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On the infinitesimal isometries
of fiber bundles

by

Tatsuo KONNO

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On the infinitesimal isometries of fiber bundles

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1. Introduction

Given a fiber bundle whose projection is a Riemannian submersion, it is important to characterize geometric objects on the total space such as curvatures, geodesics and isometries in terms of the geometry of the base space. This characterization provides us with a better understanding of the geometric objects on the total space. Let (M, g) be a connected, orientable Riemannian manifold and TM the tangent bundle over M with the Sasaki metric g^S . Then the projection $\pi: TM \rightarrow M$ illustrates a typical example of this problem. In this case, the fibers of the bundle are Euclidean spaces. Sasaki [12] proved that *the complete lift X^C of an infinitesimal isometry X of (M, g) and the vertical lift of a parallel vector field on (M, g) are infinitesimal isometries of (TM, g^S)* . Subsequently, in 1973, Tanno [14] characterized the infinitesimal isometries of the tangent bundle in terms of certain tensor fields on the base space M : *Any infinitesimal isometry Z of (TM, g^S) can be decomposed as*

$$Z = X^C + \iota T + Y^\sharp,$$

where X is an infinitesimal isometry of (M, g) , and ιT the lift of a parallel, skew-symmetric tensor field T of type $(1, 1)$ on (M, g) . The term Y^\sharp is also an infinitesimal isometry of (TM, g^S) defined in a slightly complicated manner, whose explicit definition is given below.

Another typical example of the problem is offered by tangent sphere bundles. Let λ be a positive number. The total space of the tangent sphere bundle $T^\lambda M$ over M is the set of tangent vectors of M with length λ . This yields a hypersurface of (TM, g^S) for which we denote the induced metric by g^S as well. In this case, the fibers of the bundle are spheres. A vector field Z on $T^\lambda M$ is called of fiber preserving if the local one-parameter group of local transformations generated by Z maps each fiber of $T^\lambda M$ into another one.

In the first part of this thesis, we characterize fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$ in terms of the geometry of the base manifold M . Since the infinitesimal isometries of the tangent bundle (TM, g^S) have been determined by Tanno [14], we are able to determine the conditions under which the infinitesimal isometry of $T^\lambda M$ can be extended to that of TM . Our result is stated as follows:

THEOREM 1 ([5]). *Any fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$ can be extended to an infinitesimal isometry of (TM, g^S) .*

Conversely, if an infinitesimal isometry Z of (TM, g^S) is tangent to $T^\lambda M$, then the restriction $Z|_{T^\lambda M}$ can be regarded as a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$.

In general, any infinitesimal isometry of $T^\lambda M$ is not necessarily extendable to that of TM . For example, the geodesic spray on the tangent sphere bundle over a space of constant curvature $1/\lambda^2$ is an infinitesimal isometry ([14], [15]) that does not preserve the fibers of $T^\lambda M$.

Theorem 1, together with the results of Tanno in [14], implies the following:

THEOREM 2 ([5]). *Let X be an infinitesimal isometry of (M, g) , and let T be a parallel and skew-symmetric tensor field of type $(1, 1)$ on (M, g) . Then the restriction $(X^C + \iota T)|_{T^\lambda M}$ is regarded as a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$.*

Conversely, every fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$ is of this form.

Let $\mathfrak{i}(M, g)$ and $\mathfrak{i}(T^\lambda M, g^S)$ be the Lie algebras of infinitesimal isometries of (M, g) and of $(T^\lambda M, g^S)$, respectively. Let $\mathfrak{D}^2(M)_0$ denote the set of parallel two-forms on (M, g) . Theorem 2 allows us to define for each $X \in \mathfrak{i}(M, g)$ its natural lift $\Psi_{T^\lambda M}(X) \in \mathfrak{i}(T^\lambda M, g^S)$, and for each $\phi \in \mathfrak{D}^2(M)_0$ its natural lift $\Phi_{T^\lambda M}(\phi) \in \mathfrak{i}(T^\lambda M, g^S)$. It should be noted that both lifts are fiber preserving vector fields.

In these cases, the fibers of the bundles are Euclidean spaces and spheres. On the other hand, Takagi and Yawata [13] discussed another case of this problem in which the fibers are special orthogonal groups. Let $SO(M)$ be the bundle of oriented orthonormal frames over M . For any fixed positive number λ , a Riemannian metric G on $SO(M)$ is defined by

$$G(Z, W) = {}^t\theta(Z) \cdot \theta(W) + \frac{\lambda^2}{2} \text{trace}({}^t\omega(Z) \cdot \omega(W))$$

for $Z, W \in T_u SO(M)$, $u \in SO(M)$, where θ and ω denote the canonical form and the Riemannian connection form on $SO(M)$, respectively. In their paper [13], Takagi and Yawata studied the Riemannian manifold $(SO(M), G)$ when $\lambda = \sqrt{2}$: They derived the decomposition formula of an infinitesimal isometry of $(SO(M), G)$ which is of fiber preserving, and proved that M is a space of constant curvature $1/2$, if $(SO(M), G)$ admits a horizontal infinitesimal isometry which is not of fiber preserving.

Let $\mathfrak{i}(SO(M), G)$ be the Lie algebra of infinitesimal isometries of $(SO(M), G)$. By

refining the method of the proofs in [13], we obtain the natural lifts

$$\Psi_{SO(M)}(X) \in \mathfrak{i}(SO(M), G) \quad \text{of } X \in \mathfrak{i}(M, g)$$

and

$$\Phi_{SO(M)}(\phi) \in \mathfrak{i}(SO(M), G) \quad \text{of } \phi \in \mathfrak{D}^2(M)_0$$

for every positive number λ . We note that both $\Psi_{SO(M)}(X)$ and $\Phi_{SO(M)}(\phi)$ are fiber preserving infinitesimal isometries of $(SO(M), G)$.

The total space of the tangent sphere bundle $T^\lambda M$ can be regarded as the base space of the bundle of oriented orthonormal frames $SO(M)$ (see p. 21). The main purpose of this thesis is to study the projection $SO(M) \rightarrow T^\lambda M$. We prove that there exists a natural homomorphism Ψ from the Lie algebra of fiber preserving infinitesimal isometries of the tangent sphere bundle $(T^\lambda M, g^S)$ to that of fiber preserving infinitesimal isometries of $(SO(M), G)$. We also prove that it provides the factorizations of the mappings $\Psi_{SO(M)}$ and $\Phi_{SO(M)}$ through $\Psi_{T^\lambda M}$ and $\Phi_{T^\lambda M}$. Namely, we have the following:

THEOREM 3 ([6]). *Let (M, g) be a connected, orientable Riemannian manifold and λ a positive number. Then there exists a unique homomorphism Ψ of the Lie algebra of fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$ into that of $(SO(M), G)$ such that $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$.*

In Chapter 4, for any infinitesimal isometry Z of $(T^\lambda M, g^S)$, we define a vector field $\Psi(Z)$ on $(SO(M), G)$ by using the Riemannian connection form on $SO(M)$.

When $\dim M = 2$, Theorem 3 can be further refined as follows: The tangent sphere bundle $(T^\lambda M, g^S)$ is isometric to $(SO(M), G)$, and there exists an isomorphism Ψ of $\mathfrak{i}(T^\lambda M, g^S)$ onto $\mathfrak{i}(SO(M), G)$ satisfying $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$. Moreover, we are now able to determine the structure of the Lie algebra of infinitesimal isometries of $(T^\lambda M, g^S)$, without assuming the completeness of the Riemannian manifold. Namely, we obtain the following:

THEOREM 4 ([6]). *Let (M, g) be a connected, orientable two-dimensional Riemannian manifold and λ a positive number.*

- (i) *If (M, g) is not a space of constant curvature $1/\lambda^2$, then any infinitesimal isometry*

of $(T^\lambda M, g^S)$ is of fiber preserving. Furthermore, we have

$$\mathfrak{i}(T^\lambda M, g^S)/\Psi_{T^\lambda M}(\mathfrak{i}(M, g)) \cong \Phi_{T^\lambda M}(\mathfrak{D}^2(M)_0).$$

In this case, the center of $\mathfrak{i}(T^\lambda M, g^S)$ is $\Phi_{T^\lambda M}(\mathfrak{D}^2(M)_0)$.

(ii) If (M, g) is a space of constant curvature $1/\lambda^2$, then we have

$$\mathfrak{i}(T^\lambda M, g^S)/\Psi_{T^\lambda M}(\mathfrak{i}(M, g)) \cong \underset{\mathbf{R}}{\text{span}} \{ \Phi_{T^\lambda M}(\phi), S, [\Phi_{T^\lambda M}(\phi), S]; \phi \in \mathfrak{D}^2(M)_0 \},$$

where S denotes the geodesic spray on $(T^\lambda M, g^S)$. In this case, the center of $\mathfrak{i}(T^\lambda M, g^S)$ is trivial.

It has been shown by Tanno [14] and Tashiro [15] that the geodesic spray on $T^\lambda M$ is an infinitesimal isometry of $(T^\lambda M, g^S)$ if and only if (M, g) is a space of constant curvature $1/\lambda^2$.

When (M, g) is a unit two-sphere in the Euclidean three-space with the canonical metric, it follows from Theorem 4 that the tangent sphere bundle $(T^1 M, g^S)$ is isometric to the three-dimensional real projective space with sectional curvature $1/4$, which was proved by Klingenberg and Sasaki in [3]. Podestà [11] studied the decomposition of arbitrary infinitesimal isometry of $T^1 M$, when the Ricci tensor of M is parallel and the Ricci curvatures of M are non-positive.

This thesis is organized as follows: In Chapter 2, we review relevant background materials from the Riemannian geometry of tangent bundles. In Chapter 3, we obtain the conditions under which an infinitesimal isometry of $T^\lambda M$ can be extended to that of TM , and then prove Theorems 1 and 2, which are Theorems 3.1.1 and 3.1.2. Chapter 4 is the main part of this thesis, where we first derive a useful formula to clarify the relation between the Riemannian metrics g^S and G . Then, for each infinitesimal isometry of $T^\lambda M$, we define the corresponding vector field on $SO(M)$ and obtain the necessary and sufficient condition under which the vector field becomes an infinitesimal isometry of $SO(M)$. Theorem 3 is proved here as Theorem 4.1.1, and the result is applied to the case of $\dim M = 2$ in order to prove Theorem 4 which is stated as Theorem 4.5.1.

In the Appendix, we first prove an extended version of Theorem 1 for the tangent sphere bundles over space forms, which is presented as Theorem 5.1.2, and characterize the geodesics in the total spaces in terms of the vector fields along certain curves in the base spaces satisfying appropriate conditions. Next, applying Theorem 3 to an extended

version of the results in [13], we provide other proofs of Theorem E in [14] and Theorem 2 for orientable Riemannian manifolds.

2. Tangent bundles

2.1. Tangent bundles and the Sasaki metric

Let N be a Riemannian manifold with metric h . Let $\mathfrak{F}(N)$ denote the ring of C^∞ functions on N , $\mathfrak{X}(N)$ the $\mathfrak{F}(N)$ -module of vector fields on N , and $\mathfrak{i}(N, h)$ the Lie algebra of infinitesimal isometries of (N, h) , respectively. Suppose further that N has the structure of a fiber space. A vector field Z on N is called of fiber preserving if the local one-parameter group of local transformations generated by Z maps each fiber of N into another one. We call Z vertical if it is tangent to the fiber at each point of N . The vector field Z on N is of fiber preserving if and only if the commutator product $[Z, W]$ is vertical for any vertical vector field W on N .

Let ∇ denote the Riemannian connection of an n -dimensional Riemannian manifold (M, g) , and $\pi: TM \rightarrow M$ be the bundle projection of the tangent bundle TM over M . Recall that the connection map $K: TTM \rightarrow TM$ corresponding to ∇ is defined to be

$$(2.1.1) \quad K(Z) = \lim_{t \rightarrow 0} \frac{\tau_0^t(u(t)) - u}{t} \quad \text{for } Z \in T_u TM, u \in TM,$$

where $u(t)$, $-\varepsilon < t < \varepsilon$ (for some $\varepsilon > 0$), is a differentiable curve on TM satisfying $u(0) = u$, $\dot{u}(0) = Z$. Also $\tau_0^t(u(t))$ denotes the parallel displacement of $u(t)$ from $\pi(u(t))$ to $\pi(u)$ along the geodesic arc joining $\pi(u(t))$ and $\pi(u)$ in a normal neighborhood of $\pi(u)$. We define distributions H and V on TM by

$$H_u = \text{Ker}(K|_{T_u TM}), \quad V_u = \text{Ker}(\pi_*|_{T_u TM}), \quad \text{for } u \in TM,$$

where the right hand sides of both the formulas above denote the kernels of $K|_{T_u TM}$ and $\pi_*|_{T_u TM}$, respectively. The space H_u is called the horizontal subspace of $T_u TM$ and V_u the vertical subspace of $T_u TM$. The tangent space $T_u TM$ of TM is decomposed as the direct sum $T_u TM = V_u \oplus H_u$. Then the Sasaki metric g^S on TM is defined by the formula

$$g^S(Z, W) = g(\pi_*(Z), \pi_*(W)) + g(K(Z), K(W)) \quad \text{for } Z, W \in T_u TM, u \in TM.$$

With respect to the Sasaki metric g^S , the horizontal subspace H_u and the vertical subspace V_u are orthogonal at each point u .

2.2. Infinitesimal isometries of tangent bundles

In this section, we review the results of Tanno in [14].

Given $p \in M$ and $X \in T_pM$, for any $u \in \pi^{-1}(p)$, there exist X^H_u and X^V_u in T_uTM such that

$$(2.2.1) \quad \pi_*(X^H_u) = X, \quad K(X^H_u) = 0, \quad \pi_*(X^V_u) = 0, \quad K(X^V_u) = X.$$

When X is a vector field on (M, g) , the correspondences

$$u \longmapsto (X_{\pi(u)})^H_u \quad \text{and} \quad u \longmapsto (X_{\pi(u)})^V_u$$

define vector fields on TM . We also denote these vector fields by X^H and X^V , respectively. We call X^H the horizontal lift of X and X^V the vertical lift of X . For f in $\mathfrak{F}(M)$, we define f^C in $\mathfrak{F}(TM)$ by $f^C(u) = uf$ for u in TM . Each X in $\mathfrak{X}(M)$ has the unique lift X^C to TM such that $X^C(f^C) = (Xf)^C$ for any f in $\mathfrak{F}(M)$. We call it the complete lift of X .

Let T be a tensor field of type $(1,1)$ on M . Then we define ιT and $*T$ in $\mathfrak{X}(TM)$ by

$$\iota T_u = (T(u))^V_u, \quad *T_u = (T(u))^H_u.$$

For X in $\mathfrak{X}(M)$, we define X^\sharp in $\mathfrak{X}(TM)$ by the formula $X^\sharp = X^V + *(T_X)$, where T_X is a tensor field of type $(1,1)$ on M satisfying

$$g(T_X U, V) + g(U, \nabla_V X) = 0 \quad \text{for } U, V \in \mathfrak{X}(M).$$

For a chart (U, φ) of M , a chart $(\pi^{-1}(U), \tilde{\varphi})$ of the tangent bundle TM is naturally defined by

$$(2.2.2) \quad \tilde{\varphi}\left(y^i \left(\frac{\partial}{\partial x^i}\right)_p\right) = (x^1(p), \dots, x^n(p), y^1, \dots, y^n), \quad {}^t(y^1, \dots, y^n) \in \mathbf{R}^n,$$

where $\varphi(p) = (x^1(p), \dots, x^n(p))$ for $p \in U$. Here we use the Einstein convention for the summation. Using these charts, the horizontal subspace H_u and the vertical subspace V_u , $u \in TM$, of T_uTM are expressed as

$$H_u = \left\{ a^k \left(\frac{\partial}{\partial x^k}\right)_u - \Gamma_{ij}^k y^i a^j \left(\frac{\partial}{\partial y^k}\right)_u ; {}^t(y^1, \dots, y^n) \in \mathbf{R}^n \right\},$$

$$V_u = \left\{ a^k \left(\frac{\partial}{\partial y^k}\right)_u ; {}^t(y^1, \dots, y^n) \in \mathbf{R}^n \right\},$$

and the components of the Sasaki metric g^S given by

$$g^S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} + g_{ab} \Gamma_{is}^a \Gamma_{jt}^b y^s y^t,$$

$$g^S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = g_{jb}\Gamma_{is}^b y^s, \quad g^S\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij},$$

where Γ_{ij}^k , $i, j, k = 1, \dots, n$, denote the Christoffel's symbols of the Riemannian metric g .

Putting $X = X^k \partial / \partial x^k$ and $T = (T^k_j)$ on U , we get the local expressions

$$(2.2.3) \quad \left\{ \begin{array}{l} X^H = X^k \frac{\partial}{\partial x^k} - \Gamma_{ij}^k y^i X^j \frac{\partial}{\partial y^k}, \quad X^V = X^k \frac{\partial}{\partial y^k}, \\ X^C = X^k \frac{\partial}{\partial x^k} + y^l \frac{\partial X^k}{\partial x^l} \frac{\partial}{\partial y^k}, \quad \iota T = T^k_j y^j \frac{\partial}{\partial y^k}, \\ X^\sharp = -y^r g^{kl} g_{rm} (\nabla X)^{m_l} \frac{\partial}{\partial x^k} + (X^k + \Gamma_{ij}^k g^{il} g_{rm} (\nabla X)^{m_l} y^j y^r) \frac{\partial}{\partial y^k}. \end{array} \right.$$

The general form of infinitesimal isometries of (TM, g^S) is given by

THEOREM 2.2.1 (Tanno [14]). *Let (TM, g^S) be the tangent bundle with the Sasaki metric over a Riemannian manifold (M, g) , and R the curvature tensor of ∇ . Suppose X, T and Y satisfy the following:*

- (i) X be an infinitesimal isometry of (M, g) ,
- (ii) T be a tensor field of type $(1, 1)$ on (M, g) , which satisfies
 - (ii-1) $\nabla T = 0$, and
 - (ii-2) $g(TU, V) + g(U, TV) = 0$ for $U, V \in \mathfrak{X}(M)$,
- (iii) Y be a vector field on (M, g) , which satisfies
 - (iii-1) $(\nabla^2 Y)(U, V) + (\nabla^2 Y)(V, U) = 0$ for $U, V \in \mathfrak{X}(M)$, and
 - (iii-2) $R(W, T_Y(U))V + R(W, T_Y(V))U = 0$ for $U, V, W \in \mathfrak{X}(M)$.

Then the vector field Z on TM defined by $Z = X^C + \iota T + Y^\sharp$ is an infinitesimal isometry of (TM, g^S) .

Conversely, every infinitesimal isometry of (TM, g^S) is of this form.

3. Infinitesimal isometries of tangent sphere bundles

3.1. Fiber preserving infinitesimal isometries

For each $\lambda > 0$, the tangent sphere bundle $T^\lambda M := \{u \in TM; g(u, u) = \lambda^2\}$ is a hypersurface of TM , and let $\iota: T^\lambda M \rightarrow TM$ denote the inclusion. In particular, we call $T^1 M$ the unit tangent bundle over M . We also denote by g^S the induced metric by ι on $T^\lambda M$. We define a diffeomorphism $f^\lambda: T^1 M \rightarrow T^\lambda M$ by $f^\lambda(u) = \lambda u$, $u \in T^1 M$. Put $\sigma_0 = \{u \in TM; g(u, u) = 0\}$. For Z in $\mathfrak{X}(T^1 M)$, we define \overline{Z} in $\mathfrak{X}(TM \setminus \sigma_0)$ by

$$\overline{Z}_u = (\iota \circ f^\lambda)_*(Z|_{(f^\lambda)^{-1}(u)}) \quad \text{for } u \in TM \setminus \sigma_0 \text{ and } \lambda = \sqrt{g(u, u)}.$$

For Z in $\mathfrak{X}(T^\lambda M)$, we define \overline{Z} in $\mathfrak{X}(TM \setminus \sigma_0)$ by $\overline{Z} = \overline{(f^\lambda)^{-1}(Z)}$. \overline{Z} is tangent to $T^\lambda M$ and $\overline{Z}|_{T^\lambda M} = \iota_*(Z)$. We often consider Z to be a vector field on $TM \setminus \sigma_0$ by the correspondence $Z \mapsto \overline{Z}$.

The main purpose of this section is to prove that any fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$ is extended to an infinitesimal isometry of (TM, g^S) . Namely we obtain the following.

THEOREM 3.1.1. *Let (M, g) be a Riemannian manifold and λ a positive number. If Z is an infinitesimal isometry of $(T^\lambda M, g^S)$ which preserves the fibers, then there exists an infinitesimal isometry W of (TM, g^S) such that W is tangent to $T^\lambda M$ and $W|_{T^\lambda M} = \iota_*(Z)$.*

Conversely, let Z be an infinitesimal isometry of (TM, g^S) which is tangent to $T^\lambda M$. Then there exists a fiber preserving infinitesimal isometry W of $(T^\lambda M, g^S)$ such that $\iota_(W) = Z|_{T^\lambda M}$.*

REMARK. If the infinitesimal isometry Z of (TM, g^S) is tangent to $T^\lambda M$, then it is automatically a fiber preserving vector field on (TM, g^S) . We will see it in the proof of Theorem 3.1.1.

This theorem and Theorem 2.2.1 imply the following.

THEOREM 3.1.2. *Let (M, g) be a Riemannian manifold and λ a positive number. Assume that X is an infinitesimal isometry of (M, g) , and T a parallel and skew-symmetric*

tensor field on (M, g) of type $(1, 1)$. Then the restriction $(X^C + \iota T)|_{T^\lambda M}$ is regarded as a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$.

Conversely, every fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$ is of this form.

Let Z be a vector field on TM and put

$$Z = Z^k \frac{\partial}{\partial x^k} + Z^{k+n} \frac{\partial}{\partial y^k} \quad \text{on } \pi^{-1}(U).$$

From the definition of the fiber preserving vector field, we can see that Z is of fiber preserving if and only if

$$\frac{\partial Z^k}{\partial y^l} = 0$$

holds for $k, l = 1, \dots, n$. For example, the complete lift X^C of a vector field X on M , and ιT for a tensor field T of type $(1,1)$ on M are of fiber preserving. For a fiber preserving vector field Z on TM , we define a vector field \underline{Z} on M by $(\underline{Z})_{\pi(u)} = \pi_*(Z_u)$, $u \in TM$. A fiber preserving vector field Z on $T^\lambda M$ also makes it possible to define the vector field \underline{Z} on M by $(\underline{Z})_{\pi(u)} = (\pi|_{T^\lambda M})_*(Z_u)$, $u \in T^\lambda M$.

PROPOSITION 3.1.3 ([5]). *If Z is a fiber preserving infinitesimal isometry of (TM, g^S) , then \underline{Z} is an infinitesimal isometry of (M, g) .*

PROOF. In a neighborhood of an arbitrary point $u_0 \in TM$, we use the coordinates such that $\Gamma_{ij}^k(\pi(u_0)) = 0$. Let $L_Z g^S$ denote the Lie derivative of g^S with respect to Z . From $(L_Z g^S)(\partial/\partial x^i, \partial/\partial x^j) = 0$, we have that

$$\begin{aligned} & Z^k \frac{\partial}{\partial x^k} (g_{ij} + g_{ab} \Gamma_{is}^a \Gamma_{jt}^b y^s y^t) + Z^{k+n} (g_{ab} \Gamma_{is}^a \Gamma_{jk}^b + g_{ab} \Gamma_{ik}^a \Gamma_{js}^b) y^s \\ & + \frac{\partial Z^k}{\partial x^i} (g_{kj} + g_{ab} \Gamma_{ks}^a \Gamma_{jt}^b y^s y^t) + \frac{\partial Z^{k+n}}{\partial x^i} g_{kb} \Gamma_{js}^b y^s \\ & + \frac{\partial Z^k}{\partial x^j} (g_{ik} + g_{ab} \Gamma_{is}^a \Gamma_{kt}^b y^s y^t) + \frac{\partial Z^{k+n}}{\partial x^j} g_{kb} \Gamma_{is}^b y^s = 0, \end{aligned}$$

and hence, we see immediately that $(L_{\underline{Z}} g)(\partial/\partial x^i, \partial/\partial x^j)_{\pi(u_0)} = 0$.

Q.E.D.

This Proposition is not used in this paper, but the method of the proof is applied to prove the following Lemma 3.1.4, (i).

Now we study fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$.

LEMMA 3.1.4 ([5]). *If Z is a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$, then:*

- (i) $\underline{Z} \in \mathfrak{i}(M, g)$.
- (ii) $\overline{Z}|_{T^\mu M} \in \mathfrak{i}(T^\mu M, g^S)$ for any $\mu > 0$.
- (iii) $\overline{Z} \in \mathfrak{i}(TM \setminus \sigma_0, g^S|_{TM \setminus \sigma_0})$.
- (iv) *There exists $W \in \mathfrak{i}(TM, g^S)$ such that $W|_{TM \setminus \sigma_0} = \overline{Z}$.*

PROOF. (i) In a neighborhood $\pi^{-1}(U)$ of an arbitrary point $u_0 \in T^\lambda M$, we use the coordinates such that $\Gamma_{ij}^k(\pi(u_0)) = 0$. The horizontal lifts $(\partial/\partial x^i)^H$ and $(\partial/\partial x^j)^H$ of the vector fields $\partial/\partial x^i$ and $\partial/\partial x^j$ on U are tangent to $T^\lambda M$ at any point of $\pi^{-1}(U) \cap T^\lambda M$, hence they can be regarded as the vector fields on $(\pi|_{T^\lambda M})^{-1}(U)$. Then we see that

$$(L_Z g^S) \left(\left(\frac{\partial}{\partial x^i} \right)^H_{i(u_0)}, \left(\frac{\partial}{\partial x^j} \right)^H_{i(u_0)} \right) = (L_{\underline{Z}} g) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)_{\pi(u_0)}.$$

This implies that $L_{\underline{Z}} g = 0$ at the point $\pi(u_0)$ of M proving (i) of Lemma 3.1.4.

(ii) Let $Z_\mu = \overline{Z}|_{T^\mu M}$. Suppose that A and B are arbitrary fiber preserving vector fields on $T^\lambda M$. At any point $u \in T^\mu M$, we compute that

$$\begin{aligned} & (L_{Z_\mu} g^S)(\overline{A}|_{T^\mu M}, \overline{B}|_{T^\mu M}) \\ &= Z_\mu g^S(\overline{A}|_{T^\mu M}, \overline{B}|_{T^\mu M}) - g^S([Z_\mu, \overline{A}|_{T^\mu M}], \overline{B}|_{T^\mu M}) - g^S(\overline{A}|_{T^\mu M}, [Z_\mu, \overline{B}|_{T^\mu M}]) \\ &= \overline{Z} g^S(\overline{A}, \overline{B}) - g^S([(f^\mu \circ (f^\lambda)^{-1})_* Z, (f^\mu \circ (f^\lambda)^{-1})_* A], \overline{B}|_{T^\mu M}) \\ &\quad - g^S(\overline{A}|_{T^\mu M}, [(f^\mu \circ (f^\lambda)^{-1})_* Z, (f^\mu \circ (f^\lambda)^{-1})_* B]) \\ &= \overline{Z} g^S(\overline{A}, \overline{B}) - g^S(\overline{[Z, A]}|_{T^\mu M}, \overline{B}|_{T^\mu M}) - g^S(\overline{A}|_{T^\mu M}, \overline{[Z, B]}|_{T^\mu M}) \\ &= \overline{Z} g^S(\overline{A}, \overline{B}) - g^S(\overline{[Z, A]}, \overline{B}) - g^S(\overline{A}, \overline{[Z, B]}). \end{aligned}$$

Since Z , A and B are the fiber preserving vector fields on $T^\lambda M$, there exist $Z^i, A^j, B^k \in \mathfrak{F}(\mathbf{R}^n)$ and $Z^{i+n}, A^{j+n}, B^{k+n} \in \mathfrak{F}(\mathbf{R}^{2n})$, $i, j, k = 1, \dots, n$ such that

$$\iota_*(Z) = Z^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i} + Z^{i+n}(x^1, \dots, x^n, y^1, \dots, y^n) \frac{\partial}{\partial y^i},$$

$$\iota_*(A) = A^j(x^1, \dots, x^n) \frac{\partial}{\partial x^j} + A^{j+n}(x^1, \dots, x^n, y^1, \dots, y^n) \frac{\partial}{\partial y^j},$$

$$\iota_*(B) = B^k(x^1, \dots, x^n) \frac{\partial}{\partial x^k} + B^{k+n}(x^1, \dots, x^n, y^1, \dots, y^n) \frac{\partial}{\partial y^k}.$$

We define a function r on TM by $r(u) = \sqrt{g(u, u)}$, $u \in TM$. From the definition of the extended vector fields \overline{Z} , \overline{A} and \overline{B} , we see that

$$\overline{Z} = Z^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i} + \frac{r}{\lambda} Z^{i+n} \left(x^1, \dots, x^n, \frac{\lambda y^1}{r}, \dots, \frac{\lambda y^n}{r} \right) \frac{\partial}{\partial y^i},$$

$$\overline{A} = A^j(x^1, \dots, x^n) \frac{\partial}{\partial x^j} + \frac{r}{\lambda} A^{j+n} \left(x^1, \dots, x^n, \frac{\lambda y^1}{r}, \dots, \frac{\lambda y^n}{r} \right) \frac{\partial}{\partial y^j},$$

$$\overline{B} = B^k(x^1, \dots, x^n) \frac{\partial}{\partial x^k} + \frac{r}{\lambda} B^{k+n} \left(x^1, \dots, x^n, \frac{\lambda y^1}{r}, \dots, \frac{\lambda y^n}{r} \right) \frac{\partial}{\partial y^k}.$$

Then, for any $u \in T^\mu M$, we have

$$\begin{aligned} g^S(\overline{A}, \overline{B})_u &= \left[A^j B^k (g_{jk} + g_{ab} \Gamma_{js}^a \Gamma_{kt}^b y^s y^t) + \frac{r}{\lambda} A^j B^{k+n} g_{kb} \Gamma_{js}^b y^s \right. \\ &\quad \left. + \frac{r}{\lambda} A^{j+n} B^k g_{jb} \Gamma_{ks}^b y^s + \left(\frac{r}{\lambda} \right)^2 A^{j+n} B^{k+n} g_{jk} \right]_u \\ &= \left[\left(1 - \left(\frac{r}{\lambda} \right)^2 \right) A^j B^k g_{jk} + \left(\frac{r}{\lambda} \right)^2 \left\{ A^j B^k \left(g_{jk} + g_{ab} \Gamma_{js}^a \Gamma_{kt}^b \frac{\lambda y^s}{r} \frac{\lambda y^t}{r} \right) \right. \right. \\ &\quad \left. \left. + A^j B^{k+n} g_{kb} \Gamma_{js}^b \frac{\lambda y^s}{r} + A^{j+n} B^k g_{jb} \Gamma_{ks}^b \frac{\lambda y^s}{r} + A^{j+n} B^{k+n} g_{jk} \right\} \right]_u \\ &= \left(1 - \left(\frac{\mu}{\lambda} \right)^2 \right) g(\underline{A}, \underline{B})_{\pi(u)} + \left(\frac{\mu}{\lambda} \right)^2 g^S(A, B)_{(f^{\lambda \circ (f^\mu)^{-1}})(u)}, \end{aligned}$$

and hence

$$\overline{Z} g^S(\overline{A}, \overline{B})_u = \left(1 - \left(\frac{\mu}{\lambda} \right)^2 \right) \underline{Z} g(\underline{A}, \underline{B})_{\pi(u)} + \left(\frac{\mu}{\lambda} \right)^2 Z g^S(A, B)_{(f^{\lambda \circ (f^\mu)^{-1}})(u)},$$

$$g^S(\overline{[Z, A]}, \overline{B})_u = \left(1 - \left(\frac{\mu}{\lambda} \right)^2 \right) g([Z, \underline{A}], \underline{B})_{\pi(u)} + \left(\frac{\mu}{\lambda} \right)^2 g^S([Z, A], B)_{(f^{\lambda \circ (f^\mu)^{-1}})(u)},$$

$$g^S(\overline{A}, \overline{[Z, B]})_u = \left(1 - \left(\frac{\mu}{\lambda} \right)^2 \right) g(\underline{A}, [\underline{Z}, \underline{B}])_{\pi(u)} + \left(\frac{\mu}{\lambda} \right)^2 g^S(A, [Z, B])_{(f^{\lambda \circ (f^\mu)^{-1}})(u)}.$$

Therefore, we get

$$\begin{aligned} (L_{Z_\mu} g^S)(\overline{A}|_{T^\mu M}, \overline{B}|_{T^\mu M})_u \\ = \left(1 - \left(\frac{\mu}{\lambda} \right)^2 \right) (L_{\underline{Z}g})(\underline{A}, \underline{B})_{\pi(u)} + \left(\frac{\mu}{\lambda} \right)^2 (L_Z g^S)(A, B)_{(f^{\lambda \circ (f^\mu)^{-1}})(u)} = 0. \end{aligned}$$

The last equality above follows from $Z \in \mathfrak{i}(T^\lambda M, g^S)$ and (i) of this lemma. We proved the second statement of Lemma 3.1.4.

(iii) The gradient vector field of r^2 , $\text{grad } r^2 = 2y^i \partial / \partial y^i$, is orthogonal to $T^\lambda M$ at any point of $T^\lambda M$. Since we know $L_{\bar{Z}} \text{grad } r^2 = 0$, we have that

$$(L_{\bar{Z}} g^S)(\bar{A}, \text{grad } r^2) = 0, \quad (L_{\bar{Z}} g^S)(\text{grad } r^2, \text{grad } r^2) = 0$$

for any $A \in \mathfrak{X}(T^\lambda M)$. The statement (iii) follows from this fact and (ii) of this lemma.

(iv) Let $\bar{Z} = \bar{Z}^k \partial / \partial x^k + \bar{Z}^{k+n} \partial / \partial y^k$. From (iii) of this lemma we know that

$$(L_{\bar{Z}} g^S) \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0 \quad \text{on } TM \setminus \sigma_0,$$

which implies that

$$\bar{Z}^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial \bar{Z}^{k+n}}{\partial y^i} g_{kj} + \frac{\partial \bar{Z}^{k+n}}{\partial y^j} g_{ik} = 0.$$

Since Z is the fiber preserving vector field on $T^\lambda M$, we may suppose that $(\partial / \partial y^l) \bar{Z}^k = 0$. Differentiating the left hand side of the formula above with respect to y^l , we get

$$(3.1.1) \quad \frac{\partial^2 \bar{Z}^{k+n}}{\partial y^l \partial y^i} g_{kj} + \frac{\partial^2 \bar{Z}^{k+n}}{\partial y^l \partial y^j} g_{ik} = 0.$$

Putting $i = j$ in the formula (3.1.1), we have that

$$\frac{\partial^2 \bar{Z}^{k+n}}{\partial y^l \partial y^i} g_{ki} = 0, \quad \left(\text{or } \frac{\partial^2 \bar{Z}^{k+n}}{\partial y^i \partial y^j} g_{ik} = 0 \right).$$

Therefore, putting $l = i$ in the formula (3.1.1), we have that

$$\frac{\partial^2 \bar{Z}^{k+n}}{\partial y^i \partial y^i} g_{kj} = 0.$$

Hence we get $(\partial^2 / \partial y^i \partial y^i) \bar{Z}^m = 0$ for $n < m \leq 2n$. From the definition, \bar{Z}^m is in proportion to the function r , hence it is of the form

$$\bar{Z}^m = A_k^m(x^1, \dots, x^n) \cdot y^k,$$

where A_k^m are certain functions on \mathbf{R}^n . Since \bar{Z}^m is a smooth function on $TM \setminus \sigma_0$, we see that

$$A_k^m(x^1, \dots, x^n) = \bar{Z}^m(x^1, \dots, x^n, 0, \dots, \overset{(k)}{1}, \dots, 0) \in \mathfrak{F}(\mathbf{R}^n),$$

for each m with $n < m \leq 2n$. Therefore, $\bar{Z}^m \in \mathfrak{F}(TM \setminus \sigma_0)$ is extended to a differentiable function on TM , hence \bar{Z} can be extended to the vector field W on TM such that

$$W = \begin{cases} \bar{Z} & \text{on } TM \setminus \sigma_0, \\ (i_0)_*(\underline{Z}) & \text{on } \sigma_0, \end{cases}$$

where $\iota_0: M \rightarrow TM$ denotes the natural inclusion. This satisfies the equation $L_W g^S = 0$ on σ_0 , proving (iv). Q.E.D.

Now, we will complete the proof of Theorem 3.1.1.

The necessary condition of Theorem 3.1.1 follows from (iv) of Lemma 3.1.4. So, we will prove the converse part of Theorem 3.1.1 as follows. Suppose that an infinitesimal isometry Z of (TM, g^S) is tangent to $T^\lambda M$ at any point of $T^\lambda M$. By Theorem 2.2.1, there exists an infinitesimal isometry X of (M, g) , a tensor field T of type (1,1) on M and a vector field Y on M such that Z is decomposed as $Z = X^C + \iota T + Y^\sharp$. It is easy to see that X^C and ιT are tangent to $T^\lambda M$ at any point of $T^\lambda M$. Therefore $Y^\sharp = Z - X^C - \iota T$ is also tangent to $T^\lambda M$, which implies $g^S(Y^\sharp, \text{grad } r^2)_u = 0$ for any $u \in T^\lambda M$. Putting $Y = Y^i \partial / \partial x^i$, we know

$$g^S(Y^\sharp, \text{grad } r^2)_u = g_{ij} Y^i y^j \quad \text{for any } {}^t(y^1, \dots, y^n) \in \mathbf{R}^n, \quad g_{kl} y^k y^l = \lambda^2.$$

Hence Y is identically zero on M , which implies $Z = X^C + \iota T$. Since X^C and ιT preserve fibers of TM , Z also preserves fibers of TM . By Sasaki [12, II, Lemma 1], $Z|_{T^\lambda M}$ is regarded as an infinitesimal isometry of $(T^\lambda M, g^S)$. In consequence,

$$Z|_{T^\lambda M} = (X^C + \iota T)|_{T^\lambda M}$$

is regarded as a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$. We proved Theorem 3.1.1.

We also prove Theorem 3.1.2 as follows: Let X be an infinitesimal isometry of (M, g) , and T a parallel and skew-symmetric tensor field on (M, g) of type (1, 1). From Theorem 2.2.1, the vector field $X^C + \iota T$ is an infinitesimal isometry of (TM, g^S) , which is of fiber preserving and tangent to $T^\lambda M$. So the restriction $(X^C + \iota T)|_{T^\lambda M}$ is regarded as a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$.

Conversely, we assume that Z is a fiber preserving infinitesimal isometry of $(T^\lambda M, g^S)$. From Theorem 3.1.1, Z can be extended to an infinitesimal isometry W of (TM, g^S) , which is tangent to $T^\lambda M$. By Theorem 2.2.1, there exists an infinitesimal isometry X of (M, g) , a tensor field T of type (1,1) on M and a vector field Y on M such that W is decomposed as $W = X^C + \iota T + Y^\sharp$. Since X^C and ιT are tangent to $T^\lambda M$, the vector field $Y^\sharp = W - X^C - \iota T$ is also tangent to $T^\lambda M$. Hence Y^\sharp is identically zero, which implies $W = X^C + \iota T$ and

$$Z = W|_{T^\lambda M} = (X^C + \iota T)|_{T^\lambda M}.$$

We proved Theorem 3.1.2.

3.2. An example

When M is the sphere of radius λ in the Euclidean space, there is an infinitesimal isometry of $(T^\lambda M, g^S)$, which is not of fiber preserving ([14]). In this section, we will find a Riemannian manifold M such that any infinitesimal isometry of the unit tangent bundle over M is of fiber preserving.

PROPOSITION 3.2.1. *Let (M, g) be a space of constant curvature c , where the dimension of M is greater than two and the curvature c satisfies the inequality: $-0.30 < c < 0.32$. Then every infinitesimal isometry of the unit tangent bundle over M is of fiber preserving.*

We identify $TT^\lambda M$ with $\iota_*(TT^\lambda M) \subset TTM$. Then we have

$$T_u T^\lambda M = H_u \oplus (V_u \cap T_u T^\lambda M)$$

for each u in $T^\lambda M$. It is easy to see that $Z \in \mathfrak{X}(T^\lambda M)$ is of fiber preserving if and only if

$$(\psi_s)_*(X) \in V_{\psi_s(u)} \quad \text{for any } X \in V_u \cap T_u T^\lambda M, \quad u \in T^\lambda M,$$

where ψ_s , $-\varepsilon < s < \varepsilon$ (for some $\varepsilon > 0$), denotes a local one-parameter group of local transformations generated by Z . We need the following lemma to prove Proposition 3.2.1.

LEMMA 3.2.2. *If a vector field Z on $T^\lambda M$ is not of fiber preserving, then there exist $u_0 \in T^\lambda M$, $Y_0 \in V_{u_0} \cap T_{u_0} T^\lambda M \setminus \{0\}$ and, ε_0 with $0 < \varepsilon_0 < \varepsilon$, such that the horizontal part of $(\psi_s)_*(Y_0)$ is not zero on $0 < s < \varepsilon_0$.*

PROOF. From the assumption, there exist $u \in T^\lambda M$, $Y \in V_u \cap T_u T^\lambda M \setminus \{0\}$ and $t > 0$ such that $(\psi_t)_*(Y) \notin V_{\psi_t(u)}$. Set

$$t_0 = \sup\{s; 0 \leq s \leq t, (\psi_s)_*(Y) \in V_{\psi_s(u)}\}.$$

Since ψ_s is continuous, we have $0 \leq t_0 < t$. Put

$$u_0 = \psi_{t_0}(u), \quad Y_0 = (\psi_{t_0})_*(Y), \quad \varepsilon_0 = t - t_0.$$

They satisfy the conditions stated in Lemma 3.2.2.

Q.E.D.

The shape operator A of $T^\lambda M$ in TM is computed by Blair ([1]):

$$A(X) = \begin{cases} 0 & \text{for } X \in H_u \cap T_u T^\lambda M, \\ -\lambda^{-1}X & \text{for } X \in V_u \cap T_u T^\lambda M. \end{cases}$$

Let R , R_{TM}^S and R^S denote the curvature tensor of (M, g) , (TM, g^S) and $(T^\lambda M, g^S)$, respectively. And let h denotes the second fundamental form of $T^\lambda M$ in TM . Then the Gauss equation of $T^\lambda M$ in TM is

$$g^S(R^S(X, Y)Z, W) = g^S(R_{TM}^S(X, Y)Z, W) + h(Y, Z)h(X, W) - h(X, Z)h(Y, W),$$

for $X, Y, Z, W \in TT^\lambda M$. The curvature tensor of (TM, g^S) is calculated by Kowalski ([7]).

Now we are in a position to prove Proposition 3.2.1. Suppose that there exists an infinitesimal isometry Z of $(T^\lambda M, g^S)$ which is not of fiber preserving. By Lemma 3.2.2, there are $u_0 \in T^\lambda M$, $Y_0 \in V_u \cap T_u T^\lambda M$, $g^S(Y_0, Y_0) = 1$, and $\varepsilon_0 > 0$ such that the horizontal part of $(\psi_s)_*(Y_0)$ is not zero on $0 < s < \varepsilon_0$. We define a vector field $E(s)$ along the curve $\psi_s(u_0)$ in $T^\lambda M$ by $E(s) = (\psi_s)_*(Y_0)$, $0 \leq s < \varepsilon_0$. Let

$$E(s) = h(s) + v(s), \quad h(s) \in H_{\psi_s(u_0)}, \quad v(s) \in V_{\psi_s(u_0)}$$

be the orthogonal decomposition of $E(s)$. By taking ε_0 sufficiently small if necessary, we may suppose $v(s) \neq 0$ for $0 \leq s < \varepsilon_0$, because of $v(0) = Y_0 \neq 0$. Put, for $0 \leq s < \varepsilon_0$,

$$\begin{aligned} a(s) &= \sqrt{g^S(h(s), h(s))}, \\ b(s) &= \sqrt{g^S(v(s), v(s))}, \\ X(s) &= \begin{cases} 0 & \text{for } s = 0, \\ a(s)^{-1}h(s) & \text{for } s > 0, \end{cases} \\ Y(s) &= b(s)^{-1}v(s). \end{aligned}$$

Then we have $E(s) = a(s)X(s) + b(s)Y(s)$ for $0 \leq s < \varepsilon_0$. Remark that $a(s)$ is a continuous function satisfying $a(s) > 0$ for $0 < s < \varepsilon_0$. We take a vector \hat{Y}_0 in $V_{u_0} \cap T_{u_0} T^\lambda M$ such that $g^S(Y_0, \hat{Y}_0) = 0$ and $g^S(\hat{Y}_0, \hat{Y}_0) = 1$. Put $\hat{E}(s) = (\psi_s)_*(\hat{Y}_0)$, $0 \leq s < \varepsilon_0$, and let

$$\hat{E}(s) = \hat{h}(s) + \hat{v}(s), \quad \hat{h}(s) \in H_{\psi_s(u_0)}, \quad \hat{v}(s) \in V_{\psi_s(u_0)}$$

be the orthogonal decomposition of $\hat{E}(s)$. We take ε_0 sufficiently small and put, for $0 \leq s < \varepsilon_0$,

$$\begin{aligned}\hat{a}(s) &= \sqrt{g^S(\hat{h}(s), \hat{h}(s))}, \\ \hat{b}(s) &= \sqrt{g^S(\hat{v}(s), \hat{v}(s))}, \\ \hat{X}(s) &= \begin{cases} 0 & \text{when } \hat{a}(s) = 0, \\ \hat{a}(s)^{-1}\hat{h}(s) & \text{when } \hat{a}(s) \neq 0, \end{cases} \\ \hat{Y}(s) &= \hat{b}^{-1}(s)\hat{v}(s).\end{aligned}$$

Then we have $\hat{E}(s) = \hat{a}(s)\hat{X}(s) + \hat{b}(s)\hat{Y}(s)$ for $0 \leq s < \varepsilon_0$. Since ψ_s is the isometric mapping, we see that

$$\begin{aligned}1 &= g^S(Y_0, Y_0) = g^S(E(s), E(s)) = a(s)^2 + b(s)^2, \\ 1 &= g^S(\hat{Y}_0, \hat{Y}_0) = g^S(\hat{E}(s), \hat{E}(s)) = \hat{a}(s)^2 + \hat{b}(s)^2, \\ 0 &= g^S(Y_0, \hat{Y}_0) = a(s)\hat{a}(s)g^S(X(s), \hat{X}(s)) + b(s)\hat{b}(s)g^S(Y(s), \hat{Y}(s)).\end{aligned}$$

On the other hand, from the definitions of $X(s)$, $\hat{X}(s)$, $Y(s)$, and $\hat{Y}(s)$, we have that

$$\begin{aligned}g^S(X(s), X(s)) &= \begin{cases} 0 & \text{for } s = 0, \\ 1 & \text{for } 0 < s < \varepsilon_0, \end{cases} \\ g^S(Y(s), Y(s)) &= 1, \quad g^S(\hat{X}(s), \hat{X}(s)) \leq 1, \quad g^S(\hat{Y}(s), \hat{Y}(s)) = 1.\end{aligned}$$

Since M is a space of constant curvature c , $R(U, V)W$ is of the form $R(U, V)W = c\{g(V, W)U - g(U, W)V\}$, for $U, V, W \in TM$. For each s with $0 \leq s < \varepsilon_0$, putting $k(\lambda, s) = g^S(R^S(E(s), \hat{E}(s))\hat{E}(s), E(s))$, we compute the value as follows:

$$\begin{aligned}k(\lambda, s) &= (a\hat{a})^2 \left\{ c(1 - g^S(X, \hat{X})^2) \right. \\ &\quad \left. + \frac{3}{4}c^2 \left(2g(\pi_*(X), \psi_s(u_0)) \cdot g(\pi_*(\hat{X}), \psi_s(u_0)) \cdot g^S(X, \hat{X}) \right. \right. \\ &\quad \left. \left. - g(\pi_*(X), \psi_s(u_0))^2 - g(\pi_*(\hat{X}), \psi_s(u_0))^2 \right) \right\} \\ &\quad + (a\hat{b})^2 \left\{ \frac{1}{4}c^2 \left(g(\pi_*(X), K(\hat{Y}))^2 + g(\pi_*(X), \psi_s(u_0))^2 \right) \right\} \\ &\quad + (\hat{a}\hat{b})^2 \left\{ \frac{1}{4}c^2 \left(g(\pi_*(\hat{X}), K(Y))^2 + g(\pi_*(\hat{X}), \psi_s(u_0))^2 \right) \right\} \\ &\quad + a\hat{a}\hat{b}\hat{b} \left\{ 3c \left(g(\pi_*(X), K(Y)) \cdot g(\pi_*(\hat{X}), K(\hat{Y})) - g(\pi_*(X), K(\hat{Y})) \cdot g(\pi_*(\hat{X}), K(Y)) \right) \right. \\ &\quad \left. - \frac{1}{2}c^2 \left(2g(\pi_*(X), K(Y)) \cdot g(\pi_*(\hat{X}), K(\hat{Y})) - g(\pi_*(X), K(\hat{Y})) \cdot g(\pi_*(\hat{X}), K(Y)) \right) \right. \\ &\quad \left. + g(\pi_*(X), \psi_s(u_0)) \cdot g(\pi_*(\hat{X}), \psi_s(u_0)) \cdot g^S(Y, \hat{Y}) \right\} + (b\hat{b}/\lambda)^2 (1 - g^S(Y, \hat{Y})^2).\end{aligned}$$

In particular, we know $k(\lambda, 0) = 1/\lambda^2$. If we suppose $\lambda = 1$, then we get

$$k(1, s) \leq 1 - a^2 \left\{ \left(1 - \frac{1}{2}c^2 - (|c| + 1)a^2 \right) \left| \frac{\hat{a}}{a} \right|^2 - \left(2 \left| 3c - \frac{1}{2}c^2 \right| + \frac{1}{2}c^2 \right) \left| \frac{\hat{a}}{a} \right| + 1 - \frac{1}{2}c^2 \right\}.$$

From this inequality there exists a positive number $\varepsilon_0' > 0$ ($\varepsilon_0' < \varepsilon_0$) such that, if $(6 - 2\sqrt{14})/5 < c < -6 + 2\sqrt{10}$, then $k(1, s) < 1$ for $0 < s < \varepsilon_0'$. But since ψ_s is the isometric mapping, we know that $k(1, s) = k(1, 0) = 1$, which gives a contradiction. We proved Proposition 3.2.1. Q.E.D.

4. The homomorphism between infinitesimal isometries of tangent sphere bundles and those of the bundles of orthonormal frames

4.1. Bundles of orthonormal frames

Given an orientable Riemannian manifold, we consider the bundle of oriented orthonormal frames and the tangent sphere bundle over it, which admit natural Riemannian metrics defined by the Riemannian connection. In this chapter, we show that there is a natural homomorphism between the Lie algebras of fiber preserving infinitesimal isometries of these bundles. In particular, for any orientable Riemannian manifold of dimension two, we show that the homomorphism yields an isomorphism between these Lie algebras.

Let (M, g) be a connected, orientable Riemannian manifold of dimension $n \geq 2$, and $SO(M)$ the bundle of oriented orthonormal frames over M . For any fixed positive number λ , the Riemannian metric G on $SO(M)$ is defined by

$$(4.1.1) \quad G(Z, W) = {}^t\theta(Z) \cdot \theta(W) + \frac{\lambda^2}{2} \text{trace}({}^t\omega(Z) \cdot \omega(W))$$

for $Z, W \in T_u SO(M)$, $u \in SO(M)$, where θ and ω denote the canonical form and the Riemannian connection form on $SO(M)$, respectively.

In their paper [13], Takagi and Yawata studied the Lie algebra of infinitesimal isometries of $(SO(M), G)$ with $\lambda = \sqrt{2}$ and proved that there exist the natural lifts $\Psi_{SO(M)}(X) \in \mathfrak{i}(SO(M), G)$ for each $X \in \mathfrak{i}(M, g)$ and $\Phi_{SO(M)}(\phi) \in \mathfrak{i}(SO(M), G)$ for each $\phi \in \mathfrak{D}^2(M)_0$, where $\mathfrak{i}(M, g)$ and $\mathfrak{i}(SO(M), G)$ denote the Lie algebras of infinitesimal isometries of (M, g) and $(SO(M), G)$, respectively, and $\mathfrak{D}^2(M)_0$ the set of parallel two-forms on (M, g) . Refining the proof in [13], we know that the mappings

$$\Psi_{SO(M)}: \mathfrak{i}(M, g) \rightarrow \mathfrak{i}(SO(M), G) \quad \text{and} \quad \Phi_{SO(M)}: \mathfrak{D}^2(M)_0 \rightarrow \mathfrak{i}(SO(M), G)$$

are also defined for any positive number λ .

The main purpose of this thesis is to prove that these mappings $\Psi_{SO(M)}$ and $\Phi_{SO(M)}$ are simultaneously factored through in terms of natural lifts to the tangent sphere bundle over M .

To be precise, let TM be the tangent bundle over M , and g^S the Sasaki metric on TM . For a given positive number λ , we consider the tangent sphere bundle $T^\lambda M$ over M . The total space of $T^\lambda M$ is defined to be the set of all tangent vectors at all points of M whose lengths are λ . It is a hypersurface of (TM, g^S) . We also denote the induced metric on $T^\lambda M$

by g^S . We derive the useful formula in Section 4.2, which shows the relation between the Riemannian metrics g^S and G . In Konno [5], we studied the fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$ and constructed the natural lifts $\Psi_{T^\lambda M}(X) \in \mathfrak{i}(T^\lambda M, g^S)$ for each $X \in \mathfrak{i}(M, g)$ and $\Phi_{T^\lambda M}(\phi) \in \mathfrak{i}(T^\lambda M, g^S)$ for each $\phi \in \mathfrak{D}^2(M)_0$. Regarding $SO(M)$ as the total space of a principal fiber bundle over the base manifold $T^\lambda M$, we then prove that $\Psi_{SO(M)}$ and $\Phi_{SO(M)}$ are simultaneously factored through $\Psi_{T^\lambda M}$ and $\Phi_{T^\lambda M}$, respectively. Namely, we have the following.

THEOREM 4.1.1. *Let (M, g) be a connected, orientable Riemannian manifold and λ a positive number. Then there exists a unique homomorphism Ψ of the Lie algebra of fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$ into that of $(SO(M), G)$ such that $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$.*

In Section 4.3, we define the vector field $\Psi(Z)$ on $(SO(M), G)$ for any infinitesimal isometry Z of $(T^\lambda M, g^S)$ by using the Riemannian connection form on $SO(M)$, and prove in Section 4.4 that Ψ is a homomorphism when it is restricted to the Lie algebra of fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$.

4.2. Riemannian metrics on bundles of orthonormal frames

In this section, we fix notation used in this chapter and derive the useful formula that clarify the relation between the Sasaki metric g^S on $T^\lambda M$ and the metric G on $SO(M)$ defined by (4.1.1).

When we regard $SO(M)$ as the principal fiber bundle over the base manifold M with structure group $SO(n)$, the special orthogonal group of $n \times n$ matrices, denote it simply by P . Let $\pi_P: P \rightarrow M$ denote its bundle projection, and ω_P the Riemannian connection form on P . Let (\cdot, \cdot) denote the canonical inner product on the n -dimensional real vector space \mathbf{R}^n . We regard each $u \in P$ as an isometry of $(\mathbf{R}^n, (\cdot, \cdot))$ onto $(T_{\pi_P(u)}M, g|_{\pi_P(u)})$ as follows: For $u = (X_1, \dots, X_n) \in P$,

$$u(e_i) = X_i \quad \text{where } e_i = {}^t(0, \dots, \overset{(i)}{1}, \dots, 0) \in \mathbf{R}^n, \quad 1 \leq i \leq n.$$

Let $\mathfrak{D}^2(M)$ denote the Lie algebra of two-forms on M , and $\mathfrak{D}^2(M)_0$ the Lie subalgebra of parallel two-forms in $\mathfrak{D}^2(M)$ with respect to ∇ . We shall identify $\mathfrak{D}^2(M)$ with the set of all skew-symmetric tensor fields of type (1,1) on M in the usual manner. Let $\mathfrak{o}(n)$ be

the Lie algebra of $SO(n)$. For $\phi \in \mathfrak{D}^2(M)$, we define an $\mathfrak{o}(n)$ -valued function ϕ^\sharp on P and a vector field ϕ^{LP} on P , respectively, by

$$(4.2.1) \quad \phi^\sharp(u) = u^{-1} \circ \phi_{\pi_P(u)} \circ u \quad \text{for } u \in P \text{ and } \omega_P(\phi^{LP}) = \phi^\sharp, \quad (\pi_P)_*(\phi^{LP}) = 0.$$

Given an infinitesimal isometry X of (M, g) , a vector field X^{LP} on P is defined by

$$(4.2.2) \quad X^{LP} = X^{HP} + (\nabla X)^{LP},$$

where X^{HP} denotes the horizontal lift of X . For any $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$, X^{LP} and ϕ^{LP} are the fiber preserving infinitesimal isometries of (P, G) , which can be proved in the same manner as in [13]. We define the mapping Ψ_P of $\mathfrak{i}(M, g)$ into $\mathfrak{i}(P, G)$ by $\Psi_P(X) = X^{LP}$ for $X \in \mathfrak{i}(M, g)$, and also the mapping Φ_P of $\mathfrak{D}^2(M)_0$ into $\mathfrak{i}(P, G)$ by $\Phi_P(\phi) = \phi^{LP}$ for $\phi \in \mathfrak{D}^2(M)_0$.

Let us identify $SO(n-1)$ with the subgroup of $SO(n)$ given by

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a \in SO(n-1) \right\}.$$

The set of oriented orthonormal frames over M , or $SO(M)$, can be regarded as the total space of a principal fiber bundle over the base manifold $T^\lambda M$ with structure group $SO(n-1)$. In fact, the bundle projection $\pi_Q: SO(M) \rightarrow T^\lambda M$ is defined by

$$\pi_Q(u) = \lambda \cdot X_n \quad \text{for } u = (X_1, \dots, X_n) \in SO(M),$$

and the structure group $SO(n-1)$ acts on $SO(M)$ on the right as follows:

$$ua = \left(\sum_{k_1} a^{k_1 1} X_{k_1}, \dots, \sum_{k_{n-1}} a^{k_{n-1} n-1} X_{k_{n-1}}, X_n \right) \quad \text{for } a = (a^i_j) \in SO(n-1).$$

Each a in $SO(n)$ defines a diffeomorphism $R_a: u \in SO(M) \mapsto ua \in SO(M)$. We denote this principal fiber bundle simply by Q . In [9], Nagy used Q to study the geodesics in the tangent sphere bundle over a Riemannian manifold.

We define an inner product $\langle \cdot, \cdot \rangle$ on the vector space $\mathfrak{o}(n)$ by $\langle A, C \rangle = \text{trace}({}^t A \cdot C)$ for $A, C \in \mathfrak{o}(n)$. Let $\mathfrak{o}(n-1)^\perp$ denote the orthogonal complement of $\mathfrak{o}(n-1)$ in $\mathfrak{o}(n)$, and $p: \mathfrak{o}(n) \rightarrow \mathfrak{o}(n-1)$ be the orthogonal projection. Define $\omega_Q = p \circ \omega_P$. It should be noted that ω_Q is a connection form on Q , because of $\omega_Q(A^*) = A$ for $A \in \mathfrak{o}(n-1)$ and $R_a^* \omega_Q = \text{ad}(a^{-1}) \omega_Q$ for $a \in SO(n-1)$, where A^* denotes the fundamental vector field corresponding to $A \in \mathfrak{o}(n)$.

We now define horizontal subspaces and vertical subspaces of the tangent spaces of P and Q . Let N denote either the bundle P or Q . Distributions H_N and V_N on $SO(M)$ are defined by

$$(H_N)_u = \text{Ker}(\omega_N|_{T_u SO(M)}), \quad (V_N)_u = \text{Ker}((\pi_N)_*|_{T_u SO(M)}), \quad \text{for } u \in SO(M),$$

where the right hand sides of both the formulas above denote the kernels of $\omega_N|_{T_u SO(M)}$ and $(\pi_N)_*|_{T_u SO(M)}$, respectively. The space $(H_N)_u$ is called the horizontal subspace of $T_u N$ and $(V_N)_u$ the vertical subspace of $T_u N$. At each point u in $SO(M)$, the tangent space $T_u SO(M)$ is decomposed into a direct sum $T_u SO(M) = (H_N)_u \oplus (V_N)_u$. Given a vector field Z on $T^\lambda M$, there exists a unique vector field Z^{H_N} on $SO(M)$ such that

$$(\pi_N)_*(Z^{H_N}) = Z, \quad \omega_N(Z^{H_N}) = 0,$$

which is called the horizontal lift of Z to N . For X in $\mathfrak{X}(M)$, we also define a horizontal lift of X to $T^\lambda M$. There is uniquely X^H in $\mathfrak{X}(T^\lambda M)$ such that

$$(\pi|_{T^\lambda M})_*(X^H) = X, \quad (K|_{T^\lambda M})(X^H) = 0.$$

We call X^H the horizontal lift of X to $T^\lambda M$.

A useful identity clarifying the relation between g^S and G is given by the following.

THEOREM 4.2.1. (i) *In the notation introduced above, we have*

$$G(Z, W) = g^S((\pi_Q)_*Z, (\pi_Q)_*W) + \frac{\lambda^2}{2} \langle \omega_Q(Z), \omega_Q(W) \rangle \quad \text{for } Z, W \in T(SO(M)).$$

(ii) *Let ∇^S and D denote the Riemannian connections of $(T^\lambda M, g^S)$ and $(SO(M), G)$, respectively. Then we have*

$$G(D_{X^{H_Q}} Y^{H_Q}, Z^{H_Q}) = g^S(\nabla_X^S Y, Z) \quad \text{for } X, Y, Z \in \mathfrak{X}(T^\lambda M).$$

To prove Theorem 4.2.1 we need the following lemma.

LEMMA 4.2.2. *Let Z and W be vector fields on $T^\lambda M$ and A in $\mathfrak{o}(n-1)$. Then we have $G(Z^{H_Q}, W^{H_Q}) = g^S(Z, W)$.*

PROOF. Since each tangent space of $T^\lambda M$ is decomposed as the direct sum of the horizontal subspace and the vertical one, it suffices to verify the formula for the following three cases.

Case 1. Both $Z_{\pi_Q(u)}$ and $W_{\pi_Q(u)}$ are in $H_{\pi_Q(u)}$. Since there are the vector fields X and Y on M such that $Z_{\pi_Q(u)} = X^H_{\pi_Q(u)}$, $W_{\pi_Q(u)} = Y^H_{\pi_Q(u)}$, we have

$$\begin{aligned} G(Z^{H_Q}, W^{H_Q})_u &= G((X^H)^{H_Q}, (Y^H)^{H_Q})_u = G(X^{H_P}, Y^{H_P})_u \\ &= g(X, Y)_{\pi_P(u)} = g^S(X^H, Y^H)_{\pi_Q(u)} = g^S(Z, W)_{\pi_Q(u)}. \end{aligned}$$

Case 2. $Z_{\pi_Q(u)}$ is in $H_{\pi_Q(u)}$, but $W_{\pi_Q(u)}$ is in $V_{\pi_Q(u)}$. Since there exists a vector field X on M such that $Z_{\pi_Q(u)} = X^H_{\pi_Q(u)}$, we have

$$\begin{aligned} G(Z^{H_Q}, W^{H_Q})_u &= \left\{ (\theta(X^{H_P}), \theta(W^{H_Q})) + \frac{\lambda^2}{2} \langle \omega_P(X^{H_P}), \omega_P(W^{H_Q}) \rangle \right\}_u \\ &= g((\pi_P)_*(X^{H_P}), (\pi_P)_*(W^{H_Q}))_{\pi_P(u)} + \frac{\lambda^2}{2} \langle 0, \omega_P(W^{H_Q}) \rangle_u \\ &= g(X, 0)_{\pi_P(u)} = 0 = g^S(Z, W)_{\pi_Q(u)}. \end{aligned}$$

Case 3. Both $Z_{\pi_Q(u)}$ and $W_{\pi_Q(u)}$ are in $V_{\pi_Q(u)}$. In this case, there exists A in $\mathfrak{o}(n-1)^\perp$ such that $Z^{H_Q}_u = A^*_u$. Setting

$$A = \left(\begin{array}{ccc|c} & & & \xi_1 \\ & \mathbf{0} & & \vdots \\ & & & \xi_{n-1} \\ \hline -\xi_1 & \cdots & -\xi_{n-1} & 0 \end{array} \right),$$

we have

$$G(Z^{H_Q}, Z^{H_Q})_u = \frac{\lambda^2}{2} \langle A, A \rangle = \lambda^2 \sum_{k=1}^{n-1} (\xi_k)^2.$$

Furthermore, for some $\varepsilon > 0$, putting $\exp tA = (a^i_j(t))$, $-\varepsilon < t < \varepsilon$, and $u = (X_1, \dots, X_n)$, we have

$$\begin{aligned} g^S(Z, Z) &= \left\| \frac{d}{dt} \{ (\pi_Q \circ R_{\exp tA})(u) \} \Big|_{t=0} \right\|^2 \\ &= g^S \left(\frac{d}{dt} \left\{ \lambda \sum_{k=1}^n a^k_n(t) X_k \right\} \Big|_{t=0}, \frac{d}{dt} \left\{ \lambda \sum_{l=1}^n a^l_n(t) X_l \right\} \Big|_{t=0} \right) \\ &= g \left(\lambda \sum_{k=1}^n \dot{a}^k_n(0) X_k, \lambda \sum_{l=1}^n \dot{a}^l_n(0) X_l \right) = \lambda^2 \sum_{k=1}^n \{ \dot{a}^k_n(0) \}^2 = \lambda^2 \sum_{k=1}^{n-1} (\xi_k)^2, \end{aligned}$$

and hence $G(Z^{H_Q}, Z^{H_Q})_u = g^S(Z, Z)_{\pi_Q(u)}$. Q.E.D.

We are now in a position to prove Theorem 4.2.1. Since the tangent space at $u \in SO(M)$ has the orthogonal decomposition of

$$(4.2.3) \quad T_u SO(M) = \{ X^{H_Q}_u; X \in T_{\pi_Q(u)} T^\lambda M \} \oplus \{ A^*_u; A \in \mathfrak{o}(n-1) \},$$

the statement (i) of Theorem 4.2.1 follows from Lemma 4.2.2. From Lemma 4.2.2 and the above decomposition, we know that the projection π_Q is the Riemannian submersion. Hence, by the O'Neill's formula in [10], the statement (ii) holds, proving Theorem 4.2.1.

REMARK. Let Z and W be in $\mathfrak{X}(SO(M))$. By (4.1.1) and (i) of Theorem 4.2.1, we have

$$\begin{aligned} ((\pi_Q)^*g^S)(Z, W) &= g^S((\pi_Q)_*Z, (\pi_Q)_*W) = G(Z, W) - \frac{\lambda^2}{2}\langle\omega_Q(Z), \omega_Q(W)\rangle \\ &= (\theta(Z), \theta(W)) + \frac{\lambda^2}{2}\langle\omega_P(Z), \omega_P(W)\rangle - \frac{\lambda^2}{2}\langle\omega_Q(Z), \omega_Q(W)\rangle. \end{aligned}$$

Putting $\lambda = 1$ in the formula above, we obtain

$$(\pi_Q)^*g^S = \sum_{i=1}^n \{(\theta_i)^2 + (\omega_{in})^2\},$$

where the one-forms θ_i and ω_{in} , $i = 1, \dots, n$, on $SO(M)$ are defined by $\theta_i(\cdot) = (\theta(\cdot), e_i)$ and $\omega_{in}(\cdot) = (\omega(\cdot)e_n, e_i)$, respectively. This formula is proved by Musso and Tricerri ([8, Proposition 6.1]).

4.3. Lifts of infinitesimal isometries of tangent sphere bundles

Given an infinitesimal isometry Z of $T^\lambda M$, we shall define the lift $\Psi(Z)$ of Z to $SO(M)$, and find the necessary and sufficient condition under which $\Psi(Z)$ is an infinitesimal isometry for G .

We first define $A_{ij} \in \mathfrak{o}(n)$, $i, j = 1, \dots, n$, by $A_{ij} = 0$ if $i = j$,

$$A_{ij} = \begin{pmatrix} & (i) & & (j) & \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -1 & \cdots & (i) \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots & (j) \\ & \vdots & & \vdots & \end{pmatrix} \quad \text{if } i < j,$$

and $A_{ij} = -A_{ji}$ if $i > j$.

For the A_{ij} , we recall here, without proof, the following well-known facts which will be frequently used in the argument below.

LEMMA 4.3.1. Put $A_i = A_{ni}$ for $i = 1, \dots, n-1$. When $n \geq 3$, the set $\{A_1, \dots, A_{n-1}\}$ is a basis of $\mathfrak{o}(n-1)^\perp$ and $\{A_{ij}; 1 \leq i < j \leq n-1\}$ is a basis of $\mathfrak{o}(n-1)$. Moreover, we have

$$[A_i, A_j] = A_{ij}, \quad [A_{ij}, A_k] = \delta_{ik}A_j - \delta_{jk}A_i,$$

$$[A_{ij}, A_{kl}] = \delta_{jl}A_{ik} - \delta_{jk}A_{il} - \delta_{il}A_{jk} + \delta_{ik}A_{jl}$$

for $i, j, k, l = 1, \dots, n-1$, where δ_{ij} denotes the Kronecker delta.

We define an $\mathfrak{o}(n-1)$ -valued function $F(Z)$ on $SO(M)$ by

$$(4.3.1) \quad ((F(Z)(u)) \cdot e_j, e_i) = \frac{1}{\lambda^2} G(D_{A_j^*} Z^{H_Q}, A_i^*)_u \quad \text{for } u \in SO(M).$$

To see that $F(Z)$ is $\mathfrak{o}(n-1)$ -valued, we first note that $A_i^*_u$ is in $(H_Q)_u$ from

$$\omega_Q(A_i^*_u) = (p \circ \omega_P)(A_i^*_u) = p(A_i) = 0,$$

and there exist X_i in $\mathfrak{X}(T^\lambda M)$ with $i = 1, \dots, n-1$ such that $A_i^*_u = (X_i^{H_Q})_u$. It then follows from these and (ii) in Theorem 4.2.1 that

$$\begin{aligned} ((F(Z)(u)) \cdot e_j, e_i) &= \frac{1}{\lambda^2} G(D_{A_j^*} Z^{H_Q}, A_i^*)_u = \frac{1}{\lambda^2} G(D_{X_j^{H_Q}} Z^{H_Q}, X_i^{H_Q})_u \\ &= \frac{1}{\lambda^2} g^S(\nabla^S_{X_j} Z, X_i)_{\pi_Q(u)} = -\frac{1}{\lambda^2} g^S(\nabla^S_{X_i} Z, X_j)_{\pi_Q(u)} = -((F(Z)(u)) \cdot e_i, e_j), \end{aligned}$$

which shows that $F(Z)$ is $\mathfrak{o}(n-1)$ -valued. We then define a vector field Z^{L_Q} on $SO(M)$ by

$$(4.3.2) \quad Z^{L_Q}_u = Z^{H_Q}_u + ((F(Z)(u)))^*_u \quad \text{at } u \in SO(M),$$

and get the mapping $\Psi: \mathfrak{i}(T^\lambda M, g^S) \rightarrow \mathfrak{X}(SO(M))$ by $\Psi(Z) = Z^{L_Q}$. We call Z^{L_Q} the lift of an infinitesimal isometry Z of $T^\lambda M$.

LEMMA 4.3.2. If Z is an infinitesimal isometry of $(T^\lambda M, g^S)$, then $G(Z^{L_Q}, A_{ij}^*) = G(D_{A_i^*} Z^{H_Q}, A_j^*)$.

PROOF. At each point $u \in SO(M)$, we set $F = (F(Z))(u)$ and $F^i_j = (F e_j, e_i)$. Then we have

$$\begin{aligned} G(Z^{L_Q}, A_{ij}^*)_u &= G(F^*, A_{ij}^*)_u = \frac{\lambda^2}{2} \langle F, A_{ij} \rangle = \frac{\lambda^2}{2} \text{trace}({}^t F \cdot A_{ij}) \\ &= \frac{\lambda^2}{2} \sum_{k=1}^n (A_{ij} e_k, F e_k) = \lambda^2 F^j_i = G(D_{A_i^*} Z^{H_Q}, A_j^*)_u. \end{aligned}$$

Q.E.D.

The next proposition offers a condition that Z^{LQ} is an infinitesimal isometry of $(SO(M), G)$.

PROPOSITION 4.3.3 ([6]). *Let Z be an infinitesimal isometry of $(T^\lambda M, g^S)$. Then Z^{LQ} is an infinitesimal isometry of $(SO(M), G)$ if and only if it satisfies the following equation:*

$$L_{Z^{LQ}}G(X^{HP}, A^*) = 0 \quad \text{for any } X \in \mathfrak{X}(M) \text{ and } A \in \mathfrak{o}(n-1).$$

To prove Proposition 4.3.3, we need several lemmas.

LEMMA 4.3.4. *Let Ω denote the curvature form of ∇ . For any $A, C \in \mathfrak{o}(n)$ and $\xi, \eta, \zeta \in \mathbf{R}^n$, we have the following:*

$$G([B(\xi), B(\eta)], A^*) = -\lambda^2 \langle \Omega(B(\xi), B(\eta)), A \rangle, \quad G([B(\xi), B(\eta)], B(\zeta)) = 0,$$

$$[A^*, B(\xi)] = B(A\xi), \quad [A^*, C^*] = [A, C]^*,$$

$$G(D_{B(\xi)}B(\eta), A^*) = -\frac{\lambda^2}{2} \langle \Omega(B(\xi), B(\eta)), A \rangle, \quad G(D_{B(\xi)}B(\eta), B(\zeta)) = 0,$$

$$G(D_{B(\xi)}A^*, C^*) = 0, \quad G(D_{B(\xi)}A^*, B(\eta)) = \frac{\lambda^2}{2} \langle \Omega(B(\xi), B(\eta)), A \rangle,$$

$$G(D_{A^*}B(\xi), B(\eta)) = \frac{\lambda^2}{2} \langle \Omega(B(\xi), B(\eta)), A \rangle + (A\xi, \eta),$$

$$G(D_{A^*}B(\xi), C^*) = 0, \quad D_{A^*}C^* = \frac{1}{2}[A, C]^*,$$

where $B(\xi)$ denotes the standard horizontal vector field corresponding to $\xi \in \mathbf{R}^n$.

PROOF. We only prove the first formula, because the others can be seen in a similar way as in the proof of Lemma 1 in [13]. By the structure equation of E. Cartan, we have

$$G([B(\xi), B(\eta)], A^*) = \frac{\lambda^2}{2} \langle \omega_P([B(\xi), B(\eta)]), \omega_P(A^*) \rangle = -\lambda^2 \langle \Omega(B(\xi), B(\eta)), A \rangle,$$

which shows the first one. Q.E.D.

From this lemma, it is easy to see that the tensor DA^* on $SO(M)$ is skew-symmetric with respect to G , hence A^* is an infinitesimal isometry of $SO(M)$.

To prove Proposition 4.3.3, we now find a condition which is equivalent to $L_{Z^{LQ}}G = 0$.

LEMMA 4.3.5. *If Z is an infinitesimal isometry of $(T^\lambda M, g^S)$, then $L_{Z^{LQ}}G(X^{HQ}, Y^{HQ}) = 0$ holds for any X, Y in $\mathfrak{X}(T^\lambda M)$.*

PROOF. Since π_Q is the Riemannian submersion, we have that

$$\begin{aligned} L_{Z^{LQ}}G(X^{HQ}, Y^{HQ}) &= Z^{LQ}G(X^{HQ}, Y^{HQ}) - G([Z^{LQ}, X^{HQ}], Y^{HQ}) - G(X^{HQ}, [Z^{LQ}, Y^{HQ}]) \\ &= Z^{HQ}G(X^{HQ}, Y^{HQ}) - G([Z, X]^{HQ}, Y^{HQ}) - G(X^{HQ}, [Z, Y]^{HQ}) \\ &= Zg^S(X, Y) - g^S([Z, X], Y) - g^S(X, [Z, Y]) = L_Zg^S(X, Y) = 0. \end{aligned}$$

Q.E.D.

LEMMA 4.3.6. *If Z is an infinitesimal isometry of $(T^\lambda M, g^S)$, then $L_{Z^{LQ}}G(A^*, C^*) = 0$ holds for any A, C in $\mathfrak{o}(n-1)$.*

PROOF. It suffices to show that $L_{Z^{LQ}}G(A_{ij}^*, A_{kl}^*) = 0$ for $1 \leq i, j, k, l \leq n-1$. Since A_{ij}^* and A_{kl}^* are infinitesimal isometries of $(SO(M), G)$, we have, by Lemmas 4.3.1 and 4.3.2, that

$$\begin{aligned} L_{Z^{LQ}}G(A_{ij}^*, A_{kl}^*) &= A_{ij}^*G(Z^{LQ}, A_{kl}^*) + A_{kl}^*G(A_{ij}^*, Z^{LQ}) \\ &= \delta_{ik}\{G(D_{A_j^*}Z^{HQ}, A_l^*) + G(D_{A_l^*}Z^{HQ}, A_j^*)\} \\ &\quad - \delta_{jl}\{G(D_{A_k^*}Z^{HQ}, A_i^*) + G(D_{A_i^*}Z^{HQ}, A_k^*)\}. \end{aligned}$$

The formula above vanishes, because Z is an infinitesimal isometry of $(T^\lambda M, g^S)$. Q.E.D.

LEMMA 4.3.7. *If Z is an infinitesimal isometry of $(T^\lambda M, g^S)$, then $L_{Z^{LQ}}G(A^*, C^*) = 0$ holds for any A in $\mathfrak{o}(n-1)$ and C in $\mathfrak{o}(n-1)^\perp$.*

PROOF. There exist functions a^{kl} with $k, l = 1, \dots, n-1$ on $SO(M)$ such that

$$(4.3.3) \quad Z^{LQ} = Z^{HQ} + \sum_{k < l} a^{kl} A_{kl}^*,$$

which implies that

$$\begin{aligned} G([Z^{LQ}, A^*], C^*) &= G([Z^{HQ} + \sum_{k<l} a^{kl} A_{kl}^*, A^*], C^*) \\ &= G([Z^{HQ}, A^*], C^*) + \sum_{k<l} a^{kl} G([A_{kl}^*, A^*], C^*) - \sum_{k<l} (A^* a^{kl}) G(A_{kl}^*, C^*) = 0, \end{aligned}$$

where $G([A_{kl}^*, A^*], C^*) = 0$ and $G(A_{kl}^*, C^*) = 0$ hold, because $[A_{kl}, A]$ and A_{kl} are in $\mathfrak{o}(n-1)$. By these formulas, we see that $L_{Z^{LQ}} G(A^*, C^*) = -G(A^*, [Z^{LQ}, C^*])$. Since C^* is an infinitesimal isometry, we further have

$$(4.3.4) \quad L_{Z^{LQ}} G(A^*, C^*) = C^* G(A^*, Z^{LQ}) - G([C^*, A^*], Z^{LQ}).$$

When $A = A_{ij}$, $i \neq j$, and $C = A_i$, it is verified that $C^* G(A^*, Z^{LQ}) = G([C^*, A^*], Z^{LQ})$ in the following way: From Lemma 4.3.2 and the assumption of Z , we have

$$\begin{aligned} A_i^* G(Z^{LQ}, A_{ij}^*) &= A_i^* G(D_{A_i^*} Z^{HQ}, A_{ij}^*) = -A_i^* G(D_{A_j^*} Z^{HQ}, A_i^*) \\ &= -A_i^* A_j^* G(Z^{HQ}, A_i^*) + A_i^* G(Z^{HQ}, D_{A_j^*} A_i^*), \end{aligned}$$

where $D_{A_j^*} A_i^*$ is vertical on Q by Lemmas 4.3.4 and 4.3.1. Hence the second term in the right hand side of the above formula equals zero. On the other hand, for the first term, we compute that

$$\begin{aligned} -A_i^* A_j^* G(Z^{HQ}, A_i^*) &= A_{ji}^* G(Z^{HQ}, A_i^*) - A_j^* A_i^* G(Z^{HQ}, A_i^*) \\ &= A_{ji}^* G(Z^{HQ}, A_i^*) - A_j^* G(D_{A_i^*} Z^{HQ}, A_i^*) - A_j^* G(Z^{HQ}, D_{A_i^*} A_i^*). \end{aligned}$$

Since Z is an infinitesimal isometry of $T^\lambda M$, we see $G(D_{A_i^*} Z^{HQ}, A_i^*) = 0$ by (ii) of Theorem 4.2.1. The formula $D_{A_i^*} A_i^* = 0$ holds trivially by Lemmas 4.3.1 and 4.3.4. Since A_{ji}^* is an infinitesimal isometry, we have

$$\begin{aligned} A_{ji}^* G(Z^{HQ}, A_i^*) &= G([A_{ji}^*, Z^{HQ}], A_i^*) + G(Z^{HQ}, [A_{ji}^*, A_i^*]) \\ &= G(0, A_i^*) + G(Z^{HQ}, -[A_{ij}^*, A_i^*]) \\ &= G([A_i^*, A_{ij}^*], Z^{LQ}), \end{aligned}$$

where we use (4.3.1) and the fact that $[A_{ij}^*, A_i^*] = A_j^*$ is horizontal on Q . Hence we have $A_i^* G(A_{ij}^*, Z^{LQ}) = G([A_i^*, A_{ij}^*], Z^{LQ})$, and $L_{Z^{LQ}} G(A_{ij}^*, A_i^*) = 0$ by (4.3.4).

When $A = A_{ij}$ and $C = A_k$ with $k \neq i, j$, we see from Lemma 4.3.1 that

$$(4.3.5) \quad [A_k^*, A_{ij}^*] = 0.$$

Since A_{ki}^* is an infinitesimal isometry, we have by (4.3.5) that

$$(4.3.6) \quad A_{ki}^*G(Z^{H_Q}, A_j^*) = G([A_{ki}^*, Z^{H_Q}], A_j^*) + G(Z^{H_Q}, [A_{ki}^*, A_j^*]) = 0.$$

Applying (4.3.5) to (4.3.4) and using Lemma 4.3.2, we see

$$L_{Z^{L_Q}}G(A_{ij}^*, A_k^*) = A_k^*G(D_{A_i^*}Z^{H_Q}, A_j^*),$$

and, by (4.3.6), we further have that

$$\begin{aligned} A_k^*G(D_{A_i^*}Z^{H_Q}, A_j^*) &= A_k^*A_i^*G(Z^{H_Q}, A_j^*) \\ &= A_{ki}^*G(Z^{H_Q}, A_j^*) + A_i^*A_k^*G(Z^{H_Q}, A_j^*) \\ &= A_i^*G(D_{A_k^*}Z^{H_Q}, A_j^*). \end{aligned}$$

Therefore $L_{Z^{L_Q}}G(A_{ij}^*, A_k^*)$ is symmetric with respect to i, k , and skew-symmetric with respect to i, j . Hence we must have that $L_{Z^{L_Q}}G(A_{ij}^*, A_k^*) = 0$. Q.E.D.

We are now in a position to complete the proof of Proposition 4.3.3. At each point u in $SO(M)$, the tangent space $T_uSO(M)$ is decomposed ([9]) as a direct sum:

$$(4.3.7) \quad T_uSO(M) = (H_P)_u \oplus \{A^*_u; A \in \mathfrak{o}(n-1)^\perp\} \oplus \{C^*_u; C \in \mathfrak{o}(n-1)\}.$$

Lemmas 4.3.5, 4.3.6 and 4.3.7 together with this decomposition (4.3.7) imply that Z^{L_Q} is an infinitesimal isometry of $SO(M)$ if and only if it satisfies the equation of Proposition 4.3.3. We thus proved Proposition 4.3.3.

4.4. Proof of Theorem 4.1.1.

In this section, we prove Theorem 4.1.1. Let Z be a fiber preserving infinitesimal isometry of $T^\lambda M$. We first show that the lift Z^{L_Q} is also an infinitesimal isometry of $(SO(M), G)$.

LEMMA 4.4.1. *Let Z be an infinitesimal isometry of $T^\lambda M$. Then we have*

$$L_{Z^{L_Q}}G(X^{H_P}, A_{ij}^*) = G([A_i^*, Z^{H_Q}], D_{X^{H_P}}A_j^*) - G([A_j^*, Z^{H_Q}], D_{X^{H_P}}A_i^*)$$

for any X in $\mathfrak{X}(M)$ and A_{ij} with $1 \leq i, j \leq n-1$.

PROOF. Recall that Z^{LQ} is represented as (4.3.3). We first list the following formulas:

$$(4.4.1) \quad A_i^* G([Z^{H_Q}, X^{H_P}], A_j^*) = A_i^* A_j^* G(X^{H_P}, Z^{H_Q}),$$

$$(4.4.2) \quad G([A_i^*, [Z^{H_Q}, X^{H_P}]], A_j^*) = 2G([A_i^*, Z^{H_Q}], D_{X^{H_P}} A_j^*) - X^{H_P} G(Z^{LQ}, A_{ij}^*),$$

$$(4.4.3) \quad G\left(\sum_{k<l} a^{kl} A_{kl}^*, X^{H_P}\right), A_{ij}^*) = -X^{H_P} G(Z^{LQ}, A_{ij}^*),$$

$$(4.4.4) \quad G([Z^{H_Q}, X^{H_P}], A_{ij}^*) = A_i^* A_j^* G(X^{H_P}, Z^{H_Q}) - 2G([A_i^*, Z^{H_Q}], D_{X^{H_P}} A_j^*) \\ + X^{H_P} G(Z^{LQ}, A_{ij}^*).$$

Proof of (4.4.1): Since $Z \in \mathfrak{i}(T^\lambda M, g^S)$ and $A_j^* \in \mathfrak{i}(SO(M), G)$, we have

$$\begin{aligned} A_i^* G([Z^{H_Q}, X^{H_P}], A_j^*) &= A_i^* Z^{H_Q} G(X^{H_P}, A_j^*) - A_i^* G(X^{H_P}, [Z^{H_Q}, A_j^*]) \\ &= 0 + A_i^* G(X^{H_P}, [A_j^*, Z^{H_Q}]) \\ &= A_i^* A_j^* G(X^{H_P}, Z^{H_Q}) - A_i^* G([A_j^*, X^{H_P}], Z^{H_Q}) \\ &= A_i^* A_j^* G(X^{H_P}, Z^{H_Q}) - A_i^* G(0, Z^{H_Q}). \end{aligned}$$

This shows (4.4.1).

(4.4.2) is proved as follows: Using $[A_i^*, X^{H_P}] = 0$ together with Jacobi's identity, we have

$$\begin{aligned} G([A_i^*, [Z^{H_Q}, X^{H_P}]], A_j^*) &= G([[A_i^*, Z^{H_Q}], X^{H_P}], A_j^*) \\ &= G(D_{[A_i^*, Z^{H_Q}]} X^{H_P}, A_j^*) - G(D_{X^{H_P}} [A_i^*, Z^{H_Q}], A_j^*) \\ &= -G(X^{H_P}, D_{[A_i^*, Z^{H_Q}]} A_j^*) - X^{H_P} G([A_i^*, Z^{H_Q}], A_j^*) + G([A_i^*, Z^{H_Q}], D_{X^{H_P}} A_j^*). \end{aligned}$$

Since A_j^* and A_i^* are in $\mathfrak{i}(SO(M), G)$, we have

$$\begin{aligned} -G(X^{H_P}, D_{[A_i^*, Z^{H_Q}]} A_j^*) &= G(D_{X^{H_P}} A_j^*, [A_i^*, Z^{H_Q}]), \\ -X^{H_P} G([A_i^*, Z^{H_Q}], A_j^*) &= -X^{H_P} A_i^* G(Z^{H_Q}, A_j^*) + X^{H_P} G(Z^{H_Q}, [A_i^*, A_j^*]) \\ &= -X^{H_P} G(D_{A_i^*} Z^{H_Q}, A_j^*) + \frac{1}{2} X^{H_P} G(Z^{H_Q}, [A_i^*, A_j^*]) \\ &= -X^{H_P} G(Z^{LQ}, A_{ij}^*). \end{aligned}$$

Hence (4.4.2) follows.

Proof of (4.4.3): It follows from (4.3.3) that

$$\begin{aligned}
G([\sum_{k<l} a^{kl} A_{kl}^*, X^{HP}], A_{ij}^*) &= -\sum_{k<l} (X^{HP} a^{kl}) G(A_{kl}^*, A_{ij}^*) \\
&= -X^{HP} G(\sum_{k<l} a^{kl} A_{kl}^*, A_{ij}^*) \\
&= -X^{HP} G(Z^{LQ}, A_{ij}^*),
\end{aligned}$$

which proves (4.4.3).

Proof of (4.4.4): Since A_i^* is an infinitesimal isometry, we have, by (4.4.1) and (4.4.2), that

$$\begin{aligned}
G([Z^{HQ}, X^{HP}], A_{ij}^*) &= G([Z^{HQ}, X^{HP}], [A_i^*, A_j^*]) \\
&= A_i^* G([Z^{HQ}, X^{HP}], A_j^*) - G([A_i^*, [Z^{HQ}, X^{HP}]], A_j^*) \\
&= A_i^* A_j^* G(X^{HP}, Z^{HQ}) - 2G([A_i^*, Z^{HQ}], D_{X^{HP}} A_j^*) + X^{HP} G(Z^{LQ}, A_{ij}^*).
\end{aligned}$$

This proves (4.4.4).

Using these formulas (4.4.3) and (4.4.4), we prove Lemma 4.4.1. By (4.3.3), we obtain

$$\begin{aligned}
L_{Z^{LQ}} G(X^{HP}, A_{ij}^*) &= Z^{LQ} G(X^{HP}, A_{ij}^*) - G([Z^{LQ}, X^{HP}], A_{ij}^*) - G(X^{HP}, [Z^{LQ}, A_{ij}^*]) \\
&= -G([Z^{LQ}, X^{HP}], A_{ij}^*) = -G([Z^{HQ} + \sum_{k<l} a^{kl} A_{kl}^*, X^{HP}], A_{ij}^*) \\
&= -G([Z^{HQ}, X^{HP}], A_{ij}^*) - G([\sum_{k<l} a^{kl} A_{kl}^*, X^{HP}], A_{ij}^*).
\end{aligned}$$

From (4.4.3) and (4.4.4), we see that the right hand side of the above formula equals

$$-A_i^* A_j^* G(X^{HP}, Z^{HQ}) + 2G([A_i^*, Z^{HQ}], D_{X^{HP}} A_j^*).$$

We then have

$$\begin{aligned}
L_{Z^{LQ}} G(X^{HP}, A_{ij}^*) &= -\frac{1}{2} A_i^* A_j^* G(X^{HP}, Z^{HQ}) + G([A_i^*, Z^{HQ}], D_{X^{HP}} A_j^*) \\
&\quad + \frac{1}{2} A_j^* A_i^* G(X^{HP}, Z^{HQ}) - G([A_j^*, Z^{HQ}], D_{X^{HP}} A_i^*) \\
&= -\frac{1}{2} A_{ij}^* G(X^{HP}, Z^{HQ}) + G([A_i^*, Z^{HQ}], D_{X^{HP}} A_j^*) - G([A_j^*, Z^{HQ}], D_{X^{HP}} A_i^*) \\
&= G([A_i^*, Z^{HQ}], D_{X^{HP}} A_j^*) - G([A_j^*, Z^{HQ}], D_{X^{HP}} A_i^*),
\end{aligned}$$

which shows Lemma 4.4.1.

Q.E.D.

Using Lemma 4.4.1, we next show that $L_{Z^{LQ}}G(X^{HP}, A_{ij}^*) = 0$, which is a condition that Z^{LQ} is in $\mathfrak{i}(SO(M), G)$ by Proposition 4.3.3. From Lemma 4.3.4, each $D_{X^{HP}}A_i^*$ is horizontal on P , so that, due to Lemma 4.4.1, it suffices to show that $[A_i^*, Z^{HQ}]$, $i = 1, \dots, n$, are vertical on P . Let U (resp. W) be a horizontal (resp. vertical) vector field on $T^\lambda M$. By the assumption that Z preserves the fibers of $T^\lambda M$, we have by Theorem 4.2.1 that

$$\begin{aligned} & G(D_{Z^{HQ}}W^{HQ}, U^{HQ}) - G(D_{W^{HQ}}Z^{HQ}, U^{HQ}) \\ &= g^S(\nabla^S_Z W, U) - g^S(\nabla^S_W Z, U) = g^S([Z, W], U) = 0, \end{aligned}$$

and hence we have

$$(4.4.5) \quad G(D_{Z^{HQ}}W^{HQ}, U^{HQ}) = G(D_{W^{HQ}}Z^{HQ}, U^{HQ}).$$

On a local neighborhood of $T^\lambda M$, there are horizontal (resp. vertical) vector fields U_i with $i = 1, \dots, n$ (resp. W_i with $i = 1, \dots, n-1$) and functions b^k_l with $k, l = 1, \dots, n$ (resp. a^k_l with $k, l = 1, \dots, n-1$) such that

$$B(e_j) = \sum_{l=1}^n b^l_j(U_l)^{HQ} \quad \text{for } j = 1, \dots, n,$$

$$\left(\text{resp. } A_i^* = \sum_{k=1}^{n-1} a^k_i(W_k)^{HQ} \quad \text{for } i = 1, \dots, n-1 \right).$$

Then, from (4.4.5), it turns out that

$$\begin{aligned} G(D_{A_i^*}Z^{HQ}, B(e_j)) &= \sum_{k=1}^{n-1} \sum_{l=1}^n a^k_i b^l_j G(D_{(W_k)^{HQ}}Z^{HQ}, (U_l)^{HQ}) \\ &= \sum_{k=1}^{n-1} \sum_{l=1}^n a^k_i b^l_j G(D_{Z^{HQ}}(W_k)^{HQ}, (U_l)^{HQ}) = G(D_{Z^{HQ}}A_i^*, B(e_j)), \end{aligned}$$

and hence

$$(4.4.6) \quad G([A_i^*, Z^{HQ}], B(e_j)) = 0 \quad (\text{or } G([A_i^*, Z^{LQ}], B(e_j)) = 0).$$

It follows from (4.4.6) that $[A_i^*, Z^{HQ}]$ is a vertical vector field on P . Therefore we have $L_{Z^{LQ}}G(X^{HP}, A_{ij}^*) = 0$, and Z^{LQ} is an infinitesimal isometry of $(SO(M), G)$ by Proposition 4.3.3.

For each $\phi \in \mathfrak{D}^2(M)$, there exists a unique vector field ϕ^L on $T^\lambda M$ such that

$$(\pi|_{T^\lambda M})_*(\phi^L_Y) = 0, \quad (K|_{T_Y T^\lambda M})(\phi^L_Y) = \phi(Y) \quad \text{for any } Y \in T^\lambda M,$$

where K is the connection map given by (2.1.1). Given an infinitesimal isometry X of (M, g) , the tensor field ∇X is regarded as an element of $\mathfrak{D}^2(M)$, and we then define a vector field X^L on $T^\lambda M$ by

$$(4.4.7) \quad X^L = X^H + (\nabla X)^L,$$

where X^H denotes the horizontal lift of X . We remark that $X^L = X^C|_{T^\lambda M}$ and $\phi^L = \iota P|_{T^\lambda M}$ hold for $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$, respectively. It follows from Theorem 3.1.2 that X^L and ϕ^L are fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$. We recall that $\Psi_{T^\lambda M}$ is the mapping of $\mathfrak{i}(M, g)$ into $\mathfrak{i}(T^\lambda M, g^S)$ defined by $\Psi_{T^\lambda M}(X) = X^L$ for $X \in \mathfrak{i}(M, g)$, and that $\Phi_{T^\lambda M}$ is the mapping of $\mathfrak{D}^2(M)_0$ into $\mathfrak{i}(T^\lambda M, g^S)$ defined by $\Phi_{T^\lambda M}(\phi) = \phi^L$ for $\phi \in \mathfrak{D}^2(M)_0$.

Next, we show a lemma which completes the proof of Theorem 4.1.1.

LEMMA 4.4.2. $(X^L)^{LQ} = X^{LP}$ and $(\phi^L)^{LQ} = \phi^{LP}$ for any X in $\mathfrak{i}(M, g)$ and ϕ in $\mathfrak{D}^2(M)$.

PROOF. From (2.2.1), we recall that given a vector field W on M , there exists uniquely the vector field W^V on TM , called the vertical lift of W . For any Y in TM , the vector W^V_Y at Y depends only on the the given vector $W_{\pi(Y)}$. Let V_Y denote the vertical space of $T_Y TM$. We define $I_Y := K|_Y$, which is an isomorphism from V_Y to the tangent space $T_{\pi(Y)}M$. Let $u = (Y_1, \dots, Y_n)$ be an arbitrary point in $SO(M)$. Set $\exp tA_i = (a_{(i)}^k(t))$. Then we obtain

$$\begin{aligned} (\pi_Q)_*(A_i^* u) &= (\pi_Q)_* \left(\frac{d}{dt} \{ (R_{\exp tA_i})(u) \} \Big|_{t=0} \right) = \frac{d}{dt} \{ (\pi_Q) \circ (R_{\exp tA_i})(u) \} \Big|_{t=0} \\ &= \frac{d}{dt} \left\{ \lambda \sum_{k=1}^{n-1} a_{(i)}^k(t) Y_k \right\} \Big|_{t=0} = I_{Y_n}^{-1} \left(\lambda \sum_{k=1}^{n-1} \dot{a}_{(i)}^k(0) Y_k \right) \\ &= \lambda \sum_{k=1}^{n-1} \dot{a}_{(i)}^k(0) I_{Y_n}^{-1}(Y_k) = \lambda I_{Y_n}^{-1}(Y_i) = \lambda Y_i^V_{Y_n} = (\lambda u(e_i))^V_{Y_n}, \end{aligned}$$

which implies that

$$(4.4.8) \quad A_i^* u = \{ (\lambda u(e_i))^V_{Y_n} \}^{H_{Q_u}}.$$

We shall use this in the following argument.

To prove the first formula in Lemma 4.4.2, it suffices to show that

$$(4.4.9) \quad (\pi_P)_*((X^L)^{LQ}) = (\pi_P)_*(X^{LP}), \quad \omega_P((X^L)^{LQ}) = \omega_P(X^{LP}).$$

Note that, putting $F = F(X^L)$, we get

$$(X^L)^{LQ} = (X^L)^{H_Q} + F^* = (X^H + (\nabla X)^L)^{H_Q} + F^* = X^{H_P} + ((\nabla X)^L)^{H_Q} + F^*,$$

which provides the decomposition of (4.3.7) for $(X^L)^{LQ}$. Then the first part of (4.4.9) follows from (4.2.2) and the decomposition above.

For the second part of (4.4.9), it suffices to prove the following formulas for each u in $SO(M)$ and l, i, j with $1 \leq l, i, j \leq n-1$,

$$(4.4.10) \quad (\omega_P(((\nabla X)^L)^{H_Q}_u e_n, e_l)) = (((\nabla X)^\sharp(u)) e_n, e_l),$$

$$(4.4.11) \quad (\omega_P(F^*) e_i, e_j) = (((\nabla X)^\sharp(u)) e_i, e_j),$$

where $(\nabla X)^\sharp$ is the $\mathfrak{o}(n)$ -valued function on P defined by (4.2.1).

Indeed, setting

$$(4.4.12) \quad ((\nabla X)^L)^{H_Q} = \sum_{k=1}^{n-1} \xi^k A_k^*, \quad \xi^k \in \mathfrak{F}(SO(M)),$$

we see that

$$\begin{aligned} ((\nabla X)^L)_{\pi_Q(u)} &= (\pi_Q)_* \left(\sum_{k=1}^{n-1} \xi^k(u) \cdot A_k^* u \right) = \sum_{k=1}^{n-1} \xi^k(u) \cdot (\pi_Q)_* \left(\frac{d}{dt} \{ (R_{\exp t A_k})(u) \} \Big|_{t=0} \right) \\ &= \lambda \sum_{k=1}^{n-1} \xi^k(u) \cdot I_{X_n}^{-1} \left(\sum_{l=1}^{n-1} \dot{a}_{(k)l}^n(0) X_l \right) = \lambda \sum_{k=1}^{n-1} \xi^k(u) \cdot I_{X_n}^{-1}(X_k), \end{aligned}$$

and hence

$$\begin{aligned} \xi^l(u) &= \sum_{k=1}^{n-1} \xi^k(u) \cdot g(X_k, X_l) = \sum_{k=1}^{n-1} \xi^k(u) \cdot g^S(I_{X_n}^{-1}(X_k), I_{X_n}^{-1}(X_l)) \\ &= \frac{1}{\lambda} g^S \left(\lambda \sum_{k=1}^{n-1} \xi^k(u) \cdot I_{X_n}^{-1}(X_k), I_{X_n}^{-1}(X_l) \right) = \frac{1}{\lambda} g^S(((\nabla X)^L)_{\pi_Q(u)}, I_{X_n}^{-1}(X_l)) \\ &= \frac{1}{\lambda} g(K(((\nabla X)^L)_{\pi_Q(u)}, K(I_{X_n}^{-1}(X_l))) = \frac{1}{\lambda} g((\nabla X)(\lambda X_n), X_l) \\ &= g((\nabla X)X_n, X_l) = ((u^{-1} \circ \nabla X)(u(e_n)), u^{-1}(X_l)) \\ &= ((u^{-1} \circ \nabla X)(u(e_n)), e_l) = ((u^{-1} \circ (\nabla X) \circ u) e_n, e_l) = (((\nabla X)^\sharp(u)) e_n, e_l). \end{aligned}$$

Therefore it follows that

$$\begin{aligned}
(\omega_P(((\nabla X)^L)^{H_Q}_u)e_n, e_l) &= \left(\omega_P \left(\sum_{k=1}^{n-1} \xi^k(u) \cdot A_k^* u \right) e_n, e_l \right) = \sum_{k=1}^{n-1} \xi^k(u) \cdot (A_k e_n, e_l) \\
&= \sum_{k=1}^{n-1} \xi^k(u) \cdot (e_k, e_l) = \xi^l(u) = (((\nabla X)^\sharp(u))e_n, e_l),
\end{aligned}$$

which proves (4.4.10).

Next, we show (4.4.11): Using (4.3.1), (4.4.8), (ii) of Theorem 4.2.1, and (4.4.7) in order, we obtain

$$\begin{aligned}
(\omega_P(F^*) \cdot e_i, e_j) &= (F e_i, e_j) = \frac{1}{\lambda^2} G(D_{A_i^*}(X^L)^{H_Q}, A_j^*)_u \\
&= \frac{1}{\lambda^2} G(D_{((\lambda u(e_i))^V_{X_n})^{H_Q}}(X^L)^{H_Q}, ((\lambda u(e_j))^V_{X_n})^{H_Q}) \\
&= \frac{1}{\lambda^2} g^S(\nabla^S_{\lambda u(e_i)^V} X^L, \lambda u(e_j)^V)_{X_n} \\
&= \frac{1}{\lambda^2} g^S(\nabla^S_{\lambda u(e_i)^V} X^H, \lambda u(e_j)^V)_{X_n} + \frac{1}{\lambda^2} g^S(\nabla^S_{\lambda u(e_i)^V} (\nabla X)^L, \lambda u(e_j)^V)_{X_n}.
\end{aligned}$$

Note here that the first term in the right hand side above vanishes. In fact, by (4.4.8) and Theorem 4.2.1, we see

$$\begin{aligned}
g^S(\nabla^S_{\lambda u(e_i)^V} X^H, \lambda u(e_j)^V)_{X_n} &= G(D_{A_i^*}(X^H)^{H_Q}, A_j^*)_u \\
&= G(D_{A_i^*} X^{H_P}, A_j^*)_u = -G(X^{H_P}, D_{A_i^*} A_j^*)_u,
\end{aligned}$$

because X^{H_P} is horizontal and $D_{A_i^*} A_j^*$ is vertical on P . On the other hand, we see

$$\begin{aligned}
\frac{1}{\lambda^2} g^S(\nabla^S_{\lambda u(e_i)^V} (\nabla X)^L, \lambda u(e_j)^V)_{X_n} &= g^S(\nabla^S_{X_i^V} (\nabla X)^L, X_j^V)_{X_n} \\
&= g^S\left(\frac{d}{dt} \left\{ (\nabla X)^L(tX_i + \lambda X_n) \right\} \Big|_{t=0}, X_j^V\right)_{X_n} \\
&= g^S\left(\frac{d}{dt} \left\{ I_{X_n}^{-1}(\nabla_{tX_i + \lambda X_n} X) \right\} \Big|_{t=0}, X_j^V\right)_{X_n} = g^S(I_{X_n}^{-1}(\nabla_{X_i} X), X_j^V)_{X_n} \\
&= g(K(I_{X_n}^{-1}(\nabla_{X_i} X)), K(X_j^V))_{\pi(X_n)} = g(\nabla_{X_i} X, X_j)_{\pi(X_n)} \\
&= g(\nabla_{u(e_i)} X, X_j)_{\pi(X_n)} = (u^{-1} \nabla_{u(e_i)} X, u^{-1} X_j) = (u^{-1} \nabla_{u(e_i)} X, e_j) \\
&= ((u^{-1} \circ (\nabla X) \circ u)e_i, e_j) = (((\nabla X)^\sharp(u))e_i, e_j).
\end{aligned}$$

In consequence, we obtain (4.4.11), which completes the proof of the first formula of Lemma 4.4.2. The proof of the second formula proceeds in the same way as that of the first one. Q.E.D.

Now we prove Theorem 4.1.1. From the fact proved in the beginning of this section, the mapping Ψ defined in Section 4.3 is regarded as a mapping of the Lie algebra of fiber preserving infinitesimal isometries of $T^\lambda M$ into $\mathfrak{i}(SO(M), G)$. Let Z be a fiber preserving infinitesimal isometry of $T^\lambda M$. It is easy to see that the image $\Psi(Z) = Z^{LQ}$ preserves the fibers of P . In fact, using (4.3.3), we have the following for $1 \leq i, j \leq n-1$:

$$\begin{aligned} G([Z^{LQ}, A_{ij}^*], B(e_k)) &= G([Z^{HQ} + \sum_{k<l} a^{kl} A_{kl}^*, A_{ij}^*], B(e_k)) \\ &= G([Z^{HQ}, A_{ij}^*], B(e_k)) + G([\sum_{k<l} a^{kl} A_{kl}^*, A_{ij}^*], B(e_k)) = 0. \end{aligned}$$

This formula and (4.4.6) imply

$$(4.4.13) \quad G([Z^{LQ}, A_{ij}^*], B(e_k)) = 0 \quad \text{for } 1 \leq i, j, k \leq n,$$

hence Z^{LQ} preserves the fibers of P .

The mapping Ψ is a homomorphism, that can be proved in the following way: Note that each $T^\lambda M$ is an integral manifold of the distribution $\{TT^\lambda M; \lambda > 0\}$. Using the chart (2.2.2) of the tangent bundle TM , we get by (2.2.3) that

$$(4.4.14) \quad [X^L, Y^L] = [X, Y]^L, \quad [\phi^L, \psi^L] = -[\phi, \psi]^L, \quad [X^L, \phi^L] = -[\nabla X, \phi]^L$$

for any $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$. On the other hand, in the same manner as in [13], it is verified that

$$(4.4.15) \quad [X^{LP}, Y^{LP}] = [X, Y]^{LP}, \quad [\phi^{LP}, \psi^{LP}] = -[\phi, \psi]^{LP}, \quad [X^{LP}, \phi^{LP}] = -[\nabla X, \phi]^{LP}$$

for $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$. From Theorem 3.1.2, there exist uniquely $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$ such that $Z = X^L + \phi^L$, it follows from the formulas (4.4.14), (4.4.15), and Lemma 4.4.2 that Ψ is a homomorphism. Since Ψ satisfies $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$, the uniqueness of such homomorphism follows from that of the decompositions of the fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$ and $(SO(M), G)$. This completes the proof of Theorem 4.1.1.

REMARK. When Z is a fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$, any local one-parameter group of local transformations generated by $\Psi(Z)$ is given

$$(X_1, \dots, X_n) \mapsto (f_t(X_1), \dots, f_t(X_n)) \in SO(M), \quad -\varepsilon < t < \varepsilon,$$

where (X_1, \dots, X_n) is in $SO(M)$ and f_t , $-\varepsilon < t < \varepsilon$ (for some $\varepsilon > 0$), is the local one-parameter groups of local transformations generated by Z .

4.5. The case of dimension two

In this section we assume that (M, g) is two-dimensional. Since the connection form of the bundle Q then vanishes, Theorem 4.2.1 says that $G = (\pi_Q)^*g^S$ and the mapping $\pi_Q: (SO(M), G) \rightarrow (T^\lambda M, g^S)$ is an isometry. From Proposition 4.3.3, we can define the one-to-one homomorphism $\Psi: \mathfrak{i}(T^\lambda M, g^S) \rightarrow \mathfrak{i}(SO(M), G)$ by $\Psi(Z) = Z^{LQ}$ for $Z \in \mathfrak{i}(T^\lambda M, g^S)$. Moreover, we obtain the following.

THEOREM 4.5.1. *Let (M, g) be a connected, orientable two-dimensional Riemannian manifold and λ a positive number.*

(i) *If (M, g) is not a space of constant curvature $1/\lambda^2$, then any infinitesimal isometry of $(T^\lambda M, g^S)$ is of fiber preserving, and we have*

$$\mathfrak{i}(T^\lambda M, g^S)/\Psi_{T^\lambda M}(\mathfrak{i}(M, g)) \cong \Phi_{T^\lambda M}(\mathfrak{D}^2(M)_0).$$

In this case, the center of $\mathfrak{i}(T^\lambda M, g^S)$ is $\Phi_{T^\lambda M}(\mathfrak{D}^2(M)_0)$.

(ii) *If (M, g) is a space of constant curvature $1/\lambda^2$, then we have*

$$\mathfrak{i}(T^\lambda M, g^S)/\Psi_{T^\lambda M}(\mathfrak{i}(M, g)) \cong \underset{\mathbb{R}}{\text{span}} \{ \Phi_{T^\lambda M}(\phi), S, [\Phi_{T^\lambda M}(\phi), S]; \phi \in \mathfrak{D}^2(M)_0 \},$$

where S denotes the geodesic spray on $(T^\lambda M, g^S)$. In this case, the center of $\mathfrak{i}(T^\lambda M, g^S)$ is trivial.

To prove the first part of Theorem 4.5.1, we suppose that there exists an infinitesimal isometry Z of $T^\lambda M$ which does not preserve fibers. Set $J = (\pi_Q)_*(A_1^*)$, which is a vertical infinitesimal isometry of $T^\lambda M$ satisfying $\|J\| = \lambda$. For each positive integer l , let us define infinitesimal isometry W_l of $(T^\lambda M, g^S)$ and the open set U_l of $T^\lambda M$ as follows:

$$W_1 = [J, Z], \quad W_{l+1} = [J, W_l], \quad U_l = \{Y \in T^\lambda M; (W_l)_Y \neq 0\}.$$

Then, we have the following lemma.

LEMMA 4.5.2. (i) *W_l is a horizontal infinitesimal isometry of $(T^\lambda M, g^S)$, which satisfies $g^S(W_l, W_{l+1}) = 0$ and $g^S(W_{l+1}, W_{l+1}) = -g^S(W_l, W_{l+2})$ for $l \geq 1$.*

- (ii) $U_l = T^\lambda M$, and $\|W_l\|$ is a constant function on $T^\lambda M$ for $l \geq 1$.
- (iii) $\|W_l\|^2 = \lambda^2 \langle \Omega(W_l, W_{l-1}), A_1 \rangle$ for $l \geq 2$.

PROOF. (i) Put $W_0 = Z$. Since the infinitesimal isometries constitute the Lie algebra, it is proved by induction that W_l is an infinitesimal isometry of $(T^\lambda M, g^S)$. It follows from

$$g^S(J, W_l) = g^S(J, [J, W_{l-1}]) = -\frac{1}{2}W_{l-1}g^S(J, J) = 0$$

that W_l is horizontal on $T^\lambda M$. Hence we have

$$g^S(W_l, W_{l+1}) = g^S(W_l, [J, W_l]) = -W_l g^S(W_l, J) + g^S([W_l, W_l], J) = 0.$$

Since J is an infinitesimal isometry of $(T^\lambda M, g^S)$, we have

$$g^S(W_{l+1}, W_{l+1}) = Jg^S(W_l, W_{l+1}) - g^S(W_l, [J, W_{l+1}]) = -g^S(W_l, W_{l+2}).$$

(ii) Using the second formula of (i), it is proved by an induction that $U_m \supset U_{m+1}$ for $m \geq 1$ and $U_m \subset U_{m+1}$ for $m \geq 2$. It follows that $U_m = U_2$ for $m \geq 2$.

We next show that U_2 is not empty. To do this, we suppose, on the contrary, that U_2 is an empty set, and derive a contradiction. Since we have $[J, W_1] = 0$ on $T^\lambda M$, the infinitesimal isometry W_1 preserves the fibers of $T^\lambda M$. Hence, by Theorem 3.1.2, there exist X in $\mathfrak{i}(M, g)$ and ϕ in $\mathfrak{D}^2(M)_0$ such that

$$W_1 = X^L + \phi^L = X^H + (\nabla X + \phi)^L.$$

Since W_1 is horizontal by (i), we have $\nabla X + \phi = 0$. It follows that $\nabla \nabla X = -\nabla \phi = 0$, and hence $R(Y, Y')X = 0$ on M for any $Y, Y' \in \mathfrak{X}(M)$, that is, $(\pi(U_1), g)$ is flat. But this contradicts the fact that an infinitesimal isometry $Z|_{U_1}$, which does not preserve fibers, exists on U_1 . Because, if $(\pi(U_1), g)$ is flat, then the distribution H_P is integrable, and $((\pi|_{T^\lambda M})^{-1}(\pi(U_1)), g^S)$ is also flat, which can be easily seen from the formula for the curvature tensor of $(T^\lambda M, g^S)$ (cf. Blair [1] and Section 3.2). Hence there exists an open set U_1' of U_1 such that $((\pi|_{T^\lambda M})^{-1}(\pi(U_1')), g^S)$ is isometric to an open set of \mathbf{R}^3/Γ , where Γ is the free group generated by $2\pi\lambda e_3 \in \mathbf{R}^3$, which contains a whole fiber. But, on such an open set, there exists no infinitesimal isometry which does not preserve fibers. On account of these facts, we conclude that U_2 is not empty.

Since W_l and J are infinitesimal isometries, it follows that

$$(4.5.1) \quad \begin{cases} W_l(\|W_l\|^2) = 2g^S([W_l, W_l], W_l) = 0, \\ W_{l+1}(\|W_l\|^2) = -2W_l g^S(W_{l+1}, W_l) + 2g^S(W_{l+1}, [W_l, W_l]) = 0, \\ J(\|W_l\|^2) = 2g^S([J, W_l], W_l) = 2g^S(W_{l+1}, W_l) = 0. \end{cases}$$

So, $\|W_m\|$ is a constant function on each connected component of U_m for $m \geq 2$. Then, the continuity of the vector field W_m implies that $U_m = T^\lambda M$. Hence we conclude that $U_l = T^\lambda M$ for any $l \geq 1$. This proves the assertion (ii).

(iii) Since W_l and J are in $\mathfrak{i}(T^\lambda M, g^S)$, we have by Lemma 4.3.4 that

$$\begin{aligned} g^S(W_{l+1}, W_{l+1}) &= g^S([J, W_l], W_{l+1}) = g^S(\nabla^S_J W_l, W_{l+1}) - g^S(\nabla^S_{W_l} J, W_{l+1}) \\ &= -g^S(\nabla^S_{W_{l+1}} W_l, J) + g^S(\nabla^S_{W_{l+1}} J, W_l) = \lambda^2 \langle \Omega(W_{l+1}, W_l), A_1 \rangle. \end{aligned}$$

This completes the proof of Lemma 4.5.2. Q.E.D.

It follows from (i) of Lemma 4.5.2 that $\|W_{l+1}\|/\|W_l\|$ is independent of the number l . Hence, from (ii) and (iii) of Lemma 4.5.2, we know that the Gaussian curvature of (M, g) is equal to the constant $c = \|W_{l+1}\| \cdot (\lambda^2 \|W_l\|)^{-1}$ on M .

We show that the constant c can be computed in a different way:

LEMMA 4.5.3. *For each $l \geq 1$, we have*

$$\nabla^S_{W_l} W_l = 0 \quad \text{and} \quad g^S(R^S(W_l, W_{l+1})W_{l+1}, W_l) = (c\lambda\|W_l\| \cdot \|W_{l+1}\|/2)^2.$$

PROOF. It follows from (4.5.1) and (i) of Lemma 4.5.2 that

$$(4.5.2) \quad g^S(\nabla^S_{W_l} W_l, W_l) = 0, \quad g^S(\nabla^S_{W_l} W_l, W_{l+1}) = 0, \quad g^S(\nabla^S_{W_l} W_l, J) = 0.$$

Hence we get $\nabla^S_{W_l} W_l = 0$, which implies that $(\nabla^S_{W_{l+1}} \nabla^S_{W_l})W_{l+1} = \nabla^S_{W_{l+1}} \nabla^S_{W_{l+1}} W_l$. Since any infinitesimal isometry W of $(T^\lambda M, g^S)$ satisfies the following differential equation

$$(\nabla^S_Y \nabla^S W)(Y') + R^S(W, Y)Y' = 0, \quad Y, Y' \in \mathfrak{X}(T^\lambda M),$$

we have

$$\begin{aligned} g^S(R^S(W_l, W_{l+1})W_{l+1}, W_l) &= -g^S((\nabla^S_{W_{l+1}} \nabla^S_{W_l})W_{l+1}, W_l) \\ &= -g^S(\nabla^S_{W_{l+1}} \nabla^S_{W_{l+1}} W_l, W_l) \\ &= -W_{l+1} \left(\frac{1}{2} W_{l+1} \|W_l\|^2 \right) + \|\nabla^S_{W_{l+1}} W_l\|^2 \\ &= \|\nabla^S_{W_{l+1}} W_l\|^2, \end{aligned}$$

where (4.5.1) is used. From the second formula of (4.5.2) together with the fact that W_l is an infinitesimal isometry, we know that $\nabla^S_{W_{l+1}}W_l$ is vertical on $T^\lambda M$, and hence it follows from Lemma 4.3.4 that

$$\|\nabla^S_{W_{l+1}}W_l\| = \left| \frac{1}{\lambda}g^S(\nabla^S_{W_{l+1}}W_l, J) \right| = \left| \frac{\lambda}{2}\langle \Omega(W_l, W_{l+1}), A_1 \rangle \right| = \frac{c\lambda}{2}\|W_l\| \cdot \|W_{l+1}\|.$$

Q.E.D.

On the other hand, by the formula of the curvature tensor of $(T^\lambda M, g^S)$ (Blair [1] and Section 3.2), we have the following: For an arbitrary point Y in $T^\lambda M$, put $(\pi|_{T^\lambda M})(Y) = Y^b$ and $(\pi|_{T^\lambda M})_*((W_l)_Y) = X_l$ for $l \geq 1$. Then it holds that

$$\begin{aligned} & g^S(R^S(W_l, W_{l+1})W_{l+1}, W_l)_Y \\ &= g(R(X_l, X_{l+1})X_{l+1}, X_l) + \frac{1}{4}g(R(Y^b, R(X_{l+1}, X_{l+1})Y^b)X_l, X_l) \\ &\quad + \frac{1}{4}g(R(Y^b, R(X_l, X_{l+1})Y^b)X_{l+1}, X_l) + \frac{1}{2}g(R(Y^b, R(X_l, X_{l+1})Y^b)X_{l+1}, X_l) \\ &= \left(c - \frac{3c^2\lambda^2}{4} \right) \cdot \|W_l\|^2 \cdot \|W_{l+1}\|^2. \end{aligned}$$

From Lemma 4.5.3 and the formula above, we get $c = 0$ or $c = 1/\lambda^2$. However, in the proof of Lemma 4.5.2, we see that if $c = 0$, then there exists no infinitesimal isometry Z which does not preserve the fibers. Hence (M, g) is a space of constant curvature $1/\lambda^2$, which proves the first part of Theorem 4.5.1.

Now we decompose the infinitesimal isometry Z of $T^\lambda M$ and prove the second part of Theorem 4.5.1. There exists a unique vector field S on $T^\lambda M$, called the geodesic spray on $T^\lambda M$, such that

$$(\pi|_{T^\lambda M})_*(S_Y) = Y, \quad (K|_{TT^\lambda M})(S_Y) = 0 \quad \text{for any } Y \in T^\lambda M.$$

Since the mapping π_Q is an isometry, Theorem E in [14] says that $\lambda \cdot B(e_2)$, which is the lift of S , is an infinitesimal isometry of $SO(M)$. Indeed, we can see

$$(\pi|_{T^\lambda M})_* (\{(\pi_Q)_*(\lambda B(e_2))\}_Y) = Y$$

for each Y in $T^\lambda M$. It should be noted that this formula holds for M with $n = \dim M \geq 1$. In fact, for any Y in $T^\lambda M$, there are tangent vectors Y_1, \dots, Y_{n-1} in $T_{\pi(Y)}M$ such that $(Y_1, \dots, Y_{n-1}, \lambda^{-1}Y)$ is in $SO(M)$. Set $u = (Y_1, \dots, Y_{n-1}, \lambda^{-1}Y)$. Then we have

$$(\pi|_{T^\lambda M})_* (\{(\pi_Q)_*(\lambda B(e_n))\}_Y) = ((\pi|_{T^\lambda M}) \circ \pi_Q)_*(\lambda B(e_n)_u) = (\pi_P)_*(\lambda B(e_n)_u) = Y,$$

and hence

$$(4.5.3) \quad S^{LQ} = S^{HQ} = \lambda \cdot B(e_n), \quad n = \dim M.$$

It then follows that both

$$B_1 := \frac{1}{\lambda}[J, S] = (\pi_Q)_*(B(e_1)) \quad \text{and} \quad B_2 := \frac{1}{\lambda}S = (\pi_Q)_*(B(e_2))$$

are in $i(T^\lambda M, g^S)$. Since W_l is horizontal, there exist functions b^1_l and b^2_l on $T^\lambda M$ such that $W_l = b^1_l B_1 + b^2_l B_2$. We show that both b^1_l and b^2_l are constant on $T^\lambda M$. In fact, for $m = 1, 2$, we have

$$0 = g^S(\nabla^S_J W_l, B_m) + g^S(J, \nabla^S_{B_m} W_l) = \delta_{1m}(Jb^1_l) + \delta_{2m}(Jb^2_l),$$

from which we get

$$(4.5.4) \quad Jb^m_l = 0.$$

For arbitrary vector fields Y and Y' on $T^\lambda M$, we have

$$\begin{aligned} 0 &= g^S(\nabla^S_Y W_l, Y') + g^S(Y, \nabla^S_{Y'} W_l) \\ &= g^S(\nabla^S_Y (b^1_l B_1 + b^2_l B_2), Y') + g^S(Y, \nabla^S_{Y'} (b^1_l B_1 + b^2_l B_2)) \\ &= (Yb^1_l)g^S(B_1, Y') + (Yb^2_l)g^S(B_2, Y') + (Y'b^1_l)g^S(Y, B_1) + (Y'b^2_l)g^S(Y, B_2). \end{aligned}$$

Setting $Y = Y' = B_1$ (resp. $Y = Y' = B_2$) in the formulas above, we get

$$(4.5.5) \quad B_1 b^1_l = 0 \quad (\text{resp. } B_2 b^2_l = 0).$$

Moreover, we have

$$\begin{aligned} 0 &= g^S(\nabla^S_Y W_{l+1}, Y') + g^S(Y, \nabla^S_{Y'} W_{l+1}) \\ &= g^S(\nabla^S_Y [J, W_l], Y') + g^S(Y, \nabla^S_{Y'} [J, W_l]) \\ &= g^S(\nabla^S_Y (b^2_l B_1 - b^1_l B_2), Y') + g^S(Y, \nabla^S_{Y'} (b^2_l B_1 - b^1_l B_2)) \\ &= (Yb^2_l)g^S(B_1, Y') - (Yb^1_l)g^S(B_2, Y') + (Y'b^2_l)g^S(B_1, Y) - (Y'b^1_l)g^S(B_2, Y). \end{aligned}$$

Setting $Y = Y' = B_1$ (resp. $Y = Y' = B_2$) in the formulas above, we get

$$(4.5.6) \quad B_1 b^2_l = 0 \quad (\text{resp. } B_2 b^1_l = 0).$$

These formulas (4.5.4), (4.5.5) and (4.5.6) imply that both b^1_l and b^2_l are constant on $T^\lambda M$, and hence $W_l = (\pi_Q)_*(B(b^1_l e_1 + b^2_l e_2))$.

Setting $Z' = Z - W_4$, we have

$$\begin{aligned}
[J, Z'] &= [J, Z - W_4] = [J, Z] - [J, W_4] \\
&= W_1 - [J, [J, [J, [J, [J, Z]]]] \\
&= W_1 - [J, [J, [J, [J, W_1]]] \\
&= W_1 - (\pi_Q)_*([A_1^*, [A_1^*, [A_1^*, [A_1^*, B(b^1_1 e_1 + b^2_1 e_2)]]]]) \\
&= W_1 - (\pi_Q)_*(B((A_1 \cdot A_1 \cdot A_1 \cdot A_1)(b^1_1 e_1 + b^2_1 e_2))) \\
&= W_1 - (\pi_Q)_*(B((b^1_1 e_1 + b^2_1 e_2))) \\
&= W_1 - W_1 = 0,
\end{aligned}$$

which implies that Z' is a fiber preserving infinitesimal isometry of $T^\lambda M$. It follows that there exist X in $\mathfrak{i}(M, g)$ and ψ in $\mathfrak{D}^2(M)_0$ such that $Z' = X^L + \psi^L$. Hence we decompose Z as

$$Z = W_4 + Z' = \alpha \cdot S + \beta \cdot [J, S] + X^L + \psi^L,$$

where $\alpha = \lambda^{-2} g^S([J, [J, Z]], S)$ and $\beta = \lambda^{-2} g^S([J, [J, Z]], [J, S])$.

The following formulas for the bracket products are proved in the same manner as in [13].

LEMMA 4.5.4. *Let (M, g) be a connected, orientable two-dimensional Riemannian manifold and λ a positive number. Then for $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$ it holds that*

$$[X^L, Y^L] = [X, Y]^L, \quad [\phi^L, \psi^L] = 0, \quad [X^L, \phi^L] = 0.$$

Furthermore, if (M, g) is a space of constant curvature $1/\lambda^2$, then for $m = 1, 2$, it holds that

$$[B_1, B_2] = -\frac{1}{\lambda^2} J, \quad [X^L, B_m] = 0, \quad [J, B_m] = \delta_{1m} B_1 - \delta_{2m} B_2.$$

Accordingly, these facts and Theorem 3.1.2 lead us to the second part of Theorem 4.5.1.

5. Appendix

5.1. Geodesics and infinitesimal isometries of the tangent sphere bundles over space forms

In this section, we prove an extended version of Theorem 3.1.1 for the tangent sphere bundles over space forms M and characterize the geodesics in the total spaces $T^\lambda M$ in terms of the vector fields along the curves in the base space M satisfying appropriate properties. Let $(T^\lambda M, g^S)$ be the tangent sphere bundle with Sasaki metric g^S over a Riemannian manifold (M, g) . If $(M(k), g)$ is a space of constant curvature k , then the geodesic spray ξ of $(T^\lambda M(k), g^S)$ is an infinitesimal isometry if and only if $k = 1/\lambda^2$ (Tanno [14]). It is important to note that ξ is not of fiber preserving. When $k \neq 1/\lambda^2$, we obtain the following:

THEOREM 5.1.1 ([4]). *Let $(T^\lambda M(k), g^S)$ be the tangent sphere bundle over a space of constant curvature k . If $k \neq 1/\lambda^2$, then any infinitesimal isometry of $(T^\lambda M(k), g^S)$ is of fiber preserving.*

From Theorem 3.1.1, every infinitesimal isometry Z of the tangent sphere bundle $(T^\lambda M, g^S)$ over a Riemannian manifold (M, g) can be extended to an infinitesimal isometry of the tangent bundle (TM, g^S) if Z is of fiber preserving. So, by Theorem 5.1.1 we have the following:

THEOREM 5.1.2 ([4]). *Let $(M(k), g)$ be a space of constant curvature k . Every infinitesimal isometry of the tangent sphere bundle $(T^\lambda M(k), g^S)$ can be extended to an infinitesimal isometry of the tangent bundle $(TM(k), g^S)$ if $k \neq 1/\lambda^2$.*

To prove Theorem 5.1.1 we study some properties of geodesics in $(T^\lambda M(k), g^S)$. The basic references of the geometry on the tangent bundles are Dombrowski [2] and Sasaki [12].

Let $(T^\lambda M, g^S)$ be the tangent sphere bundle with Sasaki metric g^S over a Riemannian manifold (M, g) . By π we denote the projection from $T^\lambda M$ to M . Let

$$\tilde{C} = \{(x(\sigma), y(\sigma)); 0 \leq \sigma \leq l\}$$

be a curve in $(T^\lambda M, g^S)$ with the arc-length parameter σ , where

$$y(\sigma) \in T_{x(\sigma)}M, \quad g(y(\sigma), y(\sigma)) = \lambda^2$$

It is a geodesic if and only if

$$(5.1.1) \quad (\nabla_{\dot{x}}\dot{x})(\sigma) = -R(y(\sigma), (\nabla_{\dot{x}}y)(\sigma))\dot{x}(\sigma), \quad (\nabla_{\dot{x}}\nabla_{\dot{x}}y)(\sigma) = \rho(\sigma)y(\sigma)$$

hold for some function $\rho(\sigma)$, where ∇ and R denote the Riemannian connection and Riemannian curvature tensor of (M, g) , respectively, and $\dot{x}(\sigma) = dx(\sigma)/d\sigma$ (Sasaki [12, II, p. 152]).

Geodesics on the unit tangent bundle $(T^1S^2(1), g^S)$ over the unit two-sphere $(S^2(1), g)$ were studied by Klingenberg and Sasaki ([3]). Nagy [9] studied the geodesics in the unit tangent bundle over space forms by using the generalized Frenet formulas. For any space form $(M(k), g)$, we have the following:

THEOREM 5.1.3 ([4]). *Let λ be a positive number, $(T^\lambda M(k), g^S)$ the tangent sphere bundle over a space of constant curvature k , and let*

$$\tilde{C} = \{(x(\sigma), y(\sigma)); 0 \leq \sigma \leq l\}$$

be a geodesic with the arc-length parameter σ in $(T^\lambda M(k), g^S)$. By $C = \{x(\sigma)\}$ we denote the image of the projection $\pi\tilde{C}$ of \tilde{C} . Then $\|\dot{x}\|^2 = 1 - c^2$ is constant, where $0 \leq |c| \leq 1$.

(i) *If $|c| = 1$, that is, C reduces to a point, then \tilde{C} is a (piece of) great circle in a fiber and y is rotated in a two-plane at $x(0)$.*

(ii) *If $0 < |c| < 1$, then we have the following:*

(ii-a-1) *The geodesic curvature κ_g of C is constant.*

(ii-a-2) *C satisfies*

$$(5.1.2) \quad \nabla_{\dot{x}}\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} = -\lambda^2k^2c^2\nabla_{\dot{x}}\dot{x}.$$

(ii-b-1) *If $k = 0$, then $\kappa_g = 0$ and we have the parallel orthonormal vector fields $\{E_1, E_2\}$ along C such that*

$$(5.1.3) \quad y(\sigma) = \lambda \cos\left(\frac{c\sigma}{\lambda}\right) \cdot E_1(\sigma) + \lambda \sin\left(\frac{c\sigma}{\lambda}\right) \cdot E_2(\sigma).$$

(ii-b-2) *If $k \neq 0$ and $\kappa_g = 0$, then we have the parallel orthonormal vector fields $\{E_1, E_2, \dot{x}/\|\dot{x}\|\}$ along C such that y is of the form (5.1.3).*

(ii-b-3) If $k \neq 0$ and $\kappa_g \neq 0$, then we have the parallel orthonormal vector fields $\{E_1, E_2\}$ along C such that y is of the form (5.1.3).

(ii-c) The angle $\theta(\sigma)$ between $y(\sigma)$ and $\dot{x}(\sigma)$ is given by

$$(5.1.4) \quad \cos \theta(\sigma) = \frac{\alpha}{\lambda \sqrt{1-c^2}} \sin \left[\frac{c(1-\lambda^2 k)}{\lambda} \cdot \sigma + \beta \right],$$

where α and β are constant. α^2 is given by (5.1.9) and $\alpha = 0$ for (ii-b-2).

(iii) If $c = 0$, then $C = \{x(\sigma)\}$ is a geodesic with the arc-length parameter σ , and y is a parallel vector field along C .

PROOF. By (5.1.1) the equations of geodesic in $(T^\lambda M(k), g^S)$ are given by

$$(5.1.5) \quad \nabla_{\dot{x}} \dot{x} = -kby + ka \nabla_{\dot{x}} y, \quad \nabla_{\dot{x}} \nabla_{\dot{x}} y = \rho y,$$

where we put

$$(5.1.6) \quad a = a(\sigma) = g(\dot{x}, y), \quad b = b(\sigma) = g(\dot{x}, \nabla_{\dot{x}} y).$$

We sometimes omit the parameter σ from the expression for simplicity. We put

$$(5.1.7) \quad c^2 = c^2(\sigma) = g(\nabla_{\dot{x}} y, \nabla_{\dot{x}} y).$$

By $g(y, y) = \lambda^2$, we have $g(y, \nabla_{\dot{x}} y) = 0$ and

$$g(\nabla_{\dot{x}} y, \nabla_{\dot{x}} y) + g(y, \nabla_{\dot{x}} \nabla_{\dot{x}} y) = 0,$$

in other words, we get $c^2 + \lambda^2 \rho = 0$. Differentiating (5.1.7) and using (5.1.5)₂ we see that c is constant.

Let X be a tangent vector at a point of $(M(k), g)$. By X^H and X^V , we denote the horizontal lift and the vertical lift of X to $(TM(k), g^S)$, respectively. Since the tangent vector field of \tilde{C} is expressed as

$$\frac{d\tilde{C}}{d\sigma} = \left(\frac{dx}{d\sigma}, \frac{dy}{d\sigma} \right) = \dot{x}^H + (\nabla_{\dot{x}} y)^V,$$

we have $1 = \|d\tilde{C}/d\sigma\|^2 = \|\dot{x}\|^2 + c^2$. Therefore $\|\dot{x}\|^2 = 1 - c^2$ is constant, and the parameter σ of $C = \{x(\sigma)\}$ is proportional to the arc-length.

If $|c| = 1$, that is, $\|\dot{x}\| = 0$, then \tilde{C} is a geodesic in a fiber. Since each fiber is totally geodesic and isometric to the unit $(m-1)$ -sphere, it is a (piece of) great circle. So, y is expressed as $y(\sigma) = \lambda \cos \sigma \cdot e_1 + \lambda \sin \sigma \cdot e_2$ for some orthonormal vectors $\{e_1, e_2\}$ at $x(0)$.

Next we assume $0 < |c| < 1$. Calculating $\dot{a} = \nabla_{\dot{x}}a$ and $\dot{b} = \nabla_{\dot{x}}b$, we obtain

$$(5.1.8) \quad \dot{a} = (1 - \lambda^2 k)b, \quad \dot{b} = (k - \lambda^{-2})ac^2.$$

Operating $\nabla_{\dot{x}}$ to the first equation of (5.1.5) twice, we obtain (5.1.2). By (5.1.5) we see

$$\|\nabla_{\dot{x}}\dot{x}\|^2 = k^2(c^2a^2 + \lambda^2b^2).$$

By (5.1.8) we can show that $c^2a^2 + \lambda^2b^2$ is constant, and the geodesic curvature κ_g of C given by

$$\kappa_g^2 = \frac{k^2(c^2a^2 + \lambda^2b^2)}{(1 - c^2)^2}$$

is also constant. This proves (ii-a-1) and the first part of (ii-b-1). By using (5.1.8) again, we have

$$a(\sigma) = \alpha \sin \left[\frac{c(1 - \lambda^2 k)}{\lambda} \cdot \sigma + \beta \right], \quad b(\sigma) = \frac{c\alpha}{\lambda} \cos \left[\frac{c(1 - \lambda^2 k)}{\lambda} \cdot \sigma + \beta \right],$$

where α and β are constant functions. Here, α^2 is expressed as

$$(5.1.9) \quad \alpha^2 = \frac{c^2a^2 + \lambda^2b^2}{c^2} \quad \left(\alpha^2 = \frac{(1 - c^2)^2 \kappa_g^2}{c^2 k^2}, \quad \text{if } k \neq 0 \right).$$

The angle $\theta(\sigma)$ between $y(\sigma)$ and $\dot{x}(\sigma)$ is given by

$$\cos \theta(\sigma) = g(\dot{x}/\|\dot{x}\|, y) = \frac{a(\sigma)}{\lambda\sqrt{1 - c^2}}$$

and we obtain (5.1.4).

Now we define the vector fields E_1 and E_2 along C by

$$E_1(\sigma) = \frac{1}{\lambda} \cos \left(\frac{c\sigma}{\lambda} \right) \cdot y(\sigma) - \frac{1}{c} \sin \left(\frac{c\sigma}{\lambda} \right) \cdot (\nabla_{\dot{x}}y)(\sigma),$$

$$E_2(\sigma) = \frac{1}{\lambda} \sin \left(\frac{c\sigma}{\lambda} \right) \cdot y(\sigma) + \frac{1}{c} \cos \left(\frac{c\sigma}{\lambda} \right) \cdot (\nabla_{\dot{x}}y)(\sigma).$$

Then E_1 and E_2 are parallel orthonormal vector fields along C , and define a parallel two-plane field Π along C . y is rotated in Π as

$$y(\sigma) = \lambda \cos \left(\frac{c\sigma}{\lambda} \right) \cdot E_1(\sigma) + \lambda \sin \left(\frac{c\sigma}{\lambda} \right) \cdot E_2(\sigma).$$

This proves (ii-b-1) and (ii-b-3). If $k \neq 0$ and $\kappa_g = 0$, then $a = b = 0$. So, $\{E_1, E_2, \dot{x}/\|\dot{x}\|\}$ are orthonormal and we have (ii-b-2).

Finally, if $c = 0$, then we have $\nabla_{\dot{x}}y = 0$, $\nabla_{\dot{x}}\dot{x} = 0$ and (iii). We proved Theorem 5.1.3.

The converse of Theorem 5.1.3 is given by

THEOREM 5.1.4 ([4]). *Let $(M(k), g)$ be a space of constant curvature k . Let $C = \{x(\sigma); 0 \leq \sigma \leq l\}$ be a curve of constant geodesic curvature κ_g with $\|\dot{x}\|^2 = 1 - c^2$, $0 < |c| < 1$. Assume that C satisfies*

$$(5.1.10) \quad \nabla_{\dot{x}} \nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x} = -\lambda^2 k^2 c^2 \nabla_{\dot{x}} \dot{x}.$$

(ii*-1) *When $k = 0$, we assume $\kappa_g = 0$. Let $\{E_1, E_2\}$ be parallel orthonormal vector fields along C and define the vector field y along C by*

$$(5.1.11) \quad y(\sigma) = \lambda \cos\left(\frac{c\sigma}{\lambda}\right) \cdot E_1(\sigma) + \lambda \sin\left(\frac{c\sigma}{\lambda}\right) \cdot E_2(\sigma).$$

Then $\tilde{C} = \{(x(\sigma), y(\sigma))\}$ is a geodesic in $(T^\lambda M(k), g^S)$.

(ii*-2) *When $k \neq 0$ and $\kappa_g = 0$, let $\{E_1, E_2, \dot{x}/\|\dot{x}\|\}$ be parallel orthonormal vector fields along C . Define y by (5.1.11). Then $\tilde{C} = \{(x(\sigma), y(\sigma))\}$ is a geodesic in $(T^\lambda M(k), g^S)$.*

(ii*-3) *When $k \neq 0$ and $\kappa_g \neq 0$, let $e_1 = (\nabla_{\dot{x}} \dot{x})(0)/(1 - c^2)\kappa_g$ and*

$$e_2 = \frac{1}{\lambda k c (1 - c^2) \kappa_g} \cdot (\nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x})(0).$$

Define $\{E_1, E_2\}$ along C by the parallel translation of e_1 and e_2 . Next we define y by

$$(5.1.12) \quad y(\sigma) = \lambda \cos\left(\frac{c\sigma}{\lambda} + \gamma\right) \cdot E_1(\sigma) + \lambda \sin\left(\frac{c\sigma}{\lambda} + \gamma\right) \cdot E_2(\sigma)$$

for a constant γ . Then $\tilde{C} = \{(x(\sigma), y(\sigma))\}$ is a geodesic in $(T^\lambda M(k), g^S)$.

PROOF. First we prove (ii*-1) and (ii*-2). By $\kappa_g = 0$ we have $\nabla_{\dot{x}} \dot{x} = 0$. By (3.2) we obtain $\nabla_{\dot{x}} \nabla_{\dot{x}} y = -(c/\lambda)^2 y$ and, using $g(\dot{x}, y) = g(\dot{x}, \nabla_{\dot{x}} y) = 0$ for (ii*-2), we have (5.1.5).

Next, we show (ii*-3). Since $\{(1 - c^2)\kappa_g\}^{-1} \nabla_{\dot{x}} \dot{x}$ is a unit vector field along C , we see that $\nabla_{\dot{x}} \dot{x}$ and $\nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x}$ are orthogonal. Using (5.1.10), we obtain

$$g(\nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x}, \nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x}) = \lambda^2 k^2 c^2 g(\nabla_{\dot{x}} \dot{x}, \nabla_{\dot{x}} \dot{x})$$

and $\{\lambda k c (1 - c^2) \kappa_g\}^{-1} \nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x}$ is a unit vector field along C . Therefore, $\{e_1, e_2\}$ and hence the parallel vector fields $\{E_1, E_2\}$ are orthonormal. Then $\{(x(\sigma), y(\sigma))\}$ defined by (5.1.12) satisfies the second equation of (5.1.5). The differential equation (5.1.10) yields

$$\frac{(\nabla_{\dot{x}} \dot{x})(\sigma)}{(1 - c^2) \kappa_g} = \cos \lambda k c \sigma \cdot E_1(\sigma) + \sin \lambda k c \sigma \cdot E_2(\sigma).$$

By (5.1.12) and $c^{-1}\nabla_{\dot{x}}y = -\sin(c\sigma/\lambda + \gamma)E_1 + \cos(c\sigma/\lambda + \gamma)E_2$, we obtain

$$(5.1.13) \quad \begin{aligned} \nabla_{\dot{x}}\dot{x} &= \frac{(1-c^2)\kappa_g}{\lambda} \cos\left[\frac{c(1-\lambda^2k)}{\lambda} \cdot \sigma + \gamma\right] \cdot y \\ &\quad - \frac{(1-c^2)\kappa_g}{c} \sin\left[\frac{c(1-\lambda^2k)}{\lambda} \cdot \sigma + \gamma\right] \cdot \nabla_{\dot{x}}y. \end{aligned}$$

We put $a = g(\dot{x}, y)$ and $b = g(\dot{x}, \nabla_{\dot{x}}y)$. It follows from (5.1.13) and $\nabla_{\dot{x}}\nabla_{\dot{x}}y = -(c/\lambda)^2y$ that

$$\begin{aligned} \dot{a} &= \lambda(1-c^2)\kappa_g \cos\left[\frac{c(1-\lambda^2k)}{\lambda} \cdot \sigma + \gamma\right] + b, \\ \dot{b} &= -c(1-c^2)\kappa_g \sin\left[\frac{c(1-\lambda^2k)}{\lambda} \cdot \sigma + \gamma\right] - \left(\frac{c}{\lambda}\right)^2 a. \end{aligned}$$

Solving the differential equations above with the initial condition

$$a(0) = g(\dot{x}(0), y(0)) = -\frac{(1-c^2)\kappa_g \sin \gamma}{ck},$$

we get

$$\begin{aligned} a &= -\frac{(1-c^2)\kappa_g}{ck} \sin\left[\frac{c(1-\lambda^2k)}{\lambda} \cdot \sigma + \gamma\right], \\ b &= -\frac{(1-c^2)\kappa_g}{k\lambda} \cos\left[\frac{c(1-\lambda^2k)}{\lambda} \cdot \sigma + \gamma\right]. \end{aligned}$$

So, (5.1.13) is rewritten as $\nabla_{\dot{x}}\dot{x} = -kby + ka\nabla_{\dot{x}}y$. This completes the proof of Theorem 5.1.4.

The converses of the two cases (i) and (iii) of Theorem 5.1.3 are trivial.

Now we prove Theorem 5.1.1. Assume that an infinitesimal isometry Z of $(T^\lambda M(k), g^S)$ is not of fiber preserving. Let $\{\phi_t\}$ be a (local) one-parameter group of local isometries generated by Z . Since each fiber is totally geodesic in $(T^\lambda M(k), g^S)$ and isometric to the unit $(m-1)$ -sphere, we can choose a great circle $\tilde{C} = \{(x(\sigma), y(\sigma)); 0 \leq \sigma \leq 2\pi\}$ of length 2π in a fiber and a positive number $\varepsilon > 0$, such that $\phi_t\tilde{C}$ is not contained in a fiber for t with $0 < t < \varepsilon$. Here we can assume that the domain of definition of ϕ_t contains the fiber containing \tilde{C} . In this case, for small t with $0 < t < \varepsilon$, $C_t = \pi\phi_t\tilde{C}$ is a small closed curve, and it can not be a geodesic in $(M(k), g)$. We have $0 < |c_t| < 1$. By Theorem 5.1.13, (ii-b-1), and (5.1.9) we see that $k \neq 0$ and $\alpha_t \neq 0$. It follows from (ii-c) that the angle $\theta_t(\sigma)$ between $y_t(\sigma)$ and C_t is given by

$$\cos \theta_t(\sigma) = \frac{\alpha_t}{\lambda\sqrt{1-c_t^2}} \sin\left[\frac{c_t(1-\lambda^2k)}{\lambda} \cdot \sigma + \beta_t\right].$$

As $t \rightarrow 0$, we have $|c_t| \rightarrow 1$. So we have small t such that $(1 - \lambda^2 k)c_t/\lambda$ is not an integer. This means $\theta_t(\sigma) \neq \theta_t(\sigma + 2\pi)$ for such t . This contradicts the fact that \tilde{C}_t is a closed geodesic for any t . We proved Theorem 5.1.1.

5.2. Infinitesimal isometries of frame bundles

In this section, using Theorem 4.1.1 and an extended version of the results in [13], we prove Theorem E in [14] and Theorem 3.1.2 for orientable Riemannian manifolds.

We first explain the extended version of the results of Takagi and Yawata in [13].

THEOREM 5.2.1. *Let (M, g) be a connected, orientable Riemannian manifold of dimension $n \geq 2$ and λ a non-zero number.*

(i) *For every $Y \in \mathfrak{i}(M, g)$, $\phi \in \mathfrak{D}^2(M)_0$ and $A \in \mathfrak{o}(n)$, Y^{L_P} , ϕ^{L_P} and A^* are all infinitesimal isometries of $(SO(M), G)$.*

(ii) *If $B(\xi)$ is an infinitesimal isometry for some non-zero $\xi \in \mathbf{R}^n$, then M is a space of constant curvature $1/\lambda^2$.*

Conversely, if M is a space of constant curvature $1/\lambda^2$, then $B(\xi)$ is an infinitesimal isometry for any $\xi \in \mathbf{R}^n$.

(iii) *Given an infinitesimal isometry X of $(SO(M), G)$, we have the following:*

(iii-1) *If M is complete, then there exist unique $Y \in \mathfrak{i}(M, g)$, $\phi \in \mathfrak{D}^2(M)_0$, $A \in \mathfrak{o}(n)$ and $\xi \in \mathbf{R}^n$ such that $X = Y^{L_P} + \phi^{L_P} + A^* + B(\xi)$, except when the dimension of M is 2, 3, 4 or 8.*

(iii-2) *If X is of fiber preserving on P , then there exist unique $Y \in \mathfrak{i}(M, g)$, $\phi \in \mathfrak{D}^2(M)_0$ and $A \in \mathfrak{o}(n)$ such that $X = Y^{L_P} + \phi^{L_P} + A^*$, when the dimension of M is greter than two.*

(iv) *For every $Y, Z \in \mathfrak{i}(M, g)$, $\phi, \psi \in \mathfrak{D}^2(M)_0$ and $A, C \in \mathfrak{o}(n)$, we have $[\nabla Y, \phi] \in \mathfrak{D}^2(M)_0$, and*

$$[A^*, C^*] = [A, C]^*, \quad [\phi^{L_P}, \psi^{L_P}] = -[\phi, \psi]^{L_P}, \quad [Y^{L_P}, Z^{L_P}] = [Y, Z]^{L_P},$$

$$[Y^{L_P}, \phi^{L_P}] = -[DY, \phi]^{L_P}, \quad [Y^{L_P}, A^*] = 0, \quad [\phi^{L_P}, A^*] = 0.$$

In particular, if M is a space of constant curvature $1/\lambda^2$, then $\mathfrak{D}^2(M)_0 = \{0\}$, and

$$[B(\xi), B(\eta)] = -\frac{1}{\lambda^2}(\xi \wedge \eta)^*, \quad [A^*, B(\xi)] = B(A\xi), \quad [Y^{L_P}, B(\xi)] = 0$$

for all $\xi, \eta \in \mathbf{R}^n$, when $\dim M \geq 3$.

Theorem 5.2.1 was proved by Takagi and Yawata in [13] when $\lambda = \sqrt{2}$. Since their method of the proof can be directly extended for any λ , we omit the proof of Theorem 5.2.1.

Considering the Jacobi fields along geodesics in (M, g) , Tanno [14] proved the following theorem:

THEOREM 5.2.2 (Tanno [14]). *Let (M, g) be a Riemannian manifold. The geodesic spray on $T^\lambda M$ is an infinitesimal isometry of $(T^\lambda M, g^S)$ if and only if (M, g) is a space of constant curvature $1/\lambda^2$.*

As an application of Theorem 4.1.1, we provide another proof of this theorem. Our proof is based on Theorems 4.1.1 and 5.2.1 as follows.

PROOF. We may assume that M is connected and orientable. Suppose that the geodesic spray S is an infinitesimal isometry of $T^\lambda M$. Then, from (4.5.3) and (ii) of Theorem 4.2.1, we know that

$$G(D_{A_i^*} B(e_n), B(e_j)) + G(A_i^*, D_{B(e_j)} B(e_n)) = 0 \quad \text{for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n.$$

Lemma 4.3.4 with the formula above implies that

$$(5.2.1) \quad \frac{\lambda^2}{2} \langle \Omega(B(e_n), B(e_j)), A_i \rangle + \langle A_i e_n, e_j \rangle - \frac{\lambda^2}{2} \langle \Omega(B(e_j), B(e_n)), A_i \rangle = 0,$$

where we get

$$(5.2.2) \quad \begin{aligned} \langle \Omega(B(e_n), B(e_j)), A_i \rangle &= \sum_{k=1}^n (A_i e_k, \Omega(B(e_n), B(e_j)) e_k) \\ &= -2(\Omega(B(e_n), B(e_j)) e_i, e_n). \end{aligned}$$

Applying (5.2.2) to (5.2.1), we have $(2\Omega(B(e_n), B(e_j)) e_i, e_n) = \delta_{ij}/\lambda^2$. Setting $i = j$ here, we have

$$(5.2.3) \quad (2\Omega(B(e_n), B(e_i)) e_i, e_n) = 1/\lambda^2.$$

For any $u \in SO(M)$, we put $u = (Y_1, \dots, Y_n)$. Since we know $(\pi_P)_*(B(e_k)_u) = Y_k$, that is,

$\{Y_k^{HP}\}_u = B(e_k)_u$ for $k = 1, \dots, n$, we have by (5.2.3) that

$$\begin{aligned}
g(R(Y_n, Y_k)Y_k, Y_n) &= g(u \cdot 2\Omega(Y_n^{HP}_u, Y_k^{HP}_u) \cdot \theta(Y_k^{HP}_u), Y_n) \\
&= (2\Omega(Y_n^{HP}_u, Y_k^{HP}_u) \cdot \theta(Y_k^{HP}_u), u^{-1}Y_n) \\
&= (2\Omega(B(e_n)_u, B(e_k)_u) \cdot \theta(B(e_k)_u), e_n) \\
&= (2\Omega(B(e_n)_u, B(e_k)_u)e_k, e_n) \\
&= \lambda^{-2},
\end{aligned}$$

which implies that (M, g) is a space of constant curvature $1/\lambda^2$.

Conversely, if (M, g) is a space of constant curvature $1/\lambda^2$, then, from (ii) of Theorem 5.2.1, $B(e_n)$ is an infinitesimal isometry of $(SO(M), G)$. Since we know that $B(e_n)$ preserves the fibers of Q , the geodesic spray $S = \lambda(\pi_Q)_*(B(e_n))$ is an infinitesimal isometry of $(T^\lambda M, g^S)$. This completes the proof of Theorem 5.2.2.

REMARK. In this case, the lift S^{LQ} of the geodesic spray S is a fiber preserving infinitesimal isometry of (Q, G) .

Next, we prove Theorem 3.1.2 in a different way. Assume that (M, g) is orientable and the dimension of M is greater than two. We determine the fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$, which provides another proof of Theorem 3.1.2. In what follows, we only use (iii-2) of Theorem 5.2.1, Lemma 4.4.2 and the fact that the lift Z^{LQ} of a fiber preserving infinitesimal isometry Z of $T^\lambda M$ is also a fiber preserving infinitesimal isometry of (P, G) (see the first part of Section 4.4 and (4.4.13)). From (iii-2) of Theorem 5.2.1, there exist unique $X \in \mathfrak{i}(M, g)$, $\phi \in \mathfrak{D}^2(M)_0$ and $A \in \mathfrak{o}(n)$ such that $Z^{LQ} = X^{LP} + \phi^{LP} + A^*$. By Lemma 4.4.2, we have

$$Z^{LQ} = X^{LP} + \phi^{LP} + A^* = (X^L)^{LQ} + (\phi^L)^{LQ} + A^* = (X^L + \phi^L)^{LQ} + A^*,$$

and hence

$$A^* = Z^{LQ} - (X^L + \phi^L)^{LQ} = (Z - X^L - \phi^L)^{LQ}.$$

This implies that A^* is the lift of the vector field $Z - X^L - \phi^L$ on $T^\lambda M$. Put $W = Z - X^L - \phi^L$. Let Y be an arbitrary point in $T^\lambda M$ and u an arbitrary point in $(\pi_Q)^{-1}(Y)$. Then there exist tangent vectors $Y_1, \dots, Y_{n-1} \in T_{\pi(Y)}M$ such that $u = (Y_1, \dots, Y_{n-1}, \lambda^{-1}Y)$.

Setting $\exp tA = (a^i_j(t))$ and $Y_n = \lambda^{-1}Y$, we have

$$\begin{aligned} W_Y &= (\pi_Q)_*(W^{LQ_u}) = (\pi_Q)_*(A^*_u) = (\pi_Q)_*\left(\frac{d}{dt}\{(R_{\exp tA})(u)\}\Big|_{t=0}\right) \\ &= \lambda\frac{d}{dt}\left\{\sum_{k=1}^n a^k_n(t)Y_k\right\}\Big|_{t=0} = \lambda\sum_{k=1}^n \dot{a}^k_n(0)I_Y^{-1}(Y_k). \end{aligned}$$

Since W_Y is independent of the choice of $u \in (\pi_Q)^{-1}(Y)$, it is necessary to have $\dot{a}^k_n(0) = 0$ for $k = 1, \dots, n-1$. Hence we see that A is in $\mathfrak{o}(n-1)$, and we have

$$W_Y = (\pi_Q)_*(W^{HQ_u}) = (\pi_Q)_*(W^{LQ_u}) = (\pi_Q)_*(A^*_u) = 0.$$

It follows that $Z = X^L + \phi^L$. Thus, we have determined all fiber preserving infinitesimal isometries of $T^\lambda M$, which provides another proof of Theorem 3.1.2.

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