Since $t_1 < t_2 - \varepsilon < s_1$, $\overline{I_1} \cap \overline{I_2} = \emptyset$ and $I_2 \cap J_1 \neq \emptyset$. Moreover, $t_2 - s_1 > \varepsilon/2$ because $|s_1 - t_1| < \varepsilon/2$. Then we choose an open interval J_2 such that $t_2 \in J_2$, $f(t,y) > 0$ in J_2 and $|J_2| < \varepsilon/2$. Since $t_2 - s_1 > \varepsilon/2$, $J_1 \cap J_2 = \emptyset$. Repeating these steps l times for any positive integer l, we have $\{I_q\}_{q=1}^{l}, \{J_q\}_{q=1}^{l}$, and set $I_q(y,\varepsilon) = I_q$, $J_q(y,\varepsilon) = J_q$ $(\, q = 1, \ldots, l \,)$ which satisfy the conditions **(B4)**–**(B8)**. Since

$$\left| \bigcup_{q=1}^{l} (I_q \cup J_q) \right| \;\geq\; \left| \bigcup_{q=1}^{l} I_q \right| \;=\; \sum_{q=1}^{l} |I_q| \;\geq\; \frac{\varepsilon}{2} + (l-1)\,\varepsilon,$$

let us choose l so large that $|I| + \varepsilon < (l-1)\varepsilon + \varepsilon/2$. Then $\overline{I} \subset (\bigcup_{q=1}^{l} I_q(y,\varepsilon)) \cup (\bigcup_{q=1}^{l} J_q(y,\varepsilon))$. Let t^* be the right end point of $\overline{I}$. Then, the set $\{q \in \{1, \ldots, l\}\ ; t^* \in I_q(y,\varepsilon) \cup J_q(y,\varepsilon)\}$ is non-empty. Let $M(y,\varepsilon)$ be the minimal element of this set. We choose $N(y,\varepsilon)$ in the following way.

1. $N(y,\varepsilon) = M(y,\varepsilon)$ if $t^* \in I_{M(y,\varepsilon)}(y,\varepsilon)$.

2. $N(y,\varepsilon) = M(y,\varepsilon) + 1$ if $t^* \in I_{M(y,\varepsilon)+1}(y,\varepsilon)$.

3. If $t^* \in J_{M(\varepsilon)}(y,\varepsilon) \backslash \Big(I_{M(\varepsilon)}(y,\varepsilon) \cup I_{M(\varepsilon)+1}(y,\varepsilon)\Big)$, then we take a new $I_{M(\varepsilon)+1}(y,\varepsilon)$ such that $\overline{I_{M(\varepsilon)}(y,\varepsilon)} \cap \overline{I_{M(\varepsilon)+1}(y,\varepsilon)} = \emptyset$, $J_{M(\varepsilon)}(y,\varepsilon) \cap I_{M(\varepsilon)+1}(y,\varepsilon) \neq \emptyset$, $|I_{M(\varepsilon)+1}(y,\varepsilon)| = \varepsilon$ and $t^* \in I_{M(\varepsilon)+1}(y,\varepsilon)$. We set $N(y,\varepsilon) = M(y,\varepsilon) + 1$.

And hence, $\{I_q(y,\varepsilon)\}_{q=1}^{N(y,\varepsilon)}, \{J_q(y,\varepsilon)\}_{q=1}^{N(y,\varepsilon)-1}$ satisfy the conditions **(B1)**–**(B3)**.

Next, let us construct a partition of U. $f(t,y)$ does not vanish in $J_q(y,\varepsilon)$ by **(B8)**, so there exists an open cube $Q_{q,y}$ of center y such that $f(t,x) > 0$ in $\overline{J_q(y,\varepsilon) \times Q_{q,y}}$. Let $l_{y,\varepsilon}$ be the smallest length of sides of cubes $Q_{q,y}$ $(\, q = 1, \ldots, N(y,\varepsilon) - 1 \,)$. Let $Q_{y,\varepsilon}$ be the open cube of $\boldsymbol{R}_x^d$ of center y and of length of side $l_{y,\varepsilon}/2$. We define the set $Z_{f,K}$ to be

$$Z_{f,K} = \Big\{ (t,x) \in \overline{K}\ ;\ f(t,x) = 0 \Big\}.$$

Then, from **(B2)** and **(B8)**,

$$Z_{f,K} \subset \bigcup_{y\in U} \bigcup_{q=1}^{N(y,\varepsilon)} I_q(y,\varepsilon) \times Q_{y,\varepsilon}.$$

Since $I_q(y,\varepsilon) \times Q_{y,\varepsilon}$ is open and $Z_{f,K}$ is compact, we have

$$Z_{f,K} \subset \bigcup_{j=1}^{J(\varepsilon)} \bigcup_{q=1}^{N(y_j,\varepsilon)} I_q(y_j,\varepsilon) \times Q_{y_j,\varepsilon},$$

where $J(\varepsilon)$ is finite for every ε. Let $a_{j,q}$ be the middle point of $J_{q-1}(y_j,\varepsilon) \cup I_q(y_j,\varepsilon) \cup J_q(y_j,\varepsilon)$ for every (j,q). The family $\Big\{\{I_q(y_j,\varepsilon) \times Q_{y_j,\varepsilon}\}_{q=1}^{N(y_j,\varepsilon)}\Big\}_{j=1}^{J(\varepsilon)}$ obtained above satisfies the following four conditions:

(Q1) For any $\varepsilon > 0$ and any (j,q), we have

$$|t - a_{j,q}| \le \varepsilon \quad \text{for any } t \in \overline{J_{q-1}(y_j,\varepsilon) \cup I_q(y_j,\varepsilon) \cup J_q(y_j,\varepsilon)}.$$

(Q2) $f(t,x) > 0$ in $\overline{J_q(y_j,\varepsilon) \times Q_{y_j,\varepsilon}}$ for any $\varepsilon > 0$ and any (j,q).

(Q3) $f(t,x) > 0$ on $\overline{K} \setminus \Big\{\cup_{j=1}^{J(\varepsilon)} \cup_{q=1}^{N(y_j,\varepsilon)} I_q(y_j,\varepsilon) \times Q_{y_j,\varepsilon}\Big\}$ for any $\varepsilon > 0$.

Now we construct a partition of unity on $I \times U$ subordinate to $\Big\{\{I_q(y_j,\varepsilon) \times Q_{y_j,\varepsilon}\}_{q=1}^{N(y_j,\varepsilon)}\Big\}_{j=1}^{J(\varepsilon)}$. We choose a sequence of functions $\{\phi_{j,q}(t)\}_{q=1}^{N(y_j,\varepsilon)} \subset \boldsymbol{C}_0^\infty(\boldsymbol{R})$ such that

$$\begin{aligned}
&\phi_{j,1}(t) = 1 \text{ on } I_1(y_j,\varepsilon) \cap I,\ \phi_{j,1}(t) \in \boldsymbol{C}_0^\infty(I_1(y_j,\varepsilon) \cup J_1(y_j,\varepsilon)),\\
&\phi_{j,q}(t) = 1 \text{ on } I_q(y_j,\varepsilon),\\
&\qquad \phi_{j,q}(t) \in \boldsymbol{C}_0^\infty(J_{q-1}(y_j,\varepsilon) \cup I_q(y_j,\varepsilon) \cup J_q(y_j,\varepsilon)) \quad (\, q = 2,\dots,N(y_j,\varepsilon)-1\,),\\
&\phi_{j,N(y_j,\varepsilon)}(t) = 1 \text{ on } I_{N(y_j,\varepsilon)}(y_j,\varepsilon) \cap I,\\
&\qquad \phi_{j,N(y_j,\varepsilon)}(t) \in \boldsymbol{C}_0^\infty(J_{N(y_j,\varepsilon)-1}(y_j,\varepsilon) \cup I_{N(y_j,\varepsilon)}(y_j,\varepsilon))\\
&\text{and } 0 \le \phi_{j,q}(t) \le 1 \text{ for every } (j,q).
\end{aligned}$$

Let $Q^*_{y,\varepsilon}$ be the open cube of center y and of length of side $l_{y,\varepsilon}$. We see that $Q_{y,\varepsilon} \subset\subset Q^*_{y,\varepsilon}$. We choose a sequence of functions $\{\phi_j(x)\}_{j=1}^{J(\varepsilon)} \subset \boldsymbol{C}_0^\infty(\boldsymbol{R}^d)$ such that

$$\phi_j(x) = 1 \text{ on } Q_{y_j,\varepsilon},\ \phi_j(x) \in \boldsymbol{C}_0^\infty(Q^*_{y_j,\varepsilon}) \quad \text{and} \quad 0 \le \phi_j(x) \le 1 \quad (\, j = 1,\dots,J(\varepsilon)\,).$$

Then we set

$$\varphi_{j,q}(t,x) = \phi_{j,q}(t)\,\phi_j(x) \quad \text{for } j \in \{1,\dots,J(\varepsilon)\},\ q \in \{1,\dots,N(y_j,\varepsilon)\},$$

$$\varPhi(t,x) = \sum_{j=1}^{J(\varepsilon)} \sum_{q=1}^{N(y_j,\varepsilon)} \Big(\varphi_{j,q}(t,x)\Big)^2 + \prod_{j=1}^{J(\varepsilon)} \prod_{q=1}^{N(y_j,\varepsilon)} \Big(1 - \Big(\varphi_{j,q}(t,x)\Big)^2\Big)^2$$

and

$$\widetilde{\Phi}(t,x)=\prod_{j=1}^{J(\varepsilon)}\prod_{q=1}^{N(y_j,\varepsilon)}\left(1-\Big(\varphi_{j,q}(t,x)\Big)^2\right).$$

Note that $f(t,x)$ does not vanish on the support of $\partial_t\varphi_{j,q}$ for every (j,q). Since $\Phi(t,x)$ does not vanish in $\boldsymbol{R}^{d+1}$, so $\Big(\Phi(t,x)\Big)^{-1/2}\in\boldsymbol{C}^\infty(\boldsymbol{R}^{d+1})$. We set

$$\psi_{j,q}(t,x)=\varphi_{j,q}(t,x)\Big(\Phi(t,x)\Big)^{-1/2}\qquad\text{for } j\in\{1,\dots,J(\varepsilon)\},\ q\in\{1,\dots,N(y_j,\varepsilon)\}$$

and

$$\psi_*(t,x)=\widetilde{\Phi}(t,x)\Big(\Phi(t,x)\Big)^{-1/2}.$$

We see that $\psi_{j,q}\in\boldsymbol{C}_0^\infty\left((J_{q-1}(y_j,\varepsilon)\cup I_q(y_j,\varepsilon)\cup J_q(y_j,\varepsilon))\times Q^*_{y_j,\varepsilon}\right)$ and

$$\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\psi_{j,q}(t,x)^2+\psi_*(t,x)^2=1\quad\text{on }\boldsymbol{R}^{d+1}.$$

Moreover, $\operatorname{supp}\widetilde{\Phi}\subset\boldsymbol{R}^{d+1}\setminus Z_{f,K}$, so $\operatorname{supp}\psi_*\subset\boldsymbol{R}^{d+1}\setminus Z_{f,K}$.

Now we start evaluating $\|u\|$. Let ε,δ be positive numbers which we shall choose later. By using (3.13) in Lemma 3.3, $\|u\|^2$ is estimated as follows.

$$\begin{aligned}\|u\|^2 &= \sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\|\psi_{j,q}u\|^2+\|\psi_*u\|^2\\ &\le C_1\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\{\Big|\operatorname{Re}\Big(P\psi_{j,q}u,(t-a_{j,q})\psi_{j,q}u\Big)\Big|+\sum_{k=1}^n\left\||(t-a_{j,q})f|^{1/2}L_k\psi_{j,q}u\right\|^2\right\}\\ &\quad+\|\psi_*u\|^2,\end{aligned}$$

where C_1 is independent of (ε,δ). By Schwarz' inequality and **(Q1)**, we have

$$\begin{aligned}\|u\|^2 &\le C_1\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\{\delta\|P\psi_{j,q}u\|^2+\frac{\varepsilon^2}{4\delta}\|\psi_{j,q}u\|^2+\varepsilon\sum_{k=1}^n\left\||f|^{1/2}L_k\psi_{j,q}u\right\|^2\right\}+\|\psi_*u\|^2\\ &\le 2C_1\delta\|Pu\|^2+2C_1\delta\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\|[P,\psi_{j,q}]\,u\right\|^2+2C_1\varepsilon\sum_{k=1}^n\left\|f^{1/2}L_ku\right\|^2\\ &\quad+2C_1\varepsilon\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\|f^{1/2}[L_k,\psi_{j,q}]\,u\right\|^2+C_1\frac{\varepsilon^2}{4\delta}\|u\|^2+\|\psi_*u\|^2\end{aligned}$$

Now we obtain

$$\begin{aligned}(5.2)\qquad \|u\|^2 &\le 2C_1\delta\|Pu\|^2+2C_1\delta R_{1,\varepsilon}(u)+2C_1\varepsilon\sum_{k=1}^n\left\|f^{1/2}L_ku\right\|^2+2C_1\varepsilon R_{2,\varepsilon}(u)\\ &\quad+C_1\frac{\varepsilon^2}{4\delta}\|u\|^2+\|\psi_*u\|^2,\end{aligned}$$

where

$$R_{1,\varepsilon}(u)=\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\|[P,\psi_{j,q}]\,u\right\|^2 \text{ and } R_{2,\varepsilon}(u)=\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\|f^{1/2}\,[L_k,\psi_{j,q}]\,u\right\|^2.$$

Let us evaluate each term of $R_{1,\varepsilon}(u)$. We rewrite $[P,\psi_{j,q}]$ as

$$\begin{aligned}[P,\psi_{j,q}] &= (\partial_t\psi_{j,q})+f(t,x)\left(\sum_{k=1}^{n}b_{j,k,q,\varepsilon}L_k+b_{j,0,q,\varepsilon}\right)\\ &= \Phi^{-1/2}(\partial_t\varphi_{j,q})-\frac{1}{2}\Phi^{-3/2}(\partial_t\Phi)\varphi_{j,q}+f(t,x)\left(\sum_{k=1}^{n}b_{j,k,q,\varepsilon}L_k+b_{j,0,q,\varepsilon}\right)\end{aligned}$$

and rewrite $\partial_t\Phi$ as

$$\partial_t\Phi=2\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\varphi_{j,q}(\partial_t\varphi_{j,q})+2\left(\partial_t\widetilde{\Phi}\right)\widetilde{\Phi}.$$

Then we have

$$\begin{aligned}R_{1,\varepsilon}(u) &= \sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\left\|[P,\psi_{j,q}]\,u\right\|^2\\ &\leq C(K,\varepsilon)\left(\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\|(\partial_t\varphi_{j,q})u\|^2+\sum_{k=1}^{n}\|fL_ku\|^2+\|fu\|^2+\left\|\widetilde{\Phi}u\right\|^2\right),\end{aligned}\tag{5.3}$$

where $C(K,\varepsilon)$ depends only on (K,ε).

Next, we evaluate each term of $R_{2,\varepsilon}(u)$. Let ν be a positive number which we choose later. We define $\widetilde{\phi}_{1,\nu}(t,x)$ and $\widetilde{\phi}_{2,\nu}(t,x)$ to be

$$\widetilde{\phi}_{j,\nu}(t,x)=\widetilde{\phi}_j(f(t,x)/\nu)\quad(\,j=1,2\,),$$

where $\widetilde{\phi}_j$ is the function specified in §4. Obviously, $\widetilde{\phi}_{1,\nu}+\widetilde{\phi}_{2,\nu}=1$ on $\boldsymbol{R}^{d+1}$ and $|f(t,x)|\leq 2\nu$ on $\operatorname{supp}\widetilde{\phi}_{1,\nu}$. So $\left\|f^{1/2}\,[L_k,\psi_{j,q}]\,u\right\|^2$ is evaluated as follows.

$$\begin{aligned}\left\|f^{1/2}\,[L_k,\psi_{j,q}]\,u\right\|^2 &\leq C(K,j,k,q,\varepsilon)\left\|f^{1/2}u\right\|^2\\ &\leq 2C(K,j,k,q,\varepsilon)\left(\left\|f^{1/2}\widetilde{\phi}_{1,\nu}u\right\|^2+\left\|f^{1/2}\widetilde{\phi}_{2,\nu}u\right\|^2\right)\\ &\leq \widetilde{C}(K,j,k,q,\varepsilon)\left(\nu\|u\|^2+\left\|\widetilde{\phi}_{2,\nu}u\right\|^2\right).\end{aligned}$$

Thus, we obtain

$$R_{2,\varepsilon}(u)\leq C(K,\varepsilon)\left(\nu\|u\|^2+\left\|\widetilde{\phi}_{2,\nu}u\right\|^2\right).\tag{5.4}$$

Combining (5.2) with (5.3) and (5.4), we have

$$\begin{aligned}\|u\|^2 \leq\ & 2C_1\delta\|Pu\|^2+2C_1\varepsilon\sum_{k=1}^{n}\left\|f^{1/2}L_ku\right\|^2+\delta C_2(K,\varepsilon)\sum_{j=1}^{J(\varepsilon)}\sum_{q=1}^{N(y_j,\varepsilon)}\|(\partial_t\varphi_{j,q})u\|^2\\ &+\delta C_3(K,\varepsilon)\left(\sum_{k=1}^{n}\|fL_ku\|^2+\|fu\|^2\right)+\delta C_4(K,\varepsilon)\left\|\widetilde{\Phi}u\right\|^2\\ &+C_1\frac{\varepsilon^2}{4\delta}\|u\|^2+\nu C_5(K,\varepsilon)\|u\|^2+C_6(K,\varepsilon)\left\|\widetilde{\phi}_{2,\nu}u\right\|^2+\|\psi_*u\|^2,\end{aligned}$$

where $C_l(K,\varepsilon)$ $(\,l=2,\dots,6\,)$ depends only on (K,ε). Applying (3.2) in Lemma 3.1 to the second term on the right hand side, we have by Schwarz' inequality

$$\begin{aligned}\|u\|^2 \;&\le\; 2C_1\delta\,\|Pu\|^2 + C_7\varepsilon\,\|Pu\|^2 + \delta C_3(K,\varepsilon)\left(\sum_{k=1}^n \|fL_k u\|^2 + \|fu\|^2\right)\\ &\quad + \left(C_1\frac{\varepsilon^2}{4\delta} + \frac{5C_7\varepsilon}{4} + \nu C_5(K,\varepsilon)\right)\|u\|^2 + C_8(K,\varepsilon,\delta,\nu)\sum_{\psi\in\Gamma}\|\psi u\|^2\,,\end{aligned}$$

where C_7 depends only on K, $C_8(K,\varepsilon,\delta,\nu)$ depends only on $(K,\varepsilon,\delta,\nu)$ and Γ is the set

$$\left\{\,\widetilde{\Phi},\,\widetilde{\phi}_{2,\nu},\psi_*,\,(\partial_t\varphi_{j,q})\ (\,j=1,\dots,J(\varepsilon),\ q=1,\dots,N(y_j,\varepsilon)\,)\,\right\}.$$

Given $\mu>0$, we choose δ so small that $2C_1\delta<\mu^2/8$. Next, we choose a small ε in such a way that $C_1\varepsilon^2/(4\delta)+5C_7\varepsilon/4<1/4, C_7\varepsilon<\mu^2/8$. Moreover we choose ν so small that $\nu C_5(K,\varepsilon)<1/4$. Then we have

$$\|u\|^2 \le \frac{\mu^2}{2}\|Pu\|^2 + 2\delta C_3(K,\varepsilon)\left(\sum_{k=1}^n \|fL_k u\|^2 + \|fu\|^2\right) + 2C_8(K,\varepsilon,\delta,\nu)\sum_{\psi\in\Gamma}\|\psi u\|^2\,.$$

Given $N>0$, applying this inequality to χu for u, we have

$$\begin{aligned}\|\chi u\|^2 \;&\le\; \frac{\mu^2}{2}\|P\chi u\|^2 + 2\delta C_3(K,\varepsilon)\left(\sum_{k=1}^n \|fL_k\chi u\|^2 + \|f\chi u\|^2\right)\\ &\quad + 2C_8(K,\varepsilon,\delta,\nu)\sum_{\psi\in\Gamma}\|\psi\chi u\|^2 + C_9(K,N,\chi,\varepsilon,\delta,\nu)\,\|u\|_{-N}^2\end{aligned}$$

for all $u\in \boldsymbol{C}_0^\infty(K)$, where $C_9(K,N,\chi,\varepsilon,\delta,\nu)$ depends only on $(K,N,\chi,\varepsilon,\delta,\nu)$. Let λ be a positive number which we choose later. Applying Corollary 4.2 to the second term on the right hand side, we have

$$\begin{aligned}\text{(5.5)}\quad \|\chi u\|^2 \;&\le\; \left(\frac{\mu^2}{2} + 2\delta C_3(K,\varepsilon)\lambda\right)\|P\chi u\|^2 + 2\delta C_3(K,\varepsilon)\lambda\,\|\chi u\|^2\\ &\quad + 2C_8(K,\varepsilon,\delta,\nu)\sum_{\psi\in\Gamma}\|\psi\chi u\|^2 + C_{10}(K,N,\chi,\varepsilon,\delta,\nu,\lambda)\,\|u\|_{-N}^2\,,\end{aligned}$$

where $C_{10}(K,N,\chi,\varepsilon,\delta,\nu,\lambda)$ depends only on $(K,N,\chi,\varepsilon,\delta,\nu,\lambda)$.

We evaluate $\sum_{\psi\in\Gamma}\|\psi\chi u\|^2$. Suppose that a smooth function ψ satisfies

$$\text{(5.6)}\qquad f(t,x)\neq 0 \ \text{ on supp}\psi.$$

By the hypothesis (3°), the Lie algebra generated by $\{\partial_t,\{L_j\}_{j=1}^n\}$ is of dimension $d+1$ at every point of $\boldsymbol{R}^{d+1}$. Therefore, by hypothesis (2°), P has a subelliptic estimate in some neighborhood of the support of ψ, that is to say, there exist positive constants $\kappa=\kappa(K)$ and $C=C(K,\psi)$ such that

$$\|\psi u\|_\kappa^2 \le C\left(\|P\psi u\|^2 + \|\psi u\|^2\right) \quad \text{for all } u\in \boldsymbol{C}_0^\infty(K).$$

By interpolation inequality $\|v\|^2 \le \theta \|v\|_\kappa^2 + C(N,\theta)\|v\|_{-N}^2$, we have

$$\begin{aligned}\|\psi u\|^2 &\le \theta\, C(K,\psi)\,\|P\psi u\|^2 + \theta\, C(K,\psi)\,\|\psi u\|^2 + C(K,N,\theta,\psi)\,\|u\|_{-N}^2 \\ &\le \theta\, C_1(K,\psi)\,\|Pu\|^2 + 2\theta\, C(K,\psi)\,\|[P,\psi]\,u\|^2 \\ &\qquad + \theta\, C_2(K,\psi)\,\|u\|^2 + C(K,N,\theta,\psi)\,\|u\|_{-N}^2 \\ &\le \theta\, C_1(K,\psi)\,\|Pu\|^2 + \theta\, C_3(K,\psi)\left(\|fL_j u\|^2 + \|u\|^2\right) \\ &\qquad + \theta\, C_2(K,\psi)\,\|u\|^2 + C(K,N,\theta,\psi)\,\|u\|_{-N}^2 \\ &\le \theta\, C_4(K,\psi)\,\|Pu\|^2 + \theta\, C_5(K,\psi)\,\|u\|^2 + C(K,N,\theta,\psi)\,\|u\|_{-N}^2 \\ &\qquad (\text{ by (3.3) in Lemma 3.1 }).\end{aligned}$$

Applying this to χu for u, we obtain

$$\begin{aligned}\|\psi\chi u\|^2 &\le \theta\, C_4(K,\psi)\,\|P\chi u\|^2 + \theta\, C_5(K,\psi)\,\|\chi u\|^2 \\ &\qquad + C(K,N,\chi,\psi,\theta)\,\|u\|_{-N}^2 \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).\end{aligned} \tag{5.7}$$

Each element of Γ satisfies (5.6). Indeed, $\operatorname{supp}\widetilde{\Phi} \subset \boldsymbol{R}^{d+1} \setminus Z_{f,K}$ and $\operatorname{supp}\psi_* \subset \boldsymbol{R}^{d+1} \setminus Z_{f,K}$ by **(Q3)**, so $\widetilde{\Phi}$ and ψ_* satisfy (5.6). Since $f(t,x) > \nu > 0$ on $\operatorname{supp}\widetilde{\phi}_{2,\nu}$, $\widetilde{\phi}_{2,\nu}$ also satisfies (5.6). Since $\partial_t\varphi_{j,q}(t,x) = \phi_j(x)\partial_t\phi_{j,q}(t)$ and $\operatorname{supp}\partial_t\phi_{j,q} \subset \overline{J_{q-1}(y_j,\varepsilon) \cup J_q(y_j,\varepsilon)}$, $\operatorname{supp}(\partial_t\varphi_{j,q}) \subset \overline{(J_{q-1}(y_j,\varepsilon) \cup J_q(y_j,\varepsilon)) \times Q^*_{y_j,\varepsilon}}$. So, by **(Q2)**, $\partial_t\varphi_{j,q}$ satisfies (5.6) for every (j,q).

Combining (5.5) with (5.7), we have

$$\begin{aligned}\|\chi u\|^2 \;\le\; &\left(\frac{\mu^2}{2} + 2\delta C_3(K,\varepsilon)\lambda + C_{11}(K,\varepsilon,\delta,\nu)\,\theta\right)\|P\chi u\|^2 \\ &+\Big(2\delta C_3(K,\varepsilon)\lambda + C_{12}(K,\varepsilon,\delta,\nu)\,\theta\Big)\,\|\chi u\|^2 + C_{13}(K,N,\chi,\varepsilon,\delta,\nu,\lambda,\theta)\,\|u\|_{-N}^2\,,\end{aligned}$$

where $C_{11}(K,\varepsilon,\delta,\nu)$ and $C_{12}(K,\varepsilon,\delta,\nu)$ depend only on $(K,\varepsilon,\delta,\nu)$ and $C_{13}(K,N,\chi,\varepsilon,\delta,\nu,\lambda,\theta)$ depends only on $(K,N,\chi,\varepsilon,\delta,\nu,\lambda,\theta)$.

Finally, we choose λ,θ so small that $2\delta C_3(K,\varepsilon)\lambda + C_{11}(K,\varepsilon,\delta,\nu)\theta + C_{12}(K,\varepsilon,\delta,\nu)\theta < \min\{\mu^2/4, 1/4\}$. (Note that ε,δ,ν have already been chosen.) Then we obtain

$$\|\chi u\|^2 \le \mu^2\,\|P\chi u\|^2 + \frac{4}{3}\,C_{13}(K,N,\chi,\varepsilon,\delta,\nu,\lambda,\theta)\,\|u\|_{-N}^2 \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).$$

This completes the proof in the case where $(f, I\times U)$ is of type (α-2) or of type (β).

Next we prove (5.1) in the case where I and U satisfying $K \subset\subset I\times U$ can not be chosen so that $(f, I\times U)$ is of of type (α-2) or of type (β). Let (I,U) be a pair such that $K \subset\subset I\times U$. Then, I contains the origin and $(f, I\times U)$ is of **type ($\boldsymbol{\alpha}$-1)** by the hypothesis (1°). So we write $f(t) = f(t,x)$. For any $\varepsilon > 0$, we can find open intervals I_0, I_1, I_2, J_0, J_1 satisfying **(A1)**–**(A6)** below.

(A1) $\overline{I} \subset I_0 \cap J_0 \cap I_1 \cap J_1 \cap I_2$.

(A2) I_0 contains the left end point of $\overline{I}$, I_1 contains the origin and I_2 contains the right end point of $\overline{I}$.

(A3) $\overline{I_p} \cap \overline{I_q} = \emptyset$, and $J_0 \cap J_1 = \emptyset$ if $p \neq q$.

(A4) None of $J_0 \cap I_0$, $J_1 \cap I_1$, $J_0 \cap I_1$, $J_1 \cap I_2$ is empty.

(A5) $|J_0 \cap I_1 \cap J_1| \leq 2\varepsilon$

(A6) $f(t)$ does not vanish in $J_0 \cup J_1$.

Such open intervals can be chosen in the same way as in the proof of the case of type (α-2). We define $\widetilde{I}_0, \widetilde{I}_1, \widetilde{I}_2$ by $\widetilde{I}_0 = I_0 \cup J_0, \widetilde{I}_1 = J_0 \cup I_1 \cup J_1, \widetilde{I}_2 = J_1 \cup I_2$. Then $\left(f, \widetilde{I}_0 \times U\right)$ and $\left(f, \widetilde{I}_2 \times U\right)$ are of type (α-2). We find functions $\varphi_0, \varphi_1, \varphi_2 \in \boldsymbol{C}_0^\infty(\boldsymbol{R})$ satisfying

$$\varphi_q \in \boldsymbol{C}_0^\infty(\widetilde{I}_q), \quad \varphi_q \equiv 1 \text{ on } I_q \quad (q = 0, 1, 2), \quad \sum_{q=0}^{2} {\varphi_q}^2 \equiv 1 \text{ on } \overline{I}.$$

Now we start evaluating $\|\chi u\|$. Applying Lemma 3.3 (3.12) to $\varphi_1 u$, we have

$$\begin{aligned} \|u\|^2 &= \sum_{q=0}^{2} \|\varphi_q u\|^2 \\ &\leq \|\varphi_0 u\|^2 + C_1 \left(\left| \operatorname{Re}\left(P\varphi_1 u, t\varphi_1 u\right) \right| + \|t\varphi_1 u\|^2 \right) + \|\varphi_2 u\|^2 , \end{aligned}$$

where C_1 depends only on K. Substituting χu for u into this inequality, we have

$$\begin{aligned} \|\chi u\|^2 \leq\ & \|\varphi_0 \chi u\|^2 + C_1 \left(\left| \operatorname{Re}\left(P\varphi_1 \chi u, t\varphi_1 \chi u\right) \right| + \|t\varphi_1 \chi u\|^2 \right) \\ & + \|\varphi_2 \chi u\|^2 + C(K, N, \chi) \|u\|_{-N}^2 . \end{aligned}$$

Since $\left(f, \widetilde{I}_0 \times U)\right)$ and $\left(f, \widetilde{I}_2 \times U\right)$ are of type (α-2), applying Proposition 5.1 to $\varphi_0 \chi u$ and $\varphi_2 \chi u$ for u, we have

$$\begin{aligned} \|\chi u\|^2 \leq\ & \delta \|P\varphi_0 \chi u\|^2 + C_0(K, \delta, N) \|\varphi_0 \chi u\|_{-N}^2 + \delta \|P\varphi_2 \chi u\|^2 + C_2(K, \delta, N) \|\varphi_2 \chi u\|_{-N}^2 \\ & + C_1 \left(\left| \operatorname{Re}\left(P\varphi_1 \chi u, t\varphi_1 \chi u\right) \right| + \|t\varphi_1 \chi u\|^2 \right) \end{aligned}$$

for any $\delta > 0$, where $C_0(K, \delta, N)$ depends only on $(K, \operatorname{supp}\varphi_0, \delta, N)$ and $C_2(K, \delta, N)$ depends only on $(K, \operatorname{supp}\varphi_2, \delta, N)$. Thus, from the condition **(A5)**, we have by Schwarz' inequality

$$\begin{aligned} \|\chi u\|^2 &\leq \delta \sum_{q=0}^{2} \|P\varphi_q \chi u\|^2 + \left(\frac{C_1^2}{4\delta} + C_1 \right) \|t\varphi_1 \chi u\|^2 + C_3 \|u\|_{-N}^2 \\ &\leq 2\delta \|P\chi u\|^2 + 4\varepsilon^2 \left(\frac{C_1^2}{4\delta} + C_1 \right) \|\chi u\|^2 + 2\delta \sum_{q=0}^{2} \|(\partial_t \varphi_q)\chi u\|^2 + C_3 \|u\|_{-N}^2 , \end{aligned}$$

where C_3 depends only on $(K, \delta, N, \varphi_0, \varphi_2)$. Then, for any $\mu > 0$, we choose δ so small that $2\delta < \mu^2/4$. Next we take a small ε in such a way that $4\varepsilon^2 \left(C_1^2/(4\delta) + C_1\right) < 1/2$. Then we have

$$\|\chi u\|^2 \leq \frac{\mu^2}{2} \|P\chi u\|^2 + 4\delta \sum_{q=0}^{2} \|(\partial_t \varphi_q)\chi u\|^2 + 2C_3 \|u\|_{-N}^2 .$$

Finally, since f does not vanish on $\bigcup_{q=0}^{2} \mathrm{supp}\partial_t \varphi_q$ by the condition **(A6)**, we obtain (5.1) by using (5.7) as in the proof of type (α-2). □

6 The proof of Proposition 1.3

For the proof of Theorem **A**, it suffices to show that P satisfies Conditions (I)–(V) in §2 as is mentioned in §3. Since they are local conditions, we may assume, without loss of generality, that coefficients of P are bounded as well as their derivatives of any order. We assume always that the point where $f(t)$ changes sign is $t = 0$ in the case where (f, Ω) is of type (α-1).

6.1 Verification of Condition (I)

As is mentioned in §5, the inequality (5.1) in Proposition 5.1 is an improved version of (2.1) in Condition (I). So Condition (I) is satisfied for P in question.

6.2 Verification of Condition (II)

In this subsection, we verify that P satisfies Condition (II). Let us remember that this is the following:
“ For any $K \subset\subset \Omega$, any $\beta = (0, \beta') \in \{0\} \times \boldsymbol{Z}_+^d$ ($|\beta| \neq 0$), any $\mu > 0$, any $N > 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C_2 = C_2(K, \beta, \mu, N, \chi)$ such that

$$(6.1) \qquad \left\| \langle D_x \rangle^{-|\beta|} (P\chi)_{(\beta)} u \right\| \leq \mu \left\| P\chi u \right\| + C_2 \left\| u \right\|_{-N} \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).\text{”}$$

To verify this, we will use Lemma 3.1 and Proposition 5.1.

By an asymptotic expansion of the symbol, we have $P_{(\beta)}\chi - (P\chi)_{(\beta)} \in \mathcal{S}^{-\infty}$. Since $\left\| \langle D_x \rangle^{-|\beta|} \left(P_{(\beta)}\chi - (P\chi)_{(\beta)} \right) u \right\| \leq C(K, \beta, N, \chi) \|u\|_{-N}$, it suffices to show the inequality (6.1) with $(P\chi)_{(\beta)}$ replaced by $P_{(\beta)}\chi$. Namely, we shall prove that
“ For any $K \subset\subset \Omega$, any $\beta = (0, \beta') \in \{0\} \times \boldsymbol{Z}_+^d$ ($|\beta| \neq 0$), any $\mu > 0$, any $N > 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C_2 = C_2(K, \beta, \mu, N, \chi)$ such that

$$(6.2) \qquad \left\| \langle D_x \rangle^{-|\beta|} P_{(\beta)} \chi u \right\| \leq \mu \left\| P\chi u \right\| + C_2 \left\| u \right\|_{-N} \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).\text{”}$$

The proof of (6.2) is divided into two cases $|\beta| \geq 2$ and $|\beta| = 1$. First, suppose that $|\beta| \geq 2$. Since we can regard $\langle D_x \rangle^{-|\beta|} P_{(\beta)}$ as an element of $\mathcal{S}^0_{1/2,0}$ on suppχ, we have (6.2) by Proposition 5.1.

Next we shall prove (6.2) in the case where $|\beta| = 1$. Let $p(t, x, \tau, \xi)$ be the symbol of P and $\widetilde{L_j}(x, \xi)$ the symbol of L_j. Then $p(t, x, \tau, \xi)$ is written as

$$\begin{aligned} p(t, x, \tau, \xi) &= i\tau + f(t, x) \sum_{j,k=1}^{n} a_{j,k}(t, x)\, \widetilde{L_j}(x, \xi)\, \widetilde{L_k}(x, \xi) \\ &\quad + f(t, x) \sum_{j,k=1}^{n} a_{jk} \sum_{|\alpha|=1} \widetilde{L_j}^{(\alpha)}(x, \xi)\, \widetilde{L_{k(\alpha)}}(x, \xi). \end{aligned}$$

The symbol of second order part of $P_{(\beta)}$ (principal symbol of $P_{(\beta)}$) is written as follows:

$$
\begin{aligned}
(6.3)\quad \sigma_2(P_{(\beta)}) \;&=\; f(t,x)\sum_{j,k=1}^{n}\Big\{a_{jk\,(\beta)}(t,x)\,\widetilde{L_j}(x,\xi)\,\widetilde{L_k}(x,\xi)\\
&\quad + a_{jk}(t,x)\,\widetilde{L_j}_{(\beta)}(x,\xi)\,\widetilde{L_k}(x,\xi) + a_{jk}(t,x)\,\widetilde{L_j}(x,\xi)\,\widetilde{L_k}_{(\beta)}(x,\xi)\Big\}\\
&\quad + f_{(\beta)}(t,x)\sum_{j,k=1}^{n} a_{jk}(t,x)\,\widetilde{L_j}(x,\xi)\,\widetilde{L_k}(x,\xi).
\end{aligned}
$$

Since $p_{(\beta)}(t,x,\tau,\xi) - \sigma_2(P_{(\beta)})(t,x,\tau,\xi)$ is of class $S^1_{1,0}$ and does not depend on τ, $P_{(\beta)}$ is of the form

$$
(6.4)\qquad P_{(\beta)} = \sum_{j=1}^{n} M_{j,0,\beta} f L_j + \sum_{j=1}^{n}\sum_{k=1}^{d} M_{j,k,\beta}(\partial_{x_k} f) L_j + M_{0,0,\beta},
$$

where $M_{j,k,\beta} \in \mathcal{S}^1_{1,0}(\boldsymbol{R}^{d+1})$ and their symbols do not depend on τ. Note that $M_{j,k,\beta} = 0\ (k \ge 1)$ if (f,Ω) is of type (α-1) or of type (α-2). For each (j,k), $\langle D_x\rangle^{-1} M_{j,k,\beta}$ belongs to $\mathcal{S}^0_{0,0}$, so they are bounded on $L^2(K)$ and their operator norms depend only on K. Therefore, $\left\|\langle D_x\rangle^{-1} P_{(\beta)}\chi u\right\|^2$ is evaluated as follows:

$$
\left\|\langle D_x\rangle^{-1} P_{(\beta)}\chi u\right\|^2 \le C(K,\beta,N,\chi)
\begin{cases}
\left(\displaystyle\sum_{j=1}^{n}\|fL_j\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right)\\
\qquad \text{for the case of type (α-1) or of type (α-2),}\\
\left(\displaystyle\sum_{j=1}^{n}\|fL_j\chi u\|^2 \right.\\
\qquad \left. + \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{d}\|(\partial_{x_k} f)L_j\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right)\\
\qquad \text{for the case of type (β).}
\end{cases}
$$

By means of Proposition 5.1, it suffices to show that
" For any $K \subset\subset \Omega$, any $\mu > 0$, any $N > 0$ and any $\chi \in \mathcal{S}_\Psi$, there exists a constant $C = C(K,\mu,N,\chi)$ such that

$$
(6.5)\quad \mu^2\|P\chi u\|^2 + C\|u\|_{-N}^2 \;\ge\;
\begin{cases}
\displaystyle\sum_{j=1}^{n}\|fL_j\chi u\|^2\\
\qquad \text{for the case of type (α-1) or of type (α-2),}\\
\displaystyle\sum_{j=1}^{n}\|fL_j\chi u\|^2 + \sum_{j=1}^{n}\sum_{k=1}^{d}\|(\partial_{x_k} f)L_j\chi u\|^2\\
\qquad \text{for the case of type (β)}
\end{cases}
$$

for all $u \in \boldsymbol{C}_0^\infty(K)$. "
If (f,Ω) is of type (α-2), then we have by Lemma 3.1 (3.2)

$$
(6.6)\qquad \sum_{j=1}^{n}\|fL_j\chi u\|^2 \;\le\; C(K)\sum_{j=1}^{n}\left\||f|^{1/2}L_j\chi u\right\|^2
$$

$$\leq \ C'(K)\left(|\mathrm{Re}\,(P\chi u, \chi u)| + \|\chi u\|^2\right) + C(K,N,\chi)\,\|u\|_{-N}^2 .$$

If (f,Ω) is of type (β), then we have by (3.11) and Lemma 3.1 (3.2)

$$\text{(6.7)} \qquad \sum_{j=1}^{n} \|fL_j\chi u\|^2 + \sum_{j=1}^{n}\sum_{k=1}^{d} \|(\partial_{x_k} f)L_j\chi u\|^2 \ \leq \ C(K)\sum_{j=1}^{n}\left\||f|^{1/2}L_j\chi u\right\|^2$$
$$\leq \ C'(K)\left(|\mathrm{Re}\,(P\chi u, \chi u)| + \|\chi u\|^2\right) + C(K,N,\chi)\,\|u\|_{-N}^2 .$$

If (f,Ω) is of type (α-1), then we have by Lemma 3.1 (3.1) with $\rho = 1$

$$\text{(6.8)} \qquad \sum_{j=1}^{n} \|fL_j\chi u\|^2 \ \leq \ \sum_{j=1}^{n} \||f|L_j\chi u\|^2$$
$$\leq \ C(K)\left(|\mathrm{Re}\,(P\chi u, (\mathrm{sgn}\, t)|f|\chi u)| + \|\chi u\|^2\right) + C(K,N,\chi)\,\|u\|_{-N}^2 .$$

Therefore, by Schwarz' inequality, we have from (6.6), (6.7) and (6.8)

$$\varepsilon\,\|P\chi u\|^2 + C(\varepsilon,K)\,\|\chi u\|^2 + C(K,N,\chi)\,\|u\|_{-N}^2$$
$$\geq \begin{cases} \displaystyle\sum_{j=1}^{n} \|fL_j\chi u\|^2 \\ \qquad \text{for the case of type (α-1) or of type (α-2),} \\ \displaystyle\sum_{j=1}^{n} \|fL_j\chi u\|^2 + \sum_{j=1}^{n}\sum_{k=1}^{d} \|(\partial_{x_k} f)L_j\chi u\|^2 \\ \qquad \text{for the case of type (β)} \end{cases}$$

for any $\varepsilon, N > 0$ and for all $u \in \boldsymbol{C}_0^\infty(K)$. Finally, given $\mu > 0$, we choose ε so small that $\varepsilon < \mu^2/2$ and apply Proposition 5.1 to $\mu/\sqrt{4C(\varepsilon,K)}$ in place of μ, then we obtain (6.5). The proof of (6.2) is finished.

6.3 Verification of Condition (III)

In this subsection, we verify that P satisfies Condition (III). Let us remember that this is the following:
" For any $K \subset\subset \Omega$, any $\alpha \in \boldsymbol{Z}_+^{d+1}$, any $N > 0$ and any $\chi, \chi' \in S_\Psi$ satisfying $\chi \subset\subset \chi'$, there exists a constant $C_3 = C_3(K,\alpha,N,\chi,\chi')$ such that

$$\left\|(P\chi)^{(\alpha)}u\right\| \leq C_3\left(\|P\chi' u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K)."$$

To verify this, we shall use a left parametrix of P in the microlocal domain $\mathrm{supp}(1-\sigma(\chi))$.

By an expansion formula, we divide $(P\chi)^{(\alpha)}$ into two parts as follows.

$$(P\chi)^{(\alpha)} \ \equiv \ P^{(\alpha)}\chi + \sum_{\substack{\beta+\gamma=\alpha \\ \gamma\neq 0}} C_{\beta,\gamma}P^{(\beta)}\chi^{(\gamma)} \quad \mathrm{mod} \quad S^{-\infty}$$
$$\equiv \ J_1 + J_2 \quad \mathrm{mod} \quad S^{-\infty}.$$

Notice that, if $\alpha = 0$, only J_1 appears on the right hand side. We have to show two inequalities

$$
\begin{aligned}
(6.9) \qquad \|J_1 u\| &\leq C_1(K,\alpha,N,\chi,\chi')\left(\|P\chi' u\| + \|u\|_{-N}\right) \\
(6.10) \qquad \|J_2 u\| &\leq C_2(K,\alpha,N,\chi,\chi')\left(\|P\chi' u\| + \|u\|_{-N}\right).
\end{aligned}
$$

First, let us begin with (6.10). We take $\chi'' \in \mathcal{S}_\Psi$ satisfying $\chi'' \subset\subset \chi$. Then,

$$
(6.11) \qquad \chi'' \subset\subset \chi \subset\subset \chi'
$$

$$
(6.12) \qquad \sigma(\chi)^{(\gamma)} = \sigma(\chi)^{(\gamma)}\sigma(\chi')\Big(1-\sigma(\chi'')\Big) \quad (\gamma \neq 0).
$$

Moreover, since $|\xi|^2$ is equivalent to $|\tau|$ on $\mathrm{supp}\sigma(\chi)^{(\gamma)}$, we write γ as (γ_1,γ') and have

$$
(6.13) \qquad \left|\sigma(\chi)^{(\gamma)}(\tau,\xi)\right| \leq C_\gamma \langle\tau\rangle^{-\gamma_1}\langle\xi\rangle^{-|\gamma'|} \leq C'_\gamma \langle\xi\rangle^{-2\gamma_1-|\gamma'|}.
$$

By means of (1.4) in §1, P has a left parametrix Q in the microlocal domain $\mathrm{supp}(1-\chi'')$, that is to say,

$$
(6.14) \qquad (1-\chi'')u = (1-\chi'')QPu + Ru, \quad Q \in \mathcal{S}^{-1}_{1/2,0}, \quad R \in \mathcal{S}^{-\infty}.
$$

Then, by using (6.12) and (6.14), we rewrite each term of J_2 applied to u as follows.

$$
\begin{aligned}
(6.15) \quad P^{(\beta)}\chi^{(\gamma)}u &= P^{(\beta)}\chi^{(\gamma)}\chi'(1-\chi'')u \\
&= P^{(\beta)}\chi^{(\gamma)}\chi'(1-\chi'')QPu + S_{\beta\gamma}u \quad (\text{, where } S_{\beta\gamma} \in \mathcal{S}^{-\infty}\ ) \\
&= P^{(\beta)}\chi^{(\gamma)}(1-\chi'')QP\chi' u + P^{(\beta)}\chi^{(\gamma)}\left[\chi',(1-\chi'')QP\right]u + S_{\beta\gamma}u \\
&= P^{(\beta)}\chi^{(\gamma)}QP\chi' u + S'_{\beta\gamma}u,
\end{aligned}
$$

where $S'_{\beta\gamma} = P^{(\beta)}\chi^{(\gamma)}\left[\chi',(1-\chi'')QP\right] + S_{\beta\gamma}$. Since $P^{(\beta)}\chi^{(\gamma)}\left[\chi',(1-\chi'')QP\right]$ is a smoothing operator due to (6.11), so is $S'_{\beta\gamma}$. In view of (6.13), we see that $P^{(\beta)}\chi^{(\gamma)}Q \in \mathcal{S}^0_{1/2,0}$ for $\gamma \neq 0$. Therefore,

$$
\begin{aligned}
\|J_2 u\| &\leq \sum_{\substack{\beta+\gamma=\alpha \\ \gamma\neq 0}} C_{\beta,\gamma}\left\|P^{(\beta)}\chi^{(\gamma)}u\right\| \\
&\leq \sum_{\substack{\beta+\gamma=\alpha \\ \gamma\neq 0}} C_{\beta,\gamma}\left\|P^{(\beta)}\chi^{(\gamma)}QP\chi' u\right\| + C(K,\alpha,N,\chi,\chi')\|u\|_{-N} \\
&\leq C'(K,\alpha,N,\chi,\chi')\left(\|P\chi' u\| + \|u\|_{-N}\right).
\end{aligned}
$$

Thus, (6.10) is proved.

Next, we shall show (6.9). By the same way as in the preceding subsection, there exists a constant $C = C(K,\alpha)$ such that

$$
\left\|P^{(\alpha)}v\right\| \leq C\Big(\|Pv\| + \sum_{j=1}^{n}\|fL_j v\| + \|v\|\Big) \quad \text{for all } v \in \boldsymbol{C}_0^\infty(K).
$$

Applying this inequality to χu for v, we have by (6.5) and Proposition 5.1

$$\|J_1 u\| \le C(K,\alpha,N,\chi)\left(\|P\chi u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{6.16}$$

Notice that it is not $P\chi' u$ but $P\chi u$ which appears on the right hand side of (6.16). Since $\chi \subset\subset \chi'$, $\|P\chi u\|$ is evaluated as

$$\begin{aligned} \|P\chi u\| &= \|P\chi\chi' u\| = \left\|\chi P\chi' u + [P,\chi]\,\chi' u\right\| \\ &\le \|P\chi' u\| + \left\|[P,\chi]\,\chi' u\right\|. \end{aligned} \tag{6.17}$$

Thus, we shall prove that

$$\left\|[P,\chi]\,\chi' u\right\| \le C\left(\|P\chi' u\| + \|u\|_{-N}\right).$$

We take $\widetilde{\chi} \in \mathcal{S}_\Psi$ satisfying $\chi' \subset\subset \widetilde{\chi}$. Then we have $\chi^{(\gamma)} = \chi^{(\gamma)}\widetilde{\chi}(1-\chi'')$ $(\gamma \neq 0)$. The expansion formula yields

$$[P,\chi] \equiv \sum_{0<|\gamma|<2(N+2)} \frac{(-1)^{|\gamma|}}{\gamma!}\chi^{(\gamma)}P_{(\gamma)} \quad \mathrm{mod} \quad \mathcal{S}^{-N}_{1/2,0}.$$

By the same way as in the proof of (6.15), we have $\chi^{(\gamma)}P_{(\gamma)} = \chi^{(\gamma)}P_{(\gamma)}QP\widetilde{\chi} + \widetilde{\mathcal{S}_\gamma}$ for $\gamma \neq 0$, where $Q \in \mathcal{S}^{-1}_{1/2,0}$ and $\widetilde{\mathcal{S}_\gamma} \in \mathcal{S}^{-\infty}$. Since $\chi^{(\gamma)}P_{(\gamma)}Q \in \mathcal{S}^0_{1/2,0}$ for $\gamma \neq 0$, we obtain

$$\begin{aligned} \left\|[P,\chi]\,\chi' u\right\| &\le C(K,N,\chi)\left(\sum_{0<|\gamma|<2(N+2)} \left\|\chi^{(\gamma)}P_{(\gamma)}QP\widetilde{\chi}\chi' u\right\| + \|u\|_{-N}\right) \\ &\le C'(K,N,\chi)\left(\|P\chi' u\| + \|u\|_{-N}\right) \quad (\text{ by } \chi' \subset\subset \widetilde{\chi}\ ). \end{aligned}$$

Combining this inequality with (6.16) and (6.17), we have (6.9). This completes the verification of Condition (III).

6.4 Verification of Condition (IV)

In this subsection, we shall verify that P satisfies Condition (IV). We will use Proposition 4.1 in §4. Let us remember that Condition (IV) is the following:

" For any $(t_0,x_0) \in \Omega$ and any neighborhood U of (t_0,x_0), there exist $\phi,\psi \in \boldsymbol{C}_0^\infty(U)$ such that $\phi(t,x) = 1$ in a neighborhood of (t_0,x_0), $\phi \subset\subset \psi$ and that the inequality

$$\begin{aligned} &\left\|\langle D_x\rangle^\kappa P\chi\phi u\right\| \\ &\le C_4\Big(\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big) \text{ for all } u \in \boldsymbol{C}_0^\infty(K) \end{aligned}$$

holds for any open set $K \subset\subset \Omega$, any $N > 0$ and any $\chi,\chi' \in \mathcal{S}_\Psi$ $(\chi \subset\subset \chi')$, where $C_4 = C_4(K,N,\chi,\chi',\phi,\psi)$ is a constant depending on $(K,N,\chi,\chi',\phi,\psi)$ and κ is a positive number smaller than 1 depending only on K."

Let $\phi, \psi \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^{d+1})$ be such that $\phi \subset\subset \psi$ and κ be a positive number smaller than 1 which we choose later. Let $\chi, \chi' \in \mathcal{S}_\Psi$ be such that $\chi \subset\subset \chi'$. We rewrite $\langle D_x\rangle^\kappa P\chi\phi$ as

$$\langle D_x\rangle^\kappa P\chi\phi = \langle D_x\rangle^\kappa \phi P\chi + \langle D_x\rangle^\kappa [P,\phi]\,\chi + \langle D_x\rangle^\kappa [\chi,\phi]\,P + \langle D_x\rangle^\kappa \Big[P,[\chi,\phi]\Big].$$

Since $\langle D_x\rangle^\kappa[\chi,\phi] \in \mathcal{S}_{1/2,0}^{(\kappa-1)/2}$ by (6.13) and $[\chi,\phi]P - [\chi,\phi]P\chi' \in \mathcal{S}^{-\infty}$ by $\chi \subset\subset \chi'$, we have

$$\begin{aligned}(6.18)\quad \Big\|\langle D_x\rangle^\kappa P\chi\phi u\Big\| \;\le\; & C(K,N,\chi,\chi',\kappa,\phi)\Big(\Big\|\langle D_x\rangle^\kappa\phi P\chi u\Big\| + \Big\|\langle D_x\rangle^\kappa [P,\phi]\,\chi u\Big\| \\ & + \|P\chi' u\| + \Big\|\langle D_x\rangle^\kappa\Big[P,[\chi,\phi]\Big]u\Big\| + \|u\|_{-N}\Big).\end{aligned}$$

As in the preceding subsection, we have by taking $\chi'' \in \mathcal{S}_\Psi$ such that $\chi'' \subset\subset \chi$

$$\langle D_x\rangle^\kappa\Big[P,[\chi,\phi]\Big] = \langle D_x\rangle^\kappa\Big[P,[\chi,\phi]\Big](1-\chi'')QP\chi' + R,$$

where $Q \in \mathcal{S}_{1/2,0}^{-1}$ is a left parametrix of P in the microlocal domain $\mathrm{supp}(1-\chi'')$ and $R \in \mathcal{S}^{-\infty}$. Since $\langle D_x\rangle^\kappa[P,[\chi,\phi]](1-\chi'')Q \in \mathcal{S}_{1/2,0}^{\kappa-1}$, we have

$$(6.19)\qquad \Big\|\langle D_x\rangle^\kappa\Big[P,[\chi,\phi]\Big]\,u\Big\| \le C'(K,N,\chi,\chi',\kappa,\phi)\left(\|P\chi' u\| + \|u\|_{-N}\right).$$

Next, we rewrite $[P,\phi]$ as

$$[P,\phi] \;=\; \partial_t\phi + \sum_{j=1}^n M_j f L_j + M_0 f,$$

where $M_j \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^{d+1})$ ($j = 0,\dots,n$). Then we have

$$\begin{aligned}&\Big\|\langle D_x\rangle^\kappa[P,\phi]\,\chi u\Big\| \\ &\quad\le C(K,N,\phi)\Big(\Big\|\langle D_x\rangle^\kappa(\partial_t\phi)\chi u\Big\| + \sum_{j=1}^n\Big\|\langle D_x\rangle^\kappa f L_j\chi u\Big\| + \Big\|\langle D_x\rangle^\kappa f\chi u\Big\| + \|u\|_{-N}\Big).\end{aligned}$$

Combining (6.18) with (6.19) and this inequality, we have

$$\begin{aligned}\Big\|\langle D_x\rangle^\kappa P\chi\phi u\Big\| \;\le\; & C_1(K,N,\chi,\chi',\kappa,\phi)\Big(\Big\|\langle D_x\rangle^\kappa\phi P\chi u\Big\| + \|P\chi' u\| \\ & + \Big\|\langle D_x\rangle^\kappa(\partial_t\phi)\chi u\Big\| + \sum_{j=1}^n\Big\|\langle D_x\rangle^\kappa f L_j\chi u\Big\| + \Big\|\langle D_x\rangle^\kappa f\chi u\Big\| + \|u\|_{-N}\Big).\end{aligned}$$

If we choose a κ smaller than the κ in Proposition 4.1, we obtain by applying Proposition 4.1

$$\begin{aligned}(6.20)\quad \Big\|\langle D_x\rangle^\kappa P\chi\phi u\Big\| \;\le\; & C_3(K,N,\chi,\chi',\kappa,\phi)\Big(\Big\|\langle D_x\rangle^\kappa\phi P\chi u\Big\| + \Big\|\langle D_x\rangle^\kappa(\partial_t\phi)\chi u\Big\| \\ & + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big).\end{aligned}$$

Now we shall evaluate $\|\langle D_x\rangle^\kappa(\partial_t\phi)\chi u\|$. For any given point (t_0,x_0) in Ω and any given neighborhood U of (t_0,x_0), we choose $\phi,\psi \in \boldsymbol{C}_0^\infty(U)$ in the following way.

1. If $f(t_0, x_0) \neq 0$,
   we choose ϕ, ψ such that $\mathrm{supp}\psi \cap \{\, (t,x)\,;\ f(t,x) = 0\,\} = \emptyset$ and $\phi \subset\subset \psi$.

2. If $f(t_0, x_0) = 0$,
   then we choose an open interval $I \subset \boldsymbol{R}_t$ containing t_0 and an open set $V \subset \boldsymbol{R}^d_x$ containing x_0 such that $I \times V \subset U$. We take $\psi(t,x) \in \boldsymbol{C}^\infty_0(I \times V)$ such that $f(t,x) \neq 0$ on the support of $\partial_t \psi$ and $\psi = 1$ in a neighborhood of (t_0, x_0). This is possible because the set of points where $f(t, x_0)$ does not vanish is dense in $\boldsymbol{R}_t$ and $\{t\,;\ f(t,x_0) \neq 0\}$ is open. In fact, we choose open intervals I_0, J_0, J_1 such that $J_0 \setminus I_0 \neq \emptyset$, $J_0 \cap I_0 \neq \emptyset$, $J_1 \setminus I_0 \neq \emptyset$, $J_1 \cap I_0 \neq \emptyset$, $\overline{J_0} \cap \overline{J_1} = \emptyset$, $J_0 \cup I_0 \cup J_1 \subset I$, $f(t,x_0) > 0$ in $\overline{J_0} \cup \overline{J_1}$ and $t_0 \in I_0$. Since $f(t,x_0) > 0$ in $\overline{J_0} \cup \overline{J_1}$, we choose an open cube V_0 of $\boldsymbol{R}^d_x$ of center x_0 contained in V such that $f(t,x) > 0$ in $\{J_0 \cup J_1\} \times V_0$. Let V_1 be an open cube of $\boldsymbol{R}^d_x$ of center x_0 whose length of side is a half of one of V_0. We set $\psi(t,x) = \psi_1(t)\psi_2(x)$, where $\psi_1(t) \in \boldsymbol{C}^\infty_0(J_0 \cup I_0 \cup J_1)$ such that $\psi_1(t) = 1$ in I_0 and $\psi_2(x) \in \boldsymbol{C}^\infty_0(V_0)$ such that $\psi_2(x) = 1$ on V_1. Replacing U by $I_0 \times V_1$ and repeating this argument, we get a new ψ and so we choose again a ϕ. Obviously, $\phi \subset\subset \psi$.

In either case, $f(t,x) \neq 0$ on $\mathrm{supp}\partial_t\phi$. So P has a subelliptic estimate in some neighborhood of $\mathrm{supp}\partial_t\phi$, that is to say,

" There exist positive constants $\delta = \delta(K, \mathrm{supp}\partial_t\phi)$ and $C = C(K, \partial_t\phi)$ such that

$$\left\|(\partial_t\phi)\chi u\right\|_\delta \le C\Big(\left\|P(\partial_t\phi)\chi u\right\| + \left\|(\partial_t\phi)\chi u\right\|\Big) \quad \text{for all } u \in \boldsymbol{C}^\infty_0(K)."$$

We can set $\delta = \min\{2^{1-\widetilde{k}}, 1/2\}$ for example, where $\widetilde{k}$ is the number introduced in Lemma 3.5 in §3, so we may regard δ as depending only on K. We have by choosing a κ smaller than δ above

$$\begin{aligned}
&\left\|\langle D_x\rangle^\kappa (\partial_t\phi)\chi u\right\| \\
&\quad \le \ C(K,N,\chi,\phi)\Big(\left\|(\partial_t\phi)P\chi u\right\| + \left\|[P, \partial_t\phi]\,\chi u\right\| + \|(\partial_t\phi)\chi u\|\Big) \\
&\quad \le \ C_1(K,N,\chi,\phi)\Big(\|P\chi u\| + \sum_{j=1}^n \|fL_j\chi u\| + \|\chi u\|\Big) \\
&\quad \le \ C_2(K,N,\chi,\phi)\Big(\|P\chi u\| + \|\chi u\|\Big) \quad (\text{ by Lemma 3.1 }).
\end{aligned}$$

By Proposition 5.1, we have

$$\left\|\langle D_x\rangle^\kappa (\partial_t\phi)\chi u\right\| \le C_3(K,N,\chi,\phi)\left(\|P\chi u\| + \|u\|_{-N}\right). \tag{6.21}$$

Combining (6.20) with this inequality, we obtain

$$\begin{aligned}
&\left\|\langle D_x\rangle^\kappa P\chi\phi u\right\| \\
&\quad \le \ C(K,N,\chi,\chi',\kappa,\phi)\Big(\left\|\langle D_x\rangle^\kappa \phi P\chi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big).
\end{aligned} \tag{6.22}$$

Since $\phi \subset\subset \psi$, $\langle D_x\rangle^\kappa \phi = \langle D_x\rangle^\kappa \phi\psi$. Moreover $\langle D_x\rangle^\kappa \phi \langle D_x\rangle^{-\kappa} \in \mathcal{S}^0_{0,0}$. Therefore, Condition (IV) holds from (6.22). Verification of Condition (IV) is finished.

Remark. The inequality (6.22) holds even if we replace ϕ by ψ. That is to say, the following inequality holds.

$$\begin{aligned}(6.23)\quad & \left\|\langle D_x\rangle^\kappa P\chi\psi u\right\| \\ & \le C(K,N,\chi,\chi',\kappa,\psi)\Big(\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big) \\ & \qquad \text{for all } u \in \boldsymbol{C}_0^\infty(K).\end{aligned}$$

This is because it is sufficient for inequality (6.23) to hold that $f(t,x) \neq 0$ on the support of $\partial_t\psi$. The inequality (6.23) is needed for the proof of Proposition 6.1. (See Proposition 6.1 and Lemma 6.2 in the next subsection.)

6.5 Verification of Condition (V)

In this subsection, we shall verify that P satisfies Condition (V). Let us remember that Condition (V) is the following:
" For any $K \subset\subset \Omega$, any $\beta = (0,\beta') \in \{0\} \times \boldsymbol{Z}^d_+$ ($|\beta| \neq 0$), any $\mu > 0$, any $N > 0$, any $\chi, \chi' \in \mathcal{S}_\Psi$ ($\chi \subset\subset \chi'$), there exists a constant $C_5 = C_5(K,\beta,\mu,N,\chi,\chi',\psi)$ such that

$$\begin{aligned}(6.24)\quad & \left\|\langle D_x\rangle^{\kappa-|\beta|}(\psi P\chi)_{(\beta)}u\right\| \\ & \le \mu\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| + C_5\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right) \text{ for all } u \in \boldsymbol{C}_0^\infty(K),\end{aligned}$$

where (ψ,κ) is the same as in (IV). "
To verify this, we will make use of Propositions 4.1 and 5.1.

Since $(\psi P\chi)_{(\beta)} - (\psi P)_{(\beta)}\chi \in \mathcal{S}^{-\infty}$, we shall prove that
" For any $K \subset\subset \Omega$, any $\beta = (0,\beta')$ ($\beta' \neq 0$), any $\mu > 0$, any $N > 0$ and any $\chi, \chi' \in \mathcal{S}_\Psi$ ($\chi \subset\subset \chi'$), there exists a constant $C = C(K,\beta,\mu,N,\chi,\chi',\psi)$ such that

$$\begin{aligned}(6.25)\quad & \left\|\langle D_x\rangle^{\kappa-|\beta|}(\psi P)_{(\beta)}\chi u\right\| \\ & \le \mu\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| + C\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K),\end{aligned}$$

where (ψ,κ) is the same as in (IV). "

Remark. If (f,Ω) is of type (α-1) or of type (α-2), we can prove that P satisfies the following condition (V$'$).

(V′) For any $K \subset\subset \Omega$, any $\beta = (0, \beta') \in \{0\} \times \boldsymbol{Z}_+^d$ ($|\beta| \neq 0$), any $N > 0$, any $\chi \in \mathcal{S}_\Psi$ ($\chi \subset\subset \chi'$) and any $\psi \in \boldsymbol{C}_0^\infty(\Omega)$, there exists a constant $C = C(K, \beta, N, \chi, \psi)$ such that

$$\left\| \langle D_x \rangle^{\kappa - |\beta|} (\psi P)_{(\beta)} \chi u \right\| \leq C \left(\|P\chi u\| + \|u\|_{-N} \right) \text{ for all } u \in \boldsymbol{C}_0^\infty(K),$$

where κ is the number specified in (IV).

Obviously, Condition (V) holds if Condition (V′) is satisfied.

The proof of (6.25) is divided into three cases $|\beta| \geq 3$, $|\beta| = 1$ and $|\beta| = 2$.

Case 1: $|\beta| \geq 3$

Since we can regard $\langle D_x \rangle^{\kappa - |\beta|} (\psi P)_{(\beta)}$ as an element of $\mathcal{S}^{\kappa-1}_{1/2,0}$ in the microlocal domain suppχ, so (6.25) holds by Proposition 5.1.

Case 2: $|\beta| = 2$

We rewrite $(\psi P)_{(\beta)}$ as

$$\begin{aligned} (\psi P)_{(\beta)} &= \psi_{(\beta)} P + \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1, \beta_2 \neq 0}} \psi_{(\beta_1)} P_{(\beta_2)} + \psi P_{(\beta)} \\ &= \widetilde{P}_1(\psi, \beta) + \widetilde{P}_2(\psi, \beta) + \widetilde{P}_3(\psi, \beta). \end{aligned}$$

To prove (6.25) in this case, we shall show that each of $\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_1(\psi, \beta) \chi u \right\|$, $\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_2(\psi, \beta) \chi u \right\|$ and $\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_3(\psi, \beta) \chi u \right\|$ does not exceed the right hand side of (6.25). The first term $\mu \left\| \langle D_x \rangle^{\kappa} \psi P \chi u \right\|$ on the right hand side of (6.25) is needed only to evaluate $\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_3(\psi, \beta) \chi u \right\|$.

Estimate of $\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_1(\psi, \beta) \chi u \right\|$

Since $\langle D_x \rangle^{\kappa-2} \psi_{(\beta)} \in \mathcal{S}^{(\kappa-2)/2}_{1/2,0}$ on suppχ, we have

$$\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_1(\psi, \beta) \chi u \right\| \leq C_1(K, \beta, N, \chi, \psi) \left(\|P\chi u\| + \|u\|_{-N} \right). \tag{6.26}$$

Estimate of $\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_2(\psi, \beta) \chi u \right\|$

We rewrite $\widetilde{P}_2(\psi, \beta)$ as

$$\widetilde{P}_2(\psi, \beta) = \widetilde{M_0} f + \sum_{k=1}^{d} \widetilde{M_k} (\partial_{x_k} f) + \widetilde{N},$$

where $\widetilde{M_k} \in \mathcal{S}^2_{1,0}$ ($k = 0, \ldots, d$), $\widetilde{N} \in \mathcal{S}^1_{1,0}$ and their symbols do not depend on τ. Since $\langle D_x \rangle^{\kappa-2} \widetilde{M_k} \in \mathcal{S}^{\kappa}_{1/2,0}$ on suppχ ($k = 0, \ldots, d$) and $\langle D_x \rangle^{\kappa-2} \widetilde{N} \in \mathcal{S}^{(\kappa-1)/2}_{1/2,0}$ on suppχ, we have

$$\begin{aligned} &\left\| \langle D_x \rangle^{\kappa-2} \widetilde{P}_2(\psi, \beta) \chi u \right\| \\ &\quad \leq C(K, \beta, N, \chi, \psi) \left(\left\| \langle D_x \rangle^{\kappa} f \chi u \right\| + \sum_{k=1}^{d} \left\| \langle D_x \rangle^{\kappa} (\partial_{x_k} f) \chi u \right\| + \|u\|_{-N} \right). \end{aligned}$$

Therefore, Proposition 4.1 yields

$$\left\|\langle D_x\rangle^{\kappa-2}\widetilde{P}_2(\psi,\beta)\chi u\right\| \le C_2(K,\beta,N,\chi,\psi)\left(\|P\chi u\| + \|u\|_{-N}\right). \tag{6.27}$$

Estimate of $\left\|\langle \boldsymbol{D_x}\rangle^{\kappa-2}\widetilde{\boldsymbol{P}}_3(\boldsymbol{\psi},\boldsymbol{\beta})\boldsymbol{\chi u}\right\|$

Here we shall prove that

" For any $K \subset\subset \Omega$, any $\mu > 0$, any $N > 0$ and any $\chi, \chi' \in S_\Psi$ ($\chi \subset\subset \chi'$), there exists a constant $C_3 = C_3(K,\mu,N,\chi,\chi',\psi)$ such that

$$\begin{aligned} &\left\|\langle D_x\rangle^{\kappa-2}\widetilde{P}_3(\psi,\beta)\chi u\right\| \\ &\quad\le \mu\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| + C_3\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K)." \end{aligned} \tag{6.28}$$

The proof is divided into two cases where f depends on x or not. Suppose that f does not depend on x, that is to say, $(\boldsymbol{f},\boldsymbol{\Omega})$ **is of type** $(\boldsymbol{\alpha}$**-1) or of type** $(\boldsymbol{\alpha}$**-2).** Then, $\widetilde{P}_3(\psi,\beta)$ is of the form $\psi\widetilde{P}f(t)$, where $\widetilde{P} \in S^2_{1,0}$ and $\langle D_x\rangle^{\kappa-2}\widetilde{P}\langle D_x\rangle^{-\kappa} \in S^0_{0,0}$. Then we have

$$\left\|\langle D_x\rangle^{\kappa-2}\widetilde{P}_3(\psi,\beta)\chi u\right\| \le C(K,\beta,\psi)\left\|\langle D_x\rangle^{\kappa}f\chi u\right\|.$$

By Proposition 4.1, we obtain

$$\left\|\langle D_x\rangle^{\kappa-2}\widetilde{P}_3(\psi,\beta)\chi u\right\| \le C(K,\beta,\psi)\left(\|P\chi u\| + \|u\|_{-N}\right).$$

This implies (6.28).

Next, suppose that f depends on x, that is to say, $(\boldsymbol{f},\boldsymbol{\Omega})$ **is of type** $(\boldsymbol{\beta})$**.** We rewrite $\psi P_{(\beta)}$ as $\psi P_{(\beta)} = P_{(\beta)}\psi + \left[\psi, P_{(\beta)}\right]$ and have

$$\left\|\langle D_x\rangle^{\kappa-2}\widetilde{P}_3(\psi,\beta)\chi u\right\| \le \left\|\langle D_x\rangle^{\kappa-2}P_{(\beta)}\psi\chi u\right\| + \left\|\langle D_x\rangle^{\kappa-2}\left[\psi, P_{(\beta)}\right]\chi u\right\|.$$

Since $\langle D_x\rangle^{\kappa-2}P_{(\beta)}\langle D_x\rangle^{-\kappa} \in S^0_{0,0}$ and $\langle D_x\rangle^{\kappa-2}\left[\psi, P_{(\beta)}\right] \in S^{(\kappa-1)/2}_{1/2,0}$ on suppχ, applying Proposition 5.1, we obtain

$$\begin{aligned} &\left\|\langle D_x\rangle^{\kappa-2}\widetilde{P}_3(\psi,\beta)\chi u\right\| \\ &\quad\le C_4(K)\left\|\langle D_x\rangle^{\kappa}\psi\chi u\right\| + C_5(K,N,\chi,\chi',\psi)\left(\|P\chi u\| + \|u\|_{-N}\right). \end{aligned} \tag{6.29}$$

Therefore, it suffices for the proof of (6.28) to show the following proposition.

Proposition 6.1 *Assume that* (f,Ω) *is of type* (β)*. For any* $K \subset\subset \Omega$*, any* $\mu > 0$*, any* $N > 0$ *and any* $\chi, \chi' \in S_\Psi$ $(\chi \subset\subset \chi')$*, there exists a constant* $C_6 = C_6(K,\mu,N,\chi,\chi',\psi)$ *such that*

$$\left\|\langle D_x\rangle^{\kappa}\chi\psi u\right\| \le \mu\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| + C_6\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K), \tag{6.30}$$

where (ψ,κ) *is the same as in the verification of Condition* (IV)*.*

(6.28) holds if we admit this proposition. This is because $\|\langle D_x\rangle^\kappa \psi\chi u\|$ does not exceed $\|\langle D_x\rangle^\kappa \chi\psi u\| + \|\langle D_x\rangle^\kappa [\psi,\chi] u\|$. Moreover $\langle D_x\rangle^\kappa [\psi,\chi] \in \mathcal{S}^{(\kappa-1)/2}_{1/2,0}$ and $\langle D_x\rangle^\kappa [\psi,\chi] - \langle D_x\rangle^\kappa [\psi,\chi]\chi' \in \mathcal{S}^{-\infty}$ due to $\chi \subset\subset \chi'$. So we have by Proposition 5.1

$$\left\|\langle D_x\rangle^\kappa \psi\chi u\right\| \le \left\|\chi\langle D_x\rangle^\kappa \psi u\right\| + C_7(K,N,\chi,\chi',\psi)\left(\|P\chi' u\| + \|u\|_{-N}\right).$$

Combining (6.29) with (6.30) and this inequality, we obtain (6.28).

Proof of Proposition 6.1. Applying Proposition 5.1 to $\langle D_x\rangle^\kappa \psi u$ for u, we have for any $\mu > 0$

$$\left\|\langle D_x\rangle^\kappa \chi\psi u\right\| \le \mu\left\|P\langle D_x\rangle^\kappa \chi\psi u\right\| + C_8(K,\mu,N,\chi,\psi)\,\|u\|_{-N} \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{6.31}$$

(Note that χ commutes with $\langle D_x\rangle^\kappa$.)
We shall treat the first term on the right hand side. Since $P\langle D_x\rangle^\kappa$ is rewritten as $P\langle D_x\rangle^\kappa = \langle D_x\rangle^\kappa P + [P,\langle D_x\rangle^\kappa]$, we have

$$\left\|P\langle D_x\rangle^\kappa \chi\psi u\right\| \le \left\|\langle D_x\rangle^\kappa P\chi\psi u\right\| + \left\|[P,\langle D_x\rangle^\kappa]\,\chi\psi u\right\|.$$

Applying (6.23) in the remark of the preceding subsection to the first term on the right hand side, we have

$$\begin{aligned}\left\|P\langle D_x\rangle^\kappa \chi\psi u\right\| \;\le\;& \left\|[P,\langle D_x\rangle^\kappa]\,\chi\psi u\right\| \\ &+ C_9(K,N,\chi,\chi',\psi)\Big(\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big).\end{aligned}$$

Combining this inequality with (6.31), we obtain

$$\begin{aligned}\left\|\chi\langle D_x\rangle^\kappa \psi u\right\| \;\le\;& \mu\left\|[P,\langle D_x\rangle^\kappa]\,\chi\psi u\right\| + \mu C_9\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| \\ &+ C_{10}(K,\mu,N,\chi,\chi',\psi)\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right).\end{aligned} \tag{6.32}$$

To evaluate $\|[P,\langle D_x\rangle^\kappa]\,\chi\psi u\|$, we need the following lemma.

Lemma 6.2 *Assume that (f,Ω) is of type (β). For any $K \subset\subset \Omega$, any $N > 0$ and any $\chi,\chi' \in \mathcal{S}_\Psi$ $(\chi \subset\subset \chi')$, there exists a constant $C_{11} = C_{11}(K,N,\chi,\chi',\psi)$ such that*

$$\begin{aligned}\left\|[P,\langle D_x\rangle^\kappa]\,\chi\psi u\right\| \;\le\;& C_{11}\Big(\left\|\langle D_x\rangle^\kappa \psi P\chi u\right\| + \left\|\langle D_x\rangle^\kappa \chi\psi u\right\| \\ &+ \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K),\end{aligned} \tag{6.33}$$

where (ψ,κ) is the same as in the verification of Condition (IV).

We admit Lemma 6.2 for the moment. Lemma 6.2 will be proved later. From (6.32) and (6.33), we have

$$\begin{aligned}\Big\|\langle D_x\rangle^{\kappa}\chi\psi u\Big\| \;&\le\; \mu\,(C_9+C_{11})\,\Big\|\langle D_x\rangle^{\kappa}\psi P\chi u\Big\| + \mu C_{11}\Big\|\langle D_x\rangle^{\kappa}\chi\psi u\Big\| \\ &\qquad + C_{12}(K,\mu,N,\chi,\chi',\psi)\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right).\end{aligned}$$

Finally, given $\mu' > 0$, we choose a small μ in such a way that $\mu C_{11} < 1$ and $\mu(C_9 + C_{11})/(1-\mu C_{11}) < \mu'$. Then we obtain

$$\begin{aligned}\Big\|\langle D_x\rangle^{\kappa}\chi\psi u\Big\| \;&\le\; \mu'\Big\|\langle D_x\rangle^{\kappa}\psi P\chi u\Big\| \\ &\qquad + C_{13}(K,N,\chi,\chi',\psi,\mu')\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right).\end{aligned}$$

This is equivalent to (6.30). The proof of Proposition 6.1 is completed. □

Proof of Lemma 6.2. In view of (4.18), we rewrite $[P,\langle D_x\rangle^{\kappa}]$ as

$$\Big[P,\langle D_x\rangle^{\kappa}\Big] \;=\; \sum_{j=1}^{n} M_{j,0} f L_j + \sum_{j=1}^{n}\sum_{k=1}^{d} M_{j,k}(\partial_{x_k} f)L_j + M_{0,0},$$

where $M_{j,k}\in \mathcal{S}^{\kappa}_{1/2,0}$ on $\mathrm{supp}\chi$. Then we have

$$\begin{aligned}(6.34)\quad \Big\|[P,\langle D_x\rangle^{\kappa}]\,\chi\psi u\Big\| \;&\le\; C_{14}(K,N,\chi)\Big(\sum_{j=1}^{n}\Big\|\langle D_x\rangle^{\kappa} f L_j\chi\psi u\Big\| \\ &\qquad + \sum_{j=1}^{n}\sum_{k=1}^{d}\Big\|\langle D_x\rangle^{\kappa}(\partial_{x_k} f)L_j\chi\psi u\Big\| + \Big\|\langle D_x\rangle^{\kappa}\chi\psi u\Big\|\Big).\end{aligned}$$

By Proposition 4.1, the first term on the right hand side of (6.34) is estimated as

$$\sum_{j=1}^{n}\Big\|\langle D_x\rangle^{\kappa} f L_j\chi\psi u\Big\| \le C_{15}(K,N,\chi,\psi)\Big(\Big\|\langle D_x\rangle^{\kappa} P\chi\psi u\Big\| + \|u\|_{-N}\Big).$$

Applying (6.23) to the first term on the right hand side, we have

$$\begin{aligned}(6.35)\quad &\sum_{j=1}^{n}\Big\|\langle D_x\rangle^{\kappa} f L_j\chi\psi u\Big\| \\ &\quad\le\; C_{16}(K,N,\chi,\chi',\psi)\Big(\Big\|\langle D_x\rangle^{\kappa}\psi P\chi u\Big\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big).\end{aligned}$$

Next we evaluate the second term on the right hand side of (6.34). $\langle D_x\rangle^{\kappa}(\partial_{x_k} f)L_j$ is rewritten as $\langle D_x\rangle^{\kappa}(\partial_{x_k} f)L_j = (\partial_{x_k} f)L_j\langle D_x\rangle^{\kappa} + [\langle D_x\rangle^{\kappa},(\partial_{x_k} f)L_j]$. Since $[\langle D_x\rangle^{\kappa},(\partial_{x_k} f)L_j] \in \mathcal{S}^{\kappa}_{1/2,0}$ on $\mathrm{supp}\chi$ and $|\partial_{x_k} f| \le C(K)|f|^{1/2}$ due to (4.5), we have

$$\begin{aligned}(6.36)\;\sum_{j=1}^{n}\sum_{k=1}^{d}\Big\|\langle D_x\rangle^{\kappa}(\partial_{x_k} f)L_j\chi\psi u\Big\| \;&\le\; C_{17}(K)\sum_{j=1}^{n}\Big\||f|^{1/2}L_j\langle D_x\rangle^{\kappa}\chi\psi u\Big\| \\ &\qquad + C_{18}(K,N,\chi,\psi)\Big(\Big\|\langle D_x\rangle^{\kappa}\chi\psi u\Big\| + \|u\|_{-N}\Big).\end{aligned}$$

Applying Lemma 3.1 (3.2) to the first term on the right hand side of (6.36) for $\langle D_x\rangle^\kappa \chi\psi u$ in place of u, we have

$$\begin{aligned}&\sum_{j=1}^{n} \left\| |f|^{1/2} L_j \langle D_x\rangle^\kappa \chi\psi u \right\|^2 \\ &\quad \le \; C_{19}(K,N,\chi,\psi)\Big(\Big| \operatorname{Re}\Big(P\langle D_x\rangle^\kappa \chi\psi u, \langle D_x\rangle^\kappa \chi\psi u\Big) \Big| + \left\|\langle D_x\rangle^\kappa \chi\psi u\right\|^2 + \|u\|_{-N}^2 \Big).\end{aligned}$$

Let λ be a positive number. By Schwarz' inequality, the above inequality yields

$$\begin{aligned}\sum_{j=1}^{n} \left\| |f|^{1/2} L_j \langle D_x\rangle^\kappa \chi\psi u \right\|^2 \;&\le\; \lambda^2 \left\| [P, \langle D_x\rangle^\kappa]\, \chi\psi u \right\|^2 + \lambda^2 \left\| \langle D_x\rangle^\kappa P\chi\psi u \right\|^2 \\ &\quad + C_{20}(K,N,\chi,\psi,\lambda)\Big(\left\|\langle D_x\rangle^\kappa \chi\psi u\right\|^2 + \|u\|_{-N}^2 \Big).\end{aligned}$$

Combining (6.36) with this inequality and applying (6.23) to the second term on the right hand side, we obtain

$$\begin{aligned}(6.37)\quad &\sum_{j=1}^{n}\sum_{k=1}^{d} \left\| \langle D_x\rangle^\kappa (\partial_{x_k} f) L_j \chi\psi u \right\| \\ &\le \; \lambda C_{17}(K) \left\| [P, \langle D_x\rangle^\kappa]\, \chi\psi u \right\| + \lambda C_{21}(K,N,\chi,\chi',\psi) \left\| \langle D_x\rangle^\kappa \psi P\chi u \right\| \\ &\qquad + C_{22}(K,N,\chi,\chi',\psi,\lambda)\Big(\left\|\langle D_x\rangle^\kappa \chi\psi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N} \Big).\end{aligned}$$

We have from (6.34), (6.35) and (6.37)

$$\begin{aligned}&\left\| [P, \langle D_x\rangle^\kappa]\, \chi\psi u \right\| \\ &\quad \le \; \lambda C_{17}(K) \left\| \Big[P, \langle D_x\rangle^\kappa\Big]\, \chi\psi u \right\| + C_{23}(K,N,\chi,\chi',\psi,\lambda) \\ &\qquad \times \Big(\left\| \langle D_x\rangle^\kappa \psi P\chi u \right\| + \left\|\langle D_x\rangle^\kappa \chi\psi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N} \Big).\end{aligned}$$

We obtain (6.33) by setting $\lambda = 1/(2C_{15}(K))$. □

Now we return to the proof of (6.25) in the case where $|\beta| = 2$. (6.25) follows from three inequalities (6.26), (6.27) and (6.28). Condition (V) for P in question is verified in this case.

Remark. Here we have a corollary to Proposition 6.1. This is used in the proof of (6.25) for the case where $|\beta| = 1$.

Corollary 6.3 *Assume that (f, Ω) is of type (β). For any $K \subset\subset \Omega$, any $\mu > 0$, any $N > 0$ and any $\chi, \chi' \in S_\Psi$ ($\chi \subset\subset \chi'$), there exists a constant $C = C(K,\mu,N,\chi,\chi',\psi)$ such that*

$$\begin{aligned}(6.38)\quad &\sum_{j=1}^{n}\sum_{k=1}^{d} \left\| \langle D_x\rangle^\kappa (\partial_{x_k} f) L_j \chi\psi u \right\| \\ &\le \; \mu \left\| \langle D_x\rangle^\kappa \psi P\chi u \right\| + C\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N} \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K),\end{aligned}$$

where (ψ, κ) is the same as in the verification of Condition (IV).

Proof of Corollary 6.3. Let us remember that the inequality (6.37) is the following:

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)L_j\chi\psi u\right\| \\
&\le \lambda C_{17}(K)\left\|[P,\langle D_x\rangle^{\kappa}]\,\chi\psi u\right\| + \lambda C_{21}(K,N,\chi,\chi',\psi)\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| \\
&\qquad + C_{22}(K,N,\chi,\chi',\psi,\lambda)\Big(\left\|\langle D_x\rangle^{\kappa}\chi\psi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big).
\end{aligned}$$

Applying (6.33) in Lemma 6.2 to the first term on the right hand side, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)L_j\chi\psi u\right\| \\
&\le \lambda C_{24}(K,N,\chi,\chi',\psi)\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| \\
&\qquad + C_{25}(K,N,\chi,\chi',\psi,\lambda)\Big(\left\|\langle D_x\rangle^{\kappa}\chi\psi u\right\| + \|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\Big).
\end{aligned}$$

Let λ' be a positive number and set $\lambda = \lambda'/(2C_{24})$. We apply Proposition 6.1 to $\lambda'/(2C_{25})$ in place of μ. Then we obtain

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)L_j\chi\psi u\right\| \\
&\le \lambda'\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| + C_{26}(K,\lambda',N,\chi,\chi,\psi)\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right).
\end{aligned}$$

This is equivalent to (6.38). □

We return to the proof of (6.25) which is our purpose in this subsection. The remaining case is $|\beta| = 1$.

Case 3: $|\beta| = 1$

We rewrite $(\psi P)_{(\beta)}$ as $(\psi P)_{(\beta)} = \psi_{(\beta)}P + \psi P_{(\beta)}$ and have

$$\left\|\langle D_x\rangle^{\kappa-1}(\psi P)_{(\beta)}\chi u\right\| \le \left\|\langle D_x\rangle^{\kappa-1}\psi_{(\beta)}P\chi u\right\| + \left\|\langle D_x\rangle^{\kappa-1}\psi P_{(\beta)}\chi u\right\|.$$

Since $\langle D_x\rangle^{\kappa-1}\psi_{(\beta)} \in S^{(\kappa-1)/2}_{1/2,0}$ on suppχ, the first term on the right hand side does not exceed $C(K,\beta,\chi,N)\left(\|P\chi u\| + \|u\|_{-N}\right)$. So, it suffices for the proof of (6.25) to show the following inequality:

$$\begin{aligned}
\text{(6.39)}\quad \left\|\langle D_x\rangle^{\kappa-1}\psi P_{(\beta)}\chi u\right\| &\le \mu\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| \\
&\qquad + C(K,\mu,N,\chi,\chi',\psi)\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right).
\end{aligned}$$

We rewrite $\langle D_x\rangle^{\kappa-1}\psi P_{(\beta)}\chi$ as

$$\begin{aligned}
\text{(6.40)}\quad \langle D_x\rangle^{\kappa-1}\psi P_{(\beta)}\chi &= \sum_{j=1}^{n}\langle D_x\rangle^{\kappa-1}\psi M_{j,0}fL_j\chi \\
&\qquad + \sum_{j=1}^{n}\sum_{k=1}^{d}\langle D_x\rangle^{\kappa-1}\psi M_{j,k}(\partial_{x_k}f)L_j\chi + \langle D_x\rangle^{\kappa-1}\psi M_{0,0}\chi,
\end{aligned}$$

where $M_{j,k} \in \mathcal{S}^1_{1,0}$ and their symbols do not depend on τ. Furthermore $M_{0,0}$ is rewritten as

$$M_{0,0} = \widetilde{M_0} f + \sum_{k=1}^{d} \widetilde{M_k}(\partial_{x_k} f) + \widetilde{m}, \tag{6.41}$$

where $\widetilde{M_k} \in \mathcal{S}^1_{1,0}$ ($k = 0, \dots, d$) and $\widetilde{m}$ is a multiplication by a smooth function.

Remark. **If f does not depend on x**, $M_{j,k}$ ($k \neq 0$) does not appear in (6.40), moreover $\widetilde{M_k}$ ($k \neq 0$) and $\widetilde{m}$ do not appear in (6.41).

We shall evaluate each term of (6.40) applied to u. First, we evaluate $\sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa-1} \psi M_{j,0} f L_j \chi u \right\|$. Since $\langle D_x \rangle^{\kappa-1} \psi M_{j,0} \langle D_x \rangle^{-\kappa} \in \mathcal{S}^0_{0,0}$, we have

$$\begin{aligned} \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa-1} \psi M_{j,0} f L_j \chi u \right\| &\leq C(K, \psi) \sum_{j=1}^{n} \left\| \langle D_x \rangle^{\kappa} f L_j \chi u \right\| \\ &\leq C(K, N, \chi, \psi) \left(\|P\chi u\| + \|u\|_{-N} \right) \\ &\qquad \text{(by Propositions 4.1 and 5.1).} \end{aligned} \tag{6.42}$$

Next, we evaluate $\sum_{j=1}^{n} \sum_{k=1}^{d} \left\| \langle D_x \rangle^{\kappa-1} \psi M_{j,k} (\partial_{x_k} f) L_j \chi u \right\|$. We will make use of Corollary 6.3. As is mentioned in the above remark, we may neglect to evaluate this term if f does not depend on x. Calculating the commutator between the multiplication by ψ and $M_{j,k}(\partial_{x_k} f) L_j \chi$, we have

$$\begin{aligned} \psi M_{j,k}(\partial_{x_k} f) L_j \chi &= M_{j,k}(\partial_{x_k} f) L_j \chi \psi + M_{j,k} L_j \left[\psi, \chi\right] (\partial_{x_k} f) + M_{j,k} \Big[(\partial_{x_k} f), L_j \left[\psi, \chi\right] \Big] \\ &\quad + M_{j,k} \left[\psi, L_j\right] (\partial_{x_k} f) \chi + \left[\psi, M_{j,k}\right] (\partial_{x_k} f) L_j \chi \end{aligned}$$

(Note that $\partial_{x_k} f$ commutes with $[\psi, L_j]$, because they are multiplications by functions.) $\langle D_x \rangle^{\kappa-1} M_{j,k} \langle D_x \rangle^{-\kappa}$, $\langle D_x \rangle^{\kappa-1} M_{j,k} \left[\psi, L_j\right] \langle D_x \rangle^{-\kappa}$ and $\langle D_x \rangle^{\kappa-1} \left[\psi, M_{j,k}\right]$ belong to $\mathcal{S}^0_{0,0}$. So, we have

$$\begin{aligned} &\sum_{j=1}^{n} \sum_{k=1}^{d} \left\| \langle D_x \rangle^{\kappa-1} \psi M_{j,k} (\partial_{x_k} f) L_j \chi u \right\| \\ &\leq C(K, \psi) \sum_{j=1}^{n} \sum_{k=1}^{d} \Bigg(\left\| \langle D_x \rangle^{\kappa} (\partial_{x_k} f) L_j \chi \psi u \right\| + \left\| \langle D_x \rangle^{\kappa-1} M_{j,k} L_j \left[\psi, \chi\right] (\partial_{x_k} f) u \right\| \\ &\qquad + \left\| \langle D_x \rangle^{\kappa-1} M_{j,k} \Big[\partial_{x_k} f, L_j \left[\psi, \chi\right] \Big] u \right\| + \left\| \langle D_x \rangle^{\kappa} (\partial_{x_k} f) \chi u \right\| + \left\| (\partial_{x_k} f) L_j \chi u \right\| \Bigg). \end{aligned} \tag{6.43}$$

The asymptotic expansion formula yields

$$\langle D_x \rangle^{\kappa-1} M_{j,k} L_j \left[\psi, \chi\right] \langle D_x \rangle^{-\kappa}, \quad \langle D_x \rangle^{\kappa-1} M_{j,k} \Big[\partial_{x_k} f, L_j \left[\psi, \chi\right] \Big] \in \mathcal{S}^0_{1/2,0}.$$

Since

$$\left[\psi, \chi\right] (\partial_{x_k} f) - \left[\psi, \chi\right] (\partial_{x_k} f) \chi', \quad \Big[\partial_{x_k} f, L_j \left[\psi, \chi\right] \Big] - \Big[\partial_{x_k} f, L_j \left[\psi, \chi\right] \Big] \chi' \in \mathcal{S}^{-\infty}$$

for $\chi' \in \mathcal{S}_\Psi$ satisfying $\chi \subset\subset \chi'$, we have

$$\begin{aligned}
(6.44)\qquad & \sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa-1}\psi M_{j,k}(\partial_{x_k}f)L_j\chi u\right\| \\
&\le C(K,N,\chi,\chi',\psi)\Bigg(\sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)L_j\chi\psi u\right\| + \sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)\chi' u\right\| \\
&\quad + \|\chi' u\| + \sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)\chi u\right\| + \sum_{j=1}^{n}\sum_{k=1}^{d}\left\|(\partial_{x_k}f)L_j\chi u\right\| + \|u\|_{-N}\Bigg).
\end{aligned}$$

Applying Corollary 6.3 to the first term on the right hand side, we obtain by Propositions 4.1, 5.1 and Lemma 3.1

$$\begin{aligned}
(6.45)\qquad & \sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}\psi M_{j,k}(\partial_{x_k}f)L_j\chi u\right\| \\
&\le \mu\left\|\langle D_x\rangle^{\kappa}\psi P\chi u\right\| + C(K,N,\chi,\chi',\psi,\mu)\left(\|P\chi u\| + \|P\chi' u\| + \|u\|_{-N}\right).
\end{aligned}$$

Finally, we evaluate $\left\|\langle D_x\rangle^{\kappa-1}\psi M_{0,0}\chi u\right\|$. Let us remember the definitions of $\widetilde{M_k}$ and $\widetilde{m}$ (see (6.41)). Since $\langle D_x\rangle^{\kappa-1}\psi\widetilde{M_k}\langle D_x\rangle^{-\kappa}$ and $\langle D_x\rangle^{\kappa-1}\psi\widetilde{m}$ belong to $\mathcal{S}^0_{0,0}$, we have by (6.41)

$$\left\|\langle D_x\rangle^{\kappa-1}\psi M_{0,0}\chi u\right\| \le C(K,\psi)\left(\left\|\langle D_x\rangle^{\kappa}f\chi u\right\| + \sum_{k=1}^{d}\left\|\langle D_x\rangle^{\kappa}(\partial_{x_k}f)\chi u\right\| + \|\chi u\|\right).$$

By Propositions 4.1 and 5.1, we obtain

$$(6.46)\qquad \left\|\langle D_x\rangle^{\kappa-1}\psi M_{0,0}\chi u\right\| \le C(K,N,\chi,\chi',\psi)\left(\|P\chi u\| + \|u\|_{-N}\right).$$

From (6.42), (6.45) and (6.46), we obtain (6.39). (6.25) in the case where $|\beta| = 1$ is proved. Now Condition (V) for P in question is verified.

7 Examples

In this section, we give some examples of P satisfying (1°)–(3°), especially, examples of $f(t,x)$ which satisfies Condition (1°). Since Condition (1°) is not so restrictive, the set of functions satisfying (1°) contains an $f(t,x)$ whose set of zeros is of positive Lebesgue measure.

First, we need the following lemma. This guarantees the existence of a smooth function which vanishes only on any given closed set of $\boldsymbol{R}^n$.

Lemma 7.1 (cf. Theorem 2 in §2.1 in Chapter VI of [30]) *For any closed subset K of $\boldsymbol{R}^n$, there exists a function f of class $\boldsymbol{C}^\infty$ in $\boldsymbol{R}^n$ such that*

$$f = 0 \ \text{ on } K \quad \text{and} \quad f > 0 \ \text{ in } \boldsymbol{R}^n \setminus K.$$

7.1 Examples of f independent of x

Now we show some examples of Theorem **A**. First, let us begin with the case where $f(t,x)$ does not depend on x.

1) The following operator of 3 variables is hypoelliptic:

$$P = \partial_t + f_1(t)\left(\partial_x^2 + x^2 a(t,x,y)\partial_y^2\right),$$

where $a(t,x,y)$ is a complex-valued smooth function satisfying $\operatorname{Re} a(t,x,y) > 0$ in $\boldsymbol{R}^3$ and f_1 is defined in the following way. Let $I_0 = [0,1]$, $I_1 = [0,3/8] \cup [5/8,1]$, $I_2 = [0,5/32] \cup [7/32,3/8] \cup [5/8,25/32] \cup [27/32,1]$, $\ldots$, where I_{j+1} is obtained by removing the middle open subinterval of length $4^{-(j+1)}$ from each closed interval consisting I_j. So, I_j is a disjoint union of 2^j closed intervals of equal length $(1+2^j)/2^{2j+1}$. The set $K_1 = \cap_{j=0}^\infty I_j$ is called the Harnack set. This is closed and does not contain non-empty open set. Let f_1 be a smooth function on $\boldsymbol{R}$ such that $f_1 = 0$ on K_1 and $f_1 > 0$ in $\boldsymbol{R} \setminus K_1$. Such an f_1 exists due to Lemma 7.1.

P satisfies obviously Conditions (2°) and (3°). Moreover, f_1 satisfies Condition (1°). This is because, f_1 is non-negative and the set of zeros of f_1 contains no non-empty open interval. So f_1 satisfies Condition (1°-α). Furthermore, the Lebesgue measure of the set of zeros of f_1 is equal to

$$1 - \sum_{k=1}^\infty \frac{2^{k-1}}{4^k} = \frac{1}{2} > 0.$$

2) The following operator of 3 variables is hypoelliptic:

$$P = \partial_t + f_2(t)\left(a(t,x,y)\partial_x^2 + x^2 b(t,x,y)\partial_y^2\right),$$

where $a(t,x,y)$ and $b(t,x,y)$ are complex-valued smooth functions satisfying $\operatorname{Re} a(t,x,y) > 0$ and $\operatorname{Re} b(t,x,y) > 0$ in $\boldsymbol{R}^3$ and f_2 is defined to be

$$f_2(t) = \begin{cases} f_1(t+3/8) & \text{for } t \geq 0, \\ -f_1(-t+3/8) & \text{for } t < 0, \end{cases}$$

where f_1 is as above. Obviously, f_2 is smooth possibly except at $t=0$. Since the set $\{s \in K_1\,;\, s \geq 3/8\}$ accumulates at $s = 3/8$, the right differential coefficients of f_2 of any order are equal to 0 at $t=0$ by Rolle's theorem. By the same way, the left differential coefficients of f_2 of any order are equal to 0 at $t=0$. So f_2 is smooth everywhere. f_2 has an uncountably infinite number of zeros and changes sign at the origin. And f_2 satisfies Condition ($1°$-α). Furthermore, the Lebesgue measure of the set of zeros of f_2 is positive.

7.2 Examples of f depending on x

In this subsection, we show examples for the case where $f(t,x)$ depends on x.

1) The following operator of 2 variables is hypoelliptic:

$$P = \partial_t + g_1(t,x)\partial_x^2,$$

where $g_1(t,x)$ on $\boldsymbol{R}^2$ is defined in the following way. Let f_3 be a smooth function on $\boldsymbol{R}$ such that $f_3(s) = 0$ on K_3 and $f_3 > 0$ in $\boldsymbol{R} \setminus K_3$, where K_3 is the Cantor set. There exists such an f_3 due to Lemma 7.1. We define the function $g_1(t,x)$ to be

$$g_1(t,x) = f_3(t+x).$$

f_3 is non-negative and does not vanish identically in any non-empty open interval, so does $g_1(\cdot,x)$ for every $x \in \boldsymbol{R}$. Thus g_1 satisfies Condition ($1°$-β). The set of zeros of g_1 consists of uncountably infinite number of lines. These lines accumulate at every line $t + x =$ constant.

2) The following operator of $(d+1)$-variables is hypoelliptic:

$$P = \partial_t + g_2(t,x_1,x_2,\ldots,x_d)\left(a(t,x)\partial_{x_1}^2 + b(t,x)\left(\sum_{k=2}^{d} {x_1}^k \partial_{x_k}\right)^2\right),$$

where $a(t,x)$, $b(t,x)$ are complex-valued smooth functions satisfying $(\operatorname{Re} a)(\operatorname{Re} b) > 0$ at every point of $\boldsymbol{R}^{d+1}$ and g_2 is a smooth function on $\boldsymbol{R}^{d+1}$ defined to be

$$g_2(t,x_1,x_2,\ldots,x_d) = f_1\left(t^2 + \sum_{k=1}^{d} x_k^2\right),$$

where f_1 is the same as in the preceding subsection.

Obviously, P satisfies Condition (2°). Since the set of zeros of $f_1(t^2+a)$ contains no non-empty open interval for every $a \geq 0$, g_2 satisfies Condition (1°-β). Therefore, P satisfies Condition (1°). Furthermore, the set of zeros of g_2 consists of uncountably infinite number of spheres. The Lebesgue measure of the set of zeros of g_2 is calculated as follows.

$$\int_{\{(t,x)\,;\,g_2(t,x)=0\}} dt\,dx = \iint_{\{(r,\omega)\in \boldsymbol{R}_+\times S^d\,;\,f_2(r^2)=0\}} r^d dr\, dS_\omega,$$

where $\boldsymbol{R}_+ = \{r\ ,\ 0 \leq r < +\infty\}$, S^d is the unit sphere of dimension d and dS_ω stands for the area element of S^d. The right hand side is estimated from below as follows.

$$\begin{aligned}
&\iint_{\{(r,\omega)\in \boldsymbol{R}_+\times S^d\,;\,f_2(r^2)=0\}} r^d dr\, dS_\omega \\
&= \operatorname{area}(S^d)\int_{\{r\in\boldsymbol{R}_+\,;\,f_2(r^2)=0\}} r^d dr \;=\; \frac{1}{2}\operatorname{area}(S^d)\int_{K_1} s^{(d-1)/2} ds \\
&\geq \frac{1}{2}\left(\frac{3}{8}\right)^{(d-1)/2}\operatorname{area}(S^d)\int_{K_1} ds \;=\; \frac{1}{4}\left(\frac{3}{8}\right)^{(d-1)/2}\operatorname{area}(S^d),
\end{aligned}$$

where $\operatorname{area}(S^d)$ is the area of S^d. So, the Lebesgue measure of the set of zeros of g_2 is positive.

Next, we shall prove that $\left\{\partial_{x_1}, \sum_{k=2}^d {x_1}^k \partial_{x_k}\right\}$ satisfies Condition (3°). Set $L_0 = \partial_{x_1}$ and $L_1 = \sum_{k=2}^d {x_1}^k \partial_{x_k}$. Let us remember the notation R_J (see p.35). Given a multi-index $J = (j_1, \ldots, j_l)$ with $l \geq 1$, where $j_m \in \{0,1\}$ $(\,m = 1, \ldots, l\,)$, set $\widetilde{J} = (j_1, \ldots, j_{l-1})$ and define R_J inductively by

$$R_J = L_{j_1} \quad (l=1)\ , \quad R_J = [R_{\tilde{J}}, L_{j_l}] \quad (l \geq 2).$$

(See Lemma 3.4 in §3.)

Set $J_0 = (1, \overbrace{0,0,\ldots,0}^{d})$, $J_1 = (1, \overbrace{0,0,\ldots,0}^{d-1})$, $J_2 = (1, \overbrace{0,0,\ldots,0}^{d-2})$, $\ldots$, $J_{d-2} = (1,0,0)$. Since

$$\sum_{r=0}^{l} \frac{x_1^r}{r!} R_{J_{l-r}} = (-1)^{d-l}(d-l)!\,\partial_{x_{d-l}}\ (\,l = 0, \ldots, d-2\,),$$

the Lie algebra generated by $\{L_0, L_1\}$ is of dimension d at every point of $\boldsymbol{R}^d_x$. Thus, $\{L_0, L_1\}$ satisfies Condition (3°).

8 Application 1 of Theorem A

As far as the preceding section, we studied the hypoellipticity of P of the form (**A**). We assumed that $f(t,x)$ is real-valued. In this section, we investigate the hypoellipticity of P of the form (**A**) in the case where $f(t,x)$ is complex-valued. For example, let us consider the operator $P = \partial_t + \{f(t,x) + ig(t,x)\}\,\partial_x^2$ in $\boldsymbol{R}^2$, where $f(t,x)$ satisfies Condition (1°) and $g(t,x)$ is a real-valued smooth function. P is hypoelliptic due to Theorem **A** if the quotient $g(t,x)/f(t,x)$ can be extended to a smooth function. Theorem **B** in [**2°**] in the Introduction gives a sufficient condition for hypoellipticity of P of the form (**B**) in the case where $g(t,x)/f(t,x)$ is not necessarily extended to a smooth function.

Before giving the proof and examples of Theorem **B**, let us sketch the roles of Conditions ($2^\sharp$) and ($3^\sharp$).

1. ($2^\sharp$) controls $g(t,x)$ by means of $f(t,x)$. This guarantees the following inequalities to hold.

$$\sum_{j=1}^{n} \|g(t,x)L_j u\|^2 \le C(K)\sum_{j=1}^{n}\|f(t,x)L_j u\|^2 \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{8.1}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{d}\left\|\Big(\partial_{x_k} g(t,x)\Big)L_j u\right\|^2 \le \begin{cases} C(K,\rho)\displaystyle\sum_{j=1}^{n}\Big\||f(t,x)|^\rho L_j u\Big\|^2 \\ \quad \text{for the case where } f(t,x) \text{ changes sign,} \\ C(K)\displaystyle\sum_{j=1}^{n}\Big\||f(t,x)|^{1/2} L_j u\Big\|^2 \\ \quad \text{for the case where } f(t,x) \text{ does not} \\ \quad \text{change sign} \end{cases} \tag{8.2}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where ρ is the positive number specified in ($2^\sharp$-1). These inequalities allow us to neglect effects of g in the course of estimation for P. ($2^\sharp$) is not a necessary condition for hypoellipticity of P. For example, let p, q be non-negative integers and let us consider the following operator:

$$L_{p,q} = \partial_t + \Big(t^p + i\,t^q\Big)\partial_x^2.$$

The necessary and sufficient condition for $L_{p,q}$ to be hypoelliptic is that $p \le 2q$. (see §11.) $L_{p,q}$ does not satisfy ($2^\sharp$) but satisfies ($1^\sharp$) and ($3^\sharp$) in the case where $p/2 < q < p$.

2. If the matrix $A(t,x)$ is hermitian or the function $g(t,x)$ vanishes identically, ($3^\sharp$) is equivalent to (2°).

Examples of Theorem B

1) The following operator of 3 variables is hypoelliptic:

$$P = \partial_t + \Big(f(t) + i\,g(t)\Big)\Big\{(2 + 2i\,\sin 4t)\,\partial_x^2 + (1 + i\,\sin 8t)\,(\sin x)^2\,\partial_y^2\Big\},$$

where

$$f(t) = \begin{cases} \exp\left(-\dfrac{1}{|t|}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases} \quad \text{and } g(t) = \begin{cases} \sin\left(\dfrac{1}{t}\right)\exp\left(-\dfrac{1}{|t|}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

The quotient $g(t)/f(t)$ can not be extended to a smooth function but remains bounded in $\boldsymbol{R}$. The matrix for P appearing in ($3^\sharp$) is equal to

$$A(t,x,y) = \begin{pmatrix} 2 + 2i\,\sin 4t & 0 \\ 0 & 1 + i\,\sin 8t \end{pmatrix}.$$

Thus, $\mathrm{Re}\,((f(t) + i\,g(t))\,A(t,x,y)\eta,\ f(t)\eta)$ for $\eta = {}^t(\eta_1, \eta_2) \in \boldsymbol{C}^2$ is equal to

$$|f(t)|^2 \left(4\left\{1 - (\sin 4t)\left(\sin\frac{1}{t}\right)\right\}^2 |\eta_1|^2 + \left\{1 - (\sin 8t)\left(\sin\frac{1}{t}\right)\right\}^2 |\eta_2|^2\right).$$

Since the functions $1 - (\sin 4t)(\sin 1/t)$ and $1 - (\sin 8t)(\sin 1/t)$ do not vanish in $\boldsymbol{R}$, so $A(t,x,y)$ satisfies ($3^\sharp$-1).

2) The following operator of 3 variables is hypoelliptic:

$$P = \partial_t + \Big(f(t) + i\,g(t,x,y)\Big)\Big(2\partial_x^2 + i\,\sin(x^2+y^2)\partial_x x\partial_y + 3i\,\sin(x^2+y^2)x\partial_y\partial_x + 2x^2\partial_y^2\Big),$$

where

$$f(t) = \begin{cases} (\operatorname{sgn} t)\exp\left(-\dfrac{1}{|t|}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

and

$$g(t,x,y) = \begin{cases} \dfrac{\sqrt{3}}{2}\exp\left(-\dfrac{1 + x^2 + y^2}{|t|}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

f satisfies ($1^\sharp$) and g satisfies ($2^\sharp$-1) for every ρ satisfying $0 < \rho < 1$. The matrix for P appearing in ($3^\sharp$) is equal to

$$A(t,x,y) = \begin{pmatrix} 2 & i\,\sin(x^2+y^2) \\ 3i\,\sin(x^2+y^2) & 2 \end{pmatrix}.$$

Thus, $\mathrm{Re}\,((f(t)+i\,g(t,x,y))\,A(t,x,y)\eta,\ f(t)\eta)$ for $\eta\in\boldsymbol{C}^2$ is evaluated from below as

$$\begin{aligned}&\mathrm{Re}\,((f(t)+i\,g(t,x,y))\,A(t,x,y)\eta,\ f(t)\eta)\\ \geq\ & \frac{2-\sqrt{\left\{1+3\exp\left(-\frac{2(x^2+y^2)}{|t|}\right)\right\}\sin^2(x^2+y^2)}}{2}\,|f(t)|^2\,|\eta|^2 \quad \text{for } t\neq 0.\end{aligned}$$

Since

$$\left\{1+3\exp\left(-\frac{2\,(x^2+y^2)}{|t|}\right)\right\}\sin^2\left(x^2+y^2\right)<4 \quad \text{for every } (t,x,y)\in\boldsymbol{R}^3\setminus\{t=0\},$$

so the matrix $A(t,x,y)$ satisfies ($3^\sharp$-1).

3) The following operator of 3 variables is hypoelliptic:

$$P=\partial_t+\Big(f(t,x,y)+i\,g(t,x,y)\Big)\left(\partial_x^2+x^4\partial_y^2\right),$$

where

$$f(t,x,y)\ =\ \begin{cases}\exp\left(-\dfrac{1}{(t^2+x^2+y^2)^3}\right) & \text{for } (t,x,y)\neq(0,0,0),\\ 0 & \text{for } (t,x,y)=(0,0,0)\end{cases}$$

and

$$g(t,x,y)\ =\ \begin{cases}\sin\left(\dfrac{1}{txy}\right)\exp\left(-\dfrac{1}{27t^2x^2y^2}\right) & \text{for } txy\neq 0\\ 0 & \text{for } txy=0.\end{cases}$$

$g(t,x,y)/f(t,x,y)$ can not be extended to a smooth function but remains bounded in $\boldsymbol{R}^3$. On the other hand, $(\partial_x g(t,x,y))^2/f(t,x,y)$ and $(\partial_y g(t,x,y))^2/f(t,x,y)$ can be extended to smooth functions in $\boldsymbol{R}^3$. Therefore, g satisfies ($2^\sharp$-2).

Proof of Theorem B.

The proof is done in a similar way as that of Theorem **A**. So we give it roughly. Since Proposition 1.1 holds for P in question, it suffices to show that P satisfies Conditions (I)–(IV) in §2. Let us begin by getting the inequalities analogous to those in §3. As in §3, let $f(t)=f(t,x)$ if $f(t,x)$ does not depend on x. From the hypothesis ($1^\sharp$), there exists at most one point where $f(t)$ changes sign. So we assume that the point is $t=0$. Let Ω be a bounded domain in $\boldsymbol{R}^{d+1}$. We use again the definition for (f,Ω) to be of type (α-1) or of type (α-2) or of type (β) as in §3. First, if (f,Ω) is of type (α-1), ($3^\sharp$-1) yields

$$\text{(8.3)}\qquad \mathrm{Re}\Big(\Big(f(t)+i\,g(t,x)\Big)A(t,x)\eta,\ (\mathrm{sgn}\,t)\,|f(t)|^{2\rho-1}\,\eta\Big)\geq\delta\,|f(t)|^{2\rho}\,|\eta|^2$$

for all $\rho>3/4$ and all $\eta\in\boldsymbol{C}^n$. On the other hand, if (f,Ω) is of type (α-2) or of type (β), ($3^\sharp$-2) yields

$$\text{(8.4)}\qquad \left|\mathrm{Re}\Big(\Big(f(t,x)+i\,g(t,x)\Big)A(t,x)\eta,\ f(t,x)\eta\Big)\right|\geq\delta\,|f(t,x)\eta|^2$$

for all $\eta \in \boldsymbol{C}^n$. Using above inequalities, we have the following lemma corresponding to Lemma 3.1.

Lemma 8.1 (i) *If (f,Ω) is of type (α-1), there exists, for any $K \subset\subset \Omega$ and any $\widetilde{\rho} > 3/4$, a constant C depending only on $(K,\widetilde{\rho})$ such that*

$$\sum_{j=1}^{n} \left\| |f|^{\widetilde{\rho}} L_j u \right\|^2 \le C\left\{ \left| \mathrm{Re}\left(Pu, (\mathrm{sgn}\, t)|f|^{2\widetilde{\rho}-1}u\right)\right| + \|u\|^2 \right\} \tag{8.5}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$.

(ii) *If (f,Ω) is of type (α-2) or of type (β), there exists, for any $K \subset\subset \Omega$, a constant C depending only on K such that*

$$\sum_{j=1}^{n} \left\| |f|^{1/2} L_j u \right\|^2 \le C\left\{ \left| \mathrm{Re}\,(Pu, u)\right| + \|u\|^2 \right\} \tag{8.6}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$.

(iii) *If (f,Ω) is of type (α-2) or of type (β), there exists, for any $K \subset\subset \Omega$, a constant C depending only on K such that*

$$\begin{aligned} \sum_{j=1}^{n} \|f L_j u\|^2 \ \le\ & C\Big\{ |\mathrm{Re}\,(Pu, fu)| + \sum_{k=1}^{d} \|(\partial_{x_k} f)u\|^2 \\ & + \sum_{k=1}^{d} \|(\partial_{x_k} g)u\|^2 + \|fu\|^2 + |((\partial_t f)u, u)| \Big\} \end{aligned} \tag{8.7}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$.

Proof of Lemma 8.1. We only prove (i). (The proof of (ii) (resp. (iii)) is done by using ($3^\sharp$-2) (resp. (8.4)) in place of (8.3).) Set

$$\widetilde{P} = -\partial_t + \sum_{j,k=1}^{n} L_j^*\Big(f(t,x) + i\, g(t,x)\Big) a_{jk}(t,x) L_k.$$

Since $L_j + L_j^*$ reduces to a multiplication by a smooth function, we have

$$\begin{aligned} P + \widetilde{P} \ =\ & \Big(f(t,x) + i\, g(t,x)\Big) \sum_{j=1}^{n} b_{j0}(t,x) L_j \\ & + \sum_{j=1}^{n}\sum_{l=1}^{d} b_{jl}(t,x) \Big(\partial_{x_l} f(t,x) + i\, \partial_{x_l} g(t,x)\Big) L_j, \end{aligned} \tag{8.8}$$

where $b_{jl}(t,x) \in \boldsymbol{C}^\infty(\boldsymbol{R}^{d+1})$. Note that $\partial_{x_l} f(t,x) = 0$ ($l = 1, \ldots, d$) in the case where (f,Ω) is of type (α-1). By (8.8), $-\mathrm{Re}\,(Pu, (\mathrm{sgn}\, t)|f|^{2\rho-1}u)$ is estimated from below as follows.

$$\begin{aligned} & -\mathrm{Re}\left(Pu, (\mathrm{sgn}\, t)|f|^{2\widetilde{\rho}-1}u\right) \\ & \ge\ \mathrm{Re}\left(\widetilde{P}u, (\mathrm{sgn}\, t)|f|^{2\widetilde{\rho}-1}u\right) - \left|\left(\left(P + \widetilde{P}\right)u, (\mathrm{sgn}\, t)|f|^{2\widetilde{\rho}-1}u\right)\right|. \end{aligned} \tag{8.9}$$

From (8.3), the first term on the right hand side of (8.9) is estimated from below as follows.

$$\mathrm{Re}\left(\widetilde{P}u, (\operatorname{sgn} t)|f|^{2\widetilde{\rho}-1}u\right) \geq \delta(K)\sum_{j=1}^{n}\left\||f|^{\widetilde{\rho}}L_j u\right\|^2 + \frac{1}{2}E(u), \tag{8.10}$$

where

$$E(u) = -\int (\operatorname{sgn} t)\, |f(t)|^{2\widetilde{\rho}-1}\, \partial_t\left(|u|^2\right) dt dx.$$

(See (3.4) in the proof of Lemma 3.1.) By Schwarz' inequality, the second term on the right hand side of (8.9) is evaluated as follows.

$$\begin{aligned} &\left|\left(\left(P+\widetilde{P}\right)u, (\operatorname{sgn} t)|f|^{2\widetilde{\rho}-1}u\right)\right| \\ &\leq \varepsilon\sum_{j=1}^{n}\left\||f|^{2\widetilde{\rho}}L_j u\right\|^2 + \varepsilon\sum_{j=1}^{n}\left(\left\||f|^{2\widetilde{\rho}-1}gL_j u\right\|^2 + \sum_{l=1}^{d}\left\||f|^{2\widetilde{\rho}-1}(\partial_{x_l}g)L_j u\right\|^2\right) \\ &\quad + C(K,\varepsilon)\left\|u\right\|^2. \end{aligned} \tag{8.11}$$

From ($2^\sharp$-1), we have

$$\left||f(t,x)|^{2\widetilde{\rho}-1}g(t,x)\right| \leq C_1(K,\widetilde{\rho})|f(t,x)|^{\widetilde{\rho}}. \tag{8.12}$$

Moreover, $2\widetilde{\rho}-1+\rho > \widetilde{\rho}$, where ρ is defined in ($2^\sharp$-1). So we have from ($2^\sharp$-1)

$$\left||f(t,x)|^{2\widetilde{\rho}-1}\left(\partial_{x_l}g(t,x)\right)\right| \leq C_2(K,\widetilde{\rho})|f(t,x)|^{\widetilde{\rho}}. \tag{8.13}$$

Therefore, we obtain from (8.9), (8.10), (8.11), (8.12) and (8.13)

$$\begin{aligned} &\delta(K)\sum_{j=1}^{n}\left\||f|^{\widetilde{\rho}}L_j u\right\|^2 + \frac{1}{2}E(u) \\ &\leq \left|\mathrm{Re}\left(Pu, (\operatorname{sgn} t)|f|^{2\widetilde{\rho}-1}u\right)\right| + C(K,\rho,\widetilde{\rho})\,\varepsilon\sum_{j=1}^{n}\left\||f|^{\widetilde{\rho}}L_j u\right\|^2 + C(K,\varepsilon,\widetilde{\rho})\left\|u\right\|^2. \end{aligned}$$

Setting $\varepsilon = \delta(K)/(2C(K,\widetilde{\rho}))$ and evaluating $E(u)$ as in the proof of Lemma 3.1, we obtain (8.5). □

Lemma 8.2 (i) *If (f,Ω) is of type (α-1), there exists, for any $K \subset\subset \Omega$, a constant C depending only on K such that*

$$\|u\|^2 \leq C\left(|\mathrm{Re}\,(Pu,tu)| + \|tu\|^2\right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{8.14}$$

(ii) *If (f,Ω) is of type (α-2) or of type (β), there exists, for any $K \subset\subset \Omega$ and any $a \in \boldsymbol{R}$, a constant C such that*

$$\|u\|^2 \leq C\left(|\mathrm{Re}\,(Pu,(t-a)u)| + \sum_{j=1}^{n}\left\||(t-a)f(t,x)|^{1/2}L_j u\right\|^2\right) \tag{8.15}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where C depends only on K and the diameter of $\{a\}\cup\pi_t(K)$.

If (f, Ω) is of type (α-1), ($3^\sharp$-1) yields

$$\text{Re}\Big(\big(f(t) + i\,g(t,x)\big)A(t,x)\eta,\, t\eta\Big) \ge \delta\,|tf(t)|\,|\eta|^2 \tag{8.16}$$

for all $\eta \in \boldsymbol{C}^n$. We can prove (i) by using this inequality as we used Condition (2°) in the proof of Lemma 3.3. On the other hand, the proof of (ii) is done in the same way as that of (ii) in Lemma 3.3. So we omit the proof.

Next, we have the following lemma which plays the same role as Lemma 3.5 in §3.

Lemma 8.3 *For any $K \subset\subset \Omega$, there exist a positive integer k depending only on K and a constant C depending only on K such that*

$$\left\|\langle D_x\rangle^{2^{1-k}} f(t,x)^k u\right\|^2 \le C\left(\|Pu\|^2 + \|u\|^2\right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{8.17}$$

The proof is done by using Lemma 8.1 as we used Lemma 3.1 in the proof of Lemma 3.5.

Now we give two propositions corresponding to Propositions 4.1 and 5.1 by making use of lemmas obtained above. We use Lemma 8.1 and Lemma 8.3 as we used Lemma 3.1 and Lemma 3.5 in §4 respectively. And we have the following proposition by applying (8.1) and (8.2).

Proposition 8.4 *For any $K \subset\subset \Omega$, any $N > 0$ and any $\chi \in S_\Psi$, there exist positive constants $\kappa = \kappa(K), C = C(K, N, \chi)$ such that*

$$\begin{aligned}
&\sum_{j=1}^n \|\langle D_x\rangle^\kappa f L_j \chi u\|^2 + \sum_{j=1}^n \|\langle D_x\rangle^\kappa g L_j \chi u\|^2 + \|\langle D_x\rangle^\kappa f \chi u\|^2 \\
&\quad + \|\langle D_x\rangle^\kappa g \chi u\|^2 + \sum_{k=1}^d \|\langle D_x\rangle^\kappa (\partial_{x_k} f)\chi u\|^2 + \sum_{k=1}^d \|\langle D_x\rangle^\kappa (\partial_{x_k} g)\chi u\|^2 \\
&\quad \le C\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K).
\end{aligned} \tag{8.18}$$

Furthermore, we obtain Proposition 5.1 for P in question by using Proposition 8.4, Lemmas 8.1 and 8.2. By making use of Propositions 8.4 and 5.1 in the same way as we used Propositions 4.1 and 5.1 in §6, we can verify that P satisfies Conditions (I)–(V). However, we have to evaluate the following terms:

$$\sum_{j=1}^n \|gL_j\chi u\|^2, \quad \sum_{j=1}^n \sum_{k=1}^d \|(\partial_{x_k} g)L_j \chi u\|^2, \quad \sum_{k=1}^d \|\langle D_x\rangle^\kappa (\partial_{x_k} g)\chi u\|^2$$

$$\text{and} \quad \sum_{j=1}^n \sum_{k=1}^d \|\langle D_x\rangle^\kappa (\partial_{x_k} g) L_j \chi \psi u\|^2 .$$

They are estimated as follows.

1. In the verifications of (II)–(V),

$$\begin{aligned}
&\sum_{j=1}^{n} \|gL_j\chi u\|^2 \\
&\le C(K) \sum_{j=1}^{n} \|fL_j\chi u\|^2 \quad \text{(by (8.1))} \\
&\le C(K,N,\chi)\left(\|P\chi u\|^2 + \|u\|_{-N}^2\right) \quad \text{(by Lemma 8.1 and Propsition 5.1).}
\end{aligned}$$

2. In the verifications of (II),(IV) and (V),

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{d} \|(\partial_{x_k} g)L_j\chi u\|^2 \\
&\le C(K,N,\chi)\left(\|P\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2\right) \\
&\qquad \text{(by (8.2) and Lemma 8.1)} \\
&\le C'(K,N,\chi)\left(\|P\chi u\|^2 + \|u\|_{-N}^2\right) \quad \text{(by Proposition 5.1).}
\end{aligned}$$

3. In the verifications of (IV) and (V),

$$\begin{aligned}
&\sum_{k=1}^{d} \|\langle D_x\rangle^{\kappa}(\partial_{x_k} g)\chi u\|^2 \\
&\le C(K,N,\chi)\left(\|P\chi u\|^2 + \|u\|_{-N}^2\right) \quad \text{(by Propositions 8.4 and 5.1).}
\end{aligned}$$

4. In the verification of (V),

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{d} \|\langle D_x\rangle^{\kappa}(\partial_{x_k} g)L_j\chi\psi u\|^2 \\
&\le \sum_{j=1}^{n}\sum_{k=1}^{d} \|(\partial_{x_k} g)\langle D_x\rangle^{\kappa}L_j\chi\psi u\|^2 + C(K)\,\|\langle D_x\rangle^{\kappa}\chi\psi u\|^2 \\
&\le C(K)\begin{cases} \displaystyle\sum_{j=1}^{n}\left\| |f(t)|^{\rho}\langle D_x\rangle^{\kappa}L_j\chi\psi u\right\|^2 \\ \quad \text{for the case where } f(t,x) \text{ changes sign,} \\ \displaystyle\sum_{j=1}^{n}\left\| |f(t,x)|^{1/2}\langle D_x\rangle^{\kappa}L_j\chi\psi u\right\|^2 \\ \quad \text{for the case where } f(t,x) \text{ does not change sign,} \end{cases} \quad \text{(by (8.2))} \\
&\qquad + C(K)\,\|\langle D_x\rangle^{\kappa}\chi\psi u\|^2 \\
&\le \mu\,\|\langle D_x\rangle^{\kappa}\psi P\chi u\|^2 + C(K,N,\chi,\chi',\mu)\left(\|P\chi u\|^2 + \|P\chi' u\|^2 + \|u\|_{-N}^2\right)
\end{aligned}$$

(see Proof of Corollary 6.3.)

In this way, we can evaluate these terms. So we neglect the effects of g in the verification of Conditions (I)–(V). The proof of Theorem **B** is finished. □

9 Application 2 of Theorem A

In this section, we investigate the hypoellipticity of Q of the form (**C**) which is a generalization of (**A**) if $f(t,x)$ does not depend on x. Our result is stated as Theorem **C** in the Introduction. Let us begin with stating a corollary to Theorem **C**.

Corollary 9.1 *Suppose that* $(\, a_{jk}(t,x)\,)_{j,k=1}^{n}$ *satisfies* (2°) *and that* $\{\, L_j \,\}_{j=1}^{n}$ *satisfies* (3°). *Let* $\{\, e_j \,\}_{j=1}^{n}$ *be non-negative integers. Then, the following operator is hypoelliptic:*

$$Q = \partial_t + t^{e_0} \sum_{j,k=1}^{n} a_{jk}(t,x)\, t^{e_j} L_j\, t^{e_k} L_k \quad \textit{in } \boldsymbol{R}^{d+1}. \tag{9.1}$$

Proof of Corollary 9.1.
If $e_0 = 0$ and if there exists a positive integer j such that $e_j \neq 0$, then $\{t^{e_j}\}_{j=0}^{n}$ does not satisfy $(1^\flat\text{-}3)$ for any closed interval containing $t = 0$. Therefore, we divide the proof into two cases where e_0 is equal to zero or not. Q is hypoelliptic if $e_0 \neq 0$ by applying Theorem **C** to $f_j(t) = t^{e_j}$ $(\, j = 0, \ldots, n\,)$. Next, Q is hypoelliptic even if $e_0 = 0$. This is because the Lie algebra generated by $\left\{\partial_t,\ \{\, t^{e_j} L_j \,\}_{j=1}^{n}\right\}$ is of dimension $d+1$ at every point of $\boldsymbol{R}^{d+1}$. □

Before giving the proof and examples of Theorem **C**, let us sketch the roles of Conditions $(1^\flat\text{-}1)$, $(1^\flat\text{-}2)$ and $(1^\flat\text{-}3)$.

1. $(1^\flat\text{-}1)$ and $(1^\flat\text{-}2)$ for f_0 are the same as $(1^\circ\text{-}\alpha)$ in Theorem **A**. They play the same role as $(1^\circ\text{-}\alpha)$ in Theorem **A**.

2. $(1^\flat\text{-}3)$ controls the vanishing order of f_j $(\, j = 1, \ldots, n\,)$ by means of f_0. In order to prove that Q is hypoelliptic, we have to evaluate $\left\| |f_0(t)|^{\rho} |f_j(t)|^{\delta} L_j u \right\|^2$ for some ρ, δ satisfying $\rho \le 1, \delta \le 1$ and $(\rho, \delta) \neq (1,1)$. We can evaluate $\left\| |f_0(t)|^{\rho} f_j L_j u \right\|^2$ for $3/4 < \rho$ as will be seen in Lemma 9.2 below. Thus, the degeneracy of Q with respect to t should be controlled by f_0. So we need $(1^\flat\text{-}3)$. In addition, $(1^\flat\text{-}3)$ is used to get a priori estimate (9.11) in Lemma 9.5 below. On the other hand, $(1^\flat\text{-}3)$ is not a necessary condition for hypoellipticity of Q. For example, the following operator of 3 variables is hypoelliptic:

$$\widetilde{Q} \;=\; \partial_t + \partial_x^2 + t^2 \partial_y^2.$$

$\widetilde{Q}$ satisfies $(1^\flat\text{-}1)$, $(1^\flat\text{-}2)$, (2°) and (3°) but does not satisfy $(1^\flat\text{-}3)$.

Examples of Theorem C

1) The following operator of 3 variables is hypoelliptic:

$$Q = \partial_t + f_0(t) \left(\partial_x^2 + x^6 f_1^2(t) \partial_y^2 \right),$$

where $f_0(t)$ and $f_1(t)$ are real-valued functions of class $\boldsymbol{C}^\infty$ such that

(9.2) the set of zeros of $f_0(t)$ does not contain any non-empty open interval,

(9.3) $f_0(t)$ satisfies ($1^\flat$-2) and

(9.4) there exists a positive constant C such that

$$|f_0(t)| \le C\,|f_1(t)| \quad \text{for all } t \in \boldsymbol{R}.$$

In this case, the exponent λ appearing in ($1^\flat$-3) can be chosen to be 1 due to (9.4). Moreover, form (9.2) and (9.4), the set of zeros of $f_1(t)$ does not contain any non-empty open interval. So a pair of functions (f_0, f_1) satisfies ($1^\flat$-1), ($1^\flat$-2) and ($1^\flat$-3). The following pair is an example satisfying (9.2), (9.3) and (9.4).

$$f_0(t) = \begin{cases} (\operatorname{sgn} t)\left(\sin^2 \dfrac{1}{t}\right) e^{-\frac{1}{|t|}} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

$$f_1(t) = \begin{cases} \left(\sin \dfrac{1}{t}\right) e^{-\frac{1}{|t|}} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

f_0 and f_1 have a countably infinite number of zeros which accumulate at the origin.

2) Let $\delta_0, \delta_1, \dots, \delta_d$ be positive numbers. Then the following operator of $(d+1)$ variables is hypoelliptic:

$$Q = \partial_t + g_0(t)\left(g_1(t)\partial_{x_1}^2 + \sum_{j=2}^{d} g_j(t)x_{j-1}^2\partial_{x_j}^2\right),$$

$$\text{where} \quad g_0(t) = \begin{cases} (\operatorname{sgn} t)\exp\left(-\dfrac{1}{\delta_0|t|}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

$$\text{and} \quad g_j(t) = \begin{cases} \exp\left(-\dfrac{1}{\delta_j|t|}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases} \qquad (j = 1, \dots, d).$$

For this example, $\lambda(I, j)$ appearing in ($1^\flat$-3) is equal to $2\delta_j/\delta_0$ for every $j \in \{1, \dots, d\}$ and every closed interval I containing $t = 0$.

3) Let $h_1, h_2, \dots, h_d$ be real-valued functions of class $\boldsymbol{C}^\infty$ defined in $\boldsymbol{R}$ such that the set of zeros of h_j does not contain any non-empty open interval for every j. Let $e_1, e_2, \dots, e_d$ be positive integers. Then the following operator of $(d+1)$ variables is hypoelliptic:

$$\partial_t + \prod_{k=1}^{d} h_k(t)^{2e_k} \sum_{j=1}^{d} h_j(t)^2\partial_{x_j}^2.$$

Set $f_0(t) = \prod_{k=1}^{d} h_k(t)^{2e_k}$, $f_j(t) = h_j(t)$ $(j = 1, \ldots, d)$. Then $\{f_j(t)\}_{j=0}^{d}$ satisfies ($1^\flat$-1), ($1^\flat$-2) and ($1^\flat$-3). $\lambda(I, j)$ can be chosen to be $2e_j$ for every j and every I.

Proof of Theorem C.

Let I be an arbitrary bounded open interval of $\boldsymbol{R}_t$ and $\overline{I}$ the closure of I. We shall prove that Q is hypoelliptic in $\Omega = I \times \boldsymbol{R}_x^d$. The proof is divided into two cases whether $N(\overline{I})$ is empty or not. Suppose that $N(\overline{I})$ is empty. Set $\tilde{a}_{jk}(t,x) = a_{jk}(t,x) f_j(t) f_k(t)$ $(j, k = 1, \ldots, n)$. Then $(\tilde{a}_{jk}(t,x))_{j,k=1}^{n}$ satisfies Condition (2°) on Ω. So, Q is hypoelliptic in Ω due to Theorem **A**.

Next, suppose that $N(\overline{I})$ is non-empty. The proof is done in a similar way as that of Theorem **A**. So we give it roughly. By the same way as in the proof of Proposition 1.1 for P, Proposition 1.1 holds for Q. Therefore, it suffices to show that Q satisfies Conditions (I)–(V) in §2. Without loss of generality, we may suppose that f_0 changes sign at $t = 0$. As in §3, we say that (f_0, Ω) is of type (α-1) if $\pi_t(\Omega)$ contains $t = 0$ and that (f_0, Ω) is of type (α-2) if $\pi_t(\Omega)$ does not contain $t = 0$.

First, we have the following two lemmas corresponding to Lemmas 3.1 and 3.3 respectively.

Lemma 9.2 (i) *If (f_0, Ω) is of type (α-1), there exists, for any $K \subset\subset \Omega$ and any $\rho > 3/4$, a constant C depending only on (K, ρ) such that*

$$\sum_{j=1}^{n} \| |f_0|^\rho f_j L_j u \|^2 \leq C \left\{ \left| \mathrm{Re}\left(Qu, (\operatorname{sgn} t) |f_0|^{2\rho - 1} u \right) \right| + \|u\|^2 \right\} \tag{9.5}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$.

(ii) *If (f_0, Ω) is of type (α-2), there exists, for any $K \subset\subset \Omega$, a constant C depending only on K such that*

$$\sum_{j=1}^{n} \left\| |f_0|^{1/2} f_j L_j u \right\|^2 \leq C \left\{ |\mathrm{Re}\,(Qu, u)| + \|u\|^2 \right\} \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{9.6}$$

Lemma 9.3 (i) *If (f_0, Ω) is of type (α-1), there exists, for any $K \subset\subset \Omega$, a constant C depending only on K such that*

$$\|u\|^2 \leq C \left(|\mathrm{Re}\,(Qu, tu)| + \|tu\|^2 \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{9.7}$$

(ii) *If (f, Ω) is of type (α-2), there exists, for any $K \subset\subset \Omega$ and any $a \in \boldsymbol{R}$, a constant C such that*

$$\|u\|^2 \leq C \left(|\mathrm{Re}\,(Qu, (t-a)u)| + \sum_{j=1}^{n} \left\| |(t-a) f_0|^{1/2} f_j L_j u \right\|^2 \right) \tag{9.8}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where C depends only on K and the diameter of $\{a\} \cup \pi_t(K)$.

The proof of Lemma 9.2 (Lemma 9.3) is done in the same way as that of Lemma 3.1 (Lemma 3.3) respectively, so we omit it.

The next lemma is used only for the proof of Lemma 9.5 below. We prepare a notation. Given a multi-index $J = (j_1, \ldots, j_l)$ with $l \geq 1$ and $j_m \in \{1, \ldots, n\}$ ($m = 1, \ldots, l$), we define the function $F_J(t)$ to be

$$F_J(t) \ = \ f_0(t)^l \prod_{m=1}^{l} f_{j_m}(t).$$

l is said to be the length of J and denoted by $\|J\|$.

Lemma 9.4 *For any $K \subset\subset \Omega$ and any multi-index J, there exists a constant C depending only on (K, J) such that*

$$\left\| \langle D_x \rangle^{2^{1-\|J\|-1}} F_J(t) R_J u \right\|^2 \leq C \left(\|Qu\|^2 + \|u\|^2 \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K), \tag{9.9}$$

where $\|J\|$ is the length of J and R_J is the same as in §3. (See Lemma 3.4.)

The proof is done by induction with respect to the length of J and by using Lemma 9.2 as we used Lemma 3.1 in the proof of Lemma 3.4.

Next, set

$$M = M(\overline{I}) = \max_{j \in N(\overline{I})} \frac{1}{\lambda(\overline{I}, j)}, \tag{9.10}$$

where λ has been defined in ($1^\flat$-3). M is a number which stands for what extent the functions $f_1, f_2, \ldots, f_n$ are controlled by f_0. Then, we have the following lemma which plays the same role as Lemma 3.5 in §3.

Lemma 9.5 *For any $K \subset\subset \Omega$, there exist a positive integer k depending only on K and a constant C depending only on K such that*

$$\left\| \langle D_x \rangle^{2^{1-k}} f_0(t)^{(M+1)k} u \right\|^2 \leq C \left(\|Qu\|^2 + \|u\|^2 \right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{9.11}$$

Proof of Lemma 9.5. From the hypothesis (3°), we have the following:
For any $K \subset\subset \Omega$, there exist a positive integer k and functions $b_{jJ}(x), c_j(x) \in \boldsymbol{C}^\infty(K)$ such that

$$\partial_{x_l} = \sum_{\|J\| \leq k} b_{lJ}(x) R_J(x, D_x) + c_l(x) \quad (l = 1, \ldots, d). \tag{9.12}$$

From Condition ($1^\flat$-3), we have for every multi-index $J=(j_1,\dots,j_l)$ and any $t\in I$

$$
\begin{aligned}
|F_J(t)| &\ge |f_0(t)|^l \prod_{m=1}^{l} |f_{j_m}(t)| \ge C_1(\overline{I},J)\,|f_0(t)|^l \prod_{j_m\in N(\overline{I})} C(\overline{I},m)\,|f_0(t)|^{1/\lambda(\overline{I},j_m)} \\
&\ge C_2(\overline{I},J)\,|f_0(t)|^l \prod_{m=1}^{l} |f_0(t)|^{M(\overline{I})} \ge C_3(\overline{I},J)\,|f_0(t)|^{(M+1)l}\,.
\end{aligned}
$$

This implies, together with (9.12),

$$
\begin{aligned}
\left\| \langle D_x\rangle^{2^{1-k}} f_0(t)^{(M+1)k} u \right\|^2 &\le C_1(K)\left(\sum_{\|J\|\le k} \left\| \langle D_x\rangle^{2^{1-k}-1} F_J(t) R_J u \right\|^2 + \|u\|^2 \right) \\
&\le C_2(K)\left(\|Qu\|^2 + \|u\|^2 \right) \ \text{(by (9.9))}\,.
\end{aligned}
$$

The proof is finished. □

In view of Lemma 9.5, the degeneracy of Q with respect to t is controlled by $f_0(t)$. We set $\widehat{\chi}_{j,\kappa'}(t,\xi)=\widetilde{\phi}_j\left(f_0(t)\langle\xi\rangle^{\kappa'}\right)$ $(j=1,2)$. Applying Lemmas 9.2 and 9.5, and using $\widehat{\chi}_{j,\kappa'}$ as $\widetilde{\chi}_{1,\kappa'}$ and $\widetilde{\chi}_{2,\kappa'}$ are used in getting Proposition 4.1, we obtain the following proposition.

Proposition 9.6 *For any $K\subset\subset\Omega$, any $N>0$ and any $\chi\in S_\Psi$, there exist positive constants $\kappa=\kappa(K), C=C(K,N,\chi)$ such that*

$$
\begin{aligned}
(9.13)\qquad & \sum_{j=1}^{n} \|\langle D_x\rangle^{\kappa} f_0 f_j L_j \chi u\|^2 + \|\langle D_x\rangle^{\kappa} f_0 \chi u\|^2 \\
&\le C\left(\|Q\chi u\|^2 + \|\chi u\|^2 + \|u\|_{-N}^2 \right) \quad \textit{for all } u\in \boldsymbol{C}_0^\infty(K).
\end{aligned}
$$

Furthermore, by making use of Lemmas 9.2 and 9.3, we obtain the following proposition which guarantees that Q satisfies Condition (I).

Proposition 9.7 *For any domain $K\subset\subset\Omega$, any $N>0$, any $\chi\in S_\Psi$ and any $\mu>0$, there exists a constant $C=C(K,N,\chi,\mu)$ such that*

$$
(9.14)\qquad \|\chi u\| \le \mu\,\|Q\chi u\| + C\,\|u\|_{-N} \quad \textit{for all } u\in \boldsymbol{C}_0^\infty(K).
$$

The proof of this proposition is done by constructing a partition of unity according to (f_0,Ω) as in the proof of Proposition 5.1 in §5. Thus, we omit the proof.

End of Proof of Theorem C. We can verify that Q satisfies Conditions (I)–(V) by using Lemma 9.2 and Propositions 9.6 and 9.7 as we used Lemma 3.1 and Propositions 4.1 and 5.1 in §6 respectively. So Q is hypoelliptic due to Proposition 2.1. And hence the proof of Theorem **C** is finished. □

10 Necessity of (1°) for hypoellipticity

In this section, we investigate the question of non-hypoellipticity for P of the form (**A**) under Conditions (2°), (3°). Throughout this section, we assume that $f(t,x)$ is a real-valued function of class $\boldsymbol{C}^\infty$ and **independent of $\boldsymbol{x}$**, so we set $f(t) = f(t,x)$. Our result is Theorem **D** presented in the Introduction. Before proving this theorem, we give a remark.

Remark. Condition (1°-α) is necessary and sufficient for hypoellipticity of P under Conditions (2°) and (3°) provided that f is real-analytic. This is because f satisfies one of **(F1)** and **(F2)** as follows if Condition (1°-α) does not hold.

(F1) There exists a non-empty open interval I on which f vanishes identically.

(F2) There exist $t_1, t_2 \in \boldsymbol{R}$ such that $t_1 < t_2$ and $f(t_1) > 0 > f(t_2)$.

If **(F1)** holds, P is obviously not hypoelliptic. If **(F2)** holds and if f is real-analytic, (**D**) holds and hence P is not hypoelliptic due to Theorem **D**. Indeed, let Z_f be the set of zeros of $f(t)$ in $[t_1, t_2]$. Z_f is finite because f is real-analytic. So, there exist a positive integer q and a sequence of numbers $\{s_m\}_{m=1}^q$ such that $Z_f = \{s_1, \dots, s_{q-1}\}$ and $t_1 < s_1 < s_2 < \cdots < s_{q-1} < s_q = t_2$. Since $f(t) \geq 0$ on $[t_1, s_1]$, the set $\{m \in \{1, \dots, q-1\}\ ; f(t) \geq 0 \text{ on } [t_1, s_m]\}$ is non-empty. Let l be the maximal element of this set. Then we have $f(t) \geq 0$ on $[t_1, s_l]$ and $f(t) \leq 0$ on $[s_l, s_{l+1}]$. So (**D**) is satisfied if we set $s = s_l$ and $I = (t_1, s_{l+1})$.

On the other hand, Theorem **D** does not make clear whether (1°-α) is necessary or not for hypoellipticity of P if f is not real analytic. This is because, there exists an f such that neither (**D**) nor Condition (1°-α) holds. For example, we can construct such a function in the following way. Let $g, \tilde{g}$ be functions of class $\boldsymbol{C}^\infty$ in $\boldsymbol{R}$ such that

$$\begin{cases} g > 0 \quad \text{in} \quad \{|t| < 1/2\}, & g \equiv 0 \quad \text{on} \quad \{|t| \geq 1/2\} \\ \tilde{g} > 0 \quad \text{in} \quad \{t > 0\}, & \tilde{g} \equiv 0 \quad \text{on} \quad \{t \leq 0\}. \end{cases}$$

And set

$$\begin{aligned} \Sigma_0 &= \{2\} \\ \Sigma_l &= \left\{2 + 3\delta_1 + 3^2\delta_2 + \cdots + 3^l\delta_l\ ;\ \delta_m \in \{0,2\}\ (m = 1, 2, \dots, l)\right\} \quad (l = 1, 2, \dots). \end{aligned}$$

We define a sequence of functions $\{g_l\}_{l=0}^\infty$ by

$$\begin{aligned} g_0(t) &= \tilde{g}(-t) + \tilde{g}(t-1) \\ g_l(t) &= \frac{(-1)^l}{l!} \sum_{\sigma \in \Sigma_{l-1}} g\left(3^l\left(t - \frac{2\sigma - 1}{2 \cdot 3^l}\right)\right) \quad (l \geq 1) \end{aligned}$$

and set

$$g^*(t) = \sum_{l=0}^{\infty} g_l(t).$$

We can verify that the sum $\sum_{l=1}^{\infty} g_l^{(k)}(t)$ converges uniformly in $\boldsymbol{R}$ for every $k \in \boldsymbol{Z}_+$. Therefore g^* belongs to $\boldsymbol{C}^\infty(\boldsymbol{R})$. Obviously, g^* does not satisfy Condition ($1°$-α). Moreover, since the set of zeros of g^* is the Cantor set, there exists, for any zero s of g^* and any neighborhood J of s, an $s' \in J$ such that $g^*(s') > 0$ and $s' > s$. So g^* does not satisfy (**D**).

Let us sketch the proof of Theorem **D**. The idea of proof is to construct a non-smooth function u for which $H = Pu$ is of class $\boldsymbol{C}^\infty$. The construction is based on [5]. A formal solution is obtained as a sum of distributions, but the sum is not convergent in general in the distribution sense. In the second subsection, we investigate the smoothness of each term of the sum. And we modify the formal solution to make it convergent in the distribution sense and to make its image by P smooth. In the third subsection, we verify that the modified sum is not smooth.

Let us begin with the proof of Theorem **D**. First, without loss of generality we may suppose that $s = 0$. Moreover from (**D**), we may assume that

$$-\int_0^t f(s)\,ds > 0 \quad \text{if } t \in I \setminus \{0\}. \tag{10.1}$$

This is because, if there exists an $r \in I \setminus \{0\}$ such that $-\int_0^r f(s)ds = 0$, then $f(t)$ vanishes identically on $[\min\{0,r\}, \max\{0,r\}]$. Obviously, P is not hypoelliptic in this case.

10.1 A formal solution

A formal solution is the sum of distributions obtained by solving a system of ordinary differential equations (10.2) below.

Let ξ be the dual variable of x and $a(t,x,\xi)$ be the symbol of $\sum_{j,k=1}^{n} a_{jk} L_j L_k$. Then P is written as $P = \partial_t + f(t)\,a(t,x,D_x)$. We denote by $\mathcal{F}_x$ the partial Fourier transform with respect to x. We rewrite Pu formally as follows by integration by parts.

$$Pu = \int e^{ix\cdot\xi} \left\{ \partial_t \mathcal{F}_x[u](t,\xi) + f(t) \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \Big((\partial_x^\alpha a(t,0,\xi))\, \mathcal{F}_x[u](t,\xi) \Big) \right\} đ\xi,$$

where $đ\xi = (2\pi)^{-d} d\xi$. Next, let $\psi(\xi)$ be a real-valued function of class $\boldsymbol{C}^\infty$ defined on $\boldsymbol{R}^d$ which we shall choose later. Let $\{v_j(t,\xi)\}_{j=0}^\infty$ be the solution of the following system of ordinary differential equations.

$$(10.2)\begin{cases} \dfrac{d}{dt}v_0(t,\xi) + f(t)\,a(t,0,\xi)\,v_0(t,\xi) &= 0 \\ \dfrac{d}{dt}v_j(t,\xi) + f(t)\,a(t,0,\xi)\,v_j(t,\xi) &= -f(t)\displaystyle\sum_{k=0}^{j-1}\sum_{|\alpha|=k+1}\frac{i^{|\alpha|}}{\alpha!}\times \\ & \qquad \partial_\xi^\alpha\Big(v_{j-k-1}(t,\xi)\,\partial_x^\alpha a(t,0,\xi)\Big)\ (\,j\ge 1\,) \end{cases}$$

subject to the initial condition

$$(10.3)\qquad \begin{cases} v_0(0,\xi) &= \psi(\xi) \\ v_j(0,\xi) &= 0 \qquad (\,j\ge 1\,). \end{cases}$$

We can solve (10.2)–(10.3) inductively. We see easily that $v_j(t,\xi)\in \boldsymbol{C}^\infty(\boldsymbol{R}^{d+1})$ for every j. If we denote the inverse partial Fourier transform with respect to ξ by $\mathcal{F}_\xi^{-1}$, the sum

$$(10.4)\qquad \tilde{u}(t,x) = \sum_{j=0}^{\infty}\mathcal{F}_\xi^{-1}[v_j](t,x)$$

satisfies $P\tilde{u} = 0$. (We shall show in Lemma 10.2 below that $\mathcal{F}_\xi^{-1}[v_j](t,x)$ belongs to $\mathcal{S}'(\boldsymbol{R}_\xi^d)$ for every $t\in I$.) Indeed, we can rewrite $P\tilde{u}$ formally as follows.

$$\begin{aligned} P\tilde{u} &= \int e^{ix\cdot\xi}\Bigg\{\partial_t v_0(t,\xi) + f(t)\,a(t,0,\xi)\,v_0 + \sum_{j=1}^{\infty}\Big(\partial_t v_j(t,\xi) + f(t)\,a(t,0,\xi)\,v_j \\ &\qquad + f(t)\sum_{k=0}^{j-1}\sum_{|\alpha|=k+1}\frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^\alpha\Big((\partial_x^\alpha a(t,0,\xi))\,v_{j-k-1}(t,\xi)\Big)\Big)\Bigg\}\,đ\xi. \end{aligned}$$

The right hand side is equal to 0 by (10.2). Moreover, if ψ is not rapidly decreasing, so will be $\mathcal{F}_x[u](0,\xi)$.

10.2 Modification of the formal solution

In this subsection, we shall modify the formal solution $\tilde{u}$ (see (10.4) above). Let us sketch how the modification will be done. First, we study $-\int_0^t f(s)a(s,0,\xi)ds$ and choose a suitable ψ in (10.3) (see Lemma 10.1). Next, we study the order of decay of v_j as $|\xi|\to\infty$. We will see that the order of decay of v_j grows larger as j increases (see Lemma 10.2). This means that $\mathcal{F}_\xi^{-1}[v_j]$ is more and more smooth as j increases. Finally we construct functions $\{u_j\}_{j=0}^\infty$ from $\{v_j\}_{j=0}^\infty$ such that $u=\sum_{j=0}^\infty u_j$ is convergent in the distribution sense and that Pu is smooth.

We investigate $-\int_0^t f(s)\,a(s,0,\xi)\,ds$. We set

$$F(t) = -\int_0^t f(s)\,ds,\quad G(t,\xi) = -\int_0^t f(s)\,a(s,0,\xi)\,ds.$$

Since $a(s,0,\xi)$ is a polynomial in ξ of degree 2, let $a_2(s,\xi)$ be the second degree part of $a(s,0,\xi)$ and set $a_1(s,\xi) = a(s,0,\xi) - a_2(s,\xi)$. $a_2(s,\xi)$ is written as follows:

$$a_2(s,\xi) = \sum_{j,k=1}^{n} a_{jk}(s,0)\,\sigma_p(L_j)(0,\xi)\,\sigma_p(L_k)(0,\xi),$$

where $\sigma_p(L_j)$ is the first degree part with respect to ξ of the symbol of L_j. Since the coefficients of L_j are real-valued by Condition (3°), $\sigma_p(L_j)(0,\xi)$ is purely imaginary. Then we have for every $s \in I$

$$\begin{aligned} \operatorname{Re} a_2(s,\xi) &= \operatorname{Re} \sum_{j,k=1}^{n} a_{jk}(s,0)\,\sigma_p(L_j)(0,\xi)\,\sigma_p(L_k)(0,\xi) \\ &= -\frac{1}{2}\Big(\big(A(s,0) + {}^t\bar{A}(s,0)\big)\,\sigma,\sigma\Big) \\ &\le -\frac{1}{2}\,\delta(I)\sum_{j=1}^{n} |\sigma_p(L_j)(0,\xi)|^2 \quad (\text{ by Condition } (2°)\ ). \end{aligned}$$

where σ is the column vector ${}^t(\,\sigma_p(L_1)(0,\xi),\ \ldots\ ,\ \sigma_p(L_n)(0,\xi)\,)$.

Next, in view of Condition (3°),

(10.5) there exist integers $q \in \{1,\ldots,n\}$ and $m \in \{1,\ldots,d\}$ such that the coefficient of ξ_m in $\sigma_p(L_q)(0,\xi)$ is not zero.

Let $\widetilde{\xi} = (\xi_1,\ldots,\xi_{m-1},0,\xi_{m+1},\ldots,\xi_d)$. For every $C > 0$ we define $D(C)$ to be

$$D(C) = \Big\{\xi \in \boldsymbol{R}^d\ ;\ |\xi_m|^2 \ge C|\tilde{\xi}|^2\Big\}. \tag{10.6}$$

We see immediately that

$$\xi_m^2 \ge \frac{C}{1+C}|\xi|^2 \quad \text{for every } \xi \in D(C). \tag{10.7}$$

Now we shall show the following lemma.

Lemma 10.1 *There exist positive constants (C_0, C_1, C_2) independent of (t,ξ) such that*

$$|\exp G(t,\xi)| \le \exp\Big(-C_0 F(t)|\xi|^2\Big) \quad \textit{for all } (t,\xi) \in I \times \Big(D(C_1) \cap \{|\xi| \ge C_2\}\Big). \tag{10.8}$$

Proof of Lemma 10.1. Since $|\exp G(t,\xi)| = \exp \operatorname{Re} G(t,\xi)$, it suffices to show that

$$\operatorname{Re} G(t,\xi) \le -C_0|\xi|^2 F(t) \quad \text{for all } (t,\xi) \in I \times \Big(D(C_1) \cap \{|\xi| \ge C_2\}\Big). \tag{10.9}$$

This follows from the inequality.

$$\operatorname{Re} a(s,0,\xi) \le -C_0|\xi|^2 \quad \text{for all } (s,\xi) \in I \times \Big(D(C_1) \cap \{|\xi| \ge C_2\}\Big). \tag{10.10}$$

In fact, suppose that (10.10) holds. If $t \in I$ and $t > 0$, then $-f(s) \geq 0$ on $[0, t]$ by (**D**) and we have

$$\begin{aligned}
\operatorname{Re} G(t,\xi) &= -\int_0^t f(s) \operatorname{Re} a(s,0,\xi)\, ds \\
&\leq C_0|\xi|^2 \int_0^t f(s)\, ds = -C_0|\xi|^2 F(t) \quad \text{for all } \xi \in D(C_1) \cap \{|\xi| \geq C_2\}.
\end{aligned}$$

In a similar way, we can verify that this holds also for $t < 0$. So (10.9) is satisfied and hence (10.8) holds.

Now let us prove (10.10). First, by (10.5), there exist positive constants C_3, C_4 independent of ξ such that

$$\sum_{j=1}^n |\sigma_p(L_j)(0,\xi)|^2 \geq C_3\, {\xi_m}^2 - C_4|\tilde{\xi}|^2.$$

Then, in view of the definition of $D(C)$, we choose a positive number C_5 so large that

$$C_4|\tilde{\xi}|^2 \leq \frac{C_3}{2}\, {\xi_m}^2 \text{ for all } \xi \in D(C_5).$$

Thus we have

$$\operatorname{Re} a_2(s,\xi) \leq -\frac{1}{4}\, \delta(I)\, C_3\, {\xi_m}^2 \quad \text{for all } (s,\xi) \in I \times D(C_5). \tag{10.11}$$

Since $a_1(s,\xi)$ is a polynomial in ξ of degree 1, there exist constants C_6, C_7 independent of (s,ξ) such that

$$|\operatorname{Re} a_1(s,\xi)| \leq C_6|\xi| + C_7 \quad \text{for all } (s,\xi) \in I \times \boldsymbol{R}^d.$$

We choose a positive number C_8 so large that

$$C_6|\xi| + C_7 \leq \frac{\delta(I)\, C_3\, C_5}{8(1+C_5)}\, |\xi|^2 \quad \text{for all } \xi \in \boldsymbol{R}^d \text{ satisfying } |\xi| \geq C_8.$$

Then, we have by (10.7)

$$\begin{aligned}
|\operatorname{Re} a_1(s,\xi)| &\leq \frac{\delta(I)\, C_3\, C_5}{8(1+C_5)}\, |\xi|^2 \\
&\leq \frac{1}{8}\, \delta(I)\, C_3\, {\xi_m}^2 \quad \text{for all } (s,\xi) \in I \times \Big(D(C_5) \cap \{|\xi| \geq C_8\}\Big).
\end{aligned} \tag{10.12}$$

Set $C_9 = \delta(I)C_3C_5/(8+8C_5) > 0$. We have from (10.11), (10.12) and (10.7)

$$\begin{aligned}
\operatorname{Re} a(s,0,\xi) &= \operatorname{Re} a_2(s,\xi) + \operatorname{Re} a_1(s,\xi) \;\leq\; -\frac{1}{4}\, \delta(I)\, C_3\, {\xi_m}^2 + |\operatorname{Re} a_1(s,\xi)| \\
&\leq -\frac{1}{4}\, \delta(I)\, C_3\, {\xi_m}^2 + \frac{1}{8}\, \delta(I)\, C_3\, {\xi_m}^2 \;\leq\; -\frac{1}{8}\, \delta(I)\, C_3\, {\xi_m}^2 \\
&\leq -C_9|\xi|^2
\end{aligned}$$

for all $(s,\xi) \in I \times \Big(D(C_5) \cap \{|\xi| \geq C_8\}\Big)$. This implies (10.10). And hence Lemma 10.1 is proved. $\square$

We choose a real-valued function $\psi(\xi)$ in such a way that $\psi(\xi) = \langle\xi\rangle^{-d-3}\chi(\xi)$, where $\chi(\xi)$ is an element of $S^0_{1,0}(\boldsymbol{R}^d)$ and satisfies

$$0 \leq \chi \leq 1, \quad \mathrm{supp}\chi \subset D(C_1) \cap \{|\xi| \geq C_2\} \quad \text{and} \quad \chi \equiv 1 \text{ on } D(2C_1) \cap \{|\xi| \geq 2C_2\}.$$

Note that $\psi(\xi)$ is not rapidly decreasing. From (10.2)–(10.3), we have

$$v_0(t,\xi) = \psi(\xi) \exp G(t,\xi).$$

Therefore (10.8) implies

$$|v_0(t,\xi)| \leq \langle\xi\rangle^{-d-3} \exp\Big(-C_0 F(t)|\xi|^2\Big) \quad \text{for all } (t,\xi) \in I \times \boldsymbol{R}^d.$$

By (10.1) and this inequality, $v_0(t,\xi)$ belongs to $L^1(\boldsymbol{R}^d_\xi) \cap L^2(\boldsymbol{R}^d_\xi)$ for every $t \in I$, moreover $\mathcal{F}^{-1}_\xi[v_0]$ is differentiable once with respect to t and twice with respect to x.

Now we are going to show that $\mathcal{F}^{-1}_\xi[v_j]$ is $([j/2]+1)$-times differentiable with respect to (t,x). We shall prove Lemma 10.2 below for this. Furthermore, $\mathcal{F}^{-1}_\xi[v_j]$ is differentiable once with respect to t and twice with respect to x by using Lemma 10.2 (see (10.18) below).

Lemma 10.2 *For any $(j,p,\beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+ \times \boldsymbol{Z}^d_+$, there exists a constant $C^j_{p,\beta}$ such that*

$$\Big|\partial^p_t \partial^\beta_\xi v_j(t,\xi)\Big| \leq C^j_{p,\beta} \Big(1 + F(t)\langle\xi\rangle^2\Big)^{|\beta|+2j} \langle\xi\rangle^{2p-d-3-|\beta|-j} \exp \mathrm{Re}\, G(t,\xi) \tag{10.13}$$

for all $(t,\xi) \in I \times \boldsymbol{R}^d$.

Proof of Lemma 10.2. We proceed by induction with respect to j. For $j = 0$, we treat $\partial^\beta_\xi v_0$ at first. It is written as

$$\partial^\beta_\xi v_0 = \sum_{\substack{\beta'_0+\cdots+\beta'_l=\beta \\ |\beta'_j|\geq 1,\, j=1,\ldots,l}} C_{\beta'} \Big(\partial^{\beta'_0}_\xi \psi\Big)\Big(\partial^{\beta'_1}_\xi G(t,\xi)\Big) \cdots \Big(\partial^{\beta'_l}_\xi G(t,\xi)\Big) \exp G(t,\xi),$$

and $\partial^\beta_\xi G(t,\xi)$ is evaluated as follows.

$$\Big|\partial^\beta_\xi G(t,\xi)\Big| = \Big|-\int_0^t f(s)\, \partial^\beta_\xi a(s,0,\xi)\, ds\Big| \leq C_\beta F(t)\langle\xi\rangle^{2-|\beta|}.$$

Note that $F(t)$ is non-negative by (10.1). So we have

$$\begin{aligned}
\Big|\partial^\beta_\xi v_0(t,\xi)\Big| &\leq \sum C'_{\beta'} \langle\xi\rangle^{-d-3-|\beta'_0|} F(t)^l \langle\xi\rangle^{2l-|\beta'_1+\cdots+\beta'_l|} \exp \mathrm{Re}\, G(t,\xi) \\
&= \sum_{l=1}^{|\beta|} C_{\beta,l} F(t)^l \langle\xi\rangle^{2l} \langle\xi\rangle^{-d-3-|\beta|} \exp \mathrm{Re}\, G(t,\xi) \\
&\leq C_\beta \Big(1 + F(t)\langle\xi\rangle^2\Big)^{|\beta|} \langle\xi\rangle^{-d-3-|\beta|} \exp \mathrm{Re}\, G(t,\xi).
\end{aligned}$$

Therefore (10.13) holds for $j = p = 0$. Next, we prove (10.13) in the case where $j = 0, p \geq 1$ by induction with respect to p. Let p' be a positive integer and suppose that (10.13) holds for $j = 0, 0 \leq p \leq p'$. From (10.2), we have $\partial_t^{p'+1}\partial_\xi^\beta v_0 = -\partial_t^{p'}\partial_\xi^\beta (f(t)a(t,0,\xi)v_0)$. By Leibniz' rule, this implies

$$\begin{aligned}
\left|\partial_t^{p'+1}\partial_\xi^\beta v_0\right| &= \left|\sum_{q=0}^{p'}\sum_{\alpha\leq\beta}\binom{p'}{q}\binom{\beta}{\alpha}\partial_t^q\partial_\xi^\alpha\Big(f(t)\,a(t,0,\xi)\Big)\,\partial_t^{p'-q}\partial_\xi^{\beta-\alpha}v_0\right| \\
&\leq \sum_{q=0}^{p'}\sum_{\alpha\leq\beta} C'_{q,\alpha}\langle\xi\rangle^{2-|\alpha|}\times \\
&\qquad C_{p'-q,\beta-\alpha}\left(1+F(t)\langle\xi\rangle^2\right)^{|\beta-\alpha|}\langle\xi\rangle^{2(p'-q)-d-3-|\beta-\alpha|}\exp\operatorname{Re}G(t,\xi) \\
&\qquad (\text{ by induction hypothesis }) \\
&\leq C'_{p',\beta}\left(1+F(t)\langle\xi\rangle^2\right)^{|\beta|}\langle\xi\rangle^{2(p'+1)-d-3-|\beta|}\exp\operatorname{Re}G(t,\xi).
\end{aligned}$$

So (10.13) holds for $p = p' + 1$. Now (10.13) is verified for all p provided that $j = 0$.

Next, let j' be a positive integer and suppose that (10.13) holds for $0 \leq j \leq j'$. Then (10.2)–(10.3) yield

$$\begin{aligned}
v_{j'+1}(t,\xi) &= -\sum_{k=0}^{j'}\sum_{|\alpha|=k+1}\exp G(t,\xi) \\
&\qquad\times\int_0^t f(s)\,\frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^\alpha\Big(v_{j'-k}(s,\xi)\,\partial_x^\alpha a(s,0,\xi)\Big)\exp(-G(s,\xi))\,ds.
\end{aligned}$$

Furthermore, we have by Leibniz' rule

(10.14) $\partial_\xi^\beta v_{j'+1}(t,\xi)$

$$\begin{aligned}
&= -\sum_{\gamma_1+\gamma_2+\gamma_3=\beta} C_{\gamma_1,\gamma_2,\gamma_3}\sum_{k=0}^{j'}\sum_{|\alpha|=k+1}\Big(\partial_\xi^{\gamma_1}\exp G(t,\xi)\Big) \\
&\qquad\times\int_0^t f(s)\,\frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^{\alpha+\gamma_2}\Big(v_{j'-k}(s,\xi)\,\partial_x^\alpha a(s,0,\xi)\Big)\Big(\partial_\xi^{\gamma_3}\exp(-G(s,\xi))\Big)\,ds.
\end{aligned}$$

By the same way as for v_0, we have

$$(10.15)\quad \left\{\begin{aligned}
\left|\partial_\xi^{\gamma_1}\exp G(t,\xi)\right| &\leq C_{\gamma_1}\left(1+F(t)\langle\xi\rangle^2\right)^{|\gamma_1|}\langle\xi\rangle^{-|\gamma_1|}\exp\operatorname{Re}G(t,\xi), \\
\left|\partial_\xi^{\gamma_3}\exp(-G(s,\xi))\right| &\leq C_{\gamma_3}\left(1+F(s)\langle\xi\rangle^2\right)^{|\gamma_3|}\langle\xi\rangle^{-|\gamma_3|}\exp(-\operatorname{Re}G(s,\xi)).
\end{aligned}\right.$$

Since $j' - k \leq j'$, we have by induction hypothesis

$$\begin{aligned}
(10.16)\quad &\left|\frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^{\alpha+\gamma_2}\Big(v_{j'-k}(s,\xi)\,\partial_x^\alpha a(s,0,\xi)\Big)\right| \\
&\leq C_{\alpha,\gamma_2,j',k}\left(1+F(s)\langle\xi\rangle^2\right)^{|\alpha+\gamma_2|+2(j'-k)}\langle\xi\rangle^{-d-1-|\alpha+\gamma_2|-(j'-k)}\exp\operatorname{Re}G(s,\xi).
\end{aligned}$$

Suppose that $t > 0$. Since $|f(s)| = -f(s)$ for every $s \in [0,t]$, we have by combining inequalities (10.14), (10.15) and (10.16)

$$
\begin{aligned}
&\left|\partial_\xi^\beta v_{j'+1}(t,\xi)\right| \\
&\le \sum_{\gamma_1+\gamma_2+\gamma_3=\beta}\sum_{k=0}^{j'}\sum_{|\alpha|=k+1} C_{\gamma_1,\gamma_2,\gamma_3,j',k,\alpha}\left(1+F(t)\langle\xi\rangle^2\right)^{|\gamma_1|}\langle\xi\rangle^{-d-1-|\alpha+\gamma_1+\gamma_2+\gamma_3|-(j'-k)} \\
&\qquad \times \exp \operatorname{Re} G(t,\xi)\int_0^t \left\{(-f(s))\left(1+F(s)\langle\xi\rangle^2\right)^{|\alpha+\gamma_2+\gamma_3|+2(j'-k)}\right\}ds.
\end{aligned}
$$

Since $-f(s)\ge 0$ for $s\in[0,t]$, we have $0\le F(s)\le F(t)$ for $s\in[0,t]$. Therefore we obtain

$$
\begin{aligned}
(10.17)\quad &\left|\partial_\xi^\beta v_{j'+1}(t,\xi)\right| \\
&\le C_\beta^{j'}\left(1+F(t)\langle\xi\rangle^2\right)^{|\beta|+2j'+1}\langle\xi\rangle^{-d-1-|\beta|-j'-1}\exp\operatorname{Re}G(t,\xi)\int_0^t(-f(s))\,ds \\
&= C_\beta^{j'}\left(1+F(t)\langle\xi\rangle^2\right)^{|\beta|+2j'+1}\langle\xi\rangle^{-d-1-|\beta|-(j'+1)}\Big(\exp\operatorname{Re}G(t,\xi)\Big)F(t)\langle\xi\rangle^2\langle\xi\rangle^{-2} \\
&\le C_\beta^{j'}\left(1+F(t)\langle\xi\rangle^2\right)^{|\beta|+2j'+2}\langle\xi\rangle^{-d-3-|\beta|-(j'+1)}\exp\operatorname{Re}G(t,\xi).
\end{aligned}
$$

In a similar way, we get (10.17) also for $t<0$. Now (10.13) is verified for every $(p,j)\in\{0\}\times\boldsymbol{Z}_+$. Finally, (10.2) yields

$$
\begin{aligned}
\partial_t^{p'+1}\partial_\xi^\beta v_j(t,\xi) &= -\partial_t^{p'}\partial_\xi^\beta\Big(f(t)\,a(t,0,\xi)\,v_j(t,\xi)\Big) \\
&\quad -\partial_t^{p'}\partial_\xi^\beta\left(f(t)\sum_{k=0}^{j-1}\sum_{|\alpha|=k+1}\frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^\alpha\Big(v_{j-k-1}(t,\xi)\,\partial_x^\alpha a(t,0,\xi)\Big)\right)
\end{aligned}
$$

for $j\ge 1$. Thus, again by induction with respect to p by the same way as in the proof for v_0, we have (10.13) for $p>0$. Now Lemma 10.2 is proved. □

The support of $v_j(t,\cdot)$ is contained in $\subset D(C_1)\cap\{|\xi|\ge C_2\}$ for every $(j,t)\in\boldsymbol{Z}_+\times I$ from (10.2)–(10.3) and the definition of ψ. Moreover, since

$$
\sup_{I\times\boldsymbol{R}^d}\left\{F(t)^l|\xi|^{2l}\exp(-C_0|\xi|^2F(t))\right\}<+\infty\quad\text{for every } l\in\boldsymbol{Z}_+,
$$

we have by Lemma 10.2 and (10.8)

$$
(10.18)\qquad \left|\partial_t^p\partial_\xi^\beta v_j(t,\xi)\right|\le C_{j,p,\beta}\langle\xi\rangle^{2p-d-3-j-|\beta|}\quad\text{for all } (t,\xi)\in I\times\boldsymbol{R}^d.
$$

This shows that $\mathcal{F}_\xi^{-1}[v_j](t,x)\in\boldsymbol{C}^{[j/2]+1}(I\times\boldsymbol{R}^d)$.

In what follows, we modify v_j. The formal sum $\widetilde{u}$ satisfies $P\widetilde{u}=0$ (see (10.4)). However, $\widetilde{u}$ is not necessarily convergent in the distribution sense. Therefore, we need to modify v_j consisting $\widetilde{u}$ so that the modified sum converges and in addition that its image by P is smooth. Let $h(\xi)$ be a function of class $\boldsymbol{C}^\infty$ in $\boldsymbol{R}^d$ such that

$$
h(\xi)\equiv 1 \text{ on } \{|\xi|\ge 2\},\quad \operatorname{supp}h(\xi)\subset\{|\xi|\ge 1\}.
$$

Let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a sequence of positive numbers such that $\varepsilon_0 = 1 < \varepsilon_1 < \varepsilon_2 < \cdots \to +\infty$ which we choose later. For every $(j, p, \beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+ \times \boldsymbol{Z}_+^d$ satisfying $p + |\beta| \leq [j/2]$, we have

$$\begin{aligned}
&\left| \partial_t^p \partial_x^\beta \mathcal{F}_\xi^{-1} \Big[h(\varepsilon_j^{-1} \cdot)\, v_j(t, \cdot) \Big](x) \right| \\
&= \left| \int e^{ix\cdot\xi}\, (i\xi)^\beta\, h(\varepsilon_j^{-1}\xi)\, \partial_t^p v_j(t,\xi)\, đ\xi \right| \leq C_{j,p,0} \int_{|\xi|\geq\varepsilon_j} \langle\xi\rangle^{-d-3-j+2p+|\beta|} đ\xi \quad \text{(by (10.18))} \\
&\leq C_{j,p,0} \int_{|\xi|\geq\varepsilon_j} \langle\xi\rangle^{-d-2} \langle\xi\rangle^{-1} đ\xi \leq C_{j,p,0}\varepsilon_j{}^{-1} \int \langle\xi\rangle^{-d-2} đ\xi = C_{j,p,0} C_d \varepsilon_j{}^{-1},
\end{aligned}$$

where C_d is a constant depending only on d. So we choose a $\{\varepsilon_j\}_{j=1}^{\infty}$ inductively in the following way.

$$\varepsilon_0 = 1, \quad \varepsilon_j > \max \left\{ 1 + \varepsilon_{j-1}, 2^j C_d \max_{p+|\beta|\leq[j/2]} C_{j,p,0} \right\} \quad (j \geq 1).$$

We set $u_j(t, x) = \mathcal{F}_\xi^{-1}[h(\varepsilon_j^{-1}\cdot) v_j(t, \cdot)](x)$. Then we have for every $(j, p, \beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+ \times \boldsymbol{Z}_+^d$ satisfying $p + |\beta| \leq [j/2]$

$$\text{(10.19)} \qquad \left| \partial_t^p \partial_x^\beta u_j(t, x) \right| \leq 2^{-j} \quad \text{for all } (t, x) \in I \times \boldsymbol{R}^d.$$

And hence

$$\text{(10.20)} \qquad \sum_{j=N}^{\infty} u_j(t, x) \text{ converges in } \boldsymbol{C}^{[N/2]}(I \times \boldsymbol{R}^d) \text{ for every positive integer } N.$$

Let

$$u(t, x) = \sum_{j=0}^{\infty} u_j(t, x).$$

Since $u(t, x)$ is a continuous function in $I \times \boldsymbol{R}^d$ from (10.19), it belongs to $\mathcal{D}'(I \times \boldsymbol{R}^d)$.

Now, let us show that Pu is smooth. For a positive integer N, we divide Pu into two parts as follows.

$$Pu(t, x) = P \sum_{j=0}^{N-1} u_j(t, x) + P \sum_{j=N}^{\infty} u_j(t, x) = U_{1,N}(t, x) + U_{2,N}(t, x).$$

It suffices to show the following lemma.

Lemma 10.3 *Let l be a positive integer. Then $U_{1,N}(t, x) \in \boldsymbol{C}^l(I \times \boldsymbol{R}^d)$ if $[N/2] > l$.*

In fact, given a positive integer l, we choose N so large that $[N/2] > l + 2$. By (10.20), $U_{2,N}(t, x) \in \boldsymbol{C}^{[N/2]-2}(I \times \boldsymbol{R}^d) \subset \boldsymbol{C}^l(I \times \boldsymbol{R}^d)$. Moreover, this lemma implies $U_{1,N}(t, x) \in \boldsymbol{C}^l(I \times \boldsymbol{R}^d)$. So $Pu = U_{1,N} + U_{2,N} \in \boldsymbol{C}^l(I \times \boldsymbol{R}^d)$. Since l is arbitrary, we see that $Pu \in \boldsymbol{C}^\infty(I \times \boldsymbol{R}^d)$.

Proof of Lemma 10.3. $U_{1,N}(t,x)$ is given by

$$\begin{aligned} U_{1,N}(t,x) &= \int e^{ix\cdot\xi} \sum_{j=0}^{N-1} h(\varepsilon_j^{-1}\xi)\,\partial_t v_j(t,\xi)\,đ\xi \\ &\quad + \int e^{ix\cdot\xi} f(t)\,a(t,x,\xi) \sum_{j=0}^{N-1} h(\varepsilon_j^{-1}\xi)\,v_j(t,\xi)\,đ\xi. \end{aligned}$$

By the Taylor expansion and integration by parts, we have

$$\begin{aligned} & U_{1,N}(t,x) \\ = & \int e^{ix\cdot\xi} \sum_{j=0}^{N-1} h(\varepsilon_j^{-1}\xi)\,\partial_t v_j(t,\xi)\,đ\xi \\ & + \int e^{ix\cdot\xi} f(t) \sum_{j=0}^{N-1} \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^\alpha \Big(h(\varepsilon_j^{-1}\xi)\,v_j(t,\xi)\,\partial_x^\alpha a(t,0,\xi)\Big)\,đ\xi \\ & + N \int e^{ix\cdot\xi} \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!} \int_0^1 (1-\theta)^{N-1} \partial_\xi^\alpha \Big(f(t)\,\partial_x^\alpha a(t,\theta x,\xi) \sum_{j=0}^{N-1} h(\varepsilon_j^{-1}\xi)\,v_j(t,\xi)\Big) d\theta\, đ\xi \\ = & V_{1,N}(t,x) + V_{2,N}(t,x) + V_{3,N}(t,x). \end{aligned}$$

To prove Lemma 10.3, we have to show the following:

$$V_{1,N}(t,x) + V_{2,N}(t,x) \in \boldsymbol{C}^l(I\times\boldsymbol{R}^d) \text{ if } [N/2] > l. \tag{10.21}$$

$$V_{3,N}(t,x) \in \boldsymbol{C}^l(I\times\boldsymbol{R}^d) \text{ if } [N/2] > l. \tag{10.22}$$

Let us begin with (10.22). We evaluate derivatives of the integrand of $V_{3,N}$ by using (10.18) and we have

" For any $K \subset\subset \boldsymbol{R}^d$ and any $(p,\alpha,\beta,\gamma) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+^d \times \boldsymbol{Z}_+^d \times \boldsymbol{Z}_+^d$ satisfying that $|\alpha| = N$ and $p + |\beta| \le [N/2]$, there exists a constant $C = C(K,p,\alpha,\beta,\gamma,N)$ such that

$$\left| (i\xi)^\beta \partial_t^p \partial_\xi^\alpha \partial_x^\gamma \Big(f(t)\,\partial_x^\alpha a(t,\theta x,\xi) \sum_{j=0}^{N-1} h(\varepsilon_j^{-1}\xi)\,v_j(t,\xi)\Big) \right| \le C\langle\xi\rangle^{-d-1+2p+|\beta|-N}$$

for all $(t,x,\xi,\theta) \in I\times K\times\boldsymbol{R}^d\times[0,1]$. "

This implies $V_{3,N}(t,x) \in \boldsymbol{C}^{[N/2]}(I\times\boldsymbol{R}^d)$. Consequently, $V_{3,N}(t,x) \in \boldsymbol{C}^l(I\times\boldsymbol{R}^d)$ if $[N/2] > l$. Now (10.22) is verified.

Next, we prove (10.21). We define $W_{1,N}(t,x)$ by

$$W_{1,N}(t,x) = \int e^{ix\cdot\xi} h(\xi) \sum_{j=0}^{N-1} \partial_t v_j(t,\xi)\,đ\xi. \tag{10.23}$$

Since derivatives of $h(\xi)\partial_t v_j(t,\xi) - h(\varepsilon_j^{-1}\xi)\partial_t v_j(t,\xi)$ of any order are rapidly decreasing with respect to ξ, the difference $V_{1,N} - W_{1,N}$ is smooth. Also, we define $W_{2,N}(t,x)$ by

$$W_{2,N}(t,x) = \int e^{ix\cdot\xi} h(\xi) \sum_{j=0}^{N-1} f(t) \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!}\,\partial_\xi^\alpha \Big(v_j(t,\xi)\,\partial_x^\alpha a(t,0,\xi)\Big)\,đ\xi. \tag{10.24}$$

Then, the difference $V_{2,N} - W_{2,N}$ is also smooth because derivatives of $h(\xi)\partial_\xi^\alpha\Big(v_j(t,\xi)\,\partial_x^\alpha a(t,0,\xi)\Big) - \partial_\xi^\alpha\left(h(\varepsilon_j^{-1}\xi)v_j(t,\xi)\partial_x^\alpha a(t,0,\xi)\right)$ of any order are rapidly decreasing with respect to ξ. Therefore, for the proof of (10.21), it suffices to show that

$$(10.25)\quad W_{1,N}(t,x) + W_{2,N}(t,x) \in \boldsymbol{C}^l(I \times \boldsymbol{R}^d) \quad \text{for every } N \text{ satisfying } [N/2] > l.$$

Now we prove (10.25) by making use of (10.2). Summing up both sides of (10.2) from $j = 0$ to $j = N-1$, we have

$$(10.26)\quad \int e^{ix\cdot\xi} h(\xi) \sum_{j=0}^{N-1} \left(\partial_t v_j(t,\xi) + f(t) \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!}\, \partial_\xi^\alpha\Big(v_j(t,\xi)\, \partial_x^\alpha a(t,0,\xi)\Big)\right) đ\xi$$
$$= \int e^{ix\cdot\xi} h(\xi)\, f(t) \sum_{k=0}^{N-2} \sum_{|\alpha|=k+1} \sum_{p=N-k-1}^{N-1} \frac{i^{|\alpha|}}{\alpha!}\, \partial_\xi^\alpha\Big(v_p(t,\xi)\, \partial_x^\alpha a(t,0,\xi)\Big) đ\xi.$$

The left hand side of (10.26) is equal to $W_{1,N} + W_{2,N}$. Therefore, we have to show that the right hand side of (10.26) belongs to $\boldsymbol{C}^l(I \times \boldsymbol{R}^d)$ for every N satisfying $[N/2] > l+2$. We evaluate derivatives of the integrand by using (10.18). Then we have for every $(q,\beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+^d$

$$\left| (i\xi)^\beta \partial_t^q \left(f(t) \sum_{k=0}^{N-2} \sum_{|\alpha|=k+1} \sum_{p=N-k-1}^{N-1} \frac{i^{|\alpha|}}{\alpha!}\, \partial_\xi^\alpha\Big(v_p(t,\xi)\, \partial_x^\alpha a(t,0,\xi)\Big)\right) \right|$$
$$\le \sum_{k=0}^{N-2} \sum_{|\alpha|=k+1} \sum_{p=N-k-1}^{N-1} C_{k,\alpha,\beta,p,q,N} \langle\xi\rangle^{-d-1-|\alpha|+|\beta|+2q-p} \quad \le C_{q,\beta,N} \langle\xi\rangle^{-d-1+|\beta|+2q-N}.$$

This shows that the right hand side of (10.26) belongs to $\boldsymbol{C}^{[N/2]}(I \times \boldsymbol{R}^d)$. So (10.25) holds. This completes the proof of Lemma 10.3. □

10.3 Verification of non-smoothness of the solution

From (10.19), the sum $\sum_{j=0}^\infty u_j(t,x)$ is convergent uniformly in any compact subset of I. Since $u_j(t,x)$ is continuous for every j and $u_j(0,x)$ vanishes identically for every $j \ge 1$ by (10.3), we have

$$u(0,x) = \sum_{j=0}^\infty u_j(0,x) = u_0(0,x) = \int e^{ix\cdot\xi} h(\xi)\psi(\xi)\, đ\xi.$$

If u were smooth, $u(0,x)$ should also be smooth. In what follows, we shall prove by contradiction that $u(0,x)$ is not smooth.

Set $B_r = \{x \in \boldsymbol{R}^d;\ |x| < r\}$. Suppose that $u(0,x)$ be smooth in B_1. Then for every $\varphi \in \boldsymbol{C}_0^\infty(B_1)$, $\varphi(x)u(0,x)$ belongs to $\boldsymbol{C}_0^\infty(B_1)$ and $\mathcal{F}_x[\varphi(\cdot)u(0,\cdot)](\xi)$ is rapidly decreasing.

Let $\varphi_1 \in \boldsymbol{C}_0^\infty(B_1)$ be such that $\mathcal{F}_x[\varphi_1](\xi)$ is real-valued and non-negative. There exists such a φ_1. In fact, take a $\varphi_2 \in \boldsymbol{C}_0^\infty(B_{1/3})$ such that φ_2 is real-valued and $\varphi_2(x) = \varphi_2(-x)$. Then $\mathcal{F}_x[\varphi_2](\xi)$ is real-valued. Set $\varphi_1(x) = \varphi_2 * \varphi_2(x)$, where $*$ stands for the convolution. Now, $\varphi_1 \in \boldsymbol{C}_0^\infty(B_{2/3})$, $\mathcal{F}_x[\varphi_1](\xi)$ is real-valued and non-negative, because $\mathcal{F}_x[\varphi_1](\xi) = (\mathcal{F}_x[\varphi_2](\xi))^2$.

Let us evaluate $\mathcal{F}[\varphi_1(\cdot)u(0,\cdot)](\xi)$ and prove that it is not rapidly decreasing. First, without loss of generality, we may suppose that m appeared in (10.6) is equal to 1. So $D(C)$ is defined to be

$$D(C) = \Big\{\xi \in \boldsymbol{R}^d\,;\ |\xi_1|^2 \geq C|\xi_2|^2 + \cdots + |\xi_d|^2\Big\}.$$

Let us remember that ψ is of the form

$$\psi(\xi) = \langle\xi\rangle^{-d-3}\chi(\xi),$$

where $\chi(\xi)$ is an element of $S^0_{1,0}(\boldsymbol{R}^d)$ and satisfies

$$0 \leq \chi \leq 1, \quad \mathrm{supp}\chi \subset D(C_1) \cap \{|\xi| \geq C_2\} \quad \text{and} \quad \chi \equiv 1 \text{ on } D(2C_1) \cap \{|\xi| \geq 2C_2\}.$$

Then we have

$$\begin{aligned} \mathcal{F}\Big[\varphi_1(\cdot)\,u(0,\cdot)\Big](\xi) &= \int_{\boldsymbol{R}^d_x} e^{-ix\cdot\xi}\varphi_1(x)\mathcal{F}_\xi^{-1}\Big[h(\cdot)\,\psi(\cdot)\Big](x)\,dx \\ &= \int_{\boldsymbol{R}^d_\eta} \mathcal{F}_x[\varphi_1](\eta)\,\langle\xi-\eta\rangle^{-d-3}\,h(\xi-\eta)\,\chi(\xi-\eta)\,đ\eta \end{aligned}$$

Since the integrand is non-negative and $\langle\xi-\eta\rangle \leq \sqrt{2}\langle\xi\rangle\langle\eta\rangle$, we have

$$\begin{aligned} (10.27) \qquad &\mathcal{F}\Big[\varphi_1(\cdot)\,u(0,\cdot)\Big](\xi) \\ &\geq\ 2^{-(d+3)/2}\int_{\boldsymbol{R}^d_\eta} \mathcal{F}_x[\varphi_1](\eta)\,\langle\xi\rangle^{-d-3}\langle\eta\rangle^{-d-3}\,h(\xi-\eta)\,\chi(\xi-\eta)\,đ\eta. \end{aligned}$$

Next we define three domains $D_1(\xi_1), D_2(\xi_1)$ and $D_3(\xi_1)$ by setting

$$\begin{aligned} D_1(\xi_1) &= \Big\{\eta \in \boldsymbol{R}^d\,;\ |\xi_1-\eta_1|^2 \geq 2C_1\Big(|\eta_2|^2 + \cdots + |\eta_d|^2\Big)\Big\}, \\ D_2(\xi_1) &= \Big\{\eta \in \boldsymbol{R}^d\,;\ \xi_1-\eta_1 \geq 2C_1\Big(|\eta_2|^2 + \cdots + |\eta_d|^2\Big)^{1/2}\Big\}, \\ D_3(\xi_1) &= \Big\{\eta \in \boldsymbol{R}^d\,;\ |(\xi_1,0,\ldots,0)-\eta| \geq \max\{2C_2, 2\}\Big\}. \end{aligned}$$

We see immediately that $D_2(\xi_1) \subset D_1(\xi_1)$ for every $\xi_1 \in \boldsymbol{R}$ and that

$$h(\xi-\eta)\chi(\xi-\eta)\Big|_{\xi=(\xi_1,0,\ldots,0)} = 1 \quad \text{on } D_1(\xi_1) \cap D_3(\xi_1) \text{ for every } \xi_1 \in \boldsymbol{R}.$$

Substituting $(\xi_1,0,\ldots,0)$ for ξ in (10.27), we have

$$\begin{aligned} \mathcal{F}\Big[\varphi_1(\cdot)\,u(0,\cdot)\Big](\xi_1,0,\ldots,0) &\geq 2^{-(d+3)/2}\langle\xi_1\rangle^{-d-3}\int_{D_1(\xi_1)\cap D_3(\xi_1)} \mathcal{F}_x[\varphi_1](\eta)\langle\eta\rangle^{-d-3}đ\eta \\ &\geq 2^{-(d+3)/2}\langle\xi_1\rangle^{-d-3}\int_{D_2(\xi_1)\cap D_3(\xi_1)} \mathcal{F}_x[\varphi_1](\eta)\langle\eta\rangle^{-d-3}đ\eta. \end{aligned}$$

Set

$$D_4 = \left\{ \eta \in \boldsymbol{R}^d \,;\; -\eta_1 \geq 2C_1 \left(|\eta_2|^2 + \cdots + |\eta_d|^2 \right)^{1/2} \right\}.$$

Then, $D_4 \subset D_2(\xi_1) \cap D_3(\xi_1)$ for every ξ_1 satisfying $\xi_1 > \max\{2C_2, 2\}$. Therefore we obtain for any ξ_1 satisfying $\xi_1 > \max\{2C_2, 2\}$

$$\mathcal{F}\Big[\varphi_1(\cdot)\, u(0,\cdot)\Big](\xi_1, 0, \ldots, 0) \geq 2^{-(d+3)/2} \langle \xi_1 \rangle^{-d-3} \int_{D_4} \mathcal{F}_x[\varphi_1](\eta) \langle \eta \rangle^{-d-3} \, đ\eta.$$

Since $\mathcal{F}_x[\varphi_1](\eta)$ is real-analytic by Paley-Wiener's theorem and D_4 contains an open set of $\boldsymbol{R}^d$, $\mathcal{F}_x[\varphi_1](\eta)$ does not vanish identically on D_4. Moreover $\mathcal{F}_x[\varphi_1](\eta)$ is non-negative, so $\int_{D_4} \mathcal{F}_x[\varphi_1](\eta)\langle\eta\rangle^{-d-3}\, đ\eta$ is a positive constant independent of ξ_1. This shows that $\mathcal{F}_x[\varphi_1(\cdot)u(0,\cdot)](\xi)$ is not rapidly decreasing. This is a contradiction, so $u(0,x)$ is not smooth and hence $u(t,x)$ is not smooth. The proof of Theorem **D** is completed. □

Remark. Throughout this section, we assumed that f does not depend on x. Here we mention a result on non-hypoellipticity of P of the form (**A**) in the case where f depends on x.

Corollary 10.4 *Suppose that $f(t,x)$ does not change sign, no $a_{jk}(t,x)$ depends on t and that the operator $\sum_{j,k=1}^n a_{jk} L_j L_k$ has an elementary solution $E(x,y)$ belonging to $\boldsymbol{C}^\infty(\boldsymbol{R}^n_x \times \boldsymbol{R}^n_y \setminus \{x = y\})$. Then, Condition (1°-$\beta$) is necessary and sufficient for P of the form* (**A**) *to be hypoelliptic under Conditions* (2°) *and* (3°).

An example is the following

$$L = \partial_t + f(t,x) \left(\partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_d}^2 \right).$$

If $f(t,x)$ does not change sign, L is hypoelliptic if and only if $f(t,x)$ satisfies Condition (1°-β). This is because L is hypoelliptic due to Theorem **A** if $f(t,x)$ satisfies (1°-β). If $f(t,x)$ does not satisfy (1°-β), then there exist an $x_0 \in \boldsymbol{R}^n$ and a non-empty open interval I such that $f(t,x_0)$ vanishes identically on I. Let $E(x)$ be the usual elementary solution of the Laplacian $\triangle_x$, that is,

$$E(x) = \begin{cases} \dfrac{1}{2}|\,x\,| & (d=1) \\[2mm] \dfrac{1}{2\pi}\log|\,x\,| & (d=2) \\[2mm] -\dfrac{\Gamma(\nu)}{4\pi^{d/2}}|\,x\,|^{2-d} & (d \geq 3), \text{ where } \nu = \dfrac{d-2}{2}. \end{cases}$$

Now we calculate the pairing $\langle LE(x-x_0), \varphi \rangle$ for every $\varphi \in \boldsymbol{C}_0^\infty(I \times \boldsymbol{R}^d_x)$ as follows.

$$\begin{aligned} &\langle\, LE(x-x_0), \varphi\, \rangle \\ &= \langle\, E(x-x_0), \triangle_x f(t,x)\varphi\, \rangle \quad (\text{ since } E(x-x_0) \text{ does not depend on } t \,) \\ &= \langle\, \delta(x-x_0), f(t,x)\varphi\, \rangle \;=\; \int_I f(t,x_0)\varphi(t,x_0)\, dt \;=\; 0. \end{aligned}$$

Thus, $LE(x-x_0) = 0$ in $\mathcal{D}'(I\times\boldsymbol{R}_x^d)$. Obviously, $E(x-x_0)$ is not smooth in a neighborhood of $x = x_0$. So L is not hypoelliptic in $I \times \boldsymbol{R}_x^d$. Therefore, Condition ($1°$-$\beta$) is necessary for L to be hypoelliptic.

11 Hypoellipticity of a particular operator of the form (B)

In this section, we investigate hypoellipticity of the following operator which appeared in §8

$$L_{p,q} = \partial_t + \Big(t^p + i\,t^q\Big)\partial_x^2,$$

where p, q are given non-negative integers. We gave a necessary and sufficient condition for $L_{p,q}$ to be hypoelliptic in the Introduction (see Theorem **E**). If p is large with respect to q, $i\,t^q\partial_x^2$ is dominant. $\partial_t + i\,t^q\partial_x^2$ is not hypoelliptic because this is a Schrödingerlike operator. So one can guess that $L_{p,q}$ is not hypoelliptic in this case. On the contrary, if q is large with respect to p, $t^p\partial_x^2$ is dominant. Since $\partial_t + t^p\partial_x^2$ is hypoelliptic due to Theorem **A**, we expect that $L_{p,q}$ is hypoelliptic in this case. Indeed, the following propositions hold.

Proposition 11.1 *$L_{p,q}$ is not hypoelliptic in $\boldsymbol{R}^2$ if $p \geq 2q+1$.*

Proposition 11.2 *$L_{p,q}$ is hypoelliptic in $\boldsymbol{R}^2$ if $p \leq 2q$.*

11.1 Proof of Proposition 11.1

Let us sketch the proof. First, we take a non-smooth solution w belonging to $\mathcal{D}'$ to the equation $(\partial_t + i\,t^q\partial_x^2)\,w = 0$ (see (11.3) below). Next, from this w, we construct a formal solution U of $L_{p,q}U = 0$ as a sum of distributions. And we modify U so that the modified sum $\widetilde{U}$ is convergent in the distribution sense and that its image by $L_{p,q}$ is of class $\boldsymbol{C}^\infty$. Finally, we prove that $\widetilde{U}$ is not smooth.

To simplify the proof, we restrict ourselves to the case where $p = 2q+1$. The remaining case is treated in a similar way. Set

$$\text{(11.1)} \qquad M_q = L_{2q+1,q} = \partial_t + \Big(t^{2q+1} + i\,t^q\Big)\partial_x^2,$$

$$\text{(11.2)} \qquad L_q = \partial_t + i\,t^q\partial_x^2.$$

Let a, b be numbers satisfying $0 < a < b$ and I an open interval containing $t = 0$. We shall show that M_q is not hypoelliptic in $I \times (a,b)$. For this, it suffices to prove that there exists a $\widetilde{U} \in \mathcal{D}'(I \times (a,b)) \setminus \boldsymbol{C}^\infty(I \times (a,b))$ such that $M_q\widetilde{U} \in \boldsymbol{C}^\infty(I \times (a,b))$.

Let k be the smallest integer such that $k - (q+1)/2 > -1$. We define $w \in \mathcal{D}'(I \times (a,b))$ to be

$$\text{(11.3)} \quad \langle w, \varphi\rangle = \frac{1}{k!}\int_a^b\int_0^\infty t^k w_0(t,x)\left({}^tL_q^k\varphi\right)dt\,dx \quad \text{for every } \varphi \in \boldsymbol{C}_0^\infty(I \times (a,b)),$$

where

$$w_0(t,x) = |t|^{-(q+1)/2} \exp\left(-\frac{i\,(q+1)\,x^2}{4t^{q+1}} \right)$$

and ${}^tL_q = -\partial_t + i\,t^q\partial_x^2$. We see that $L_q w_0 = 0$ in $\{t > 0\}$. First, let us show that

$$L_q w = 0 \tag{11.4}$$

as an element of $\mathcal{D}'(I \times (a,b))$. We calculate the pairing $\langle L_q w, \varphi\rangle$ for $\varphi \in \boldsymbol{C}_0^\infty(I \times (a,b))$.

$$\begin{aligned} \langle L_q w, \varphi\rangle &= \left\langle w, {}^tL_q\varphi \right\rangle = \frac{1}{k!}\int_a^b \left\{ \int_0^\infty t^k w_0(t,x) \left({}^tL_q^{k+1}\varphi\right) dt \right\} dx \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{k!}\int_a^b \left\{ \int_\varepsilon^\infty t^k w_0(t,x) \left({}^tL_q^{k+1}\varphi\right) dt \right\} dx \\ &\qquad \text{(by Lebesgue's convergence theorem).} \end{aligned}$$

Set

$$I_\varepsilon = \frac{1}{k!}\int_a^b \int_\varepsilon^\infty t^k w_0(t,x) \left({}^tL_q^{k+1}\varphi\right) dt\,dx \quad \text{for } \varepsilon > 0.$$

Since integration by parts yields

$$L_q^{k+1}\left\{ \frac{t^k}{k!}\,w_0 \right\} = L_q^k \left\{ L_q\left(\frac{t^k}{k!}\,w_0 \right) \right\} = L_q^k \left\{ \frac{t^{k-1}}{(k-1)!}\,w_0 \right\} = \cdots = L_q w_0 = 0$$

in $\{t > 0\}$, I_ε is rewritten as

$$I_\varepsilon = \sum_{l=0}^k J_{l,\varepsilon},$$

where $J_{l,\varepsilon} = \dfrac{1}{(k-l)!}\displaystyle\int_a^b \varepsilon^{k-l-(q+1)/2} \exp\left(-\frac{i\,(q+1)\,x^2}{4\varepsilon^{q+1}} \right) \left({}^tL_q^{k-l}\varphi\right)(\varepsilon,x)\,dx$ ($l = 0,\dots,k$).

It reveals that

$$J_{l,\varepsilon} = O\Big(\varepsilon^{k-l+(q+1)/2}\Big). \tag{11.5}$$

In fact, $J_{l,\varepsilon}$ is rewritten as follows.

$$\begin{aligned} J_{l,\varepsilon} &= -\frac{1}{(k-l)!}\int_a^b \varepsilon^{k-l-(q+1)/2} \left\{ \frac{2\varepsilon^{q+1}}{i\,(q+1)\,x}\,\partial_x \exp\left(-\frac{i\,(q+1)\,x^2}{4\varepsilon^{q+1}} \right) \right\} \left({}^tL_q^{k-l}\varphi\right)(\varepsilon,x)\,dx \\ &= -\frac{2\,\varepsilon^{k-l+(q+1)/2}}{i\,(q+1)\,(k-l)!}\int_a^b \left\{ \partial_x \exp\left(-\frac{i\,(q+1)\,x^2}{4\varepsilon^{q+1}} \right) \right\} \frac{{}^tL_q^{k-l}\varphi(\varepsilon,x)}{x}\,dx. \end{aligned}$$

Since $0 < a < b$, $({}^tL_q^{k-l}\varphi(\varepsilon,x))/x \in \boldsymbol{C}_0^\infty((a,b))$. Therefore we have by integration by parts

$$J_{l,\varepsilon} = \frac{2\,\varepsilon^{k-l+(q+1)/2}}{i\,(q+1)\,(k-l)!}\int_a^b \exp\left(-\frac{i\,(q+1)\,x^2}{4\varepsilon^{q+1}} \right) \partial_x \left(\frac{{}^tL_q^{k-l}\varphi(\varepsilon,x)}{x} \right) dx.$$

This shows (11.5). Since $k-l+(q+1)/2 > 0$ ($l = 0,\dots,k$), $\lim_{\varepsilon\downarrow 0} I_\varepsilon = 0$. And hence (11.4) holds.

Now we prepare three lemmas to construct a formal solution U of $M_qU = 0$. Lemma 11.3 is used only to have (11.8) below. The construction starts from (11.8). Lemma 11.4 is needed only for proving Lemma 11.5. We use Lemma 11.5 to construct successively a formal series solution U.

Lemma 11.3 *Let w be as in* (11.3)*. Then, the following equality holds as an element of $\mathcal{D}'(I \times (a,b))$.*

$$M_q\Big(e^{(q+1)x^2/8}w\Big) = \Big(t^{2q+1} + i\,t^q\Big)\Big(\partial_x^2 e^{(q+1)x^2/8}\Big)\,w - \frac{i(q+1)}{2}\,t^q e^{(q+1)x^2/8}w - \frac{i(q+1)^2}{4}\,t^q x^2 e^{(q+1)x^2/8}w. \tag{11.6}$$

Proof of Lemma 11.3. Given $\varphi \in \boldsymbol{C}_0^\infty(I \times (a,b))$, set

$$\begin{aligned}\widetilde{J} &= \Big\langle\Big(t^{2q+1} + i\,t^q\Big)\Big(\partial_x^2 e^{(q+1)x^2/8}\Big)\,w, \varphi\Big\rangle - \frac{i(q+1)}{2}\Big\langle t^q e^{(q+1)x^2/8}w, \varphi\Big\rangle \\ &\quad - \frac{i(q+1)^2}{4}\Big\langle t^q x^2 e^{(q+1)x^2/8}w, \varphi\Big\rangle\end{aligned}$$

and

$$\widetilde{I}_\varepsilon = \frac{1}{k!}\int_a^b \left\{\int_\varepsilon^\infty t^k w_0(t,x)\,{}^t\!L_q^k\left\{e^{(q+1)x^2/8}\Big({}^t\!M_q\varphi\Big)\right\}dt\right\}dx.$$

We see that

$$\Big\langle M_q\Big(e^{(q+1)x^2/8}w\Big), \varphi\Big\rangle = \lim_{\varepsilon\downarrow 0}\widetilde{I}_\varepsilon.$$

Since $M_q = L_q + t^{2q+1}\,\partial_x^2$ due to (11.1) and (11.2), (11.6) holds in $\{t > 0\}$ for w_0 in place of w. So the difference $\widetilde{I}_\varepsilon - \widetilde{J}$ is equal to a sum of integrals on $[a,b]$. By the same way as we evaluated $J_{l,\varepsilon}$, we see that this sum of integrals tends to 0 as $\varepsilon \downarrow 0$. Therefore, (11.6) holds in the distribution sense. □

We define the distribution $\widetilde{w}$ to be

$$\widetilde{w} = \exp\left(\frac{(q+1)\,x^2}{8}\right)w. \tag{11.7}$$

Remark. In the case where $p \ge 2q+2$, $\widetilde{w}$ is not needed for the proof of Proposition 11.1. Lemma 11.5 below holds if we replace $\widetilde{w}$ by w. Thus, the construction of U goes well without $\widetilde{w}$ in this case.

Rewriting (11.6) for $\widetilde{w}$, we have

$$M_q\,\widetilde{w} = t^{2q+1}\,C_1(q,x)\,\widetilde{w} + t^q\,C_2(q,x)\,\widetilde{w}, \tag{11.8}$$

where $C_1(q,x) = \dfrac{q+1}{4} + \dfrac{(q+1)^2}{16}\,x^2$ and $C_2(q,x) = -i\left(\dfrac{q+1}{4} + \dfrac{3(q+1)^2}{16}\,x^2\right)$.
Next, we have the following lemma which generalizes (11.8).

Lemma 11.4 *Let $\widetilde{w}$ be as in* (11.7). *For any $F(x) \in \boldsymbol{C}^\infty([a,b])$ and any positive integer l, there exist two functions $C_1(q,F,x)$ and $C_2(q,F,x)$ belonging to $\boldsymbol{C}^\infty([a,b])$ such that*

$$\begin{aligned} M_q\Big(t^l F(x)\,\widetilde{w}\Big) \;&=\; t^{2q+1+l}\,C_1(q,F,x)\,\widetilde{w} + t^{q+l}\,C_2(q,F,x)\,\widetilde{w} \\ &\quad + t^{l-1}\Big((q+1)\,x\,F' + lF\Big)\widetilde{w} \\ &\quad \text{as an element of } \mathcal{D}'(I\times(a,b)). \end{aligned} \tag{11.9}$$

Proof of Lemma 11.4. Set

$$\begin{cases} C_1(q,F,x) = C_1(q,x)F(x) + \dfrac{q+1}{2}\,x\,F'(x) + i\,F''(x) \\ C_2(q,F,x) = C_2(q,x)F(x) - i\,\dfrac{q+1}{2}\,x\,F'(x) + F''(x), \end{cases}$$

where $C_1(q,x)$ and $C_2(q,x)$ are as in (11.8). Then (11.9) holds in $\{t>0\}$. By the same way as in the proof of Lemma 11.3, we obtain (11.6) in the distribution sense. □

For every non-negative integer l, let V_l be the vector space consisting of linear combinations of $C(x)\,t^m\widetilde{w}$ where $C(x) \in \boldsymbol{C}^\infty([a,b])$ and m is an integer not smaller than l. As is mentioned above, the following lemma allows us to construct successively the summands of U.

Lemma 11.5 *Given a non-negative integer l, there exists, for any $v \in V_l$, a $u \in V_{l+1}$ such that*

$$M_q\,u + v \in V_{l+1}.$$

Proof of Lemma 11.5. Let $\widetilde{w}$ be as in (11.7). For any $v \in V_l$, there exist a finite number of functions $G_1(x),\dots,G_r(x)$ belonging to $\boldsymbol{C}^\infty([a,b])$ such that

$$v - \sum_{j=1}^{r} t^l\,G_j(x)\,\widetilde{w} \in V_{l+1}.$$

So, it suffices to show that, there exists, for any $G \in \boldsymbol{C}^\infty([a,b])$, a $\widetilde{u} \in V_{l+1}$ such that $M_q\,\widetilde{u} + t^l\,G(x)\,\widetilde{w} \in V_{l+1}$. Given $G(x) \in \boldsymbol{C}^\infty([a,b])$, we set

$$F(x) = -x^{-(l+1)/(q+1)} \int_a^x \frac{y^{(l+1)/(q+1)-1}}{q+1}\,G(y)\,dy \quad \text{and} \quad \widetilde{u} = t^{l+1}F(x)\,\widetilde{w}.$$

Since $0<a<b$, the interval $[a,b]$ does not contain $x=0$. Consequently, $\widetilde{u} \in V_{l+1}$. Then, $M_q\,\widetilde{u} + t^l\,G(x)\,\widetilde{w} \in V_{l+1}$ by (11.9). Lemma 11.5 is proved. □

Now let us construct a formal solution U of $M_qU = 0$. To do this, we are going to find a sequence of distributions $\{v_j\}_{j=0}^{\infty}$ such that

$$v_l \in V_l \quad \text{and} \quad M_q \sum_{j=0}^{l} v_j \in V_l \quad \text{for every } l \in \boldsymbol{Z}_+. \tag{11.10}$$

If such a sequence $\{v_j\}_{j=0}^{\infty}$ is chosen, set

$$U = \sum_{j=0}^{\infty} v_j.$$

Then, this is a formal solution of $M_qU = 0$. We choose $\{v_j\}_{j=0}^{\infty}$ in the following way.

Set $v_0 = \widetilde{w}$. Then, $M_qv_0 \in V_0$ from (11.8). We take a $v_1 \in V_1$ such that $M_qv_1 + M_qv_0 \in V_1$. This is possible by applying Lemma 11.5 to $l = 0$ and $v = M_qv_0$. Next, we take a $v_2 \in V_2$ such that $M_qv_2 + M_q(v_0 + v_1) \in V_2$. This is also possible by Lemma 11.5 because $M_q(v_0 + v_1) \in V_1$. We repeat this procedure and obtain $\{v_j\}_{j=0}^{\infty}$.

Next, we modify U obtained above. The sum U is not necessarily convergent in the distribution sense. So we construct $\{u_j\}_{j=0}^{\infty}$ from $\{v_j\}_{j=0}^{\infty}$ so that the sum $\sum_{j=0}^{\infty} u_j$ converges in the distribution sense and that its image by M_q belongs to $\boldsymbol{C}^{\infty}(I \times (a, b))$.

For every non-negative integers (j, m), $t^j\widetilde{w} \in \boldsymbol{C}^m(\overline{I} \times [a, b])$ if $j - 1 - q/2 \geq m(q + 2)$. Thus, we have

$$V_j \subset \boldsymbol{C}^{[(j-1-q/2)/(q+2)]}(\overline{I} \times [a, b]) \quad \text{for every } j > \frac{q+3}{2}, \tag{11.11}$$

where $[s]$ denotes the largest integer not exceeding s. Let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a sequence of positive numbers such that $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$ which we choose later. Let $h(s)$ be a smooth function of class $\boldsymbol{C}^{\infty}$ such that

$$h(s) \equiv 1 \text{ on } \{|s| \leq 1/2\}, \quad \operatorname{supp} h(s) \subset \{|s| \leq 1\} \quad \text{and} \quad 0 \leq h(s) \leq 1.$$

We have for every $j > (q + 3)/2$ and every non-negative integers α, β satisfying $\alpha + \beta \leq [(j - 1 - q/2)/(q + 2)]$

$$\left|\partial_t^{\alpha}\partial_x^{\beta}\Big(h(t/\varepsilon_j)\, v_j\Big)\right| \leq {\varepsilon_j}^{j-\alpha(q+2)-\beta(q+1)-(q+1)/2} C_{j\alpha\beta} \quad \text{for all } (t, x) \in \overline{I} \times [a, b], \tag{11.12}$$

where $C_{j\alpha\beta}$ does not depend on the choice of $\{\varepsilon_j\}_{j=0}^{\infty}$. So we choose a $\{\varepsilon_j\}_{j=0}^{\infty}$ inductively in the following way.

$$\begin{cases} \varepsilon_j = 1 & \text{if } 0 \leq j \leq [(q+3)/2], \\ 0 < \varepsilon_j < \displaystyle\min_{\alpha+\beta \leq \left[\frac{j-1-q/2}{q+2}\right]} \left\{ \Big(2^j (C_{j\alpha\beta} + 1)\Big)^{-4}, \frac{\varepsilon_{j-1}}{2} \right\} & \text{if } j \geq [(q+3)/2] + 1. \end{cases} \tag{11.13}$$

Set $u_j = h(t/\varepsilon_j)\, v_j$ ($j \geq 0$). We define $\widetilde{U}$ to be

$$\widetilde{U} = \sum_{j=0}^{\infty} u_j. \tag{11.14}$$

By (11.12) and (11.13), we have for every $j > (q+3)/2$ and every non-negative integers α, β satisfying $\alpha + \beta \leq [(j-1-q/2)/(q+2)]$

$$\left| \partial_t^{\alpha} \partial_x^{\beta} u_j(t,x) \right| \leq 2^{-j} \quad \text{on } \overline{I} \times [a,b].$$

And hence, for any non-negative integer l,

$$\text{the sum} \quad \sum_{j=[q/2]+2+l(q+2)}^{\infty} u_j \text{ converges uniformly in } \boldsymbol{C}^l(\overline{I} \times [a,b]). \tag{11.15}$$

Now let us prove that $\widetilde{U}$ (see (11.14)) converges in $\mathcal{D}'(I \times (a,b))$. We divide it into two parts as follows.

$$\widetilde{U} = \sum_{j=0}^{\infty} u_j = \sum_{j=0}^{[q/2]+1} u_j + \sum_{j=[q/2]+2}^{\infty} u_j.$$

The second sum on the right hand side converges uniformly in $\boldsymbol{C}^0(\overline{I} \times [a,b])$ by (11.15). Thus, $\widetilde{U}$ converges in $\mathcal{D}'(I \times (a,b))$.

Next, we prove that $M_q \widetilde{U} \in \boldsymbol{C}^{\infty}(I \times (a,b))$. It suffices to show that $M_q \widetilde{U} \in \boldsymbol{C}^l(I \times (a,b))$ for any non-negative integer l. Given l, we set $N = [q/2] + 2 + (l+2)(q+2)$. We divide $M_q \widetilde{U}$ into two parts as follows.

$$M_q \widetilde{U} = M_q \sum_{j=0}^{N-1} u_j + M_q \sum_{j=N}^{\infty} u_j.$$

The second sum $\sum_{j=N}^{\infty} u_j$ converges in $\boldsymbol{C}^{l+2}(\overline{I} \times [a,b])$ from (11.15) with $l+2$ in place of l. So $M_q \sum_{j=N}^{\infty} u_j \in \boldsymbol{C}^l(\overline{I} \times [a,b])$. The remaining question is to prove that $M_q \sum_{j=0}^{N-1} u_j \in \boldsymbol{C}^l(\overline{I} \times [a,b])$. It is rewritten as follows.

$$\begin{aligned}
\sum_{j=0}^{N-1} M_q\, u_j &= \sum_{j=0}^{N-1} M_q\, h(t/\varepsilon_j)\, v_j \\
&= \sum_{j=0}^{N-1} \varepsilon_j^{-1} h'(t/\varepsilon_j)\, v_j + \sum_{j=0}^{N-1} \Big(h(t/\varepsilon_j) - h(t/\varepsilon_{N-1}) \Big) M_q\, v_j \\
&\qquad + h(t/\varepsilon_{N-1}) \sum_{j=0}^{N-1} M_q\, v_j \\
&= P_{1,N} + P_{2,N} + P_{3,N}.
\end{aligned}$$

The support of $P_{1,N}$ and that of $P_{2,N}$ are contained in $\{2^{-1}\varepsilon_{N-1} \leq t \leq 1\}$. So $P_{1,N}$ and $P_{2,N}$ belong to $\boldsymbol{C}^{\infty}(I \times (a,b))$ because v_j is smooth except at $t = 0$ for every j. Moreover, we have from (11.10) and (11.11)

$$P_{3,N} = h(t/\varepsilon_{N-1}) \sum_{j=0}^{N-1} M_q\, v_j \in V_{N-1} \subset \boldsymbol{C}^l(I \times (a,b)).$$

Therefore, $M_q\widetilde{U} \in \boldsymbol{C}^l(I \times (a,b))$. Since l is arbitrary, $M_q\widetilde{U} \in \boldsymbol{C}^\infty(I \times (a,b))$.

Finally, let us show that $\widetilde{U} \notin \boldsymbol{C}^\infty(I \times (a,b))$. The proof is done by contradiction. Suppose that $\widetilde{U} \in \boldsymbol{C}^\infty(I \times (a,b))$. Since the support of u_j is contained in $\{t \geq 0\}$ for every j, so is the support of $\widetilde{U}$. Thus,

$$\partial_t^m \widetilde{U}(0,x) = 0 \quad \text{in } (a,b) \quad \text{for every } m \in \boldsymbol{Z}_+. \tag{11.16}$$

Let δ be a positive number smaller than 1 and let $\varphi_1(t), \varphi_2(x)$ be functions such that

$$\begin{cases} \varphi_1 \in \boldsymbol{C}_0^\infty(I), & 0 \leq \varphi_1 \leq 1, \quad \varphi_1 \equiv 1 \quad \text{in a neighborhood of } t = 0, \\ \varphi_2 \in \boldsymbol{C}_0^\infty((a,b)), & 0 \leq \varphi_2 \leq 1, \quad \displaystyle\int_a^b \varphi_2\, dx > 0. \end{cases}$$

Set

$$\varphi_\delta(t,x) = \varphi_1(t/\delta)\, \varphi_2(x) \in \boldsymbol{C}_0^\infty(I \times (a,b)) \quad \text{and} \quad \zeta(t,x) = \exp\left(\frac{i\,(q+1)\,x^2}{4t^{q+1}}\right).$$

Then, (11.16) implies

$$\left\langle t^k \zeta \widetilde{U}, \varphi_\delta \right\rangle = O\left(\delta^m\right) \quad \text{for every } m \in \boldsymbol{Z}_+, \tag{11.17}$$

where k is as in (11.3). We shall show that (11.17) does not hold. First, $t^k u_0$ belongs to $L^1_{\mathrm{loc}}(I \times (a,b))$, $t^k \zeta u_0$ is real-valued and $t^k \zeta u_0 = h(t)\, t^{k-(q+1)/2}$ in $\{t > 0\} \cap (I \times (a,b))$. So, we have for sufficiently small δ

$$\begin{aligned} \left\langle t^k \zeta u_0, \varphi_\delta \right\rangle &= \int_{t>0} \int_a^b h(t)\, t^{k-(q+1)/2} \varphi_1(t/\delta)\, \varphi_2(x)\, dt\, dx \\ &\geq \int_{t>0} \int_a^b h(t)\, \varphi_1(t/\delta)\, \varphi_2(x)\, dt\, dx \\ &\qquad (\text{ since } t^{k-(q+1)/2} \geq 1 \text{ on the domain of integration }) \\ &\geq \delta \int_{t>0} \varphi_1(t)\, dt \int_a^b \varphi_2(x)\, dx \;=\; C_1 \delta, \end{aligned}$$

where C_1 is a positive constant independent of δ. So we have

$$\left\langle t^k \zeta u_0, \varphi_\delta \right\rangle \geq C_1 \delta. \tag{11.18}$$

On the other hand, $t^k u_j$ belongs to $L^1_{\mathrm{loc}}(I \times (a,b))$ for $j \geq 1$, and

$$\left| t^k u_j(t,x) \right| \leq C_{j00} |t|^{1/4} {\varepsilon_j}^{1/4} \quad \text{on } \overline{I \times (a,b)} \quad (\, j \geq 1\,).$$

If $j > (q+3)/2$, C_{j00} is the same as that in (11.12). From (11.13), ${\varepsilon_j}^{1/4} < 2^{-j}C_{j00}^{-1}$ if $j > (q+3)/2$. Thus, we have

$$\left| t^k\zeta \sum_{j=1}^{\infty} u_j \right| \le C_2|t|^{1/4} \quad \text{on } \overline{I \times (a,b)}.$$

This yields

$$\left|\left\langle t^k\zeta \sum_{j=1}^{\infty} u_j, \varphi_\delta \right\rangle\right| \le C_3\delta^{5/4},$$

where C_3 is a positive constant independent of δ. Combining this inequality with (11.18), we have

$$\left|\left\langle t^k\zeta\widetilde{U}, \varphi_\delta \right\rangle\right| \ \ge\ C_1\delta - C_3\delta^{5/4} \ \ge\ (C_1/2)\,\delta \quad \text{for } 0 < \delta < (1/(2C_3))^4.$$

This contradicts (11.17). So $\widetilde{U} \notin \boldsymbol{C}^\infty(I \times (a,b))$. Proposition 11.1 is now proved. □

11.2 Proof of Proposition 11.2

Let Ω be a bounded open set of $\boldsymbol{R}^2$. We sketch the proof for $L_{p,q}$ to be hypoelliptic in Ω if $p \le 2q$. Let (τ, ξ) be the dual variables of (t, x) and S_Ψ be the class of operators defined in §1. For a $\Xi \in S_\Psi$, we divide the space $\boldsymbol{R}^2_{\tau,\xi}$ into two microlocal domains supp$\sigma(\Xi)$, supp$(1-\sigma(\Xi))$ and prove that $L_{p,q}$ is hypoelliptic in each domain.

First, we shall prove that $L_{p,q}$ has a left parametrix Q in the microlocal domain supp$(1-\sigma(\Xi))$ for every $\Xi \in S_\Psi$. The proof of the next lemma is done in the same way as that of Proposition 1.1 in §1. So we omit it.

Lemma 11.6 *Suppose that $p \le 2q$. Then, for any $\Xi \in S_\Psi$, there exist $Q \in S^{-1}_{1/2,0}$ and $R \in S^{-\infty}$ such that*

$$(1 - \Xi) = QL_{p,q} + R. \tag{11.19}$$

Remark. This holds even if $p > 2q > 0$.

$L_{p,q}$ is hypoelliptic in supp$(1-\sigma(\Xi))$ for every $\Xi \in S_\Psi$ from this lemma. This is because, given $\Xi \in S_\Psi$, $(1-\Xi)u = QL_{p,q}u + Ru \in \boldsymbol{C}^\infty(\Omega)$ if $u \in \mathcal{D}'(\Omega)$ and if $L_{p,q}u \in \boldsymbol{C}^\infty(\Omega)$. Since $u = (1-\Xi)u + \Xi u$, $L_{p,q}\Xi u \in \boldsymbol{C}^\infty(\Omega)$ for every $\Xi \in S_\Psi$ provided that $L_{p,q}u \in \boldsymbol{C}^\infty(\Omega)$. So it suffices for the proof of Proposition 11.2 to show the following proposition.

Proposition 11.7 *Let u be an element of $\mathcal{D}'(\Omega)$. If $L_{p,q}\Xi u \in \boldsymbol{C}^\infty(\Omega)$ for any $\Xi \in S_\Psi$, then $\Xi u \in \boldsymbol{C}^\infty(\Omega)$ for every $\Xi \in S_\Psi$.*

To prove this, we prepare three lemmas. Set

$$\kappa(p) = \frac{2}{p+1}. \tag{11.20}$$

The proof of Proposition 11.7 starts from the inequality (11.21) below which is an a priori estimate with weight $\langle\xi\rangle^{\kappa(p)}$. This lemma will also be used to prove Lemmas 11.10 and 11.11 below.

Lemma 11.8 *For any open set $K \subset\subset \Omega$, any $N > 0$ and any $\Xi \in S_\Psi$, there exists a positive constant $C = C(p, K, N, \Xi)$ such that*

$$\left\|\langle D_x\rangle^{\kappa(p)}\Xi u\right\|^2 \leq C\left(\|L_{p,q}\Xi u\|^2 + \|u\|_{-N}^2\right) \quad \textit{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{11.21}$$

Proof of Lemma 11.8. We define an ordinary differential operator $Q_{p,q}(\xi)$ on $\boldsymbol{R}_t$ with real parameter ξ to be

$$Q_{p,q}(\xi) = \frac{d}{dt} - \left(t^p + i\,t^q\right)\xi^2.$$

For the proof of (11.21), it suffices to prove the following lemma.

Lemma 11.9 *For any non-empty bounded open interval I, there exists a positive constant $C = C(p, I)$ such that*

$$|\xi|^{4/(p+1)}\int |v|^2 dt \leq C\int |Q_{p,q}(\xi)\,v|^2\,dt \quad \textit{for all } (v,\xi) \in \boldsymbol{C}_0^\infty(\boldsymbol{R}) \times \{|\xi| \geq 1\}, \tag{11.22}$$

where C is independent of ξ and v.

We admit this lemma for the moment. We denote by $\mathcal{F}_x$ the partial Fourier transform with respect to x. Substituting v by $\mathcal{F}_x u(t,\xi)$ in (11.22), we have for any ξ satisfying $|\xi| \geq 1$

$$\int \left|\langle\xi\rangle^{\kappa(p)}\mathcal{F}_x u\right|^2 dt \leq 2^{\kappa(p)} C\int |Q_{p,q}(\xi)\,\mathcal{F}_x u|^2\,dt.$$

On the other hand, we have for any ξ satisfying $|\xi| \leq 1$

$$\int \left|\langle\xi\rangle^{\kappa(p)}\mathcal{F}_x u\right|^2 dt \leq 4^{N+\kappa(p)}\int \left|\langle\xi\rangle^{-2N}\mathcal{F}_x u\right|^2 dt.$$

Therefore, we have for any $\xi \in \boldsymbol{R}$

$$\int \left|\langle\xi\rangle^{\kappa(p)}\mathcal{F}_x u\right|^2 dt \leq C(p,K,N)\left(\int |Q_{p,q}(\xi)\,\mathcal{F}_x u|^2\,dt + \int \left|\langle\xi\rangle^{-2N}\mathcal{F}_x u\right|^2 dt\right),$$

where $C(p, K, N)$ is a constant depending only on (p, K, N). Integrating both sides with respect to ξ, we get by Plancherel's formula

$$\left\|\langle D_x\rangle^{\kappa(p)} u\right\|^2 \leq C(p,K,N)\left(\|L_{p,q}u\|^2 + \left\|\langle D_x\rangle^{-2N}u\right\|^2\right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K).$$

We obtain (11.21) by applying this inequality to Ξu in place of u. Lemma 11.8 is now proved. □

Proof of Lemma 11.9. We prove this lemma only in the case where $\boldsymbol{p}$ **is even**. The case where p is odd is treated in a similar way. (11.22) follows from the inequality

$$\int |w|^2 ds \leq C_1 \int \left| \frac{dw}{ds} - s^p w \right|^2 ds \quad \text{for all } w \in \boldsymbol{C}_0^\infty(\boldsymbol{R}), \tag{11.23}$$

where C_1 does not depend on w. Indeed, (11.21) is obtained by applying (11.23) to

$$w(s) = \exp\left(-\frac{i\,|\xi|^{(2p-2q)/(p+1)}}{q+1} s^{q+1} \right) v(s/|\xi|^{2/(p+1)}).$$

Let us prove (11.23). Given $w \in \boldsymbol{C}_0^\infty(\boldsymbol{R})$, set

$$\lambda = \int_s^{+\infty} \exp\left(-\frac{2\mu^{p+1}}{p+1} \right) d\mu \quad \text{and} \quad \theta(\lambda) = \exp\left(-\frac{s^{p+1}}{p+1} \right) w(s).$$

Then, the following inequality yields (11.23).

$$\rho(s) = \exp\left(\frac{2s^{p+1}}{p+1} \right) \int_s^{+\infty} \exp\left(-\frac{2\mu^{p+1}}{p+1} \right) d\mu \leq C_2 \quad (-\infty < s < \infty), \tag{11.24}$$

where C_2 does not depend on s. This is because, since $\theta \in \boldsymbol{C}_0^\infty((0,\infty))$,

$$\begin{aligned}
\int |w|^2 ds &= \int_0^{+\infty} |\theta(\lambda)|^2 \exp\left(\frac{4s^{p+1}}{p+1} \right) d\lambda \;\leq\; C_2^2 \int_0^{+\infty} \left| \frac{\theta(\lambda)}{\lambda} \right|^2 d\lambda \quad (\text{ by (11.24) }) \\
&\leq 4C_2^2 \int_0^{+\infty} \left| \frac{d\theta}{d\lambda} \right|^2 d\lambda \quad (\text{ by Hardy's inequality (see Theorem } \mathbf{327} \text{ in [8].) }) \\
&= 4C_2^2 \int \left| \frac{dw}{ds} - s^p w \right|^2 ds.
\end{aligned}$$

Thus, it suffices for the proof of Lemma 11.9 to verify (11.24). Now let us show (11.24). Obviously, $\rho(s)$ is bounded in $\{|s| < 1\}$. Suppose that $s \leq -1$. Since $1 \leq \mu^p$ for $\mu \in [s, -1]$,

$$\begin{aligned}
\rho(s) &\leq \exp\left(\frac{2s^{p+1}}{p+1} \right) \left(\int_{-1}^{+\infty} \exp\left(-\frac{2\mu^{p+1}}{p+1} \right) d\mu + \int_s^{-1} \mu^p \exp\left(-\frac{2\mu^{p+1}}{p+1} \right) d\mu \right) \\
&\leq \exp\left(\frac{2s^{p+1}}{p+1} \right) \left(\int_{-1}^{+\infty} \exp\left(-\frac{2\mu^{p+1}}{p+1} \right) d\mu - \frac{1}{2} e^{2/(p+1)} \right) + \frac{1}{2} < +\infty.
\end{aligned}$$

A similar argument applies to the case where $s \geq 1$ because $1 \leq \mu^p$ for $\mu \in [s, +\infty]$. And hence Lemma 11.9 holds. □

The next lemma is needed only for proving Lemma 11.11. We define a non-negative number $r(p)$ to be

$$r(p) = \left[\frac{p+1}{2} \right], \tag{11.25}$$

where $[s]$ denotes the largest integer not exceeding s, so $r(p) = p/2$ if p is even.

Lemma 11.10 *For any $K \subset\subset \Omega$, any $N > 0$ and any $\Xi \in S_\Psi$, there exists a positive constant $C = C(p, K, N, \Xi)$ such that*

$$\left\| \langle D_x \rangle^{\kappa(p)/2} t^{r(p)} \partial_x \Xi u \right\|^2 \le C \left(\| L_{p,q} \Xi u \|^2 + \| u \|_{-N}^2 \right) \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K), \tag{11.26}$$

where $\kappa(p)$ is as in (11.20).

Proof of Lemma 11.10. We prove (11.26) only in the case where $\boldsymbol{p}$ **is even**. (The proof of the case where p is odd goes by using (11.28) in place of (11.27) below.) First, the following inequality holds.

$$\left\| t^{p/2} \partial_x u \right\|^2 \le |\operatorname{Re}(L_{p,q} u, u)| \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \tag{11.27}$$

In fact, this follows from the equality

$$\left\| t^{p/2} \partial_x u \right\|^2 = -|\operatorname{Re}(L_{p,q} u, u)|.$$

(If p is odd, the following inequality holds.

$$\left\| t^{(p+1)/2} \partial_x u \right\|^2 \le |\operatorname{Re}(L_{p,q} u, tu)| \quad \text{for all } u \in \boldsymbol{C}_0^\infty(K). \quad) \tag{11.28}$$

Applying (11.27) to $\langle D_x \rangle^{\kappa(p)/2} \Xi u$ for u, we have

$$\begin{aligned}
\left\| \langle D_x \rangle^{\kappa(p)/2} t^{r(p)} \partial_x \Xi u \right\|^2 &\le \left| \operatorname{Re}\left(L_{p,q} \langle D_x \rangle^{\kappa(p)/2} \Xi u, \langle D_x \rangle^{\kappa(p)/2} \Xi u \right) \right| + C(K, N, \Xi) \| u \|_{-N}^2 \\
&\le \left| \operatorname{Re}\left(\langle D_x \rangle^{\kappa(p)/2} L_{p,q} \Xi u, \langle D_x \rangle^{\kappa(p)/2} \Xi u \right) \right| + C(K, N, \Xi) \| u \|_{-N}^2 \\
&\le \left| \operatorname{Re}\left(L_{p,q} \Xi u, \langle D_x \rangle^{\kappa(p)} \Xi u \right) \right| + C(K, N, \Xi) \| u \|_{-N}^2 \\
&\le \| L_{p,q} \Xi u \|^2 + \left\| \langle D_x \rangle^{\kappa(p)} \Xi u \right\|^2 + C(K, N, \Xi) \| u \|_{-N}^2 .
\end{aligned}$$

This implies (11.26) by Lemma 11.8. □

Remark. Lemma 11.8 and Lemma 11.10 hold even if $p > 2q$.

The following lemma is used in the proof of Proposition 11.7 to evaluate the commutator of $L_{p,q} \Xi$ with multiplication by a function applied to u.

Lemma 11.11 *Suppose that $p \le 2q$. Let $\varphi, \psi \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^2)$ be such that $\varphi \subset\subset \psi$. Then, for any K, any $N > 0$ and any $\Xi, \tilde{\Xi} \in S_\Psi$ ($\Xi \subset\subset \tilde{\Xi}$), there exists a constant $C = C(K, N, \Xi, \tilde{\Xi}, \varphi, \psi)$ such that*

$$\begin{aligned}
&\left\| \langle D_x \rangle^{\kappa(p)/2} L_{p,q} \Xi \varphi u \right\| \\
&\quad \le C \left(\left\| \langle D_x \rangle^{\kappa(p)/2} \psi L_{p,q} \Xi u \right\| + \| L_{p,q} \Xi u \| + \left\| L_{p,q} \tilde{\Xi} u \right\| + \| u \|_{-N} \right)
\end{aligned} \tag{11.29}$$

for all $u \in \boldsymbol{C}_0^\infty(K)$, where $\kappa(p) = 2/(p+1)$.

Proof of Lemma 11.11. First, $L_{p,q}\Xi\varphi u$ is rewritten as follows.

$$\begin{aligned} L_{p,q}\Xi\varphi u &= L_{p,q}\varphi\Xi u + L_{p,q}\Big[\Xi,\varphi\Big]u \\ &= \varphi\psi L_{p,q}\Xi u + \Big[L_{p,q},\varphi\Big]\Xi u + \Big[\Xi,\varphi\Big]L_{p,q}u + \Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big]u. \end{aligned}$$

Then the left hand side of (11.29) is evaluated as follows.

$$\begin{aligned} (11.30)\qquad & \Big\|\langle D_x\rangle^{\kappa(p)/2}L_{p,q}\Xi\varphi u\Big\| \\ &\le C(K,\Xi,\varphi)\,\Big\|\langle D_x\rangle^{\kappa(p)/2}\psi L_{p,q}\Xi u\Big\| + \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[L_{p,q},\varphi\Big]\Xi u\Big\| \\ &\quad + \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[\Xi,\varphi\Big]L_{p,q}u\Big\| + \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big]u\Big\|. \end{aligned}$$

We treat the second term on the right hand side. $\Big[L_{p,q},\varphi\Big]$ is rewritten as

$$\Big[L_{p,q},\varphi\Big] = (\partial_t\varphi) + 2(\partial_x\varphi)\Big(t^p + i\,t^q\Big)\partial_x + (\partial_x^2\varphi)\Big(t^p + i\,t^q\Big).$$

Since $p \le 2q$, $r(p) \le \min\{p,q\}$, where $r(p)$ is as in (11.25). Thus, we have

$$\begin{aligned} \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[L_{p,q},\varphi\Big]\Xi u\Big\| &\le C(K,\varphi)\Big(\Big\|\langle D_x\rangle^{\kappa(p)/2}\Xi u\Big\| + \Big\|\langle D_x\rangle^{\kappa(p)/2}t^{r(p)}\partial_x\Xi u\Big\|\Big) \\ &\le C(K,N,\Xi,\varphi)\,\Big(\|L_{p,q}\Xi u\| + \|u\|_{-N}\Big) \\ &\qquad\text{(by Lemmas 11.8 and 11.10).} \end{aligned}$$

Combining (11.30) with this inequality, we have

$$\begin{aligned} (11.31)\qquad & \Big\|\langle D_x\rangle^{\kappa(p)/2}L_{p,q}\Xi\varphi u\Big\| \\ &\le \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[\Xi,\varphi\Big]L_{p,q}u\Big\| + \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big]u\Big\| \\ &\quad + C(K,\Xi,N,\varphi)\Big(\Big\|\langle D_x\rangle^{\kappa(p)/2}\psi L_{p,q}\Xi u\Big\| + \|L_{p,q}\Xi u\| + \|u\|_{-N}\Big). \end{aligned}$$

The first term on the right hand side of (11.31) is estimated as

$$(11.32)\qquad \Big\|\langle D_x\rangle^{\kappa(p)/2}\Big[\Xi,\varphi\Big]L_{p,q}u\Big\| \le C(K,N,\Xi,\tilde{\Xi},\varphi)\,\Big(\Big\|L_{p,q}\tilde{\Xi}u\Big\| + \|u\|_{-N}\Big)$$

because $\Big[\Xi,\varphi\Big]L_{p,q} - \Big[\Xi,\varphi\Big]L_{p,q}\tilde{\Xi} \in \mathcal{S}^{-\infty}$ and $\langle D_x\rangle^{\kappa(p)/2}\Big[\Xi,\varphi\Big] \in \mathcal{S}^0_{1/2,0}$. It remains to evaluate the second term on the right hand side of (11.31). We choose a $\hat{\Xi} \in S_\Psi$ such that $\hat{\Xi} \subset\subset \Xi$. Since $\hat{\Xi} \subset\subset \Xi \subset\subset \tilde{\Xi}$,

$$\Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big] - \Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big](1-\hat{\Xi})\tilde{\Xi} \in \mathcal{S}^{-\infty}.$$

So, we have by Lemma 11.6

$$\Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big] = \Big[L_{p,q},\Big[\Xi,\varphi\Big]\Big]QL_{p,q}\tilde{\Xi} + R,$$

where $Q \in S^{-1}_{1/2,0}$ is a left parametrix of $L_{p,q}$ in the microlocal domain $\mathrm{supp}(1-\widehat{\Xi})$ and $R \in S^{-\infty}$. Furthermore, $\Big[L_{p,q}, \big[\Xi, \varphi\big]\Big]Q \in S^{-1}_{1/2,0}$. Therefore, we get

$$\Big\| \langle D_x \rangle^{\kappa(p)/2} \Big[L_{p,q}, \big[\Xi, \varphi\big]\Big] u \Big\| \le C(K, \Xi, \tilde{\Xi}, N, \varphi) \left(\big\| L_{p,q} \tilde{\Xi} u \big\| + \|u\|_{-N} \right).$$

Combining (11.31) with (11.32) and the above inequality, we obtain (11.29). □

Proof of Proposition 11.7. Suppose that $L_{p,q}\Xi u \in \boldsymbol{C}^\infty(\Omega)$ for every $\Xi \in S_\Psi$. It suffices for the proof of Proposition 11.7 to show that $\langle D_x \rangle^s \, \Xi \varphi u \in L^2(\boldsymbol{R}^2)$ for every $(s, \varphi, \Xi) \in \boldsymbol{R} \times \boldsymbol{C}_0^\infty(\Omega) \times S_\Psi$. If this is done, we see that $\Xi \varphi u \in \boldsymbol{C}^\infty(\Omega)$ for every $(\varphi, \Xi) \in \boldsymbol{C}_0^\infty(\Omega) \times S_\Psi$ by Sobolev's imbedding theorem, so our assertion holds.

Let $s > 0$, $\Xi, \tilde{\Xi} \in S_\Psi$ ($\Xi \subset\subset \tilde{\Xi}$) and $\varphi, \psi \in \boldsymbol{C}_0^\infty(\Omega)$ ($\varphi_0 \subset\subset \psi$) be any given. Set $\kappa = \kappa(p)/2$. If $u \in \mathcal{D}'(\Omega)$, there exists an $N > 0$ such that $\psi u \in H_{-N}(\boldsymbol{R}^2)$. Let us choose a positive integer l larger than $2(s+N+2)/\kappa$. We find sequences $\{\varphi_j\}_{j=0}^l, \{\psi_j\}_{j=0}^l \subset \boldsymbol{C}_0^\infty(\Omega)$ and $\{\Xi_j\}_{j=0}^l \subset S_\Psi$ such that

$$\varphi = \varphi_0 \subset\subset \psi_1 \subset\subset \varphi_1 \subset\subset \psi_1 \subset\subset \cdots \subset\subset \varphi_l \subset\subset \psi_l = \psi$$

$$\Xi = \Xi_0 \subset\subset \Xi_1 \subset\subset \cdots \subset\subset \Xi_{l-1} \subset\subset \Xi_l = \tilde{\Xi}.$$

Our aim here is to show the inequality

$$(11.33) \qquad \Big\| \langle D_x \rangle^s \, \Xi \varphi_0 u \Big\| \le C \left(\sum_{j'=0}^{l} \Big\| \langle D_x \rangle^s \, \psi L_{p,q} \Xi_{j'} u \Big\| + \|\psi u\|_{-N} \right).$$

As in §2 of [22] we introduce a pseudo-differential operator $\Lambda_{s,k,\varepsilon} = \langle D_x \rangle^s (1 + \varepsilon \langle D_x \rangle)^{-k}$ for real s, $\varepsilon > 0$ and $k \ge 0$. Note that $L_{p,q}\Xi$ commutes with $\Lambda_{s,k,\varepsilon}$ for every (Ξ, s, k, ε). First, applying Lemma 11.8 to $\Lambda_{s-\kappa,k,\varepsilon}\varphi_0 u$ with $k = 2(s+N+2)$, we have

$$\begin{aligned} (11.34) \qquad \Big\| \Lambda_{s,k,\varepsilon} \, \Xi \varphi_0 u \Big\| \; &\le \; C \Big(\Big\| L_{p,q} \Xi \, \Lambda_{s-\kappa,k,\varepsilon} \, \varphi_0 u \Big\| + \|\psi u\|_{-N} \Big) \\ &= \; C \Big(\Big\| \langle D_x \rangle^\kappa \, L_{p,q} \Xi_0 \, \Lambda_{s-2\kappa,k,\varepsilon} \, \varphi_0 u \Big\| + \|\psi u\|_{-N} \Big), \end{aligned}$$

where C is independent of ε. Here and in what follows we denote different constants independent of ε by the same notation C.

Next, let j, j' be non-negative integers satisfying $0 \le j' \le j \le l-1$. The expansion formula yields

$$\Xi_{j'} \Lambda_{s-(j+2)\kappa,k,\varepsilon} \, \varphi_0 - \Xi_{j'} \sum_{0 \le |\alpha| \le 2(s+N+2)} \varphi_{j\,(\alpha)} \, \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon} / \alpha! \quad \in S^{-N-\kappa}_{1/2,0},$$

so we have

$$\tag{11.35} \begin{aligned} &\left\| \langle D_x\rangle^{\kappa} L_{p,q} \Xi_{j'} \Lambda_{s-(j+2)\kappa,k,\varepsilon}\, \varphi_j u \right\| \\ &\quad \le\; C \left(\sum_{0\le|\alpha|\le 2(s+N+2)} \left\| \langle D_x\rangle^{\kappa} L_{p,q} \Xi_{j'}\, \varphi_{j\,(\alpha)}\, \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon}\, u \right\| + \|\psi u\|_{-N} \right). \end{aligned}$$

Since $\varphi_{j\,(\alpha)} \subset\subset \psi_j$ and $\varphi_{j\,(\alpha)}\, \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon} - \varphi_{j\,(\alpha)}\, \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon}\, \varphi_{j+1}$, we have by applying Lemma 11.11 to the right hand side of (11.35)

$$\begin{aligned} &\left\| \langle D_x\rangle^{\kappa} L_{p,q} \Xi_{j'} \Lambda_{s-(j+2)\kappa,k,\varepsilon}\, \varphi_j u \right\| \\ &\quad \le\; C \Bigg\{ \sum_{0\le|\alpha|\le 2(s+N+2)} \Bigg(\left\| \langle D_x\rangle^{\kappa} \psi_j L_{p,q} \Xi_{j'} \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon}\, \varphi_{j+1} u \right\| \\ &\qquad\qquad + \left\| L_{p,q} \Xi_{j'} \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon}\, \varphi_{j+1} u \right\| + \left\| L_{p,q} \Xi_{j'+1} \Lambda^{(\alpha)}_{s-(j+2)\kappa,k,\varepsilon}\, \varphi_{j+1} u \right\| \Bigg) + \|\psi u\|_{-N} \Bigg\} \\ &\quad \le\; C \Bigg(\left\| \Lambda_{s-(j+2)\kappa,k,\varepsilon}\, \psi L_{p,q} \Xi_{j'} u \right\| + \left\| \langle D_x\rangle^{\kappa} L_{p,q} \Xi_{j'} \Lambda_{s-(j+3)\kappa,k,\varepsilon}\, \varphi_{j+1} u \right\| \\ &\qquad\qquad + \left\| \langle D_x\rangle^{\kappa} L_{p,q} \Xi_{j'+1} \Lambda_{s-(j+3)\kappa,k,\varepsilon}\, \varphi_{j+1} u \right\| + \|\psi u\|_{-N} \Bigg). \end{aligned}$$

Combining (11.34) with the above estimate, we have

$$\begin{aligned} &\left\| \Lambda_{s,k,\varepsilon}\, \Xi \varphi_0 u \right\| \\ &\quad \le\; C \left(\sum_{j'=0}^{l} \left\| \Lambda_{s-2\kappa,k,\varepsilon}\, \psi L_{p,q} \Xi_{j'} u \right\| + \sum_{j'=0}^{l} \left\| \Lambda_{s-(l+1)\kappa,k,\varepsilon}\, L_{p,q} \Xi_{j'} \psi u \right\| + \|\psi u\|_{-N} \right). \end{aligned}$$

Since $\|\Lambda_{s-2\kappa,k,\varepsilon}\, \psi L_{p,q} \Xi_{j'} u\| \le \|\langle D_x\rangle^{s} \psi L_{p,q} \Xi_{j'} u\|$ and the family $\{\Lambda_{s-(l+1)\kappa,k,\varepsilon}\, L_{p,q} \Xi_{j'}\}_{0<\varepsilon<1}$ is bounded in $S^{-N}_{1/2,0}$ for every j, we obtain

$$\left\| \Lambda_{s,k,\varepsilon}\, \Xi \varphi_0 u \right\| \le C \left(\sum_{j'=0}^{l} \left\| \langle D_x\rangle^{s} \psi L_{p,q} \Xi_{j'} u \right\| + \|\psi u\|_{-N} \right).$$

$\langle D_x\rangle^{s} \psi L_{p,q} \Xi_{j'} u \in L^2(\boldsymbol{R}^2)$ for every j' because $L_{p,q}\Xi u \in \boldsymbol{C}^{\infty}(\Omega)$ for every $\Xi \in S_{\Psi}$. So the right hand side is bounded uniformly with respect to ε, we finally obtain (11.33) and $\langle D_x\rangle^{s} \Xi \varphi u \in L^2(\boldsymbol{R}^2)$ by letting ε tend to 0. Since (s, Ξ, φ) are arbitrary, $\Xi u \in \boldsymbol{C}^{\infty}(\Omega)$ for every $\Xi \in S_{\Psi}$. This completes the proof of Proposition 11.7. And hence Proposition 11.2 holds. □

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