ISSN 1343-9499

TOHOKU
MATHEMATICAL
PUBLICATIONS

Number 17

Model-theoretic studies on subsystems of second order arithmetic

by

Takeshi YAMAZAKI

September 2000

This series aims to publish material related to the activities of the Mathematical Institute of Tohoku University. This may include:

1. Theses submitted to the Institute by grantees of the degree of Doctor of Science.
2. Proceedings of symposia as well as lecture notes of the Institute.

A primary advantage of the series lies in quick and timely publication. Consequently, some of the material published here may very likely appear elsewhere in final form.

Tohoku Mathematical Publications

Mathematical Institute
Tohoku University
Sendai 980-8578, Japan

Model-theoretic studies on subsystems of second order arithmetic

A thesis presented

by

Takeshi Yamazaki

to

The Mathematical Institute

for the degree of

Doctor of Science

Tohoku University

Sendai, Japan

March 2000

Abstract

This research is motivated by the following theme originated with H. Friedman: very often, if a theorem τ of ordinary mathematics is proved from the "right" set existence axioms, τ is equivalent to those axioms over some weaker system in which τ itself is not provable. This theme is referred to as *Reverse mathematics*.

Here, we focus on three subsystems RCA_0, WKL_0 and ACA_0 of second order arithmetic and a second order system BTFA of 0-1 strings. By RCA_0, we mean the system of recursive comprehension axioms with Σ^0_1 induction. WKL_0 consists of RCA_0 plus weak König's lemma which asserts that every infinite 0-1 tree has a path. The first-order part of WKL_0 is the same as that of RCA_0. ACA_0 consists of RCA_0 plus arithmetical comprehension axioms. The first order part of ACA_0 is just first order Peano arithmetic PA. The acronym **BTFA** stands for base theory for feasible analysis. BTFA is conservative over Polynomial Time Computable Arithmetic PTCA with respect to the Π^0_2 sentences.

In chapter 2, we study models of $\mathsf{RCA}_0+\Pi^0_\infty$-BCT and WKL_0. Π^0_∞-BCT is a version of the Baire category theorem introduced by Brown and Simpson. We show the following conservation result: for any arithmetical formula $\varphi(X,Y)$, if WKL_0 or $\mathsf{RCA}_0 + \Pi^0_\infty$-BCT proves $\forall X \exists ! Y \varphi(X,Y)$, then so does RCA_0. Note that $\exists ! Z \psi(Z)$ means that there exists a unique Z satisfying $\psi(Z)$.

In chapter 3, we first show within RCA_0 that the existence of Haar measure on separable compact groups is equivalent to WKL_0. To prove the existence of Haar measure in WKL_0, we give a non-standard construction of Haar measure by using the self-embedding theorem for WKL_0. Next we show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for intuitionistic logic: any countable theory Γ in intuitionistic predicate logic has a Kripke model such that for any sentence φ, φ is forced in the model if and only if φ is intuitionistically deducible from Γ. Finally, we develop some basic real analysis within BTFA and show a version of the maximum principle is equivalent to some weak comprehension scheme.

Acknowledgments

A special word of gratitude and admiration is due to my supervisor, Professor Kazuyuki Tanaka, for his assistance, encouragement and patience, through thousands of constructive and stimulating discussions. My interest in foundations of mathematics, especially, my thesis topics, has been inspired by him. In addition to working with me on my thesis, he fostered a very lively atmosphere among the logic students, visitors and e-mail companions.

I would like to thank several expert logicians who gave me invaluable suggestions and comments on my thesis. First of all, I must thank Professor Steve Simpson for pointing out an error in an early version of my proof in Section 2.2, and kindly showing me how to fix it. Professor Fernando Ferreira gave me several useful comments on Section 3.3. I am also indebted to Professor Akito Tuboi for his advice to work on the problem in Section 3.2.

I have benefited from far more people than I can list here. I am also grateful to my parents, Yoshiaki and Sachiko Yamazaki, and my brother Satoshi for their encouragement during my study at Tohoku University.

Contents

0. Introduction

This thesis is a contribution to foundations of mathematics. Almost all of the problems studied in this thesis are motivated by the following core question: what are the appropriate axioms for mathematics?

Long ago, Hilbert and Bernays [2] pointed out that most or all of ordinary mathematics can be developed within the formal system Z_2 of second order arithmetic, which deal with sets of natural numbers as well as natural numbers. However, in many particular cases, the set existence axioms of Z_2 are very strong, including as they do the full comprehension scheme.

Subsequent investigations by Weyl and many others revealed that small subsystems of Z_2, employing much weaker set existence axioms, are sufficient for the development of the bulk of ordinary mathematics. We have in mind especially the following five subsystems (cf. [23]):

RCA_0. Here the acronym RCA stands for recursive comprehension axiom. Roughly speaking, the axioms of RCA_0 are only strong enough to prove the existence of recursive sets (though they do not rule out the existence of nonrecursive sets). It is strong enough to prove some of elementary facts about countable algebraic structures and continuous functions of a real variable.

WKL_0. This system consists of RCA_0 plus weak König's lemma (WKL) which states that every infinite 0-1 tree has a path. An equivalent statement to WKL is the compactness of the Cantor space. Although the first-order part of WKL_0 is the same as that of RCA_0, WKL_0 proves many theorems which RCA_0 does not, e.g., the Heine-Borel covering lemma, the existence of a prime ideal of countable commutative rings, the local existence theorem for solutions of ordinary differential equations, the Hahn-Banach theorem for separable Banach spaces, and so on. These results have important implications in the foundations of mathematics, especially related to Hilbert's program [22].

ACA_0. Here ACA stands for *arithmetical comprehension axiom*. The first order part of ACA_0 is just first order Peano arithmetic PA. ACA_0 permits a smooth theory of sequential convergence and isolates the same portion of mathematical practice which was identified as "predicative analysis" by Weyl in his famous monograph *Das Kontinum*.

ATR_0. Here ATR stands for *arithmetical transfinite recursion*. The principal axiom of ATR_0 says that arithmetical comprehension can be iterated along any countable well ordering. ATR_0 is just strong enough to accommodate the development of a good theory of countable well orderings, Borel sets, analytic sets, etc.

Π^1_1-CA_0. This is the system of Π^1_1 *comprehension*. It is properly stronger than ATR_0 and yields an improved theory of countable well orderings, etc. Both ATR_0 and Π^1_1-CA_0 have numerous mathematical consequences in the realms of algebra, analysis, classical descriptive theory, and countable combinatorics.

By further investigations into the core question, H. Friedman has revealed the the following theme: very often, if a theorem τ of ordinary mathematics is proved from the "right" set existence axioms, τ is equivalent to those axioms over some weaker system in which τ itself is not provable. This theme is known as *Reverse mathematics*. For example, we can prove within RCA_0 that the Bolzano-Weierstrass theorem is equivalent to arithmetical comprehension axiom, that is, ACA_0 is the right subsystem of Z_2 to prove Bolzano-Weierstrass theorem.

An important research direction for the future on reverse mathematics is to observe the theme of reverse mathematics over weaker base theories rather than RCA_0. Simpson-Smith [24] and Hatzikiriakou [13] study reverse mathematics over RCA^*_0, which is roughly RCA_0 minus Σ^0_1 induction plus Σ^0_0 induction plus exponentiation. Ferreira [8] proposed to develop Reverse Mathematics over BTFA or $\mathsf{BTFA}+\Sigma^b_\infty$-WKL, which are second order systems of 0-1 strings known to be conservative over Polynomial Time Computable Arithmetic PTCA with respect to the Π^0_2 sentences. It can be regarded as a modern analogy of Hilbert's program to examine what part of infinitistic mathematics can be reduced to feasible reasoning.

Chapter 1 is devoted to define the systems RCA_0, WKL_0, ACA_0 and BTFA. In Section 1.2, we study new variants of axioms of choice, which will not be used in the other chapters. So, the reader who is already familiar with the popular subsystems of second order arithmetic may skip directly to Chapters 2 and 3.

In Chapter 2, we do some model theoretic studies on $\mathsf{RCA}_0+\Pi^0_\infty$-BCT and WKL_0. Π^0_∞-BCT is a version of the Baire category theorem introduced by Brown-Simpson [4]. We show two conservation results inspired by the results due to Harrington [23] and Brown-Simpson [4].

Theorem. *Let $\varphi(X,Y)$ be an arithmetical formula with exactly the free variables shown.*

(1) *If WKL_0 proves $\forall X\exists! Y\varphi(X,Y)$, then so does RCA_0.*

(2) *If* $\mathsf{RCA}_0 + \Pi^0_\infty$-BCT *proves* $\forall X \exists! Y \varphi(X, Y)$*, then so does* RCA_0.

Note that $\exists! Z \psi(Z)$ means that there exists a unique Z satisfying $\psi(Z)$. Part (1) answers Tanaka's problem [25]. We owe Professor S. G. Simpson (by private communications) a great deal for constructing the present proof of (1).

In Section 3.1, we show within RCA_0 that the existence of Haar measure on separable compact group is equivalent to WKL_0. In Section 3.2, we show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for intuitionistic logic: any countable theory Γ of intuitionistic predicate logic has a Kripke model such that for any sentence φ, φ is forced in the model if and only if φ is intuitionistically deducible from Γ. In Section 3.3, we show that the intermediate value theorem on $[0, 1]$ is provable in BTFA, and a version of the maximum principle is equivalent to Σ^b_1-comprehension axiom within BTFA.

The works in Section 2.1 and 3.1 will appear in [30] and [27], respectively. Other works in this thesis have been presented in several workshops and preprints.

1 Subsystems of second order arithmetic

In this chapter, we give rigorous definitions of the systems treated in this thesis. In Section 1.2, we also introduce some finite versions of axioms of choice, and compare them with popular axioms.

1.1 RCA_0, WKL_0 and ACA_0

The language $\mathcal{L}_2$ of second-order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$. Numerical terms are built up from numerical variables and constant symbols $0, 1$ by means of binary operations $+$ and $\cdot$. Atomic formulas are $s = t$, $s < t$ and $s \in X$, where s and t are numerical terms. *Bounded* (Σ^0_0 or Π^0_0) formulas are constructed from atomic formulas by propositional connectives and bounded numerical quantifiers $(\forall x < t)$ and $(\exists x < t)$, where t does not contain x. A Σ^0_n formula is of the form $\exists x_1 \forall x_2 \ldots x_n \theta$ with θ bounded, and a Π^0_n formula is of the form $\forall x_1 \exists x_2 \ldots x_n \theta$ with θ bounded. All the Σ^0_n and Π^0_n formulas are the *arithmetical* (Σ^1_0 or Π^1_0) formulas. A Σ^1_n formula is of the form $\exists X_1 \forall X_2 \ldots X_n \varphi$ with φ arithmetical, and a Π^1_n formula is of the form $\forall X_1 \exists X_2 \ldots X_n \varphi$ with φ arithmetical.

The semantics of $\mathcal{L}_2$ are given by the following definition.

Definition 1.1 An $\mathcal{L}_2$-*structure* is an ordered 7-tuple

$$(M, S, +_M, \cdot_M, 0_M, 1_M, <_M),$$

where M is a set which serves as the range of the number variables, S is a set of subsets of M serving as the range of set variables, $+_M$ and $\cdot_M$ are binary operations on M, 0_M and 1_M are distinguished elements of M, and $<_M$ is a binary relation on M. We always assume that the sets M and S are disjoint and nonempty. $(M, S, +_M, \cdot_M, 0_M, 1_M, <_M)$ is simply denoted by (M, S) throughout this thesis. Formulas of $\mathcal{L}_2$ are interpreted in (M, S) in the obvious way.

We also write M for an $\mathcal{L}_1$ structure $(M, +_M, \cdot_M, 0_M, 1_M, <_M)$. If M is the set (or structure) of standard natural numbers, an $\mathcal{L}_2$-structure (M, S) is called an ω-*structure* or an ω-*model*.

Definition 1.2 The system of RCA_0 consists of

(1) the ordered semiring axioms for $(\omega, +, \cdot, 0, 1, <)$,

(2) Δ^0_1-CA:

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ is Σ^0_1, $\psi(x)$ is Π^0_1, and X does not occur freely in $\varphi(x)$,

(3) Σ^0_1 induction scheme:

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x),$$

where $\varphi(x)$ is a Σ^0_1 formula.

The acronym RCA stands for recursive comprehension axiom. Roughly speaking, the set existence axioms of RCA_0 are strong enough to prove the existence of recursive sets.

If X and Y are set variables, we use $X \subseteq Y$ and $X = Y$ as abbreviations for the formulas $\forall n(n \in X \to n \in Y)$ and $\forall n(n \in X \leftrightarrow n \in Y)$. We define $\mathbb{N}$ to be the unique set X such that $\forall n(n \in X)$.

Within RCA_0, we define a *pairing map* $(m, n) = (m+n)^2 + m$. We can prove within RCA_0 that for all m, n i, j in $\mathbb{N}$, $(m, n) = (i, j)$ if and only if $m = i$ and $n = j$. Moreover, using Δ^0_1-CA, we can prove that for any X and Y, there exists a set $X \times Y \subseteq \mathbb{N}$ such that

$$\forall n(n \in X \times Y \leftrightarrow \exists x \leq n \exists y \leq n(x \in X \wedge y \in Y \wedge (x, y) = n)).$$

For X and Y, a *function* $f : X \to Y$ is defined to be a set $F \subseteq X \times Y$ such that $\forall x \forall y_0 \forall y_1((x, y_0) \in F \wedge (x, y_1) \in F \to y_0 = y_1)$ and $\forall x \in X \exists y \in Y (x, y) \in F$. We write $f(x) = y$ for $(x, y) \in F$.

Within RCA_0, the universe of functions is closed under composition, primitive recursion (i.e., given $f : X \to Y$ and $g : \mathbb{N} \times X \times Y \to Y$, there exists a unique $h : \mathbb{N} \times X \to Y$ defined by $h(0, m) = f(m)$, $h(n+1, m) = g(n, m, h(n, m))$ and the least number operator (i.e., given $f : \mathbb{N} \times X \to \mathbb{N}$ such that for all $m \in X$ there exists $n \in \mathbb{N}$ such that $f(n, m) = 1$, there exists a unique $g : X \to \mathbb{N}$ defined by $g(m) =$the least n such that $f(n, m) = 1$).

Definition 1.3 ACA_0 is the system which consists of RCA_0 plus ACA (arithmetical comprehension axioms) :

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(x)$ is arithmetical and X does not occur freely in $\varphi(x)$.

ACA_0 permits a smooth theory of sequential convergence and corresponds to Weyl's program of predicativity. For any sentence σ of the language of Peano Arithmetic (PA), σ is a theorem of ACA_0 if and only if σ is a theorem of PA. ACA_0 is finite axiomatizable although PA is not. The following lemma will be useful in showing that ACA is needed in order to prove various theorems of ordinary mathematics.

Lemma 1.4 *The following are pairwise equivalent over* RCA_0*.*

(1) ACA_0

(2) Σ^0_1-CA: $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$ *restricted to* Σ^0_1 *formulas* $\varphi(x)$ *in which* X *does not occur freely in* $\varphi(x)$*.*

(3) *For any* 1-1 *function* $f : \mathbb{N} \to \mathbb{N}$, $\mathrm{rng}(f)$ *exists, that is,* $\exists X \forall n (n \in X \leftrightarrow \exists m f(m) = n)$

Proof. See Chapter III [23]. $\square$

Within RCA_0, we define $2^{<\mathbb{N}}$ to be the set of (codes for) finite sequences of 0's and 1's. A set $T \subseteq 2^{<\mathbb{N}}$ is said to be a *tree* (or precisely 0-1 *tree*) if any initial segment of a sequence in T is also in T. A *path* through T is a function $f : \mathbb{N} \to \{0, 1\}$ such that for each n, the sequence $f[n] = \langle f(0), f(1), \ldots, f(n-1)\rangle$ belongs to T.

Definition 1.5 WKL_0 *is the system which consists of those of* RCA_0 *plus weak König's lemma: every infinite* 0-1 *tree* T *has a path.*

In particular, ω-models of WKL_0 are known as Scott systems and extensively studied by not a few people, e.g. Kaye [17]. The first-order part of WKL_0 is the same as that of RCA_0. Furthermore, WKL_0 is conservative over Primitive Recursive Arithmetic (PRA) with respect to Π^0_1 sentences. On the other hand, WKL_0 is strong enough to prove many theorems of ordinary mathematics, for example, Hine-Borel covering lemma, maximum principle for continuous functions on [0,1], Brouwer's fixed point theorem and so on. In Section 3.1, we show that WKL_0 proves the existence of Haar measure on separable compact group.

We finally define *weak weak König's lemma* to be the following axiom: if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : lh(\sigma) = n\}|}{2^n} = 0.$$

Weak weak König's lemma is a consequence of weak König's lemma. WWKL_0 is the system consisting of RCA_0 plus weak weak König's lemma. We prove within WWKL_0 that any Borel measure on any compact metric space X, that is, a positive bounded linear functional of $C(X)$, is countably additive. It is known that $\mathsf{RCA}_0 \subsetneq \mathsf{WWKL}_0 \subsetneq \mathsf{WKL}_0$. See [29] for details.

1.2 Σ^0_k-BAC_0 and Σ^0_k-BDC_0

In this subsection, we define bounded axioms of choice (BAC) and bounded dependent choice (BDC). Then we prove that they are equivalent to some induction axioms over RCA_0. To begin with, we recall the usual versions of axioms of separation SP, axioms of choice AC and axioms of dependent choice DC.

Definition 1.6 The following definitions are made in RCA_0. Let Γ be a set of $\mathcal{L}_2$ formulas.

(1) Γ-SP is the set of universal closures of formulas of the form

$$\forall n(\varphi(n) \to \neg\psi(n)) \to \exists Z \forall n((\varphi(n) \to n \in Z) \land (n \in Z \to \neg\psi(n))),$$

where $\varphi(n)$ and $\psi(n)$ belong to Γ and have no free occurrence of variable Z.

(2) Γ-AC is the set of universal closures of formulas of the form

$$\forall n \exists X \eta(n, X) \to \exists Z \forall n \eta(n, (Z)_n),$$

where $\eta(n, X)$ belongs to Γ and has no free occurrence of variable Z.

(3) Γ-DC is the set of universal closures of formulas of the form

$$\forall n \forall X \exists Y \xi(n, X, Y) \to \exists Z \forall n \xi(n, (Z)_n, (Z)_{n+1}),$$

where $\xi(n, X, Y)$ belongs to Γ and has no free occurrence of variable Z.

For a set Λ of sentences, let Λ_0 denote the subsystem of second-order arithmetic which consists of RCA_0 plus Λ. Then the following equivalences are well-known [23].

Lemma 1.7 (1) $\mathsf{RCA}_0 \equiv \Pi^0_1$-$\mathsf{SP}_0 \equiv \Sigma^0_1$-$\mathsf{AC}_0 \equiv \Sigma^0_1$-$\mathsf{DC}_0$.

(2) $\mathsf{WKL}_0 \equiv \Sigma^0_1$-$\mathsf{SP}_0 \equiv \Pi^0_1$-$\mathsf{AC}_0 \equiv \Pi^0_1$-$\mathsf{DC}_0$.

(3) $\mathsf{ACA}_0 \equiv \Pi^0_2$-$\mathsf{SP}_0 \equiv \Sigma^0_2$-$\mathsf{AC}_0 \equiv \Sigma^0_2$-$\mathsf{DC}_0$.

Now, we introduce the finite versions of the above axioms.

Definition 1.8 The following definitions are made in RCA_0. Let Γ be a set of $\mathcal{L}_2$ formulas.

(1) Γ-BSP is the set of universal closures of formulas of the form

$$\forall n(\varphi(n) \to \neg\psi(n)) \to \forall l \exists Z \forall n < l \ ((\varphi(n) \to n \in Z) \land (n \in Z \to \neg\psi(n))),$$

where $\varphi(n)$ and $\psi(n)$ belong to Γ and have no free occurrence of variable Z.

(2) Γ-BAC is the set of universal closures of formulas of the form

$$\forall n \exists X \eta(n, X) \to \forall l \exists Z \forall n < l \ \eta(n, (Z)_n),$$

where $\eta(n, X)$ belongs to Γ and has no free occurrence of variable Z.

(3) Γ-BDC is the set of universal closures of formulas of the form

$$\forall n \forall X \exists Y \xi(n, X, Y) \to \forall l \exists Z \forall n < l \ \xi(n, (Z)_n, (Z)_{n+1}),$$

where $\xi(n, X, Y)$ belongs to Γ and has no free occurrence of variable Z.

Definition 1.9 The following definitions are made in RCA_0. Δ^0_k-BCA is the set of universal closures of formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \neg\psi(n)) \to \forall l \exists Z \forall n < l(\varphi(n) \leftrightarrow n \in Z),$$

where $\varphi(n)$ and $\psi(n)$ are Σ^0_k and have no free occurrence of variable Z.

Lemma 1.10 (1) *For each* $k = 1, 2, \ldots,$ $\Delta^0_k\text{-}\mathsf{BCA}_0 \subseteq \Pi^0_k\text{-}\mathsf{BSP}_0 \subseteq \Sigma^0_k\text{-}\mathsf{BAC}_0 \subseteq \Sigma^0_k\text{-}\mathsf{BDC}_0$.

(2) $\Sigma^0_2\text{-}\mathsf{BAC}_0 \equiv (\Sigma^0_1 \land \Pi^0_1)\text{-}\mathsf{BAC}_0$ *and* $\Sigma^0_2\text{-}\mathsf{BDC}_0 \equiv (\Sigma^0_1 \land \Pi^0_1)\text{-}\mathsf{BDC}_0$.

(3) *For each* $k = 2, 3, \ldots,$ $\Pi^0_k\text{-}\mathsf{BAC}_0 \equiv \Sigma^0_{k+1}\text{-}\mathsf{BAC}_0$ *and* $\Pi^0_k\text{-}\mathsf{BDC}_0 \equiv \Sigma^0_{k+1}\text{-}\mathsf{BDC}_0$.

Proof. (1) it is trivial. (2) Obviously, $\Sigma^0_2\text{-}\mathsf{BAC}_0 \Rightarrow (\Sigma^0_1 \land \Pi^0_1)\text{-}\mathsf{BAC}_0$. We show that $(\Sigma^0_1 \land \Pi^0_1)\text{-}\mathsf{BAC}_0 \Rightarrow \Sigma^0_2\text{-}\mathsf{BAC}_0$. Let $\varphi(x, n, X)$ be a Π^0_1 formula. Suppose that $\forall n \exists X \exists x \varphi(x, n, X)$. Denote $\varphi'(n, X, Y)$ by $(Y \neq \emptyset \land \forall x \in Y \varphi(x, n, X))$. Then $\forall n \exists X \exists Y \varphi'(n, X, Y)$. By $(\Sigma^0_1 \land \Pi^0_1)$-BAC, $\forall l \exists X \exists Y \forall n < l \ \varphi'(n, (X)_n, (Y)_n)$. That is, $\forall l \exists X \forall n < l \exists Y (Y \neq \emptyset \land \forall x \in Y \varphi(x, n, (X)_n))$. Then $\forall l \exists Z \forall n < l \exists x \varphi(x, n, (Z)_n)$. Similarly, we can show the equivalences of the rest of (2) and (3). $\square$

Theorem 1.11 $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 \equiv \Delta^0_2\text{-}\mathsf{BCA}_0 \equiv \Pi^0_2\text{-}\mathsf{BSP}_0$.

Proof. Within RCA_0. We first show that Δ^0_2-BCA $\Rightarrow$ BΣ^0_2. Suppose that Δ^0_2-BCA. Then it suffices to show that the least number principle holds for any Σ^0_2 formula which is equivalent to a Π^0_2 formula. Let φ and ψ be Σ^0_2 and Π^0_2 formulas. Suppose that $\exists x\varphi(x)$ and $\forall x\big(\varphi(x) \leftrightarrow \psi(x)\big)$. Fix an n such that $\varphi(n)$. By Δ^0_2-BCA, there exists $X = \{x \leq n : \varphi(x)\}$. Since it is obvious that X has the least number, the least number principle holds for φ.

Next we prove that B$\Sigma^0_2 \Rightarrow \Pi^0_2$-BSP. Suppose that B$\Sigma^0_2$. Let φ and ψ are Π^0_1 formulas. Assume that $\forall x\big(\exists y\varphi(x,y) \vee \exists z\psi(x,z)\big)$. By B$\Sigma^0_2$, for any n, there exists l such that

$$\forall x \leq n\big(\exists y < l\ \varphi(x,y) \vee \exists z < l\ \psi(x,z)\big).$$

Let $X = \{x \leq n : \exists z < l\ \psi(x,z)\}$. Then, for each $x \leq n$, if $\forall y\neg\varphi(x,y)$ then $\exists z < l\ \psi(x,z)$, i.e., $x \in X$, and if $\forall z\neg\psi(x,z)$ then $\exists y < l\ \varphi(x,y)$, i.e., $x \notin X$. Therefore, X separates φ and ψ for $x \leq n$.

It is easy to show that Π^0_2-BSP$\rightarrow \Delta^0_2$-BCA by (1) of Lemma 1.7. $\square$

Theorem 1.12 *For each $k \geq 2$, $\mathrm{I}\Sigma^0_k \subseteq \Sigma^0_k\text{-}\mathsf{BDC}_0$.*

Proof. For simplicity, we may assume $k = 2$ since the other cases can be treated similarly. We work within Σ^0_2-BDC_0. Let φ be a Π^0_1 formula such that $\exists y\varphi(0,y)$ and $\forall x(\exists y\varphi(x,y) \rightarrow \exists y\varphi(x+1,y))$.

Let $\psi(n,X,Y)$ be a Σ^0_2 formula which says that either $n = 0 \wedge Y \neq \emptyset \wedge \forall y \in Y\varphi(n,y)$ or $n > 0 \wedge ((X \neq \emptyset \wedge \forall y \in X\varphi(n-1,y)) \rightarrow (Y \neq \emptyset \wedge \forall y \in Y\varphi(n,y)))$. Then $\forall n\forall X\exists Y\psi(n,X,Y)$.

By Σ^0_2-BDC, for any $l \in \mathbb{N}$, there exists Z such that $\forall n \leq l\psi(n,(Z)_n,(Z)_{n+1})$. Therefore we can prove that $\forall n \leq l\forall y \in (Z)_{n+1}\varphi(n,y)$ by Π^0_1 induction. We can also show that $\forall n \leq l((Z)_{n+1} \neq \emptyset)$. Then $\forall n \leq l\ \exists y\varphi(n,y)$. Therefore Σ^0_2 induction holds. $\square$

Lemma 1.13 (1) *For any Π^0_1 formula $\psi(X)$, we can find a Π^0_1 formula $\hat{\psi}$ such that* WKL_0 *proves $\hat{\psi} \leftrightarrow \exists X\psi(X)$.*

(2) *For any Σ^0_2 formula $\psi(X)$, we can find a Σ^0_2 formula $\hat{\psi}$ such that* WKL_0 *proves $\hat{\psi} \leftrightarrow \exists X\psi(X)$.*

Proof. See VIII.2.4 [23]. $\square$

Theorem 1.14 *The followings hold.*

(1) Σ^0_2-$\mathsf{BAC}_0 \subseteq \mathsf{WKL}_0 + \mathrm{B}\Sigma^0_2$.

(2) Σ^0_2-$\mathsf{BDC}_0 \subseteq \mathsf{WKL}_0 + \mathrm{I}\Sigma^0_2$.

Proof. To prove (1), assume that WKL_0+$\mathrm{B}\Sigma^0_2$. Let φ be a Π^0_1 formula. Suppose that $\forall n \exists X \exists x \varphi(n, x, X)$. By Lemma 1.13 and $\mathrm{B}\Sigma^0_2$, for each k, there exists l such that $\forall n \leq k \exists x < l \exists X \varphi(n, x, X)$. That is, $\forall n \leq k \exists X \exists x < l\ \varphi(n, x, X)$. By Π^0_1-BAC, $\exists Z \forall n \leq k \exists x < l\ \varphi(n, x, (Z)_n)$. Then $\exists Z \forall n \leq k \exists x \varphi(n, x, (Z)_n)$. To prove (2), assume that WKL_0+$\mathrm{I}\Sigma^0_2$. Let φ be a Σ^0_2 formula. Suppose that $\forall n \forall X \forall Y \varphi(n, X, Y)$. Let $\varphi'(m)$ denote $\exists Z \forall n \leq m\ \varphi(n, (Z)_m, (Z)_{m+1})$. Then $\varphi'(0)$ and $\forall m(\varphi'(m) \to \varphi'(m+1))$. By Lemma 1.13 and Σ^0_2-induction, we have $\forall m \varphi'(m)$. Then $\forall l \exists Z \forall n < l \varphi(n, (Z)_n, (Z)_{n+1})$. $\square$

We define a weak version of BAC, called UBAC.

Definition 1.15 The following definition is made in RCA_0. Let Γ be a set of $\mathcal{L}_2$ formulas. Γ-UBAC is the set of universal closures of formulas of the form

$$\forall n \exists ! X \varphi(n, X) \to \forall l \exists ! Z \forall n < l\ \varphi(n, (Z)_n),$$

where $\varphi(n, X)$ belongs to Γ and has no free variable Z.

Corollary 1.16 Σ^0_2-$\mathsf{UBAC}_0 \equiv \mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$.

We finally raise some open problems. I do not know whether or not the following equivalences and inclusions hold.

(1) $\mathsf{RCA}_0 \equiv \Pi^0_1$-$\mathsf{BAC}_0$, Σ^0_2-$\mathsf{BAC}_0 \equiv \mathsf{RCA}_0$+$\mathrm{B}\Sigma^0_2$ and Σ^0_2-$\mathsf{BDC}_0 \equiv \mathsf{RCA}_0$+$\mathrm{I}\Sigma^0_2$.

(2) Σ^0_2-$\mathsf{BDC}_0 \subseteq \Pi^0_2$-$\mathsf{BAC}_0$.

1.3 Σ^b_1-NIA and BTFA

In this section, we introduce two second-order systems Σ^b_1-NIA and BTFA of finite 0-1 sequences. Σ^b_1-NIA (the acronym NIA stands for notation induction axiom) consists of basic axioms and **NP**-notation induction, and it is essentially equivalent to Buss' S^1_2.

We discuss Σ^b_1-NIA in Section 2.1.2. BTFA, which stands for base theory for feasible analysis, is introduced by F. Ferreira [8] to answer Sieg's problem: find a mathematically significant subsystem of analysis whose class of provably recursive functions consists only of the computationally "feasible" ones. In Section 3.3, we will study Reverse Mathematics over BTFA.

The language $\mathcal{L}_S$ of second-order systems of 0-1 strings consists of three constant symbols ε, 0, and 1, two binary function symbols $\frown$(for concatenation, usually omitted) and $\times$($x \times y$ means the word x concatenated with itself the length of y times) and a binary relation symbol $\subseteq$ (for initial subwordness). BASIC is the set of the following fourteen basic axioms:

$$
\begin{array}{ll}
x\varepsilon = x, & x \times \varepsilon = \varepsilon, \\
x(y0) = (xy)0, & x \times y0 = (x \times y)x, \\
x(y1) = (xy)1, & x \times y1 = (x \times y)x, \\
x0 = y0 \to x = y, & x1 = y1 \to x = y, \\
x \subseteq \varepsilon \leftrightarrow x = \varepsilon, & x0 \neq y1, \\
x0 \neq \varepsilon, & x1 \neq \varepsilon, \\
x \subseteq y0 \leftrightarrow x \subseteq y \vee x = y0, & x \subseteq y1 \leftrightarrow x \subseteq y \vee x = y1.
\end{array}
$$

The class of s.w.q. formulas (where s.w.q. stands for "subword quantification") is the smallest class of formulas containing the atomic formulas and closed under the Boolean operations and quantifications of the form $\forall x \subseteq^* t$ or $\exists x \subseteq^* t$ where $x \subseteq^* y$ means $\exists z \subseteq y(z \frown x \subseteq y)$ and t is a term in which the variable x does not occur. The relation $x \leq y$ is defined by $1 \times x \subseteq 1 \times y$ to express that the length of x is less than or equal to the length of y. We write $|x|$ for $1 \times x$. Σ^b_1-formula is a formula of the form $\exists x \leq t\varphi$ where φ is a s.w.q. formula and t is a term in which the variable x does not occur. We notice that Σ^b_1-formulas define exactly the NP sets in the standard model. The class of Σ^b_∞-formulas is the smallest class containing the s.w.q. formulas and closed under Boolean operations and bounded quantification, i.e., quantification of the form $\exists x \leq t(\dots)$ or $\forall x \leq t(\dots)$, where the variable x does not occur in the term t.

Definition 1.17 **(0)** ∇^b_1-CA is the set of universal closures of formulas of the form

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where φ and ψ are Σ^b_1-formulas and X does not occur in φ.

(1) ($)-CA is the set of universal closures of formulas of the form

$$\forall x(\exists y\varphi(x,y) \leftrightarrow \forall z\neg\psi(x,z)) \to \exists X\forall x(x \in X \leftrightarrow \exists y\varphi(x,y)),$$

where φ and ψ are Σ^b_1-formulas and X does not occur in φ.

(2) Σ^b_1-CA is the set of universal closures of formulas of the form

$$\exists X\forall x(x \in X \leftrightarrow \varphi(x)),$$

where φ is a Σ^b_1-formula and X does not occur in φ.

(3) Σ^b_1-NIA (notation induction axioms) is the set of universal closures of formulas of the form

$$\varphi(\varepsilon) \wedge \forall x(\varphi(x) \to \varphi(x0) \wedge \varphi(x1)) \to \forall x\varphi(x),$$

where φ is a Σ^b_1-formula.

(4) $\mathrm{B}\Sigma^b_\infty$ is the set of universal closures of formulas of the form

$$\forall x \leq a\exists y\varphi(x,y) \to \exists z\forall x \leq a\exists y \leq z\varphi(x,y),$$

where φ is a Σ^b_∞-formula.

Definition 1.18 **(1)** Σ^b_1-NIA is the second-order system of finite 0-1 sequences which consists of BASIC plus Σ^b_1-NIA.

(2) BTFA is the second-order system of finite 0-1 sequences which consists of Σ^b_1-NIA plus $\mathrm{B}\Sigma^b_\infty$ plus ($)-CA.

It is known that Σ^b_1-NIA and Buss' S^1_2 are mutually interpretable.

It is obvious that the smallest model of BTFA is $(2^\omega, \Delta^0_1)$, though it is unknown whether or not ($)-CA implies Σ^b_1-CA. The following theorem is essential.

Theorem 1.19 **(1)** *Σ^b_1-$\mathsf{NIA} + \mathrm{B}\Sigma^b_\infty$ is conservative over Σ^b_1-NIA with respect to the Π^0_2-formulas.*

(2) *BTFA is conservative over Σ^b_1-$\mathsf{NIA} + \mathrm{B}\Sigma^b_\infty$ with respect to the Π^1_1-formulas.*

For a proof of this theorem and other related results, see Ferreira [8].

2 Some conservation results over RCA_0

A celebrated metamathematical theorem due to L. Harrington asserts that WKL_0 is conservative over RCA_0 for the arithmetical (in fact, Π^1_1) sentences. In other words, if an arithmetical theorem can be obtained by some analytical methods involving the compactness argument over computable mathematics, it is already provable without it. This result can be viewed as a computable analogue of the Gödel-Kreisel theorem on set theory, which asserts that if an arithmetical sentence can be proved in ZF with the axiom of choice, it is already provable without it.

It is natural to think of extending Harrington's conservation result to analytical sentences, since the Gödel-Kreisel theorem has been extended to the Σ^1_2 sentences by J. Shoenfield. However, it is obvious that WKL_0 is not Σ^1_1 conservative over RCA_0, since an instance of weak König's lemma is Σ^1_1.

In this context, we claim that if WKL_0 proves $\forall X \exists ! Y \varphi(X,Y)$ with φ arithmetical, so does RCA_0. Note that $\exists ! X \varphi(X)$ means that there exists a unique X satisfying $\varphi(X)$. This claim answers a problem posed by Tanaka [25] and attempted by some others [5]. We also discuss some other conservation results analogous to this result.

In Section 2.1, we generalize a result of Brown and Simpson [4] to prove that $\mathsf{RCA}_0+\Pi^0_\infty$-BCT is conservative over RCA_0 with respect to the set of formulas in the form $\exists ! X \varphi(X)$ where φ is arithmetical. We also consider the conservation of Π^0_∞-BCT over Σ^b_1-NIA+∇^b_1-CA. In Section 2.2, we first show that for any countable model (M,S) of RCA_0, there exists a countable model (M,S') of WKL_0 such that $S\cap S'$ consists of all Δ^0_1 subsets of M. By combining this result with a certain forcing argument with universal tree, we finally prove that if WKL_0 proves $\forall X \exists ! Y \varphi(X,Y)$ with φ arithmetical, so does RCA_0.

2.1 Baire category theorem

In [4], Brown and Simpson proved that a version of the Baire category theorem, Π^0_∞-BCT can be added to RCA_0 without increasing the Π^1_1-theorems, i.e., $\mathsf{RCA}_0+\Pi^0_\infty$-BCT is Π^1_1-conservative over RCA_0. In this section, we generalize their result in two ways. Firstly, we prove that $\mathsf{RCA}_0+\Pi^0_\infty$-BCT is conservative over RCA_0 with respect to the set of formulas in the form $\exists ! X \varphi(X)$ where φ is arithmetical. Secondly, we consider the conservation of Π^0_∞-BCT over Σ^b_1-NIA+∇^b_1-CA, which is a second-order system of finite

0-1 sequences introduced by Ferreira [8].

We recall some basic concepts from Brown and Simpson [4]. There are two versions of the Baire category theorem, BCT-I and BCT-II. Uryson's lemma, for instance, follows from BCT-I, which is provable in RCA_0. By contrast, usual proofs for the inverse mapping theorem and the open mapping theorem use BCT-II, which is not provable in RCA_0, but in RCA_0^+=RCA_0+Π^0_∞-BCT. Π^0_∞-BCT is an assertion that for any sequence of arithmetical dense sets of finite 0-1 sequences, there exists an infinite 0-1 sequence which meets each set in the sequence. Brown and Simpson proved that WKL_0^+=WKL_0+Π^0_∞-BCT is Π^1_1-conservative over RCA_0. It is unknown whether or not the inverse mapping theorem etc. are provable in RCA_0. We hope that the result in this section can help showing that the inverse mapping theorem is provable in RCA_0. But, at the moment, only a very limited case of the theorem follows from our result.

2.1.1 Π^0_∞-BCT and unique existence

To begin with, we give a rigorous definition of Π^0_∞-BCT.

Definition 2.1 The following definition is made in RCA_0. Π^0_∞-BCT is the following scheme:

$$\forall n \forall \sigma \in 2^{<\mathbb{N}} \exists \tau \in 2^{<\mathbb{N}} \big(\sigma \subseteq \tau \wedge \varphi(n,\tau)\big) \to \exists X \forall n \exists k \varphi(n, X[k]),$$

where $\varphi(x,y)$ is an arithmetical formula with no free variable X.

The main theorem in this section is as follows.

Theorem 2.2 *Let $\varphi(X,Y)$ be an arithmetical formula with exactly the free variables shown. If $\mathsf{RCA}_0 + \Pi^0_\infty$-BCT proves $\forall X \exists! Y \varphi(X,Y)$, then RCA_0 also proves it.* (*In addition, RCA_0 proves $\forall X \exists Y (Y \in \Delta^0_1(\{X\}) \wedge \varphi(X,Y))$.*)

As usual, $\exists! Z\psi(Z)$ is an abbreviation for $\exists Z \psi(Z) \wedge \forall V \forall W (\psi(V) \wedge \psi(W) \to V = W)$.

We first show the following lemma.

Lemma 2.3 *Let (M,S) be a countable model of RCA_0. Then there exist three subcollections $S_1,\ S_2,\ S_3$ of $\mathfrak{P}(M)$ such that*

(1) $(M, S_i) \models \mathsf{RCA}_0 + \Pi^0_\infty$-BCT $(i = 1,2,3)$,

(2) $S_1 \bigcap S_2 = S$,

(3) $S_1 \bigcup S_2 \subseteq S_3$.

Proof. We use a forcing construction inspired by Brown and Simpson [4]. Let (M, S) be any countable model of RCA_0. A set $D \subseteq 2^{<M}$ is *dense* if for all $\sigma \in 2^M$ there is a $\tau \in D$ such that $\sigma \subseteq \tau$. We say that $D \subseteq 2^{<M}$ is (M, S)-*definable* or *definable* if it can be defined by an arithmetical formula (i.e. formula with no set quantifier) over (M, S) with parameters from $M \bigcup S$. A set $G \subset M$ is (M, S)-*generic* if for each definable dense set D there is a $\sigma \in D \bigcap \{G[k] : k \in M\}$. Brown and Simpson [4] show that for any countable model (M, S) of RCA_0, there exists a sequence $\{G_i\}_{i<\omega}$ of subsets of M such that for any i, (M, S_i) is a model of RCA_0 and G_i is (M, S_i)-generic where S_i is the class of Δ^0_1-definable sets with parameters from $M \bigcup S \bigcup \{G_j\}_{j<i}$.

For each i, S^i_1 (S^i_2) is defined to be the class of Δ^0_1-definable sets with parameters from $M \bigcup S \bigcup \{\, G_j : j \leq 2i+1$ and j is odd (even)$\,\}$. Then we show that the lemma holds with $S_1 = \bigcup S^i_1$, $S_2 = \bigcup S^i_2$ and $S_3 = \bigcup S_i$. The condition (1) follows from Brown and Simpson's argument, and (3) is obvious. Thus, we only need to show (2).

Suppose that $S_1 \bigcap S_2 \neq S$. Then, take an $A \in S^i_1 \setminus S$ ($S^j_2 \setminus S$) for some i (j). We choose the least such i (j). We may also assume $j \leq i$. Then $A \in S^i_1 \bigcap S^j_2 \setminus S^{i-1}_1$. Let $\exists k \varphi_n(x, k, G_1[k], \ldots, G_{2i+1}[k])$ be the n-th Σ^0_1-formula in (M, S^i_1) where φ_n is Σ^0_0. We define $E_{n,m}$ by

$$\begin{aligned}\sigma \in E_{n,m} \text{ iff } \exists x \forall \tau \supset \sigma \quad & ((x \in A \not\leftrightarrow \exists k \leq lh(\tau)\varphi_n(x, k, G_1[k], \ldots, G_{2i-1}[k], \tau[k])) \\ & \vee (x \in A \not\leftrightarrow \forall l \leq lh(\tau)\neg\varphi_m(x, l, G_1[l], \ldots, G_{2i-1}[l], \tau[l]))\,).\end{aligned}$$

Firstly we want to show that for any n and m, $E_{n,m} \bigcap \{G_{2i+1}[k] : k \in M\}$ is not empty. We define $D_{n,m}$ by

$$\sigma \in D_{n,m} \text{ iff } \sigma \in E_{n,m} \vee \neg\exists\tau \in E_{n,m}(\sigma \subseteq \tau).$$

Since $A \in S^j_2$, $D_{n,m}$ is a (M, S_{2i+1})-definable dense set. Take $\sigma_0 \in D_{n,m} \bigcap \{G_{2i+1}[k] : k \in M\}$. Suppose that $\sigma_0 \notin E_{n,m}$. (Otherwise, we are done.) Then for all τ such that $\sigma_0 \subseteq \tau$,

$$\begin{aligned}\forall x \exists \tau' \supset \tau \quad & (\,(x \in A \leftrightarrow \exists k \leq lh(\tau')\varphi_n(x, k, G_1[k], \ldots, G_{2i-1}[k], \tau'[k])) \wedge \\ & (x \in A \leftrightarrow \forall l \leq lh(\tau')\neg\varphi_m(x, l, G_1[l], \ldots, G_{2i-1}[l], \tau'[l]))\,).\end{aligned}$$

Therefore, for any $x \in M$,

$$x \in A \leftrightarrow \exists \tau \supset \sigma_0 \exists k \leq lh(\tau)\varphi_n(x, k, G_1[k], \ldots, G_{2i-1}[k], \tau[k])$$

$$\leftrightarrow \forall\tau \supset \sigma_0 \forall k \leq lh(\tau) \neg\varphi_m(x, k, G_1[k], \ldots, G_{2i-1}[k], \tau[k]).$$

Then A is Δ^0_1-definable set in (M, S^{i-1}_1). This is contrary to $A \notin S^{i-1}_1$. Hence, $E_{n,m} \bigcap \{G_{2i+1}[k] : k \in M\}$ is not empty.

Since $A \in S^i_1$, there exist n' and m' such that

$$\begin{aligned} A &= \{ x \ : \exists k \varphi_{n'}(x, k, G_1[k], \ldots, G_{2i+1}[k]) \} \\ &= \{ x \ : \forall l \neg\varphi_{m'}(x, l, G_1[l], \ldots, G_{2i+1}[l]) \} \end{aligned}$$

Take $\sigma_1 \in E_{n',m'} \bigcap G_{2i+1}$. Then, there is x_0 such that for all $\tau \supset \sigma_1$,

$$\begin{aligned} (1)\ x_0 \in A \quad &\to \forall k \leq lh(\tau)\neg\varphi_{n'}(x_0, k, G_1[k], \ldots, G_{2i-1}[k], \tau[k]) \vee \\ &\quad \exists l \leq lh(\tau)\varphi_{m'}(x_0, l, G_1[l], \ldots, G_{2i-1}[l], \tau[l]); \\ (2)\ x_0 \notin A \quad &\to \exists k \leq lh(\tau)\varphi_{n'}(x_0, k, G_1[k], \ldots, G_{2i-1}[k], \tau[k]) \vee \\ &\quad \forall l \leq lh(\tau)\neg\varphi_{m'}(x_0, l, G_1[l], \ldots, G_{2i-1}[l], \tau[l]). \end{aligned}$$

If $x_0 \in A$, then there is k_0 such that $\varphi_{n'}(x_0, k_0, G_1[k_0], \ldots, G_{2i+1}[k_0])$. Let τ be an initial segment of G_{2i+1} such that τ is an end-extension of σ_1 and $G_{2i+1}[k_0]$. By (1), $\exists l \leq lh(\tau)\varphi_{m'}(x_0, l, G_1[l], \ldots, G_{2i-1}[l], \tau[l])$. Since $\tau \subseteq G_{2i+1}$, $\exists l\varphi_{m'}(x_0, l, G_1[l], \ldots, G_{2i+1}[l])$. This leads to a contradiction. The case of $x_0 \notin A$ can be treated similarly. Then $S_1 \bigcap S_2 = S$. $\square$

Now, we are finishing the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\varphi(X, Y)$ be an arithmetical formula with exactly the free variables shown. Suppose that $\mathsf{RCA}_0 + \Pi^0_\infty$-BCT proves $\forall X \exists! Y \varphi(X, Y)$ and RCA_0 can not prove it. Then by Gödel's completeness theorem, there exists a countable model (M, S) of RCA_0 such that $\neg\exists! Y \varphi(A, Y)$ holds in (M, S) for some $A \in S$. First suppose that $\exists Y \varphi(A, Y)$ holds in (M, S). Then there exists more than one set in S which satisfies φ. By Brown and Simpson's argument, there is a model (M, S') of $\mathsf{RCA}_0 + \Pi^0_\infty$-BCT such that $S \subseteq S'$. Obviously, S' has at least two distinct sets which satisfy φ. Hence $\mathsf{RCA}_0 + \Pi^0_\infty$-BCT does not prove $\forall X \exists! Y \varphi(X, Y)$, which is a contradiction.

Next assume that $\forall Y \neg\varphi(A, Y)$ holds within (M, S). Let $S_0 = \Delta^0_1(\{A\})$. Then $\forall Y \neg\varphi(A, Y)$ holds within (M, S_0). By the above lemma, there exist three subcollections S_1, S_2, S_3 of $\mathfrak{P}(M)$ such that:

(1) $(M, S_i) \models \mathsf{RCA}_0 + \Pi^0_\infty$-BCT ($i$=1, 2, 3);

(2) $S_1 \bigcap S_2 = S_0$;

(3) $S_1 \bigcup S_2 \subseteq S_3$.

By (1), there exist $B_1 \in S_1$ and $B_2 \in S_2$ such that $(M, S_i) \models \varphi(A, B_i)$ for $i = 1, 2$. By (2), $A_1 \neq A_2$, and by (3), $(M, S_3) \models \varphi(A, B_i)$ for $i = 1, 2$. Since $(M, S_3) \models \forall X \exists! Y \varphi(X, Y)$ (by $(M, S_3) \models \mathsf{RCA}_0 + \Pi^0_\infty$-BCT), this is a contradiction. □

2.1.2 A conservation result over Σ^b_1-NIA

In this subsection, we apply Brown and Simpson's argument to Σ^b_1-NIA. Path(X) is the Π^0_1-formula,

$$\begin{aligned} \forall x \forall y (x \in X \wedge y \subset x \to y \in X) \quad &\wedge \quad \forall u \exists x \equiv u (x \in X) \\ &\wedge \quad \forall x \forall y (x \in X \wedge y \in X \to y \subset x \vee y \subset x). \end{aligned}$$

where $x \equiv y$ means that x and y have the same length. The natural translation of Π^0_∞-BCT into $\mathcal{L}_s$ is given as follows.

Definition 2.4 The following definition is made in Σ^b_1-NIA. Π^0_∞-$\mathrm{BCT_s}$ is the set of universal closures of formulas of the form

$$\forall x \forall y \exists z \big(y \subset z \wedge \varphi(x, z) \big) \to \exists X (\mathrm{Path}(X) \wedge \forall x \exists y \in X \varphi(x, y))$$

where $\varphi(x, y)$ is an arithmetical formula with no free variable X.

Let (M, S) be a countable model of Σ^b_1-NIA+∇^b_1-CA. A set $D \in S$ is *dense* if for any x there is a $y \in D$ such that $x \subset y$. We say that D is (M, S)-*definable* or *definable* if it can be defined by an arithmetical formula (i.e. formula with no set quantifier) over (M, S) with parameters from $M \bigcup S$. A set $G \subset M$ is (M, S)-*generic* if Path(G) is true in $(M, S \cup \{G\})$ and $D \bigcap G \neq \emptyset$ for all definable dense sets D.

Lemma 2.5 *Let (M, S) be a countable model of Σ^b_1-NIA + ∇^b_1-CA. Then there exists a collection S' of subsets of M such that*

(1) *(M, S') is a countable model of Σ^b_1-NIA + ∇^b_1-CA;*

(2) *$S \subset S'$;*

(3) *there is a $G \in S'$ such that Path(G) holds and G intersects all (M,S)-definable dense sets D.*

Proof. Let G be (M,S)-generic, and let S' be the set of ∇_1^b definable sets with parameters from $M \bigcup S$. Then $S \subset S'$ and $\mathrm{Path}(G)$ and for all (M,S)-definable dense sets D, $G \cap D \neq \emptyset$. Moreover, it can be shown that $(M,S') \models \nabla_1^b$-CA.

Using induction on the number of symbols, we can prove that for any Σ_1^b-formula $\varphi(x,G)$ with parameters from $M \cup S'$ and any $a \in M$, there exists a Σ_1^b-formula $\psi_a(x)$ with only parameters from $M \bigcup S$ such that $(M,S') \models \forall x \leq a(\varphi(x,G) \leftrightarrow \psi_a(x))$.

To prove Σ_1^b-notation induction in (M,S'), we fix any Σ_1^b-formula $\varphi(x)$ and suppose that

$$(M,S') \models \varphi(\varepsilon) \wedge \forall x(\varphi(x) \to \varphi(x0) \wedge \varphi(x1)).$$

Fix any $a \in M$. By the above claim, there exists a $\psi_a(x)$ with parameters from $M \bigcup S$ such that $(M,S') \models \forall x \leq a(\varphi(x) \leftrightarrow \psi_a(x))$. Let $\psi'_a(x)$ be $x \leq a \to \psi_a(x)$. Then

$$(M,S) \models \psi'_a(\varepsilon) \wedge \forall x(\psi'_a(x) \to \psi'_a(x0) \wedge \psi'_a(x1)).$$

Since (M,S) is a countable model of Σ_1^b-NIA, $(M,S) \models \forall x \psi'_a(x)$. Then $(M,S) \models \psi'_a(a)$, that is $(M,S') \models \varphi(a)$. Therefore, $(M,S') \models \forall x \varphi(x)$. By the above, Σ_1^b-notation induction holds in (M,S'). $\square$

Lemma 2.6 *Let (M,S) be a countable model of Σ_1^b-NIA + ∇_1^b-CA. Then there exists a model (M,S') of Σ_1^b-NIA + ∇_1^b-CA + Π_∞^0-BCT$_\mathrm{s}$ such that $S \subset S'$.*

Proof. Apply lemma 2.5 repeatedly to obtain an increasing sequence $\langle S_i : i \in \omega \rangle$ such that

(1) $S_0 = S$,

(2) (M,S_i) is a model of Σ_1^b-NIA + ∇_1^b-CA,

(3) there is a $G \in S_{i+1}$ such that $(M,S') \models \mathrm{Path}(G)$ and G intersects all (M,S_i)-definable dense sets D.

Let $S' = \bigcup_\omega S_i$. Then $(M,S') \models \Sigma_1^b$-$\mathsf{NIA}$ + ∇_1^b-CA + Π_∞^0-BCT$_s$. $\square$

Theorem 2.7 Σ^b_1-$\mathsf{NIA}+\nabla^b_1$-CA$+\Pi^0_\infty$-BCT$_s$ *is a conservative extension of* Σ^b_1-$\mathsf{NIA}+\nabla^b_1$-CA *with respect to* Π^1_1*-sentences.*

We could not manage to extend the above theorem in a way similar to Section 2.1.1. Thus, we state it as an open problem.

Open problem 1 *Let* φ *be an arithmetical formula. If* Σ^b_1-$\mathsf{NIA}+\nabla^b_1$-CA$+\Pi^0_\infty$-BCT$_s$ *proves* $\exists! X\varphi(X)$*, then so does* Σ^b_1-$\mathsf{NIA}+\nabla^b_1$-CA.

2.2 Weak König's lemma

In this section we prove that if WKL_0 proves $\forall X\exists! Y\varphi(X,Y)$ with φ arithmetical, so does RCA_0. Note that $\exists! X\varphi(X)$ means that there exists a unique X satisfying $\varphi(X)$. This result answers a problem posed by Tanaka [25].

Let us note an application of our result. The fundamental theorem of algebra, which asserts that any complex polynomial of any positive degree has a unique factorization into linear terms, can be stated in the form $\forall X\exists! Y\varphi(X,Y)$ by using a canonical expression (i.e., the binary expansion) for the complex numbers. Most of popular proofs of the theorem use some analytical methods which can be easily formalized in WKL_0 but not in RCA_0. However, by our conservation result, it can be concluded without elaborating a computable solution that the fundamental theorem of algebra is already provable in RCA_0.

By contrast, take a look at the statement that any continuous real function on the closed unit interval $[0,1]$ has a maximum value. This sentence cannot be expressed in the form $\forall X\exists! Y\varphi(X,Y)$ with φ arithmetical. The point is that we can not determine arithmetically whether or not a set encodes a *total* continuous function in the terms of Simpson [23].

2.2.1 A non-ω hard core theorem

In this subsection, we first review the tree forcing argument originated by Jockusch-Soare [16] and used by L.Harrington for his conservation result on WKL_0. We then reinforce this argument with some other machinery to prove that for any countable model (M,S) of RCA_0, there exists a countable model (M,S') of WKL_0 such that $S\cap S'$ is the set of Δ^0_1 subsets of M. The following exposition of the tree forcing argument is based on Section IX.2 of Simpson [23]. See also VIII.2 of [23] for an account of hard core theorems.

We say that $X \subseteq M$ is Δ^0_1 definable over (M,S), denoted $X \in \Delta^0_1(S)$, if there exist a Σ^0_1 formula φ and a Π^0_1 formula ψ both with parameters from $M \cup S$ such that

$$X = \{m \in M : (M,S) \models \varphi(m)\} = \{m \in M : (M,S) \models \psi(m)\}.$$

We write Δ^0_1 for $\Delta^0_1(\emptyset)$. It is easy to see that if (M,S) is a model of RCA_0, $\Delta^0_1(S) = S$.

Lemma 2.8 *Let (M,S) be an $\mathcal{L}_2$-structure which satisfies the axioms of ordered semirings and Σ^0_1 induction. Then $(M, \Delta^0_1(S))$ is a model of* RCA_0.

Proof. See the proof of Lemma IX.1.8 [23]. □

We now define basic notions of the tree forcing. Let (M,S) be a model of RCA_0. Let $\mathcal{T}_S$ be the set of all $T \in S$ such that

$$(M,S) \models T \text{ is an infinite 0-1 tree.}$$

For any $T \in \mathcal{T}_S$ and $P \subseteq M$, we say that P is a *path through* T if, for any $n \in M$, $P[n] \in T$. Here $P[n] \in T$ means that there exists $\sigma \in M$ such that $(M,S) \models \sigma \in T$ and $lh(\sigma) = n$, and for all $m <_{(M,S)} n$, $m \in P$ if and only if $(M,S) \models \sigma(m) = 1$. We say that $D \subseteq \mathcal{T}_S$ is *dense* if for each $T \in \mathcal{T}_S$, there exists $T' \in D$ such that $T' \subseteq T$. For a set $\mathcal{D}$ of dense sets, P is said to be *generic* for $\mathcal{D}$ if for each set $D \in \mathcal{D}$, there exists $T \in D$ such that P is a path through T.

Lemma 2.9 *Let (M,S) be a countable model of* RCA_0*, and $\mathcal{D}$ a countable set of dense subsets of $\mathcal{T}_S$. Then each $T \in \mathcal{T}_S$ has a generic path for $\mathcal{D}$.*

Proof. If $\mathcal{D} = \{D_i : i \in \omega\}$ is a set of dense sets, we can easily construct a sequence of trees T_i $(i \in \omega)$ such that $T_0 = T, T_{i+1} \subseteq T_i$ and $T_{i+1} \in D_i$ for each $i \in \omega$. Then a path $P \subseteq \bigcap T_i$ is what we want. □

Lemma 2.10 *Let (M,S) be a model of* RCA_0*. For any tree $T \in \mathcal{T}_S$, there exists a path P through T such that $(M, S \cup \{P\}) \models \Sigma^{\mathbf{0}}_{\mathbf{1}}$ induction.*

Proof. Let (M, S) be a model of RCA_0. Since Σ^0_1 induction is provably equivalent to bounded Σ^0_1 comprehension (cf. Remark II.3.11 [23]), it suffices to prove that any tree $T \in \mathcal{T}_S$ has a path P such that for each $m \in M$, $\{n \in M : n <_M m \wedge (M, S \cup \{P\}) \models \varphi(n, P)\} \in S$, where $\varphi(x, X)$ is a Σ^0_1 formula with parameters from $M \cup S$.

Let $\{\varphi_e(x, X) : e \in \omega\}$ enumerate all the Σ^0_1-formulas with parameters from $M \cup S$ and no free variables other than x and X. Without loss of generality, we may assume that $\varphi_e(x, X)$ is of the form $\exists y \theta_e(x, X[y])$ with $\theta_e(x, s) \in \Sigma^0_0$, where $X[y]$ denotes the sequence $\langle f(0), f(1), \ldots, f(y-1)\rangle$ with the characteristic function f of X.

For each $e \in \omega$ and $m \in M$, let $D^0_{e,m}$ be the set of all $T \in \mathcal{T}_S$ such that for any $n <_M m$, (M, S) satisfies either

1. $\forall s \in T \neg\theta_e(n, s)$, or
2. $\exists w \forall s \in T(\mathrm{lh}(s) = w \to \exists y \leq w \theta_e(n, s[y]))$,

where $\mathrm{lh}(s)$ denotes the length of sequence s, and $s[y]$ is the initial subsequence of s with the length y. Then it is not difficult to see that $D^0_{e,m}$'s are dense. (See Lemma IX.2.4 [23].)

Let $T \in \mathcal{T}_S$ be given. By Lemma 2.9, we can take a path P through T which is generic for $\{D^0_{e,m} : e \in \omega, m \in M\}$. Fix any $e \in \omega$ and $m \in M$. Since P is generic, there is a tree $T' \in D^0_{e,m}$ which has a path P. Then, it is easy to see

$\{n \in M : n <_M m \wedge (M, S \cup \{P\}) \models \varphi_e(n, P)\}$

$= \{n \in M : n <_M m \wedge (M, S) \models \exists w \forall s \in T'(\mathrm{lh}(s) = w \to \exists y \leq w \theta_e(n, s[y]))\}$.

The set on the right-hand side belongs to S by bounded Σ^0_1 comprehension for (M, S). Thus $(M, S \cup \{P\})$ also satisfies bounded Σ^0_1 comprehension. □

Lemma 2.11 *Let (M, S) be a countable model of RCA_0. For any infinite 0-1 tree $T \in S$, there exists a countable model (M, S') of RCA_0 such that $S \subseteq S'$ and T has a path in S'.*

Proof. It is obvious from Lemmas 2.8 and 2.10. □

Lemma 2.12 *Let (M, S) be a countable model of RCA_0. Then there exists a countable model (M, S') of WKL_0 such that $S \subseteq S'$.*

Proof. Use Lemma 2.11 repeatedly. □

Theorem 2.13 (L. Harrington) *For any Π^1_1-sentence φ, if φ is a theorem of WKL_0, then φ is already a theorem of RCA_0. Especially, the first order part of WKL_0 is the same as that of RCA_0, i.e., $\mathsf{I}\Sigma_1$ (Peano arithmetic with induction restricted to the Σ_1-formulas).*

Proof. It easily follows Lemma 2.12 by the help of Gödel's completeness theorem. □

We now recall another important characterization of models of WKL_0.

Lemma 2.14 *Let (M,S) be a countable model of RCA_0. Let T be a tree in $\mathcal{T}_S$. Then, for each $A \subseteq M$ such that $A \notin S$, there exists a path P through T such that A is not in $\Delta^0_1(S \cup \{P\})$ and that $(M, S \cup \{P\}) \models \Sigma^{\mathbf{0}}_{\mathbf{1}}$ induction.*

Proof. Let (M,S) be a countable model of RCA_0 and A a subset of M such that $A \notin S$. Let $\{\varphi_e(x,X) : e \in \omega\}$ enumerates all the Σ^0_1-formulas with parameters from $M \cup S$ and no free variables other than x and X.

We first claim that for each $T \in \mathcal{T}_S$ and each pair $(e,d) \in \omega^2$, there exists a path Z through T such that $(M, \{Z\}) \models \Sigma^{\mathbf{0}}_{\mathbf{1}}$ induction and that (e,d) is not a $\Delta^0_1(S \cup \{Z\})$-index of A, i.e., $A \neq \{m : (M, S \cup \{Z\}) \models \varphi_e(m,Z)\}$ or $A \neq \{m : (M, S \cup \{Z\}) \models \neg\varphi_d(m,Z)\}$.

By way of contradiction, deny the claim. Then there exist a tree $T \in \mathcal{T}_S$ and a pair $(e,d) \in \omega^2$ such that if Z is a path through T and $(M, S \cup \{Z\}) \models \Sigma^{\mathbf{0}}_{\mathbf{1}}$ induction, then (e,d) is a $\Delta^0_1(S \cup \{Z\})$-index of A. By Lemma 2.12, we can construct a countable model (M,S') of WKL_0 such that $S \subseteq S'$. Then, for any path Z through T such that $Z \in S'$,

$$m \in A \Leftrightarrow (M, S \cup \{Z\}) \models \varphi_e(m,Z)$$

$$\Leftrightarrow (M,S') \models \varphi_e(m,Z).$$

Hence, we have

$$m \in A \Leftrightarrow (M,S') \models \forall Z(Z \text{ is a path through } T \to \varphi_e(m,Z)).$$

Since "Z is a path through T" is expressed as a Π^0_1 formula, "Z is a path through T $\to \varphi_e(m,Z)$" is Σ^0_1, and so the whole formula $\forall Z(Z \text{ is a path through } T \to \varphi_e(m,Z))$ is logically equivalent in (M,S') to a Σ^0_1 formula $\varphi'(m)$ (with parameters from $M \cup S$) by virtue of compactness of the Cantor space (cf. Lemma V.III.2.4 [23]). Since for any $m \in M$, $(M,S') \models \varphi'(m)$ if and only if $(M,S) \models \varphi'(m)$, we finally have

$$m \in A \Leftrightarrow (M,S) \models \varphi'(m).$$

Similarly, we have

$$m \in A \Leftrightarrow (M,S') \models \exists Z(Z \text{ is a path through } T \wedge \neg\varphi_d(m,Z))\},$$

and so by compactness, there exists a Π^0_1-formula $\psi'(m)$ with parameters from $M \cup S$ such that

$$m \in A \Leftrightarrow (M,S) \models \psi'(m).$$

Therefore, A is in $\Delta^0_1(S)$, hence in S since (M,S) is a model of RCA_0. This contradicts with our assumption. Thus the claim is proved.

From now, we may assume that for each $e \in \omega$, Σ^0_1-formula $\varphi_e(x,X)$ is of the form $\exists y \theta_e(x, X[y])$ with $\theta_e(x,s) \in \Sigma^0_0$. For each $(e,d) \in \omega^2$, we define $D^A_{e,d}$ to be the set of all $T \in \mathcal{T}_S$ such that one of the followings holds for some $m \in M$:

A1. $m \in A \wedge (M,S) \models \forall s \in T \neg\theta_e(m,s)$,

A2. $m \notin A \wedge (M,S) \models \exists w \forall s \in T(\mathrm{lh}(s) = w \rightarrow \exists y \leq w \theta_e(m, s[y]))$,

A3. $m \in A \wedge (M,S) \models \exists w \forall s \in T(\mathrm{lh}(s) = w \rightarrow \exists y \leq w \theta_d(m, s[y]))$,

A4. $m \notin A \wedge (M,S) \models \forall s \in T \neg\theta_d(m,s)$.

To show that for each $(e,d) \in \omega^2$, $D^A_{e,d}$ is dense, we choose any $T \in \mathcal{T}_S$. By the above claim, there exists a path Z through T and $m \in M$ such that one of the following conditions holds:

B1. $m \in A \wedge (M, S \cup \{Z\}) \models \forall y \neg\theta_e(m, Z[y])$,

B2. $m \notin A \wedge (M, S \cup \{Z\}) \models \exists y \theta_e(m, Z[y])$,

B3. $m \in A \wedge (M, S \cup \{Z\}) \models \exists y \theta_d(m, Z[y])$,

B4. $m \notin A \wedge (M, S \cup \{Z\}) \models \forall y \neg\theta_d(m, Z[y])$.

First suppose that condition B1 holds. Let $T' = \{s \in T : \forall t \subseteq s \neg\theta_e(x,t)\}$. Then, $T' \in \mathcal{T}_S$, since $T' \in S$ and T' is an infinite subtree of T. It is also clear that A1 holds with T' (instead of T). Thus $T' \in D^A_{e,d}$. Next suppose that condition B2 holds. Take an initial segment t of Z such that $\theta_e(m,t)$. Then $T' = \{s \in T : s \subseteq t \text{ or } t \subseteq s\}$ clearly satisfies A2, hence $T' \in D^A_{e,d}$. The other two cases can be treated similarly. Hence, in any case, there exists a subtree T' of T such that $T' \in D^A_{e,d}$, which means that $D^A_{e,d}$ is dense.

Given a $T \in \mathcal{T}_S$, we take a path P through T which is generic for $\{D^0_{e,m} : e \in \omega, m \in M\} \cup \{D^A_{e,d} : (e,d) \in \omega^2\}$, where $D^0_{e,m}$'s are the dense sets defined in the proof of Lemma 2.10. Then, $(M, S \cup \{P\})$ satisfies Σ^0_1 induction by the proof of Lemma 2.10. By way of contradiction, we assume that A is in $\Delta^0_1(S \cup \{P\})$, that is, there exist e and d such that

$$A = \{m : (M, S \cup \{P\}) \models \exists y \theta_e(x, P[y])\} = \{m : (M, S \cup \{P\}) \models \forall y \neg\theta_d(x, P[y])\}.$$

Since P is generic, there exists $T' \in D^A_{e,d}$ with a path P. First suppose that condition A1 of the definition of $D^A_{e,d}$ holds for T'. Then there exists $m \in A$ such that $(M, S \cup \{P\}) \models \forall y \neg\theta_e(m, P[y])$, since $P \subset T'$. This contradicts with the above equation for A. Suppose that condition A2 holds for T'. Then there exists $m \notin A$ such that $(M, S \cup \{P\}) \models \exists y \theta_e(m, P[y])$, which is also absurd. Similarly, conditions A3 and A4 lead to a contradiction. Thus, we have shown that A is not in $\Delta^0_1(S \cup \{P\})$. This completes the proof. □

Lemma 2.15 *Let (M, S) be a countable model of RCA_0, and C a countable set of subsets of M such that $C \cap S = \emptyset$. Then any tree $T \in \mathcal{T}_S$ has a path P such that $C \cap \Delta^0_1(S \cup \{P\}) = \emptyset$ and that $(M, S \cup \{P\}) \models \Sigma^0_1$ induction.*

Proof. Let (M, S) and C be as in the above statement. We define dense sets $D^0_{e,m}$ and $D^A_{e,d}$ as in the proofs of Lemmas 2.10 and 2.14. By Lemma 2.9, for each $T \in \mathcal{T}_S$, we can take a path P through T which is generic for $\{D^0_{e,m} : e \in \omega, m \in M\} \cup \{D^A_{e,d} : A \in C \wedge (e, d) \in \omega^2\}$. Then by the proofs of Lemmas 2.10 and 2.14, it is easy to see that $C \cap \Delta^0_1(S \cup \{P\}) = \emptyset$ and that $(M, S \cup \{P\}) \models \Sigma^0_1$ induction. □

Lemma 2.16 *Let (M, S) be a countable model of RCA_0, and C a countable set of subsets of M such that $C \cap S = \emptyset$. Then there exists a countable model (M, S') of WKL_0 such that $S \subseteq S'$ and $S' \cap C = \emptyset$.*

Proof. Use the above lemma repeatedly. □

The next corollary is a generalized version of Kreisel's hard core theorem.

Theorem 2.17 *Let (M, S) be a countable model of RCA_0. Then there exists a countable model (M, S') of WKL_0 such that $S \cap S' = \Delta^0_1$.*

Proof. By replacing S and C in Lemma 2.16 by Δ^0_1 and $S \setminus \Delta^0_1$, respectively, we obtain the theorem. □

Corollary 2.18 *Let M be a countable model of $\mathsf{I}\Sigma_1$. Then there exist uncountably many countable sets S of subsets of M such that (M, S) satisfies WKL_0.*

Proof. If there were only countably many of such S's, then, putting $C =$ (the union of all of them)-Δ^0_1, by Lemma 2.16 we could obtain another model (M, S') of WKL_0 such that $S' \cap C = \emptyset$, which is a contradiction. □

Finally, we remark that in our theorem, $(M, S \cup S')$ may not satisfy Σ^0_1 induction. In fact, there are two models (M, S) and (M, S') of Σ^0_1 induction such that $(M, S \cup S')$ does not satisfy Σ^0_1 induction. This fact is easily obtained from the following two theorems.

Theorem 2.19 (Mytilinaios [20]) *Let $M \models \mathsf{I}\Sigma_1$, and let W be a Σ_1 definable but not Δ_1 definable subset of M. Then there exist two Σ_1 definable subsets A, B of M such that W is in $\Delta^0_1(A, B)$, but neither in $\Delta^0_1(A)$ nor in $\Delta^0_1(B)$.*

Theorem 2.20 (Groszek et al. [11]) *Let $M \models \mathsf{B}\Sigma_2$, and let A be a Σ_1 definable subset of M. Then $(M, \{A\}) \models \Sigma^0_1$ induction, or A is complete.*

Note that $\mathbf{B}\Sigma_2$ denotes the collection axioms for the Σ_2 formulas.

Theorem 2.21 *Let $M \models \mathsf{B}\Sigma_2 \wedge \neg\mathsf{I}\Sigma_2$. Then there exist two Σ_1 definable subsets A, B of M such that both $(M, \{A\})$ and $(M, \{B\})$ satisfies Σ^0_1 induction, but $(M, \{A, B\})$ does not.*

Proof. Let $M \models \mathsf{B}\Sigma_2 \wedge \neg\mathsf{I}\Sigma_2$, and let W be a complete Σ_1 subset of M. Then $(M, \{W\})$ does not satisfy Σ^0_1 induction, since $M \models \neg\mathsf{I}\Sigma_2$ and any Σ_2 set is $\Sigma^0_1(W)$. By Theorem 2.19, there exist two incomplete Σ_1 subsets A, B of M such that W is in $\Delta^0_1(A, B)$. By Theorem 2.20, both $(M, \{A\})$ and $(M, \{B\})$ satisfies Σ^0_1 induction. But $(M, \{A, B\})$ does not satisfy Σ^0_1 induction, since $(M, \{W\})$ does not. □

2.2.2 Weak König's lemma and unique existence

To prove our main theorem, we use a formalized forcing argument with universal trees. We adopt the formalized forcing notion $\Vdash_1$ due to Avigad [1], which will capture the notion of truth in a generic model of RCA_0. We first define $\Vdash_{1/2}$ with an associated generic path G, then define $\Vdash_1$ with names for Δ^0_1 definable sets in G. The idea of forcing with universal trees is due to Professor S. G. Simpson (by private communications). We owe him a great deal for the argument of this subsection.

The following definition of $\Vdash_{1/2}$ is made in RCA_0. The 1/2-conditions are just the infinite subtrees of $2^{<\mathbb{N}}$. We have two types of 1/2-names: $\check{X} = \{\langle 0, x\rangle : x \in X\}$ for any set X and, $\check{G} = \{\langle 1, 0\rangle\}$ for the new generic path. $T \Vdash_{1/2} Name(X)$ means that X is a 1/2-name. For an atomic φ, $T \Vdash_{1/2} \varphi$ is defined as follows: $T \Vdash_{1/2} t_1 = t_2$ if $t_1 = t_2$ where t_1 and t_2 are terms; $T \Vdash_{1/2} t \in \check{X}$ if $t \in X$; $T \Vdash_{1/2} t \in \check{G}$ if $\exists m(\forall \sigma \in T(lh(\sigma) = m \to \sigma(t) = 1))$. We then extend this notion to all the formulas inductively as follows:

(1) $T \Vdash_{1/2} \neg\varphi$ if $\forall T' \subseteq T(T' \not\Vdash_{1/2} \varphi)$;

(2) $T \Vdash_{1/2} \varphi_1 \wedge \varphi_2$ if $T \Vdash_{1/2} \varphi_1 \ \wedge \ T \Vdash_{1/2} \varphi_2$;

(3) $T \Vdash_{1/2} \varphi_1 \vee \varphi_2$ if $\forall T' \subseteq T \exists T'' \subseteq T(T'' \Vdash_{1/2} \varphi_1 \ \vee \ T'' \Vdash_{1/2} \varphi_2)$;

(4) $T \Vdash_{1/2} \varphi_1 \to \varphi_2$ if $\forall T' \subseteq T(T' \Vdash_{1/2} \varphi_1 \to \exists T'' \subseteq T'(T'' \Vdash_{1/2} \varphi_2))$;

(5) $T \Vdash_{1/2} \exists x\varphi$ if $\forall T' \subseteq T \exists T'' \subseteq T' \exists x(T'' \Vdash_{1/2} \varphi)$;

(6) $T \Vdash_{1/2} \forall x\varphi$ if $\forall x(T \Vdash_{1/2} \varphi)$;

(7) $T \Vdash_{1/2} \exists X\varphi$ if $\forall T' \subseteq T \exists T'' \subseteq T' \exists X(T'' \Vdash_{1/2} Name(X) \wedge \varphi)$;

(8) $T \Vdash_{1/2} \forall X\varphi$ if $\forall X(T \Vdash_{1/2} Name(X) \to \varphi)$.

The forcing notion $\Vdash_{1/2}$ thus defined satisfies the monotonicity, substitution property and

$$T \Vdash_{1/2} \varphi \Leftrightarrow T \Vdash_{1/2} \neg\neg\varphi.$$

Lemma 2.22 *Let (M, S) be a model of RCA_0. Then $(M, S) \models T \Vdash_{1/2} \Sigma^0_1$ induction, for any $T \in \mathcal{T}_S$.*

Proof. See [1]. □

Next we define $\Vdash_1$ within RCA_0. The 1-conditions are the same as 1/2-conditions. A 1-name is defined to be $\langle X, \psi_1, \psi_2\rangle$ where ψ_1 and ψ_2 are (codes of) Σ^0_1 and Π^0_1 formulas to encode a $\mathbf{\Delta^0_1}$ definable set in X and G. So $T \Vdash_1 Name(\langle X, \psi_1, \psi_2\rangle)$ is defined to be $T \Vdash_{1/2} \forall x(\psi_1(x) \leftrightarrow \psi_2(x))$. $T \Vdash_1 t \in \langle X, \psi_1, \psi_2\rangle$ if $T \Vdash_{1/2} \psi_1(t)$. For any formula φ, $T \Vdash_1 \varphi$ is defined in the same way as $T \Vdash_{1/2} \varphi$ by (1) through (8). The canonical 1-name $\check{G}$ is defined to be $\langle \emptyset, x \in G, x \in G\rangle$ and, for any $X \in S$, $\check{X}$ be $\langle X, x \in X, x \in X\rangle$. $\Vdash_1$ also satisfies most of the properties of $\Vdash_{1/2}$, e.g., Lemma 2.22. Moreover, we have the following lemma. See [1] for a proof.

Lemma 2.23 *Let (M, S) be a model of RCA_0. Then $(M, S) \models T \Vdash_1 \mathsf{RCA}_0$, for any $T \in \mathcal{T}_S$.*

From now on, we write $\Vdash$ for $\Vdash_1$. Let (M,S) be a model of RCA_0. For $X \subseteq M$, $S[X]$ denotes the structure $(M, \Delta^0_1(S \cup \{X\}))$. We simply say that P is *generic* if P is generic for the set of (M,S)-definable dense sets. The next two lemmas are standard.

Lemma 2.24 *Let (M,S) be a countable model of RCA_0. Let $\varphi(G)$ be a sentence of $\mathcal{L}_2(M \cup S \cup \{G\})$. If P is a generic path, then $S[P] \models \varphi(P)$ if and only if there exists $T \in \mathcal{T}_S$ such that P is a path through T and $(M,S) \models T \Vdash \varphi(\check{G})$.*

Lemma 2.25 *Let (M,S) be a countable model of RCA_0. Let $\varphi(G)$ be a sentence for $\mathcal{L}_2(M \cup S \cup \{G\})$. Then $(M,S) \models T \Vdash \varphi(\check{G})$ if and only if $S[P] \models \varphi(P)$ for all generic paths P through T.*

Let $[T]$ be the set of paths P through T such that $S[P] \models \mathsf{RCA}_0$. We put $\mathcal{P} = [2^{<M}]$. Then, in most cases, $S \subsetneq \mathcal{P} \subsetneq 2^M$. By the proofs of Lemmas 2.8 and 2.10, if P is generic, then $P \in \mathcal{P}$. Let $\mathcal{B}$ be the set of boolean expressions built from atoms σ in $2^{<M}$ by means of the usual set operations $\cup$, $\cap$ and ^{c}. For $\sigma \in 2^{<M}$, let $[\sigma] = \{P \in \mathcal{P} : P[lh(\sigma)] = \sigma\}$. Then, for any expression $b \in \mathcal{B}$, $[b]$ is defined to be the subset of $\mathcal{P}$ which b denotes in the obvious way.

A mapping F from $[T]$ to $[T']$ is said to be (M,S)-*continuous* if S contains a function $f : \mathcal{B} \to \mathcal{B}$ (called a *code* for F) such that for any $b \in \mathcal{B}$,

$$[f(b)] \cap [T] = F^{-1}([b] \cap [T']).$$

Then, we can easily see that $F(P) \in \Delta^0_1(S \cup \{P\})$.

Lemma 2.26 *Let $F : [T] \to [T']$ be an (M,S)-continuous function. Then*

(1) If $T_1 \in \mathcal{T}_S$ is a subtree of T, then there exists a subtree $T'_1 (\in \mathcal{T}_S)$ of T' such that $F([T_1]) = [T'_1]$.

(2) If $T'_1 \in \mathcal{T}_S$ is a subtree of T', then there exists a subtree $T_1 (\in \mathcal{T}_S)$ of T such that $[T_1] = F^{-1}([T'_1])$.

Proof. Let $F : [T] \to [T']$ be an (M,S)-continuous function with code f.

To prove (1), let $T_1 \in \mathcal{T}_S$ be a subtree of T . Let $\psi(\sigma)$ be a Π^0_1 formula over (M,S) which means that

$$\sigma \in T' \text{ and } [T_1] \cap [f(\sigma)] \neq \emptyset.$$

Then we can show that for any $P \in \mathcal{P}$, $P \in F([T_1]) \Leftrightarrow \forall n\psi(P[n])$. To see that $P \in F([T_1]) \Rightarrow \forall n\psi(P[n])$, assume that $P = F(Q)$ where $Q \in [T_1]$. Fix any $n \in M$. Since $Q \in [f(P[n])] \cap [T_1]$, we have $[f(P[n])] \cap [T_1] \neq \emptyset$, that is, $\psi(P[n])$. Next we show that $\forall n\psi(P[n]) \Rightarrow P \in F([T_1])$. Assume that $\forall n\psi(P[n])$. For any $n \in M$, $[f(P[n])] \cap [T_1] \neq \emptyset$. Since $S[P] \models \mathsf{RCA}_0$, there exists a model (M, S') of WKL_0 such that $\Delta^0_1(S \cup \{P\}) \subseteq S'$ by Lemma 2.12. Then, there exists $Q \in S'$ such that $Q \in \bigcap_{n\in M}[f(P[n])] \cap [T_1]$. Obviously, $F(Q) = P$.

By the normal form theorem, we write $\psi(\sigma)$ as $\forall m\theta(m, \sigma)$, where θ is Σ^0_0. Let T'_1 be the set of $\tau \in 2^{<M}$ such that $\tau \in T_1$ and $\forall\sigma \subseteq \tau\forall m \leq lh(\tau)\theta(m, \sigma)$. Then $T'_1 \in \mathcal{T}_S$ and $P \in [T'_1] \Leftrightarrow \forall n\psi(P[n])$, that is, $[T'_1] = F([T_1])$.

For (2), let $T'_1 \in \mathcal{T}_S$ be a subtree of T'. Let $\psi'(\sigma)$ be a Π^0_1 formula which means that $\sigma \in T$ and $\forall n\exists\tau \in T'_1(lh(\tau) = n \wedge [T] \cap [f(\tau) \cap \sigma] \neq \emptyset)$. Then, for any $P \in \mathcal{P}$, $P \in F^{-1}([T'_1]) \Leftrightarrow \forall n\psi'(P[n])$. In the same way as (1), we have a subtree T_1 of T in $\mathcal{T}_S$ such that $[T_1] = F^{-1}([T'_1])$. □

Lemma 2.27 *Let $F : [T] \to [T']$ be an (M, S)-continuous function. Then there exists a generic path P through T such that $F(P)$ is a generic path through T'.*

Proof. Let $F : [T] \to [T']$ be an (M, S)-continuous function. Let $\mathcal{D}$ be the set of (M, S)-definable dense sets of $\mathcal{T}_S$. Let $\langle D_i : i \in \omega\rangle$ be an enumeration of $\mathcal{D}$. By Lemma 2.26, construct two descending sequences $\langle T_i : i \in \omega\rangle$ and $\langle T'_i : i \in \omega\rangle$ of trees in $\mathcal{T}_S$ such that $T_0 = T$ and $T'_0 = T'$ and, for any $i \in \omega$, $F([T_{i+1}]) = [T'_{i+1}]$ and, both T_{2i+1} and T'_{2i+2} belong to D_i. Then, we have a path P through T_i for all $i \in \omega$. By the construction, P and $F(P)$ are generic. □

A tree $T \in \mathcal{T}_S$ is said to be *universal*, if for any $T' \in \mathcal{T}_S$, there exists an (M, S)-continuous F from $[T]$ to $[T']$. Obviously, any subtree of a universal tree is also universal, whenever it belongs to $\mathcal{T}_S$.

Lemma 2.28 *Let (M, S) be a countable model of RCA_0 such that $S = \Delta^0_1(\{A\})$ for some $A \in S$. Then there exists a universal tree in $\mathcal{T}_S$.*

Proof. Let (M, S) be a countable model of RCA_0. Assume that $S = \Delta^0_1(\{A\})$. For any consistent first-order theory Γ, let T_Γ be an infinite tree such that $[T_\Gamma] =$ the set of the characteristic functions of consistent and complete extensions of Γ which is closed under logical consequence. It is known that for any 0-1 infinite tree T, there exists a first-order theory Γ_T such that there exists an (M,S)-homeomorphism from $[T_{\Gamma_T}]$ to $[T]$. (See Section IV.3.2 [23] for details.)

Let Q_A be an $\mathcal{L}_1(R)$-theory whose axioms consist of Robinson arithmetic Q plus $\{R(n) : n \in A\} \cup \{\neg R(n) : n \notin A\}$ with a new unary relation symbol R. Then Q_A is consistent since it has a weak model. (See Theorem II.8.10 [23] for details.)

We now show that T_{Q_A} is universal. Fix any $T \in \mathcal{T}_S$. Let $B = \{\sigma : \exists n\big(\exists \tau \in T(lh(\tau) = n \wedge \sigma^\frown\langle 1\rangle \subseteq \tau) \wedge \forall \tau' \in T(lh(\tau') = n \rightarrow \sigma^\frown\langle 0\rangle \not\subseteq \tau')\big)\}$ and $C = \{\sigma : \exists n\big(\exists \tau \in T(lh(\tau) = n \wedge \sigma^\frown\langle 0\rangle \subseteq \tau) \wedge \forall \tau' \in T(lh(\tau') = n \rightarrow \sigma^\frown\langle 1\rangle \not\subseteq \tau')\big)\}$. Then B and C are disjoint Σ^0_1 definable sets. We write $T_{B,C}$ for a tree T' such that $[T'] = \{X \in \mathcal{P} : B \subseteq X \subseteq C^c\}$. (Cf. Lemma IV.4.4 [23].) Let $b^i_n = \bigcup\{\tau : \tau(n) = i \wedge lh(\tau) = n+1\}$ for $i = 0, 1$ and $n \in M$. Let F be an (M,S)-continuous function from $[T_{B,C}]$ to $[T]$ with code f such that for each $\sigma \in 2^{<M}$,

$$f(\sigma) = \Big(\bigcap_{\sigma'^\frown\langle 1\rangle \subseteq \sigma} b^1_{\sigma'}\Big) \cap \Big(\bigcap_{\sigma'^\frown\langle 0\rangle \subseteq \sigma} b^0_{\sigma'}\Big).$$

We can show that there exists a formula Φ of $\mathcal{L}_1(R)$ with one free variable such that

$$n \in B \rightarrow \mathsf{Q}_A \vdash \Phi(\underline{\mathrm{n}}), \qquad n \in C \rightarrow \mathsf{Q}_A \vdash \neg\Phi(\underline{\mathrm{n}}),$$

where $\underline{\mathrm{n}}$ is the numeral for n. (Cf. Theorem III.1.23 [12].) Let g be a function in S such that $g(n)$ is the Gödel number of $\Phi(\underline{\mathrm{n}})$. Then, let F' be an (M,S)-continuous function from $[T_{\mathsf{Q}_A}]$ to $[T_{B,C}]$ with code f' such that for each $\sigma \in 2^{<M}$,

$$f'(\sigma) = \Big(\bigcap_{\substack{n<lh(\sigma)\\ \sigma(n)=1}} b^1_{g(n)}\Big) \cap \Big(\bigcap_{\substack{n<lh(\sigma)\\ \sigma(n)=0}} b^0_{g(n)}\Big).$$

Therefore $F(F')$ is an (M,S)-continuous function from $[T_{\mathsf{Q}_A}]$ to $[T]$. $\square$

G is said to be *universally generic* over (M,S) if there exists a universal tree $T \in \mathcal{T}$ such that G is a generic path through T.

Lemma 2.29 *Let (M,S) be a countable model of RCA_0 such that $S = \Delta^0_1(\{A\})$ for some $A \in S$. Let φ be a Σ^1_1 sentence of $\mathcal{L}_2(M,S)$. If G is universally generic over (M,S) and P is generic over (M,S), then $S[P] \models \varphi \Rightarrow S[G] \models \varphi$. In particular, if G_1 and G_2 are* *universally generic over (M,S), then $S[G_1]$ and $S[G_2]$ satisfy the same Σ^1_1 sentences of $\mathcal{L}_2(M,S)$.*

Proof. Let (M,S) be a countable model of RCA_0 such that $S=\Delta^0_1(\{A\})$ for some $A\in S$. Let φ be a Σ^1_1 sentence of $\mathcal{L}_2(M,S)$. Let G be a universally generic path over (M,S) and P be a generic path over (M,S). Suppose that $S[P]\models\varphi$. Then $(M,S)\models T\Vdash\varphi$ for some $T\in\mathcal{T}_S$ such that $P\in[T]$. By way of contradiction, we assume that $S[G]\not\models\varphi$. Then there exists a universal tree T' such that $G\in[T']$ and $T'\Vdash\neg\varphi$. Let F be an (M,S)-continuous function F from $[T']$ to $[T]$. By Lemma 2.27, there exists a generic path G' through T' such that $F(G')$ is a generic path through T. Then $S[F(G')]\models\varphi$. Since $S[F(G')]\subseteq S[G']$, we have $S[G']\models\varphi$, which contradicts with $T'\Vdash\neg\varphi$. □

We now recall another important characterization of models of WKL_0.

Theorem 2.30 *There is a Π^0_1 formula $\psi(X,Y)$ with no free variables other than X and Y such that for any model (M,S) of WKL_0 and for any $A\in S$,*

(1) *there exists $W\in S$ such that $(M,S)\models\psi(A,W)$, and*

(2) *if $(M,S)\models\psi(A,W)$, then $(M,\{(W)_n : n\in M\})\models\mathsf{WKL}_0$ and $A\in\{(W)_n : n\in M\}$, where $(W)_n=\{k\in M:(k,n)\in W\}$ for each $n\in M$.*

Proof. See Lemma VII.2.9 [23]. □

The above theorem essentially says that WKL_0 proves the existence of a structure satisfying WKL_0. We notice that this assertion does not conflict with Gödel's second incompleteness theorem, since the structure need not be equipped with the satisfaction relation. Though the essence of the theorem is a kind of folklore, the above particular statement is due to S. Simpson [23]. See also [17], [19] for other accounts.

Theorem 2.31 *Let $\varphi(X,Y)$ be an arithmetical formula with exactly the free variables shown. If WKL_0 proves $\forall X\exists!Y\varphi(X,Y)$, then so does RCA_0. (Then, RCA_0 also proves $\forall X\exists Y(Y\in\Delta^0_1(\{X\})\wedge\varphi(X,Y))$.)*

Proof. Let $\varphi(X,Y)$ be an arithmetical formula with exactly the free variables shown. Suppose that WKL_0 proves $\forall X\exists!Y\varphi(X,Y)$. By way of contradiction, we assume RCA_0

does not prove it. Then by Gödel's completeness theorem, there exists a countable model (M, S) of RCA_0 in which $\neg\exists! Y \varphi(A, Y)$ holds for some $A \in S$.

Case 1. First suppose that $\exists Y \varphi(A, Y)$ holds in (M, S). Then there exists more than one set in S which satisfies φ. By Lemma 2.12, there exists a model (M, S') of WKL_0 such that $S \subseteq S'$. Since S' has at least two distinct sets which satisfy φ, WKL_0 does not prove $\forall X \exists! Y \varphi(X, Y)$, which is a contradiction.

Case 2. Next assume that $\forall Y \neg\varphi(A, Y)$ holds in (M, S). Let $S_0 = \Delta^0_1(\{A\})$. Then $\forall Y \neg\varphi(A, Y)$ holds in (M, S_0). By Lemma 2.15, there exist universally generic paths G and H over (M, S_0) such that $\Delta^0_1(\{A, G\}) \cap \Delta^0_1(\{A, H\}) = S_0$.

By Theorem 2.30, there exists a Π^0_1 formula $\psi(Y)$ with no set parameter but A such that for any countable model (M, S') of WKL_0 such that $S_0 \subseteq S'$,

(1) there exists $W \in S'$ such that $(M, S') \models \psi(A, W)$, and

(2) if $(M, S') \models \psi(A, W)$, then $(M, \{(W)_n : n \in M\}) \models \mathsf{WKL}_0$ and $A \in \{(W)_n : n \in M\}$.

By the normal form theorem, we write $\psi(Y)$ as $\forall n \theta(Y[n])$ where θ is Σ^0_0. Let T be the tree of $\tau \in 2^{<M}$ such that $\forall \sigma \subseteq \tau \theta(\sigma)$. Then $T \in S_0$, and by (1) and Lemma 2.12, T is infinite.

Since G and H are universally generic, both $S_0[G]$ and $S_0[H]$ satisfy that T has paths. Let B_1 and B_2 be paths through T in $S_0[G]$ and $S_0[H]$, respectively. Let $S_1 = \{(B_1)_n : n \in M\}$ and $S_2 = \{(B_2)_n : n \in M\}$. Then, (M, S_1) and (M, S_2) are countable models of WKL_0. Let $Y_i \in S_i$ be such that (M_i, S) satisfies $\varphi(A, Y_i)$, $(i = 1, 2)$. Therefore, $S_0[G] \models \varphi(A, Y_1)$ and $S_0[H] \models \varphi(A, Y_2)$. In the same way as case 1, $S_0[G]$ and $S_0[H]$ satisfy $\exists! Y \varphi(A, Y)$.

Then, by Lemma 2.29, for each n in M,

$$n \in Y_1 \Leftrightarrow S_0[G] \models \exists Y(\varphi(A, Y) \wedge n \in Y)$$

$$\Leftrightarrow S_0[H] \models \exists Y(\varphi(A, Y) \wedge n \in Y) \Leftrightarrow n \in Y_2.$$

Then $Y_1{=}Y_2$, that is, Y_1 in (M, S_0). Since φ is arithmetical and (M, S_1) satisfies $\varphi(A, Y_1)$, (M, S_0) satisfies $\varphi(A, Y_1)$. This is a contradiction. □

Finally we show a variant of Theorem 2.31.

Theorem 2.32 *Let $\varphi(X,Y)$ be an arithmetical formula with exactly the free variables shown. If* $\mathsf{WKL}_0+\Sigma^0_\infty$ *induction proves $\forall X \exists! Y \varphi(X,Y)$, then so does* $\mathsf{RCA}_0+\Sigma^0_\infty$ *induction.*

Proof. This proof does not use a homogeneous tree argument. Let $\varphi(X,Y)$ be an arithmetical formula with exactly the free variables shown. Suppose that $\mathsf{WKL}_0+\Sigma^0_\infty$ induction proves $\forall X \exists! Y \varphi(X,Y)$. By way of contradiction, we assume $\mathsf{RCA}_0+\Sigma^0_\infty$ induction does not prove it. Then by Gödel's completeness theorem, there exists a countable model (M,S) of $\mathsf{RCA}_0+\Sigma^0_\infty$ induction in which $\neg\exists! Y \varphi(A,Y)$ holds for some $A \in S$.

We may assume that $S = \Delta^0_1(\{A\})$ and no set in S satisfies φ. Let S' be the set of arithmetical definable sets with parameters from $M \cup S$. Since a formalized version of Kreisel's hard core theorem is provable within ACA_0 (Corollary VIII.2.26 [23]), there exist W, $W' \in S'$ such that $(M,\{(W)_n\})$ and $(M,\{(W')_n\})$ are models of WKL_0 and $\{(W)_n\} \cap \{(W')_n\} = S$. Since there exists no set C in S such that $(M,S) \models \varphi(A,C)$, S' must have at least two sets B_1 and B_2 such that $(M,S') \models \varphi(A,B_1)$ and $\varphi(A,B_2)$. So, $(M,S') \models \neg\exists! Y \varphi(n,A,Y)$. Since $(M,S') \models \mathsf{ACA}_0$, this contradicts with the assumption that WKL_0 proves $\forall x \forall X \exists! Y \varphi(x,X,Y)$. Thus the proof is completed. □

3 Reverse mathematics

This chapter is a contribution to *Reverse Mathematics*, an ongoing program to determine which set existence axioms are needed to prove particular theorems of ordinary mathematics [23]. In Section 3.1, we show within RCA_0 that weak König's lemma is necessary and sufficient to prove that any (separable) compact group has a Haar measure. Within WKL_0, a Haar measure is constructed by a non-standard method based on a theorem due to Tanaka [26] that every countable non-standard model of WKL_0 has a proper initial part isomorphic to itself. In Section 3.2, we investigate the logical strength of completeness theorems for intuitionistic logic along the program of reverse mathematics. In Section 3.3, we develop a basic part of real analysis within weaker second-order subsystems than RCA_0, e.g., BTFA. Among others, we show within BTFA that a version of the maximum principle is equivalent to Σ^b_1-CA.

3.1 The existence of Haar measure

Haar measure has an important role in the foundations of real analysis, and also relates to a famous problem of Hilbert (i.e., the fifth of his twenty-three problems). The existence of Haar measure was first shown by Haar in 1933 for locally compact groups which are second countable, and subsequently by von Neumann for compact groups. As explained in a classical textbook [21] of Pontryagin, von Neumann's proof essentially depends on the Arzela-Ascoli lemma, which can not be proved within WKL_0. Succeedingly, Weil, Cartan and others invented simpler proofs and yet for more general families of groups. Although some proofs do not use the Arzela-Ascoli argument, none seems to be free from the notion of sup or limit, which also requires the existence axioms beyond WKL_0.

Later, Bishop [3] modifies Cartan's proof by a certain approximation trick to obtain his constructive version. By contrast, Hauser [14] and others simplify Weil's proof by way of non-standard analysis. Inspired by both of these disparate proofs (constructive and non-standard), we here manage to construct a Haar measure in WKL_0.

3.1.1 Haar measure and its finite approximations

We are working within RCA_0 unless otherwise stated. A *complete separable metric space* $\widehat{A}$ is coded by a set $A \subseteq \mathbb{N}$ together with a pseudo-metric $d : A \times A \to \mathbb{R}$. A *point* in $\widehat{A}$ is a sequence $\langle a_n : n \in \mathbb{N}\rangle$ from A such that $d(a_n, a_{n+i}) < 2^{-n}$ for each $n, i \in \mathbb{N}$. A complete separable metric space is *compact* if there exists an infinite sequence $\langle\langle a_{i,j} \in A : i \leq n_j\rangle : j \in \mathbb{N}\rangle$ of finite sequences of points in A such that for each j, $\langle a_{i,j} : i \leq n_j\rangle$ is a 2^{-j}-net, i.e., $\forall a \in \widehat{A} \exists i \leq n_j [d(a, a_{i,j}) < 2^{-j}]$. RCA_0 proves that the unit interval $[0, 1]$ is compact in this sense, but does not that $[0, 1]$ has the Heine-Borel property.

A triple $b = \langle a, r, s\rangle \in A \times \mathbb{Q} \times \mathbb{Q}$ with $0 \leq s < r$ encodes a *basic function* $b : \widehat{A} \to \mathbb{R}$ defined by

$$b(x) = \begin{cases} 1 & \text{if } d(a, x) \leq s, \\ \dfrac{r - d(a, x)}{r - s} & \text{if } s < d(a, x) < r, \\ 0 & \text{if } d(a, x) \geq r. \end{cases}$$

Then a finite sequence $p = \langle\langle q_n, b_n\rangle : n \leq m\rangle$ encodes a *polynomial* $p(x) = \sum_{n=0}^{m} q_n b_n(x)$, where q_n's are rationals and b_n's are basic functions.

Let P be the set of all (codes for) polynomials. Assuming that $\widehat{A}$ is compact, P can be seen as a countable vector space over $\mathbb{Q}$ equipped with the sup-norm $||p|| = \sup\{|p(x)| : x \in \widehat{A}\}$. Finally, by $C(\widehat{A})$, we mean the separable Banach space $\widehat{P}$. A point in $C(\widehat{A})$ can be regarded as a continuous function $f : \widehat{A} \to \mathbb{R}$ in the obvious way, and moreover it has a modulus h of uniform continuity, i.e., such that for each n and each x, $y \in \widehat{A}$, $d(x, y) \leq 2^{-h(n)} \to |f(x) - f(y)| \leq 2^{-n}$.

Now we define a compact group as follows.

Definition 3.1 The following definition is made in RCA_0. A compact metric space $\widehat{G}$ is called a compact group if it is equipped with an element $e \in G$ and continuous functions ${}^{-1} : \widehat{G} \to \widehat{G}$, $\cdot : \widehat{G} \times \widehat{G} \to \widehat{G}$ such that $(\widehat{G},\ e,\ {}^{-1},\ \cdot)$ satisfies the axioms of groups.

Let $\widehat{G}$ be a compact group. A *measure* μ on $\widehat{G}$ is defined to be a positive bounded linear functional on $C(\widehat{G})$ such that $\mu(1) = 1$. For each $f \in C(\widehat{G})$ and $s \in \widehat{G}$, let f^s denote the continuous function defined by $f^s(x) = f(sx)$. Then a measure μ on $\widehat{G}$ is called *left-invariant* if $\mu(f^s) = \mu(f)$ whenever $f, f^s \in C(\widehat{G})$. For example, the unit circle S^1 is regarded as a compact group with a left-invariant measure in RCA_0. Finally, the countable additivity of measure is defined as usual. See [29].

Definition 3.2 The following definition is made in RCA_0. A measure μ on $\widehat{G}$ is called a Haar measure if μ is a countably additive left-invariant measure.

We shall use the symbol $C(\widehat{G})^+$ informally to denote the set of positive functions in $C(\widehat{G})$. A standard construction of a Haar measure calls for the concept of least upper bound such as

$$(f : g) = \inf\{\sum_{i=0}^{n} a_i : f \leq \sum_{i=0}^{n} a_i g^{s_i} \text{ for some } s_i \in \widehat{G} \text{ and } a_i \geq 0\}$$

for f, $g \in C(\widehat{G})^+$. But, in RCA_0 (or WKL_0), the existence of "inf" cannot be guaranteed.

From now on, we assume that the group operations of $\widehat{G}$ are uniformly continuous. Then, an approximation to $(f : g)$ exists in RCA_0 as follows.

Lemma 3.3 *The following is provable in* RCA_0. *Let $\widehat{G}$ be a compact group with uniformly continuous operations. Choose any f, $g \in C(\widehat{G})^+$. Then for each positive real $\varepsilon \in \mathbb{R}_{>0}$, there exists a finite sequence $\langle a_i : i \leq n\rangle$ of non-negative reals such that*

(1) *there exists a sequence $\langle s_i : i \leq n\rangle$ from G such that $f \leq \sum_{i=0}^{n} a_i g^{s_i}$, and*

(2) *if a sequence $\langle c_i : i \leq m\rangle$ of non-negative reals and a sequence $\langle v_i : i \leq m\rangle$ of points in $\widehat{G}$ satisfy $f \leq \sum_{i=0}^{m} c_i g^{v_i}$, then $\sum_{i=0}^{n} a_i \leq \sum_{i=0}^{m} c_i + \varepsilon$*

Proof. Working in RCA_0. Take any f, $g \in C(\widehat{G})^+$. Since $g \neq 0$, there are $r > 0$ and $t \in G$ such that $2r < g(t)$. Since the operations are uniformly continuous, there exists $\delta_1 > 0$ such that $d(x, y) \leq \delta_1 \to r < g(tx^{-1}y)$.

Let $\langle t'_i : i \leq k\rangle$ be a δ_1-net of $\widehat{G}$, i.e., $\forall x \in \widehat{G} \exists i \leq k[d(t'_i, x) \leq \delta_1]$. Then for each $x \in \widehat{G}$, there exists $i \leq k$ such that $r < g(t' t_i^{-1} x)$. Now, we write t_i by tt'^{-1}_i. Then for each $x \in \widehat{G}$, there exists $i \leq k$ such that $r < g(t_i x)$. Hence we have,

$$\text{(i) } r < \sum_{i=0}^{k} g(t_i x).$$

Without loss of generality, we may assume that the t_i's are taken from G.

Choose a rational number M such that $||f|| < M$. So, we have

$$\text{(ii) } f(x) \leq r^{-1}M\sum_{i=0}^{k} g(t_i x).$$

Fix any $\varepsilon > 0$. Then take $\delta < (1+r^{-2}M(k+1)^2)^{-1}\varepsilon$. Since $g(xy)$ is uniformly continuous and $\widehat{G}$ is compact, there exists a finite sequence $\langle u_i : i \leq l\rangle$ from G such that

$$\text{(iii) } \forall s \in \widehat{G}\exists i \leq l\forall x \in \widehat{G}[g(sx) \leq g(u_i x) + \delta].$$

Choose $J \in \mathbb{N}$ such that $J > \delta^{-1}(l+1)r^{-1}M(k+1)$. By bounded Π^0_1-comprehension, we define a set Φ as follows:

$$\begin{aligned}\Phi = \{\langle j_0, \ldots, j_l\rangle \in \{0, \ldots, J\}^{l+1} :\\ \forall x \in G[f(x) \leq \delta(l+1)^{-1}\sum_{i=0}^{l} j_i g(u_i x) + \delta r^{-2}M(k+1)\sum_{i=0}^{k} g(t_i x)]\}.\end{aligned}$$

Then $\Phi \neq \phi$, since

$$\begin{aligned} f(x) &\leq r^{-1}M\sum_{i=0}^{k} g(t_i x) \text{ (by (ii))}\\ &\leq r^{-1}M(k+1)\sum_{i=0}^{l} g(u_i x) + \delta r^{-1}M(k+1) \text{ (by (iii))}\\ &\leq \delta(l+1)^{-1}\sum_{i=0}^{l} J g(u_i x) + \delta r^{-2}M(k+1)\sum_{i=0}^{k} g(t_i x) \text{ (by (i) and the choice of } J \text{).}\end{aligned}$$

Choose $\langle j_i : i \leq l\rangle \in \Phi$ with the least $\sum_{i=0}^{l} j_i$. And, let

$$a_i = \begin{cases} \delta(l+1)^{-1}j_i, & \text{if } i \leq l,\\ \delta r^{-2}M(k+1), & \text{if } l+1 \leq i \leq k+l+1.\end{cases}$$

Then $\langle a_i : i \leq k+l+1\rangle$ satisfies property (1) of Lemma 3.3 with $\langle s_i\rangle = \langle u_i\rangle^\frown\langle t_i\rangle$.

Next, to show $\langle a_i\rangle$ satisfies the property (2), assume that $f(x) \leq \sum_{i=0}^{m} c_i g(v_i x)$. By (iii), there exists $\langle d_i : i \leq l\rangle$ such that $\sum_{i=0}^{m} c_i = \sum_{i=0}^{l} d_i$ and

$$\sum_{i=0}^{m} c_i g(v_i x) \leq \sum_{i=0}^{l} d_i g(u_i x) + \delta\sum_{i=0}^{l} d_i.$$

First consider the case $\sum_{i=0}^{m} c_i (= \sum_{i=0}^{l} d_i) \leq r^{-1}M(k+1)$. Since $1 < r^{-1}\sum_{i=0}^{k} g(t_i x)$ by (i),

$$f(x) \leq \sum_{i=0}^{l} d_i g(u_i x) + \delta r^{-2}M(k+1)\sum_{i=0}^{k} g(t_i x).$$

For each $i \leq l$, let $j'_i = \min\{j \in \mathbb{N} : j \leq J \wedge d_i \leq \delta(l+1)^{-1}j\}$. Clearly, $\langle j'_i : i \leq l\rangle \in \Phi$. So,

$$\begin{aligned}
\sum_{i=0}^{k+l+1} a_i &= \delta(l+1)^{-1}\sum_{i=0}^{l} j_i + \delta r^{-2}M(k+1)^2 \\
&\leq \delta(l+1)^{-1}\sum_{i=0}^{l} j'_i + \delta r^{-2}M(k+1)^2 \\
&\leq \sum_{i=0}^{l}(d_i + \delta(l+1)^{-1}) + \delta r^{-2}M(k+1)^2 (\text{ since } \delta(l+1)^{-1}(j'_i - 1) < d_i \text{ }) \\
&\leq \sum_{i=0}^{l} d_i + \delta(1 + r^{-2}M(k+1)^2) \leq \sum_{i=0}^{m} c_i + \varepsilon.
\end{aligned}$$

Secondly, consider the case $r^{-1}M(k+1) \leq \sum_{i=0}^{m} c_i$. By (ii), $\langle c'_i : i \leq k\rangle = \langle r^{-1}M : i \leq k\rangle$ satisfies $f(x) \leq \sum_{i=0}^{m} c'_i g(t_i x)$ and $\sum_{i=0}^{m} c'_i = r^{-1}M(k+1)$. Hence, by the above argument, $\sum_{i=0}^{k+l+1} a_i \leq \sum_{i=0}^{m} c'_i + \varepsilon = r^{-1}M(k+1) + \varepsilon \leq \sum_{i=0}^{m} c_i + \varepsilon$. $\square$

For each $\varepsilon \in \mathbb{R}_{>0}$ and each f, $g \in C(\widehat{G})^+$, $(f : g)^\varepsilon$ is defined to be $\sum_{i=0}^{n} a_i$ where a_i's are given in the above lemma. We may assume that $(f : g)^\varepsilon$ is rational.

Lemma 3.4 *The followings are provable in* RCA_0. *Choose any* f_1, f_2 *and* $g \in C(\widehat{G})^+$. *Then,*

(1) *for each* $\varepsilon > 0$, $(f_1 : g)^\varepsilon \geq \dfrac{||f_1||}{2||g||}$;

(2) *if* $f_1 \leq f_2$, *then for each* $\varepsilon > 0$, $(f_1 : g)^\varepsilon \leq (f_2 : g)^\varepsilon + \varepsilon$;

(3) *for each* $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, $(f_1 : g)^\varepsilon \leq (f_1 : f_2)^{\varepsilon_1}(f_2 : g)^{\varepsilon_2} + \varepsilon$;

(4) *for each* $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, $(f_1 + f_2 : g)^\varepsilon \leq (f_1 : g)^{\varepsilon_1} + (f_2 : g)^{\varepsilon_2} + \varepsilon$;

(5) *for each* $\varepsilon, \lambda > 0$, $|(\lambda f_1 : g)^\varepsilon - \lambda(f_1 : g)^\varepsilon| \leq (\lambda + 1)\varepsilon$;

(6) *for each* $\varepsilon > 0$ *and each* $s \in \widehat{G}$, $|(f_1^s : g)^\varepsilon - (f_1 : g)^\varepsilon| \leq \varepsilon$.

Proof. (1) Since $f_1 \neq 0$, there exists $\alpha \in \widehat{G}$ such that $\dfrac{||f_1||}{2} < f(\alpha)$. If $f_1(x) \leq \sum_i a_i g(s_i x)$, then

$$\frac{||f_1||}{2} < f(\alpha) \leq \sum_i a_i g(s_i\alpha) \leq ||g|| \sum_{i=0}^{n} a_i.$$

Hence $(f_1 : g)^\varepsilon \geq \dfrac{||f_1||}{2||g||}$.

(2) is trivial. For (3), assume that $(f_1 : f_2)^\varepsilon = \sum_i a_i$ and $(f_2 : g)^{\varepsilon_2} = \sum_j c_j$ where $f_1(x) \leq \sum_i a_i f_2(s_i x)$ and $f_2(x) \leq \sum_j c_j g(t_j x)$. Then $f_1(x) \leq \sum_{i,j} a_i c_j g(s_i t_j x)$. Since $\sum_{i,j} a_i c_j \leq \sum_i a_i \sum_j c_j$, (3) holds. (4), (5) and (6) can be treated similarly. □

Now we define $I_g^\varepsilon(f)$ by $\dfrac{(f : g)^\varepsilon}{(1 : g)^\varepsilon}$. We say g is *small of order c* if $g(x) = 0$ whenever $d(x, e) \geq c$. We are going to show that $I_g^\varepsilon(f)$ is "approximate" to the Haar measure when g is sufficiently small.

Lemma 3.5 *The following is provable in* RCA_0. *For each $\varepsilon \in \mathbb{R}_{>0}$ and each $f_0, \dots, f_n \in C(\widehat{G})^+$, there exists $c \in \mathbb{R}_{>0}$ such that if $g \in C(\widehat{G})^+$ with $||g|| = 1$ is small of order c, then for each sufficiently small $\varepsilon' \in \mathbb{R}_{>0}$ and for each $0 \leq \lambda_i \leq 1$,*

$$\sum_{i=0}^n \lambda_i I_g^{\varepsilon'}(f_i) \leq I_g^{\varepsilon'}(\sum_{i=0}^n \lambda_i f_i) + \varepsilon.$$

Proof. Fix any $\varepsilon < \dfrac{1}{2}$. Let $h_j^\lambda = \dfrac{f_j}{\sum_{i=0}^n \lambda_i f_i + \varepsilon}$. It is easy to see that all the h_j^λ's have a common modulus of uniform continuity independent from the choice of j and $\lambda = \langle \lambda_i : i \leq n \rangle$ (with $0 \leq \lambda_i \leq 1$). Take $M > \sum_{i=0}^n ||f_i|| + 3$. Then there exists $c > 0$ such that for each λ such that $0 \leq \lambda_i \leq 1$,

$$(1)\ \forall j{\leq}n \forall s, x \in \widehat{G}(d(sx, e) < c \to |h_j^\lambda(s^{-1}) - h_j^\lambda(x)| < \frac{\varepsilon}{(n+1)M}).$$

Suppose that a $g \in C(\widehat{G})^+$ is small of order c and $||g|| = 1$. For each $\langle c_k \rangle$, $\langle s_k \rangle$ such that $\sum_{i=0}^n \lambda_i f_i + \varepsilon \leq \sum_{k=0}^m c_k g^{s_k}$, we have

$$\begin{aligned} f_j = h_j^\lambda \cdot (\sum_{i=0}^n \lambda_i f_i + \varepsilon) &\leq h_j^\lambda \sum_{k=0}^m c_k g^{s_k} \\ &\leq \sum_{k=0}^m c_k (h_j^\lambda(s_j^{-1}) + \frac{\varepsilon}{(n+1)M}) g^{s_k} \end{aligned}$$

by (1). Choose any $\varepsilon' < \dfrac{\varepsilon}{\max(M, n+1)}$. Then by Lemma 3.3 (2),

$$(f_j : g)^{\varepsilon'} \leq \sum_{k=0}^m c_k (h_j^\lambda(s_j^{-1}) + \frac{\varepsilon}{(n+1)M}) + \frac{\varepsilon}{(n+1)}.$$

Therefore,

$$\sum_{i=0}^{n} \lambda_i (f_i : g)^{\varepsilon'} \leq \sum_{k=0}^{m} c_k (1 + \frac{\varepsilon}{M}) + \varepsilon.$$

Now take a $\langle c_j \rangle$ such that $(\sum_{i=0}^{n} \lambda_i f_i + \varepsilon : g)^{\varepsilon'} = \sum_j c_j$ and $\sum_{i=0}^{n} \lambda_i f_i + \varepsilon \leq \sum_j c_j g^{s_j}$ for some $\langle s_j \rangle$. So by Lemma 3.4,

$$\sum_{i=0}^{n} \lambda_i (f_i : g)^{\varepsilon'} \leq [(\sum_{i=0}^{n} \lambda_i f_i : g)^{\varepsilon'} + \varepsilon (1 : g)^{\varepsilon'} + \varepsilon](1 + \frac{\varepsilon}{M}) + \varepsilon.$$

Dividing the both sides of the above inequality by $(1 : g)^{\varepsilon'}$, we have

$$\begin{aligned} (2) \quad \sum_{i=0}^{n} \lambda_i I_g^{\varepsilon'}(f_i) &\leq (I_g^{\varepsilon'}(\sum_{i=0}^{n} \lambda_i f_i) + \varepsilon + \frac{\varepsilon}{(1 : g)^{\varepsilon'}})(1 + \frac{\varepsilon}{M}) + \frac{\varepsilon}{(1 : g)^{\varepsilon'}} \\ &\leq I_g^{\varepsilon'}(\sum_{i=0}^{n} \lambda_i f_i) + \varepsilon (\frac{I_g^{\varepsilon'}(\sum_{i=0}^{n} \lambda_i f_i)}{M} + 6) (\text{since } \frac{1}{2} \leq (1 : g)^{\varepsilon'} \text{by Lemma 3.4}). \end{aligned}$$

Since $\sum_{i=0}^{n} \lambda_i f_i \leq \sum_{i=0}^{n} ||f_i||$, by Lemma 3.4,

$$(\sum_{i=0}^{n} \lambda_i f_i : g)^{\varepsilon'} \leq \sum_{i=0}^{n} ||f_i|| (1 : g)^{\varepsilon'} + \varepsilon.$$

Then,

$$I_g^{\varepsilon'}(\sum_{i=0}^{n} \lambda_i f_i) \leq \sum_{i=0}^{n} ||f_i|| + 2\varepsilon \leq M.$$

By (2) and the above inequality, we finally obtain

$$\sum_{i=0}^{n} \lambda_i I_g^{\varepsilon'}(f_i) \leq I_g^{\varepsilon'}(\sum_{i=0}^{n} \lambda_i f_i) + 7\varepsilon.$$

□

Lemma 3.6 *The following is provable in* RCA_0. *Let* C *be a finite subset of* $C(\widehat{G})^+$. *Given* $\varepsilon \in \mathbb{R}_{>0}$, *then there exists a* $g \in C(\widehat{G})^+$ *with* $||g|| = 1$ *such that for each* $f_1, f_2 \in C$ *and each sufficiently small* $\varepsilon' \in \mathbb{R}_{>0}$,

(1) *if* $f_1 \leq f_2$, *then* $I_g^{\varepsilon'}(f_1) \leq I_g^{\varepsilon'}(f_2) + \varepsilon$;

(2) $|I_g^{\varepsilon'}(f_1 + f_2) - (I_g^{\varepsilon'}(f_1) + I_g^{\varepsilon'}(f_2))| < \varepsilon$;

(3) *if* $f_1^s = f_2$ *with* $s \in \widehat{G}$, *then* $|I_g^{\varepsilon'}(f_1) - I_g^{\varepsilon'}(f_2)| < \varepsilon$;

(4) *if* $\lambda f_1 = f_2$ *with* $\lambda \in \mathbb{R}$, *then* $|\lambda I_g^{\varepsilon'}(f_1) - I_g^{\varepsilon'}(f_2)| < (\lambda + 1)\varepsilon$.

Proof. Fix any $\varepsilon \in \mathbb{R}_{>0}$. By Lemma 3.5, we can choose a $g \in C(\widehat{G})^+$ with $||g|| = 1$ such that for each sufficiently small $\varepsilon' \in \mathbb{R}_{>0}$ ($\varepsilon' < \dfrac{\varepsilon}{2}$),

$$I_g^{\varepsilon'}(f_1) + I_g^{\varepsilon'}(f_2) \leq I_g^{\varepsilon'}(f_1 + f_2) + \varepsilon$$

for each $f_1, f_2 \in C$. Since $\dfrac{1}{2} \leq (1 : g)^{\varepsilon'}$ by Lemma 3.4,

$$I_g^{\varepsilon'}(f_1 + f_2) \leq I_g^{\varepsilon'}(f_1) + I_g^{\varepsilon'}(f_2) + \varepsilon$$

Similarly, by (6) (resp.(5)) of Lemma 3.4, if $f_1^s = f_2$ for some $s \in \widehat{G}$ (resp. $\lambda f_1 = f_2$ for some $\lambda \in \mathbb{R}$),

$$|I_g^{\varepsilon'}(f_1) - I_g^{\varepsilon'}(f_2)| < \varepsilon \text{ (resp. } |\lambda I_g^{\varepsilon'}(f_1) - I_g^{\varepsilon'}(f_2)| < (\lambda + 1)\varepsilon).$$

□

3.1.2 A non-standard method in WKL_0

By $V = (M, S)$, we denote a structure of second-order arithmetic, where M is an ordered semiring and S consists of subsets of M. For an initial segment I of M, we put $S\lceil I = \{X \cap I : X \in S\}$ and $V\lceil I = (I, S\lceil I)$.

In [26], we have shown

Theorem 3.7 (the self-embedding theorem) *Let V be a countable non-standard model of WKL_0. Then there exists a proper initial part $V\lceil I$ of V and an isomorphism $f : V \to V\lceil I$.*

Fix a countable non-standard model V of WKL_0. By the above theorem, V has an initial part isomorphic to itself. Since the initial part and V are isomorphic to each other, they may exchange their roles, and thus we can say that V has an isomorphic extension ${}^*V = ({}^*M, {}^*S)$. We shall use ${}^*V$ as a non-standard universe.

Let f be a function from $\mathbb{N}$ to $\mathbb{R}$ in V. Rigorously, f is coded by its graph $F \subseteq \mathbb{N} \times \mathbb{R} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{Q}$. Then, F must satisfy the following conditions: for each $m \in M$,

(1.1)$\forall i \leq m \forall n \leq m \exists q \in \mathbb{Q}(\langle i, n, q\rangle \in F)$;

(1.2)$\forall i \leq m \forall n \leq m \forall q_1, q_2 \leq m(\langle i, n, q_1\rangle, \langle i, n, q_2\rangle \in F \to q_1 = q_2)$;

(1.3)$\forall i \leq m \forall n_1, n_2 \leq m \forall q_1, q_2 \leq m(\langle i, n_1, q_1\rangle, \langle i, n_2, q_2\rangle \in F \wedge n_1 \leq n_2$
$\qquad \to |q_1 - q_2| < 2^{-n_1})$,

where q_1, q_2 in the bounded quantifiers are treated as their codes. Let ${}^*\mathbb{Q}$ be the set of rationals in ${}^*V$. Take a set ${}^*F \in {}^*S$ such that $F = {}^*F\lceil M$. Fix a non-standard element $\alpha \in {}^*M$. Then, for each $m \in M$, it holds in ${}^*V$ that

(2.1)$\forall i \leq m \forall n \leq m \exists q \leq \alpha(q \in {}^*\mathbb{Q} \wedge \langle i, n, q\rangle \in {}^*F)$;

(2.2)$\forall i \leq m \forall n \leq m \forall q_1, q_2 \leq m(\langle i, n, q_1\rangle, \langle i, n, q_2\rangle \in {}^*F \rightarrow q_1 = q_2)$;

(2.3)$\forall i \leq m \forall n_1, n_2 \leq m \forall q_1, q_2 \leq m(\langle i, n_1, q_1\rangle, \langle i, n_2, q_2\rangle \in {}^*F \wedge n_1 \leq n_2$
$\rightarrow |q_1 - q_2| < 2^{-n_1})$.

Since (2.1), (2.2) and (2.3) are Σ_0^0, by overspill, there exists a non-standard element $\beta \in {}^*M$(with $\beta \leq \alpha$) such that in ${}^*V$

(3.1)$\forall i \leq \beta \forall n \leq \beta \exists q \leq \alpha(q \in {}^*\mathbb{Q} \wedge \langle i, n, q\rangle \in {}^*F)$;

(3.2)$\forall i \leq \beta \forall n \leq \beta \forall q_1, q_2 \leq \beta(\langle i, n, q_1\rangle, \langle i, n, q_2\rangle \in {}^*F \rightarrow q_1 = q_2)$;

(3.3)$\forall i \leq \beta \forall n_1, n_2 \leq \beta \forall q_1, q_2 \leq \beta(\langle i, n_1, q_1\rangle, \langle i, n_2, q_2\rangle \in {}^*F \wedge n_1 \leq n_2$
$\rightarrow |q_1 - q_2| < 2^{-n_1})$.

For a set $X \in {}^*S$, let $X(m)$ denote $\{n \in X : n \leq m\}$. Then put ${}^*F_0 = {}^*F \cap ({}^*\mathbf{M}(\beta) \times {}^*\mathbf{M}(\beta) \times {}^*\mathbb{Q}(\alpha))$. Since ${}^*F_0$ is a finite subset of ${}^*F$ in ${}^*V$, we can define ${}^*F_1 : {}^*\mathbf{M}(\beta) \times {}^*\mathbf{M}(\beta) \rightarrow {}^*\mathbb{Q}(\alpha)$ as follows:

$${}^*F_1(i, n) = \min\{q : {}^*V \models (i, n, q) \in {}^*F_0\},$$

where "min" means the minimum with respect to codes for rationals. Then we obtain a function ${}^*f$ from ${}^*\mathbf{M}(\beta)$ to ${}^*\mathbb{Q}(\alpha)$ (called an *extension* of f) such that ${}^*f(i) = {}^*F_1(i, \beta)$. Occasionally, we regard ${}^*f(i)$ as a ${}^*V$-finite sequence $\langle {}^*F_1(i, 0), \ldots, {}^*F_1(i, \beta)\rangle$. By noticing that for each $i, n \in M$ and $q \in \mathbb{Q}$, ${}^*V \models {}^*F_1(i, n) = q$ iff $V \models (i, n, q) \in F$, we easily obtain the following lemma.

Lemma 3.8 *Let ${}^*f$ be an extension of a real-valued function f. Then for each $i \in M$ and $q, q' \in \mathbb{Q}$,*

(1) *if $V \models q < f(i) < q'$ then ${}^*V \models q < {}^*f(i) < q'$;*

(2) *Regarding ${}^*f(i)$ as a ${}^*V$-finite sequence, $f(i)$ is an initial segment of ${}^*f(i)$.*

Let $\widehat{A}$ be a complete separable metric space with metric d. Since d is a real-valued function on $A \times A$, d has an extension ${}^*d$. Moreover we can take β such that ${}^*d$ is a pseudo-metric on ${}^*A(\beta) \times {}^*A(\beta)$. We use ${}^*A$ in the place of ${}^*A(\beta)$ for simplicity. Then, we can think that the pseudo-metric space ${}^*A$ includes $\widehat{A}$ in the following sense: For each $x = \langle a_n \rangle \in \widehat{A}$, there exists ${}^*a \in {}^*A$ such that for each $n \in M$, ${}^*V \models |{}^*d(a_n, {}^*a)| < 2^{-n}$. Using this ${}^*d$ instead of $|\ |$ in (2.3) and (3.3), we can define an extension of a sequence from $\widehat{A}$.

Let $d_{2^{<\mathbb{N}}}$ be a metric on $2^{<\mathbb{N}}$ defined by

$$d_{2^{<\mathbb{N}}}(\sigma,\tau) = |\sum_{i<lh(\sigma)} \sigma(i)2^{-i-1} - \sum_{j<lh(\tau)} \tau(j)2^{-j-1}|.$$

The space $\widehat{2^{<\mathbb{N}}}$ can be regarded as a closed unit interval $[0,1]$. Then, for each function f from M to $[0,1]$, an extension ${}^*f$ is a function from a proper initial segment of ${}^*V$ to a set of ${}^*V$-finite 0-1 sequences. By adding suitably many 0's at the end of a sequence, we may suppose that every sequence in the range of ${}^*f$ has the same length $\beta \in {}^*M \setminus M$. Hence, ${}^*[0,1]$ is defined by a set of ${}^*V$-finite 0-1 sequences with length β. Similarly, with a metric $d_k \stackrel{\text{def}}{=} kd_{2^{<\mathbb{N}}}$, ${}^*[0,k]$ is defined by a set of ${}^*V$-finite 0-1 sequences.

Since a continuous function f from $\widehat{A}$ to $\widehat{B}$ is uniquely determined by a sequence $\langle f(a) : a \in A \rangle$, we define an *extension* ${}^*f$ of f to be ${}^*\langle f(a) : a \in A \rangle$. The extension of a continuous function plays a leading part in the arguments of next section.

3.1.3 Haar measure and WKL_0

In this subsection, we describe a construction of Haar measure by a non-standard method within WKL_0. Fix a countable non-standard model V of WKL_0. Choose a compact group $\widehat{G}$ (in V). Then we have, as continuous functions, the norm $||\ ||$ on $C(\widehat{G})$, the operation $Ab : C(\widehat{G}) \to C(\widehat{G})$ defined by $Ab(f)(x) = |f(x)|$ and $L : C(\widehat{G}) \times \widehat{G} \to C(\widehat{G})$ by $L(f,s) = f^s$. Notice that any continuous function is uniformly continuous within WKL_0. And a continuous function has an extension in ${}^*V$, which we shall often denote by the same symbol.

Lemma 3.9 *Let $V = (M,S)$ be a countable non-standard model of WKL_0 and $\widehat{G}$ be a compact group (in V). Then there exists $I : P \to \mathbb{R}_{\geq 0}$ such that:*

(1) *for each non-negative $f = \langle p_i : i \in \mathbb{N} \rangle \in C(\widehat{G})$ with $||f|| \leq 1$, $I(f) \stackrel{\text{def}}{=} \lim_i I(p_i)$ exists.*

(2) *for $f_1, f_2, f_3 \in C(\widehat{G})^+$ with $||f_i|| \leq 1$ $(i = 1,2,3)$,*

(2.1) *if* $f_3 = f_1 + f_2$*, then* $I(f_3) = I(f_1) + I(f_2)$*;*

(2.2) *if* $f_1^s = f_2$ *with* $s \in \widehat{G}$*, then* $I(f_1) = I(f_2)$*;*

(2.3) *if* $\lambda f_1 = f_2$ *with* $\lambda \in \mathbb{R}_{>0}$*, then* $\lambda I(f_1) = I(f_2)$*;*

(2.4) $I(1) = 1$.

Proof. We first define a Σ^0_0-formula $\varphi(\sigma, m)$ with parameters from ${}^*V$ which roughly means that σ is a 2^{-m}-approximation of Haar measure on $\{p \in P(m) : p \text{ is positive}, ||p|| < 2\}$. More precisely, φ asserts the following: σ is a finite sequence from ${}^*[0, 2]$ with length m and, for each $p_1, p_2, p_3 \in {}^*P(m)$ with $||p_i|| < 2$ and each $l \leq m$,

(i) if $||Ab(p_3) - (Ab(p_1) + Ab(p_2))|| < 2^{-l}$, then $|\sigma(p_3) - (\sigma(p_1) + \sigma(p_2))| < 6 \cdot 2^{-l}$;

(ii) if $||Ab(L(p_1, s)) - Ab(p_2)|| < 2^{-l}$ with $s \in {}^*G(m)$, then $|\sigma(p_1) - \sigma(p_2)| < 5 \cdot 2^{-l}$;

(iii) if $||Ab(rp_1) - Ab(p_2)|| < 2^{-l}$ with $r \in {}^*\mathbb{Q}_{>0}(m)$, then $|r\sigma(p_1) - \sigma(p_2)| < (r + 5)2^{-l}$;

(iv) $\sigma(1) = 1$.

Similarly to the proof of Lemma 3.3, the following claim can be shown in WKL_0: Given any $g \in C(\widehat{G})^+$ and any finite sequence $\langle f_i : i \leq n \rangle$ from $C(\widehat{G})^+$, for each $\varepsilon \in \mathbb{R}_{>0}$, there exists a rational sequence $\langle q_i : i \leq n \rangle$ such that $q_i = (f_i : g)^\varepsilon$ for each $i \leq n$.

Fix any $m \in M$. Letting $C = \{Ab(p) : p \in P(m) \text{ and } ||p|| < 2\}$ and $\varepsilon = 2^{-m}$ in Lemma 3.6, we take g and ε' to satisfy the assertion of the lemma. Then there exists a finite sequence σ of length m such that $\sigma(p) = I_g^{\varepsilon'}(Ab(p))$ if $p \in P(m)$ and $||p|| < 2$, and $\sigma(p) = 1$ otherwise. We shall see that σ satisfies $\varphi(\sigma, m)$.

To show that the condition (i) of $\varphi(\sigma, m)$ holds, assume that $p_1, p_2, p_3 \in {}^*P(m)$ with $||p_i|| < 2$ and $||Ab(p_3) - (Ab(p_1) + Ab(p_2))|| < 2^{-l}$ $(l \leq m)$. By Lemma 3.8, the same assertion holds in V. Then using Lemma 3.6, the following two inequalities hold in V:

$$|I_g^{\varepsilon'}(Ab(p_1) + Ab(p_2)) - (I_g^{\varepsilon'}(Ab(p_1)) + I_g^{\varepsilon'}(Ab(p_2)))| < 2^{-m},$$

$$|I_g^{\varepsilon'}(Ab(p_3)) - I_g^{\varepsilon'}(Ab(p_1) + Ab(p_2))| < 2^{-l} + 2^{-m+2}.$$

Then,

$$|I_g^{\varepsilon'}(Ab(p_3)) - (I_g^{\varepsilon'}(Ab(p_1)) + I_g^{\varepsilon'}(Ab(p_2)))| < 6 \cdot 2^{-l}.$$

Again by Lemma 3.8, the last inequality holds in ${}^*V$. Similarly, (ii) to (iv) hold.

Therefore, for each $m \in M$,

$$^*V \models \exists \sigma \varphi(\sigma, m).$$

By overspill, there exists $\gamma \in {}^*M \setminus M$ and σ_0 such that $\varphi(\sigma_0, \gamma)$ holds in $^*V$.

Let $I = \sigma_0 \lceil M$. Since $^*[0,2]$ is regarded as a set of $^*V$-finite 0-1 sequences, I is a $[0,2]$-valued function defined all over P. $I(1) = 1$ is trivial. By the definition of φ and Lemma 3.8, for any positive $p_1, p_2, p_3 \in P$ with $||p_i|| < 2$ and each $l \in M$,

(a) if $||p_3 - (p_1 + p_2)|| < 2^{-l}$, then $|I(p_3) - (I(p_1) + I(p_2))| < 6 \cdot 2^{-l}$;

(b) if $||p_1 - p_2^s|| < 2^{-l}$ with $s \in G$, then $|I(p_1) - I(p_2)| < 5 \cdot 2^{-l}$;

(c) if $||rp_1 - p_2|| < 2^{-l}$ with $r \in \mathbb{Q}_{>0}$, then $|rI(p_1) - I(p_2)| < (r+5)2^{-l}$.

Fix $f \in C(\widehat{G})^+$ with $||f|| \leq 1$. Since f can be taken as a sequence $\langle p_i : i \in M \rangle$ of positive polynomials with $||p_i|| < 2$, $\lim_i I(p_i) = \langle I(p_{i+4})_{i+4} : i \in M \rangle$ is a real by (a). The other properties of I can be easily shown by (a) to (c). □

From the above result, we can easily obtain the following theorem.

Theorem 3.10 *The following is provable in* WKL_0. *Any compact group has a Haar measure.*

Proof. Fix a countable non-standard model V of WKL_0 and fix a compact group $\widehat{G}$ (in V). For each $f \in C(\widehat{G})$, $f^+ = \max(f, 0)$ (resp. $f^- = -\min(f, 0)$) is a non-negative point of $C(\widehat{G})$. Moreover, we can obtain functions: $f \mapsto f^+$, $f \mapsto f^-$. Therefore, using I of Lemma 3.9, we can define a left-invariant measure $\mu : P \to \mathbb{R}$ as follows:

$$\mu(f) = (||f|| + 1)(I(\frac{f^+}{||f|| + 1}) - I(\frac{f^-}{||f|| + 1})).$$

The countable additivity of a measure is provable in WWKL_0 [29]. Thus the proof is completed.□

Now, we have our main results.

Theorem 3.11 *The following assertions are pairwise equivalent over* RCA_0.

(1) WKL_0;

(2) *Any compact group has a Haar measure.*

Proof. Since (1)→(2) is shown by Theorem 3.10, we only need to prove (2)→(1). We reason in RCA_0. Deny WKL_0 and let $T \subset 2^{<\mathbb{N}}$ be an infinite tree with no path.

Let $G = \{\sigma \in T : \sigma = \emptyset \vee (\sigma \neq \emptyset \to \sigma(lh(\sigma)-1) = 1)\}$ and $\widehat{G}$ be a complete separable metric space coded by G and d_{Seq_2}. (d_{Seq_2} was defined at the end of the last section.) If there were $x \in \widehat{G} \setminus G$, x would be a path through T. So, $\widehat{G}$=G. Take a bijection $h : \mathbf{Z} \to G$. Then it induces a group operation on G. Since $\widehat{G}$=G, the operation must be continuous on $\widehat{G}$. For each $n \in \mathbb{N}$, $G_n = \{\sigma \in G : \mathrm{lh}(\sigma) \leq n\}$ is 2^{-n}-net. Hence, $\widehat{G}$ is a compact group.

Now, if $\widehat{G}$ had a Haar measure μ, $\mu(\widehat{G}) = 0$ in the case that $\mu(\{e\}) = 0$, and $\mu(\widehat{G}) = \infty$ otherwise, either of which is a contradiction. So $\widehat{G}$ does not possess a Haar measure. □

3.1.4 Some variations

In this subsection, we eliminate the non-standard argument from the previous construction of Haar measure to obtain some refined assertion. Among others, we show within RCA_0 the existence of a left-invariant measure on a compact group with a modulus of uniform continuity for its operations.

Lemma 3.12 *The following is provable in* RCA_0. *Let* $\widehat{G}$ *be a compact group with a modulus of uniform continuity. Given* $f \in C(\widehat{G})^+$ *and* $\varepsilon \in \mathbb{Q}_{>0}$, *we can effectively find* $c > 0$ *such that if a positive* $p \in P$ *with* $||p|| = 1$ *is small of order* c, *there exist two finite sequences* $\langle c_i : i \leq k\rangle$ *and* $\langle d_i : i \leq k\rangle$ *of nonnegative rationals and a sequence* $\langle s_i : i \leq k\rangle$ *from* G *such that*

$$\max(||f - \sum_{i=0}^{k} c_i p^{s_i}||, ||1 - \sum_{i=0}^{k} d_i p^{s_i}||) < \varepsilon.$$

Proof. Fix any $f \in C(\widehat{G})^+$ and $\varepsilon \in \mathbb{Q}_{>0}$. We may suppose that $\varepsilon < \frac{1}{2}$. Then we can effectively find $c > 0$ such that $d(y^{-1}x, e) \leq c \to |f(x) - f(y)| \leq \frac{\varepsilon}{2}$. We shall see that this c satisfies the condition of lemma.

Assume that a positive $p \in P$ with $||p|| = 1$ is small of order c. Let $\tilde{p}(x) = p(x^{-1})$ and $\eta < \frac{\varepsilon}{6[(f : \tilde{p})^1 + 1]}$. Since p is uniformly continuous, $c' > 0$ such that $d(y^{-1}x, e) \leq c' \to |p(x) - p(y)| \leq \eta$. We take a $\frac{c'}{2}$-net $\langle s_i : i \leq m\rangle$ from G and $h_0, \ldots, h_m \in C(\widehat{G})^+$ such that $\sum_{i=0}^{m} h_i = 1$, and $h_i(x) = 0$ wherever $c' \leq d(s_i x, e)$ (Lemma II.7.3 [23]. Then, for each $x, s \in \widehat{G}$,

$$\begin{aligned} h_i(s)f(s)(p(s^{-1}x) - \eta) &\leq h_i(s)f(s)p(s_i x) \\ &\leq h_i(s)f(s)(p(s^{-1}x) + \eta). \end{aligned}$$

Since p is small of order c,

$$(f(x) - \frac{\varepsilon}{2})p(s^{-1}x) \leq f(s)p(s^{-1}x) \leq (f(x) + \frac{\varepsilon}{2})p(s^{-1}x).$$

Therefore,

$$\begin{aligned} (f(x) - \frac{\varepsilon}{2})p(s^{-1}x) - \eta f(s) &\leq f(s)(p(s^{-1}x) - \eta) \\ &\leq \sum_{i=0}^{m} h_i(s)f(s)p(s_i x) \\ &\leq \sum_{i=0}^{m} h_i(s)f(s)(p(s^{-1}x) + \eta) \\ &\leq (f(x) + \frac{\varepsilon}{2})p(s^{-1}x) + \eta f(s). \end{aligned}$$

Using Lemma 3.4, for each positive $p' \in P$ and each $\varepsilon' > 0$,

$$(f(x) - \frac{\varepsilon}{2})(\tilde{p} : p')^{\varepsilon'} - \eta(f : p')^{\varepsilon'} \leq (\sum_{i=0}^{m} p(s_i x)h_i f : p')^{\varepsilon'} + (||f|| + 8 + \eta + \frac{\varepsilon}{2})\varepsilon'.$$

Similarly,

$$(\sum_{i=0}^{m} p(s_i x)h_i f : p')^{\varepsilon'} \leq (f(x) + \frac{\varepsilon}{2})(\tilde{p} : p')^{\varepsilon'} + \eta(f : p')^{\varepsilon'} + (||f|| + 8 + \eta + \frac{\varepsilon}{2})\varepsilon'.$$

Therefore,

$$\begin{aligned} (1)\ f(x) - \frac{\varepsilon}{2} - \frac{\eta(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} - \frac{M_1\varepsilon'}{(\tilde{p} : p')^{\varepsilon'}} \\ \leq \frac{1}{(\tilde{p} : p')^{\varepsilon'}}(\sum_{i=0}^{m} p(s_i x)h_i f : p')^{\varepsilon'} \\ \leq f(x) + \frac{\varepsilon}{2} + \frac{\eta(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} + \frac{M_1\varepsilon'}{(\tilde{p} : p')^{\varepsilon'}}, \end{aligned}$$

where $M_1 = ||f|| + 9$. Again using Lemma 3.4, since $0 \leq p(s_i x) \leq 1$,

$$(2)\ I_{p'}^{\varepsilon'}(\sum_{i=0}^{m} p(s_i x)h_i f) \leq \sum_{i=0}^{m} p(s_i x)I_{p'}^{\varepsilon'}(h_i f) + \frac{(3m+2)\varepsilon'}{(1 : p')^{\varepsilon'}}.$$

By (1) and (2),

$$(3)\ f(x)-\frac{\varepsilon}{2}-\frac{\eta(f:p')^{\varepsilon'}}{(\tilde{p}:p')^{\varepsilon'}}-\frac{M_1\varepsilon'}{(\tilde{p}:p')^{\varepsilon'}}\leq\frac{1}{I_{p'}^{\varepsilon'}(\tilde{p})}\sum_{i=0}^{m}p(s_ix)I_{p'}^{\varepsilon'}(h_if)+\frac{(3m+2)\varepsilon'}{(\tilde{p}:p')^{\varepsilon'}}.$$

By Lemma 3.5, we take a sufficiently small ε' and a positive polynomial p with $||p||=1$ such that

$$(4)\ \sum_{i=0}^{m}p(s_ix)I_{p'}^{\varepsilon'}(h_if)\leq I_{p'}^{\varepsilon'}(\sum_{i=0}^{m}p(s_ix)h_if)+\frac{\varepsilon}{6[(1:\tilde{p})^1+1]}.$$

By (1) and (4),

$$(5)\ \frac{1}{I_{p'}^{\varepsilon'}(\tilde{p})}\sum_{i=0}^{m}I_{p'}^{\varepsilon'}(h_if)p(s_ix)\leq f(x)+\frac{\varepsilon}{2}+\frac{\eta(f:p')^{\varepsilon'}}{(\tilde{p}:p')^{\varepsilon'}}+\frac{M_1\varepsilon'}{(\tilde{p}:p')^{\varepsilon'}}+\frac{\varepsilon}{6[(1:\tilde{p})^1+1]I_{p'}^{\varepsilon'}(\tilde{p})}.$$

We may suppose that $\varepsilon'<\dfrac{\varepsilon}{12(M_1+3m+2)}$.

Since $\dfrac{(f:p')^{\varepsilon'}}{(\tilde{p}:p')^{\varepsilon'}}\leq(f:\tilde{p})^1+1$ and $\dfrac{(1:p')^{\varepsilon'}}{(\tilde{p}:p')^{\varepsilon'}}\leq(1:\tilde{p})^1+1$ by Lemma 3.4,

$$|f(x)-\sum_{i=0}^{m}\frac{I_{p'}^{\varepsilon'}(h_if)}{I_{p'}^{\varepsilon'}(\tilde{p})}p(s_ix)|<\varepsilon,$$

for all $x\in\widehat{G}$. Similarly, $||1-\sum_{i=0}^{m}d_ip^{s_i}||<\varepsilon$ for some $\langle d_i\rangle$. $\square$

Let $\dot{P}=\{\langle\langle q_i,b_i\rangle:i\leq m\rangle\in P:b_i$'s are basic functions and $q_i\in\mathbb{Q}_{\geq 0}\}$. Then we define a Σ_1^0-formula $\varphi(p,b,s,k,\sigma,\tau)$ to denote that $p\in\dot{P}$ and b is a basic function, two sequences σ, τ from $\mathbb{Q}_{>0}$ and one sequence s from G have all the same length and

$$\max(||p-\sum\sigma(i)b^{s(i)}||,||1-\sum\tau(i)b^{s(i)}||)<2^{-k}.$$

Since $\forall p\in\dot{P}\forall k\exists b,s,\sigma,\tau\varphi(p,b,s,k,\sigma,\tau)$ by Lemma 3.12, there exists a function $F_1:\dot{P}\times\mathbb{N}\to\mathbb{Q}$ such that

$$F_1:\langle p,k\rangle\mapsto\frac{\sum_{i=0}^{n}c_i}{\sum_{i=0}^{n}d_i}\text{ such that }\varphi(p,b,s,k,\langle c_i:i\leq n\rangle,\langle d_i:i\leq n\rangle)\text{ holds for some }b,s.$$

We write $t_k^p=F_1(\langle p,k\rangle)$. Let F_2 be a function from $\dot{P}\times\mathbb{N}$ to $\mathbb{Q}$ such that

$$\langle p,k\rangle\mapsto t^p_{k+m_p+1},$$

where m_p is a number given effectively by $p\in\dot{P}$ such that $||p||+3\leq 2^{m_p}$. The following lemma can be shown easily.

Lemma 3.13 *The following is provable in* RCA_0*. For each* $\varepsilon > 0$ *and each* $k \in \mathbb{N}$*, there exists* $c > 0$ *such that if a basic function* b *is small of order* c*, then* $|I_b^{\varepsilon'}(p) - t^p_{k+m_p+1}| \leq 2^{-k} + \varepsilon$ *for each sufficiently small* $\varepsilon' > 0$*.*

Theorem 3.14 *The following is provable in* RCA_0*. Let* $\widehat{G}$ *be a compact group with a modulus of uniform continuity. Then there exists a unique left-invariant measure on* $\widehat{G}$*.*

Proof. By Lemma 3.13, for each n, j $(n \leq j)$ and each $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that $|I_b^{\varepsilon'}(p) - t^p_{n+m_p+1}| \leq 2^{-n-1} + \varepsilon$ and $|I_b^{\varepsilon'}(p) - t^p_{j+m_p+1}| \leq 2^{-j-1} + \varepsilon$. Then, $|t^p_{n+m_p+1} - t^p_{j+m_p+1}| \leq 2^{-n} + 2\varepsilon$. Since ε is at random, $|t^p_{n+m_p+1} - t^p_{j+m_p+1}| \leq 2^{-n}$. Hence, $\langle F_2(p,k) : k \in \mathbb{N}\rangle$ is real. We define $F : \dot{P} \to \mathbb{R}$ by $F(p) = \langle F_2(p,k) : k \in \mathbb{N}\rangle$.

Since, for each $p \in P$, p can be expressed by $\langle\langle q_i, b_i\rangle : i \leq m\rangle$ such that $\forall i \leq l(0 \leq q_i)$ and $\forall i > l(q_i < 0)$, we define

$$\mu(p) = F(\langle\langle q_i, b_i\rangle : i \leq k\rangle) - F(\langle\langle -q_i, b_i\rangle : k < i \leq m\rangle).$$

Then μ is a left-invariant measure by Lemma 3.13.

Let μ' be another left-invariant measure. Fix any $p \in \dot{P}$ and any $k > 1$. We take finite sequences $\langle c_i\rangle$ and $\langle d_i\rangle$ of positive rationals such that $t^p_k = \dfrac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i}$ with $\max(||p - \sum_i c_i b_p^{s_i}||, ||1 - \sum_i d_i b_p^{s_i}||) < 2^{-k}$. Then, $|\mu'(p) - \sum_i c_i \mu''(b_p)| < 2^{-k}$ and $|1 - \sum_i d_i \mu'(b_p)| < 2^{-k}$. Therefore,

$$t_k = \frac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i} < \frac{\mu'(p) + 2^{-k}}{\mu'(b_p)\sum_{i=0}^n d_i} < \frac{\mu'(p) + 2^{-k}}{1 - 2^{-k}}.$$

Similarly, $\dfrac{\mu'(p) - 2^{-k}}{1 + 2^{-k}} < \dfrac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i} = t_k$. Since $\mu(p) = \lim_k t_k$, $\mu'(p) = \mu(p)$. Then $\widehat{G}$ has a unique invariant-measure. □

By Theorem 1 in [12], Theorem 3.14 leads to the following corollary.

Theorem 3.15 *The following assertions are pairwise equivalent over* RCA_0*.*

(1) WWKL_0.

(2) *Any compact group whose operations have a modulus of uniform continuity has a unique Haar measure.*

3.2 Completeness for Intuitionistic Logic

In this section, we investigate the logical strength of completeness theorems for intuitionistic logic along the program of reverse mathematics. Several kinds of models have been invented for intuitionistic logic, e.g., lambda calculus models, Kripke-Beth's models, topological models, etc. We here treat only with Kripke models of intuitionistic logic. The completeness theorem of classical logic asserts that if Γ is consistent then T has a model. On the other hand, the (strong) completeness theorem for intuitionistic logic asserts that any countable theory in intuitionistic predicate logic can be characterized by a single Kripke model. The standard proof can be regarded as a generalization of Henkin construction for classical logic, in which the maximal filters of classical Lindenbaum Boolean algebras is replaced by presheaves of prime filters of intuitionistic Lindenbaum distributive lattices [28]. We show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for intuitionistic logic. The proof of the strong completeness theorem for intuitionistic logic in ACA_0 is essentially due to Ishihara-Khoussainov-Nerode [15] and Gabbay [10], which are the pioneer works on recursive model theory for intuitionistic logic.

The following definitions are made in RCA_0. A (*first order*) *language* here consists of countably many relation symbols and constant symbols, but no function symbols. Logical symbols are given as usual. We use $\perp$ (the falsity) to define the negation $\neg\varphi$ as $\varphi \to \perp$ [28]. We identify terms and formulas with their Gödel numbers under a fixed primitive recursive coding. Then let $Form$ and Snt be the sets of formulas and sentences respectively. A set of sentences is often called a *theory*. By $\Gamma \vdash_i \varphi$, we mean that φ is intuitionistically deducible from Γ, which is formally Σ^0_1-definable. Let Γ and Δ be two theories. The pair (Γ, Δ) is *consistent* if there are no finite sets $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ such that $\vdash_i \wedge\Gamma_0 \to \vee\Delta_0$. Here, we set $\wedge\emptyset \equiv \top$, $\vee\emptyset \equiv \perp$. Γ is *consistent* if $(\Gamma, \emptyset)$ is consistent.

3.2.1 C-saturated theory

In this subsection, we show that WKL_0 is equivalent over RCA_0 to a version of saturation lemma for intuitionistic logic.

Definition 3.16 The following definition is made in RCA_0. Let C be a set of constants. A theory Γ is *C-saturated* if it satisfies the following conditions:

(1) $\Gamma \not\vdash_i \bot$;

(2) if $\Gamma \vdash_i \sigma$ then $\sigma \in \Gamma$;

(3) if $\sigma \vee \tau \in \Gamma$ then $\sigma \in \Gamma$ or $\tau \in \Gamma$;

(4) if $\exists x \varphi(x) \in \Gamma$ then $\exists c \in C((\varphi(c)) \in \Gamma)$,

where a formula $\varphi(x)$ has no other free variable than x.

Lemma 3.17 (The saturation lemma) *The following is provable in* WKL_0. *Suppose that a sentence* σ_0 *is not intuitionistically deducible from a theory* Γ. *Let* C *be an infinite set of constants not in the language* $\mathcal{L}$ *of* Γ. *Then there is a* C*-saturated set* Γ' *of sentences in the language* $\mathcal{L}(C)$ *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \not\vdash_i \sigma_0$.

Proof. Let Γ be a theory such that $\Gamma \not\vdash_i \sigma_0$. Put $t \in T_\Gamma$ if and only if $\forall \sigma < lh(t)(t(\sigma) = 1 \to \sigma \in Snt_{\mathcal{L}(C)})$ and $\forall \sigma < lh(t)(\sigma \in \Gamma \to t(\sigma) = 1)$ and $\forall p < lh(t)\big((p$ is an intuitionistic proof $\wedge \forall i < lh(p)(p(i)$ is a nonlogical axiom of $p \to t(p(i)) = 1)) \to \forall i < lh(p)(p(i) \in Snt_{L \cup C} \to t(p(i)) = 1)\big)$ and $\forall \sigma < lh(t) \forall \tau < lh(t)(t(\sigma \vee \tau) = 1 \to t(\sigma) = 1 \vee t(\tau) = 1)$ and $\forall n < lh(t) \forall m < lh(t)((n = \ulcorner \exists x \varphi(x) \urcorner$, $m = \ulcorner \varphi(c_\varphi) \urcorner$ and $t(n) = 1) \to t(m) = 1)$ and $\sigma_0 < lh(t) \to t(\sigma_0) = 0$, where $c_\varphi \in C$ is the Henkin-constant of an $\mathcal{L}(C)$-formula $\varphi(x)$.

T_Γ exists by Δ^0_1 comprehension. Clearly T_Γ is an infinite 0-1 tree since $T_\Gamma \not\vdash_i \sigma_0$. By weak König's lemma, T_Γ has a path P. Let $\Gamma' = \{x \in Snt : P(x) = 1\}$. By the construction of T_Γ, Γ' is a C-saturation of Γ. □

Theorem 3.18 *The following assertions are pairwise equivalent over* RCA_0.

(1) WKL_0.

(2) *The saturation lemma for intuitionistic predicate logic.*

(3) *The saturation lemma for intuitionistic propositional logic with countably many atoms.*

Proof. Lemma 3.17 gives the implication (1)→(2). The implication (2)→(3) is straightforward. It remains to prove (3)→(1).

Now consider intuitionistic propositional logic with countably many atomic formulas $\langle a_n : n \in \mathbb{N} \rangle$. A set Δ of formulas is *saturated* if and only if Δ satisfies the conditions of C-saturation except (4).

The saturation lemma for intuitionistic propositional logic asserts that if $\Gamma \not\vdash_i \sigma$ then there exists a saturated set Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \not\vdash_i \sigma$. We want to prove weak

König's lemma from the saturation lemma. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree. For each $n \in \mathbb{N}$, define a propositional formula

$$\sigma_n \equiv \bigvee \{ \bigwedge \{ a_i^{s(i)} : i < n \} : s \in T, lh(s) = n \}$$

where $a_i^1 = a_i \mathrm{C} a_i^0 = \neg a_i$. Let $\Gamma = \{\sigma_n : n \in \mathbb{N}\}$. Since T contains sequences of length n, Γ is classically consistent, hence also, intuitionistically consistent. From the saturation lemma, it follows that Γ has a saturation Γ'. Let $P(n) = 1$ if $a_n \in \Gamma'$; $P(n) = 0$ if $\neg a_n \in \Gamma'$. Then P is a path through T. This completes the proof of (3)$\to$(1). $\square$

In a classical Henkin-type completeness proof for intuitionistic logic, we give a Kripke model of Γ as a set of saturated sets containing Γ. This can not be formalized in the language of second-order arithmetic. We assume that any theory X is accompanied by the set C of all constant symbols of X. Let $K(X, \Gamma)$ be a formula which means that $\Gamma \subseteq X$ and X is C-saturated. By using $K(X, \Gamma)$, we can show the following unnatural version of strong completeness theorem in WKL_0 by the usual argument.

Corollary 3.19 *The following is provable in* WKL_0. *Let* Γ *be a theory such that* $\Gamma \nvdash_i \bot$. *Then* $K(\cdot, \Gamma)$ *satisfies the conditions of a Kripke model with the partial order* $\subseteq$, $X \Vdash \sigma$ *interpreted by* $\sigma \in X$ *and the domain of* X *as the set of all constant symbols of* X, *where* $K(\cdot, \Gamma)$ *means the class of* X *such that* $K(X, \Gamma)$. *In addition,*

$$\forall Y (K(Y, \Gamma) \to \Gamma \subseteq Y) \land \forall \sigma \big(\Gamma \nvdash_i \sigma \to \exists X (K(X, \Gamma) \land \sigma \notin X)\big).$$

Since the above completeness theorem implies the saturation lemma, it is equivalent over RCA_0 to WKL_0.

3.2.2 The strong completeness theorem

In this subsection, we first define Kripke models in the usual way and show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for Intuitionistic Logic.

Definition 3.20 The following definition is made in RCA_0. Let $K (\subseteq \mathbb{N})$ be a non-empty set of possible worlds, and $\leq_K$ a partial order on K. Let D be a function assigning a domain to each world of K. Let $\Vdash$ be a binary relation on $K \times Snt_{\mathcal{K}}$, where $Snt_{\mathcal{K}}$ is the set of closed sentences in the language extended with the names for elements in $\bigcup_{k \in K} D(k)$.

Then $\mathcal{K} = (K, \leq_K, D, \Vdash)$ is a (code for a) *Kripke model* if $\mathcal{K}$ obeys the familiar conditions: $\forall k, k' \in K$

(1) if $k \leq_K k'$ then $D(k) \subseteq D(k')$;

(2) if $k \Vdash \sigma$ then σ is a sentence in $L \cup D(k)$;

(3) if $k \leq_K k'$ and $k \Vdash \sigma$ then $k' \Vdash \sigma$;

(4) $k \nVdash \perp$;

(5) $k \Vdash \sigma \wedge \tau$ if and only if $k \Vdash \sigma \wedge k \Vdash \tau$;

(6) $k \Vdash \sigma \vee \tau$ if and only if $k \Vdash \sigma \vee k \Vdash \tau$;

(7) $k \Vdash \sigma \rightarrow \tau$ if and only if $\forall k'' \in K(k \leq_K k'' \rightarrow (k'' \Vdash \sigma \rightarrow k'' \Vdash \tau))$;

(8) $k \Vdash \exists x \varphi(x)$ if and only if $k \Vdash \varphi(c)$ for some $c \in D(k)$;

(9) $k \Vdash \forall x \varphi(x)$ if and only if $\forall k'' \geq_K k \forall c \in D(k'')(k'' \Vdash \varphi(c))$,

where $k \Vdash \sigma$ means $(k, \sigma) \in \Vdash$.

Definition 3.21 The following definition is made in RCA_0. Let σ_0 be a $\mathcal{L}$-sentence. A theory Γ of $\mathcal{L}$ is *$\mathcal{L}$-maximal with respect to* σ_0 if it satisfies the following conditions:

(1) $\Gamma \nvdash_i \sigma_0$;

(2) if $\Gamma \vdash_i \sigma$ then $\sigma \in \Gamma$;

(3) if $\sigma \vee \tau \in \Gamma$ then $\sigma \in \Gamma \vee \tau \in \Gamma$;

(4) if $\exists x \varphi(x) \in \Gamma$ then $\varphi(c) \in \Gamma$ for some constant c in L;

(5) if $\Gamma \cup \{\sigma\} \nvdash_i \sigma_0$ then $\sigma \in \Gamma$.

Lemma 3.22 *The following is provable in* ACA_0. *Suppose that a sentence* σ_0 *is not intuitionistically deducible from a theory* Γ. *Let* C *be a infinite set of constants not in* L. *Then there is a* $\mathcal{L}(C)$*-maximal* Γ' *with respect to* σ_0 *such that* $\Gamma \subseteq \Gamma'$.

Proof. Let $\langle \tau_n : n \in \mathbb{N} \rangle$ be a one-to-one enumeration of all sentences in $L \cup C$. We may assume that $\sigma \vee \tau$, $\sigma \rightarrow \tau$ and $\varphi(c_\varphi)$ appear after σ, τ and $\exists x \varphi(x)$ in the enumeration. By arithmetical comprehension, there exists a set $\Gamma^* = \{\sigma : \Gamma \vdash_i \sigma\}$. Define a function $f : \mathbb{N} \rightarrow \{0, 1\}$ by primitive recursion as follows:

$$f(n) = \begin{cases} 1 & \text{if } \Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\tau_n\} \nvdash_i \sigma_0 \\ 0 & \text{otherwise} \end{cases}$$

Let $\Gamma' = \{\tau_n : f(n) = 1\}$. We prove that Γ' is an $L \cup C$-maximal theory with respect to σ_0. We need to prove that all the conditions of Definition 3.21. It is obvious that Γ' satisfies the conditions (1), (2) and (5) of Definition 3.21 and $\Gamma \subseteq \Gamma'$.

Suppose that $\sigma \notin \Gamma'$ and $\tau \notin \Gamma'$. Since $\sigma \vee \tau$ is enumerated after σ and τ, $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\sigma\} \vdash_i \sigma_0$ and $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\tau\} \vdash_i \sigma_0$. Then $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\sigma \vee \tau\} \vdash_i \sigma_0$, that is, $\sigma \vee \tau \notin \Gamma'$. This implies that Γ' satisfies the conditions (3) of Definition 3.21.

Finally, we show that Γ' satisfies condition (4) of Definition 3.21. Suppose that $\exists x \varphi(x) \in \Gamma'$ and $\varphi(c_\varphi) = \tau_n$ where c_φ is the Henkin constant of $\varphi(x)$. Since $\exists x \varphi(x)$ is enumerated before $\varphi(c_\varphi)$, we have $\exists x \varphi(x) \in \{\tau_l : f(l) = 1 \wedge l < n\}$. Then $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\varphi(c_\varphi)\} \not\vdash_i \sigma_0$, that is, $\varphi(c_\varphi) \in \Gamma'$. □

Remark. In the proof of Lemma 3.22, arithmetical comprehension axioms are sufficient to show the existence of $\Gamma^* = \{a' : \Gamma \vdash_i a'\}$. Fix any $\mathcal{L}$-theory Γ and any set C of new constants. Then, in ACA_0, an $\mathcal{L}(C)$-maximal extension Γ' of Γ with respect to σ_0 is $\Gamma^* \oplus C$-recursive. We identify a $\Gamma^* \oplus C$-recursive set Δ with its $\Gamma^* \oplus C$-recursive index i_Δ. Then we can show the following generalization of Lemma 3.22.

Lemma 3.23 *The following is provable in* ACA_0. *Suppose that an $\mathcal{L}$-sentence σ_0 is not intuitionistically deducible from an $\mathcal{L}$-theory Γ. Let $\langle C_n : n \in \mathbb{N}\rangle$ be an infinite sequence of pairwise disjoint infinite sets of constants not in $\mathcal{L}$. We put $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_{n+1} = \mathcal{L}_n \cup C_n$, and $C = \bigcup_{n \in N} C_n$. Then there is a partial function Φ from the set of $\Gamma^* \oplus C$-recursive theories to itself which satisfies the following: if a $\Gamma^* \oplus C$-recursive theory Δ in $\mathcal{L}_n$ is closed under intuitionistic deduction and an $\mathcal{L}_n$-sentence ψ is not intuitionistically deducible from Δ, then $\Phi(n, \psi, \Delta)$ is an $\mathcal{L}_{n+1}$-maximal extension of Δ with respect to ψ.*

Lemma 3.24 (The strong complete theorem) *The following is provable in* ACA_0. *Let Γ be an intuitionistically consistent theory. Then there exists a Kripke model $(K, \leq_K, \Vdash)$ such that $\forall \sigma (\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K \ k \Vdash \sigma)$.*

Proof. Fix any intuitionistically consistent theory Γ. By Lemma 3.23, we have $\langle F_n : n \in \mathbb{N}\rangle$ such that

$F_0 = \{\Phi(0, \psi, \Delta) :$ an $\mathcal{L}_0$-sentence ψ is not deducible from Δ and Δ is a finite extension of $\Gamma\}$

and

$$F_n = \{\Phi(n, \psi, \Delta) : \text{an } \mathcal{L}_n\text{-sentence } \psi \text{ is not deducible from } \Delta \text{ and } \Delta$$
$$\text{is a finite extension of some theory in } F_n\}.$$

Let K be the set of (indices of) theories in $\bigcup_{n \in \mathbb{N}} F_n$. For each $\Delta, \Delta' \in K$, we have $\Delta \leq_K \Delta'$ if $\Delta \subseteq \Delta'$. If $\Delta \in F_n$, then $D(\Delta) =$ the set of all constants of $\mathcal{L}_{n+1}$. $\Delta \Vdash \sigma$ if $\sigma \in \Delta$. Then it is easy to check that $(K, \leq_K, D, \Vdash)$ is a Kripke model by the usual way. By the construction of K, we see that

$$\forall \sigma (\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K \ k \ \Vdash \sigma).$$

This completes the proof of Lemma 3.24.

Theorem 3.25 *The following assertions are pairwise equivalent over* RCA_0.

(1) ACA_0.

(2) *The strong completeness theorem for intuitionistic predicate logic.*

(3) *The strong completeness theorem for intuitionistic propositional logic with countably many atoms.*

Proof. Lemma 3.24 gives the implication (1)→(2). The implication (2)→(3) is straightforward. It remains to prove (3)→(1).

Now consider intuitionistic propositional logic with countably many atomic formulas $\langle a_n : n \in \mathbb{N} \rangle$. A triple $(K, \leq_K, \Vdash)$ is a *Kripke model* if and only if it satisfies the conditions of Definition 3.20 except (1), (2), (8) and (9).

The strong completeness theorem for intuitionistic propositional logic asserts that if a set Γ of formulas is intuitionistically consistent then there exists a Kripke model $(K, \leq_K, \Vdash)$ such that $\forall \sigma (\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K \ k \Vdash \sigma)$. It is enough to show Σ^0_1-comprehension from the strong completeness theorem.

Let $\varphi(n)$ be a Σ^0_1 formula. Write $\varphi(n)$ as $\exists x \theta(x, n)$ where $\theta(x, n)$ is Σ^0_0. Let

$$\Gamma = \{a_n \vee a_n \vee \ldots \vee a_n : \exists m \leq (a_n \vee a_n \vee \ldots \vee a_n) \theta(m, n)\}.$$

Γ exists by Δ^0_1-comprehension. Clearly, Γ is intuitionistically consistent. By the strong completeness theorem, there exists a Kripke model $(K, \leq_K, \Vdash)$ such that $\forall \sigma (\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K \ k \Vdash \sigma)$. Then

$$\forall k \in K (k \Vdash a_n) \leftrightarrow \Gamma \vdash_i a_n \leftrightarrow a_n \vee a_n \vee \ldots \vee a_n \in \Gamma \leftrightarrow \varphi(n).$$

The left hand side of this equivalence is Π^0_1. Hence by Δ^0_1-comprehension, we obtain $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$. This completes the proof of (3)→(1). □

3.3 Reverse mathematics on weak base theory

For the main-stream researches of Reverse Mathematics, the system RCA_0 is presupposed as a base theory in which most of basic concepts of ordinary mathematics (e.g., reals, continuous functions) are defined. However, it has been claimed by several people that the phenomena of Reverse Mathematics depend on the base theory, so that necessary axioms for a theorem may be changing if one replaces RCA_0 by a weaker system. Actually, Simpson and Smith [24] already studied Reverse Mathematics over RCA^*_0, which is roughly RCA_0 minus Σ^0_1 induction plus Σ^0_0 induction plus exponentiation. F.Ferreira [8] proposed to do Reverse Mathematics over BTFA (or BTFA+Σ^b_∞-WKL), a second-order systems of 0-1 strings whose provably total functions are the polynomial time computable functions.

In this section, we carry out Ferreira's plan [6] and show, for instance, that the intermediate value theorem on $[0, 1]$ is provable in BTFA, and a version of the maximum principle is equivalent to Σ^b_1-CA within BTFA.

3.3.1 Basics of real analysis

We begin with defining a real number and a (uniformly) continuous function on the reals in BTFA. We here have two sorts of natural numbers, i.e., tally natural numbers and dyadic natural numbers. A *tally natural number* is defined by a string of 1's, i.e., ε, 1, 11, ... Let $\mathbb{N}$ be the set of tally natural numbers. We can define $0_\mathbb{N}$, $\leq_\mathbb{N}$, $+_\mathbb{N}$ and $\cdot_\mathbb{N}$ by ε, $\subseteq$, $^\frown$ and $\times$, respectively. Then it is easy to show in BTFA that $\mathbb{N}$ is an ordered semi-ring. We use n, m, l, k, ... as variables over $\mathbb{N}$. A tally natural number is used to express the length of a string or the index of a sequence. A string σ is a *dyadic natural number* if $\sigma = 1\tau$ for some string τ of 0's and 1's, or $\sigma = 0$. In the standard model, σ can be seen as the ordinary dyadic notation of a natural number. The set of all dyadic natural numbers is denoted by $\mathbb{N}_2$. Also we can define $0_{\mathbb{N}_2}$, $\leq_{\mathbb{N}_2}$, $+_{\mathbb{N}_2}$ and $\cdot_{\mathbb{N}_2}$ in the usual way (cf. Ferreira [9]), and show in BTFA that $\mathbb{N}_2$ is an ordered semi-ring. We should notice that there exists a natural embedding of $\mathbb{N}$ into $\mathbb{N}_2$, but not vice versa. Without misunderstanding, we omit subscripts of $+_\mathbb{N}$, $\leq_{\mathbb{N}_2}$, etc.

A triple (i, n, σ) denotes a *dyadic rational number* $(-1)^i 2^n \sum_{j<lh(\sigma)} \sigma(j) 2^{-j-1}$, where $i = 0$ or 1 and $\sigma(j)$ is the j'th element of 0-1 sequence σ. Let $\mathbb{D}'$ be the set of dyadic rational numbers. Then we define $=_{\mathbb{D}'}$, $\leq_{\mathbb{D}'}$, $+_{\mathbb{D}'}$, $\cdot_{\mathbb{D}'}$, etc. in the usual way. We have a natural embedding of $\mathbb{N}_2$ into $\mathbb{D}'/=_{\mathbb{D}'}$. Let $\mathbb{D}$ be the set of $\mathbb{D}' \cap [0,1]$ and $\mathbb{D}_n$ the set of all elements $(0, m, \sigma)$ of $\mathbb{D}$ where the length of σ is $m + n$.

To simplify the notation, we write σ for $(0, n, 0^n\sigma) \in \mathbb{D}$, where 0^n is the string of 0's whose length is n. Moreover, we write 2^{-n} for $0^{n-1}1$.

Definition 3.26 The following definitions are made in BTFA. A function $f : \mathbb{N} \to \mathbb{D}'$ is a real number if $|f(n) - f(m)| \leq 2^{-n}$ for each $n \leq m$. Two real numbers f and g are said to be equal if $\forall n \in \mathbb{N}(|f(n) - g(n)| \leq 2^{-n+1})$.

The relations $<$, $\leq$ and basic operations on the real numbers are defined as usual. Note that $=$, $\leq$ on the real numbers can be defined by a formula of the form $\forall\sigma\varphi(\sigma)$ where φ is Π^b_1.

Definition 3.27 The following definition is made in BTFA. $F = (\langle f_n : n \in \mathbb{N}\rangle, h)$ is a (code for a) *(uniformly) continuous function from* $[0,1]$ *to* $[0,1]$ if F satisfies the following four conditions:

(1) $h : \mathbb{N} \to \mathbb{N}$ is an increasing function, called a modulus function for F,

(2) $\langle f_n : n \in \mathbb{N}\rangle$ is a sequence of piecewise linear functions $f_n : \mathbb{D} \to \mathbb{D}$ whose break points are in $\mathbb{D}_{h(n)}$,

(3) $|f_n(d) - f_n(d + 2^{-h(n)})| \leq 2^{-n}$ for each $n \in \mathbb{N}$ and $d \in \mathbb{D}$,

(4) $|f_n(d) - f_{n+m}(d)| \leq 2^{-n}$ for each $n, m \in \mathbb{N}$ and $d \in \mathbb{D}$.

We now define the value $F(x)$ for each $x \in [0,1]$. First suppose that x is not equal to any $\sigma \in \mathbb{D}$. For each n, there exists a unique string σ_n such that $|x - \sigma_n| \leq 2^{-h(n+1)-1} \wedge \sigma_n \in \mathbb{D}_{h(n+1)}$. By (\$)-CA, put $F(x) = \langle f_{n+1}(\sigma_n) : n \in \mathbb{N}\rangle$. If $x = \tau$ for some $\tau \in \mathbb{D}$, put $F(x) = \langle f_{|\tau|+n}(\tau) : n \in \mathbb{N}\rangle$.

It is easy to extend the above definition to define a continuous function from any bounded closed interval to any bounded closed interval. Also a continuous function of several variables can be defined in an obvious way. The identity function, the constant function, $+$, $\cdot$, x^n, etc. are all continuous. The continuous functions are closed under the composition.

Remark. The above definition is inspired by Ko [18]. But it is also possible to adopt another definition of continuous functions such as given in Simpson [23].

The following lemma can be used to show that functions defined by power series, e.g., $\exp(x)$ and $\sin(x)$, are continuous on $[0,1]$.

Lemma 3.28 *The following is provable in* BTFA*. Let $\langle F_n : n \in \mathbb{N}\rangle$ be a sequence of continuous functions $F_n : [0,1] \to [0, 2^{-n}]$ with the modulus function h_n. Suppose that there exists $h : \mathbb{N} \to \mathbb{N}$ such that $h_n(k) \le h(k+n)$ for each $n, k \in \mathbb{N}$. Then $F = \sum_{k\in\mathbb{N}} F_k$ is continuous.*

Proof. We reason in BTFA. Let $F_n = (\langle f^n_m : m \in \mathbb{N}\rangle, h_n)$. Let $\sigma\lceil n$ denote the initial segment of σ whose length is n. Since we can compute $\sum_{k=0}^{m} \sigma_k\lceil n$, then we have a continuous function $F = (\langle \sum_{k=0}^{n} f^k_{2n}\lceil 2n : n \in \mathbb{N}\rangle, h')$ where $h'(i) = h(3i)$. □

3.3.2 The intermediate value theorem and the maximum principle

Before proving the intermediate value theorem, we show a useful lemma.

Lemma 3.29 *The following is provable in* BTFA*. Let g, h_0 and h_1 be functions and t be a term. Assume that there is a term t' such that $g(\tau) \le t'(\tau)$ for each τ. Then, there exists f such that*

(1) $f(\varepsilon, \tau) = g(\tau)$

(2) $f(\sigma 0, \tau) = h_0(f(\sigma, \tau), \sigma, \tau)\lceil t(\sigma 0, \tau)$

(3) $f(\sigma 1, \tau) = h_1(f(\sigma, \tau), \sigma, \tau)\lceil t(\sigma 1, \tau)$

Proof. By modifying the proof of proposition 7 in Ferreira [7], f is obtained by a formula of the form $\exists y \varphi$ with $\varphi \in \Sigma^b_1$, which just describes the course of values. By (\$)-CA, f exists. □

Theorem 3.30 *the following is provable in* BTFA*. Let F be a continuous function from $[0,1]$ to $[0,1]$ such that $F(0) < 1/2 < F(1)$. Then, there exists a real $x \in (0,1)$ such that $F(x) = 1/2$.*

Proof. We may assume that $F(\sigma) \ne 0$ for all $\sigma \in \mathbb{D}$. Then by (\$)-CA there exists a set X consisting of all $\sigma \in D$ such that $F(\sigma) > 0$. By the above lemma, we define $g : \mathbb{N} \to \mathbb{D}$

by

$$g(n) = \begin{cases} 0 & \text{if } n = \varepsilon, \\ g(n-1)0 & \text{if } n \neq \varepsilon \text{ and } g(n-1)1 \in X, \\ g(n-1)1 & \text{otherwise} \end{cases}$$

By Σ_1^b-NI,

$$\forall n \in \mathbb{N} \forall m \in \mathbb{N}[n \leq m \to g(n) \subseteq g(m) \land g(n) \equiv n+1].$$

Thus g is a real. By Σ_1^b-NI again, $\forall n \in \mathbb{N}[F(g(n)) < 1/2 < F(g(n) + 2^{-n})]$. Therefore, $F(x) = 1/2$ where $x = \langle g(n) : n \in \mathbb{N} \rangle$. □

If the modulus function for a continuous function F is of the form $|t|$ where t is a term, then we say that F has a polynomial modulus function.

We now prove a lemma saying that a weak version of the maximum principle can be shown in BTFA adding a very weak comprehension scheme.

Lemma 3.31 *The following is provable in* BTFA *plus* Σ_1^b-CA. *For each continuous function F on* $[0, 1]$ *with a polynomial modulus function, then* $\sup_{0 \leq y \leq 1} F(y)$ *exists.*

Proof. Let $F = (\langle f_n : n \in \mathbb{N} \rangle, h)$. By Σ_1^b-CA,

$$X_n^l = \{\tau \lceil l : \exists \sigma \in \mathbb{D}_{h(n)} f_n(\sigma) = \tau\}$$

exists.

We define $\varphi(l, n, \sigma)$ by

$$\sigma \in X_n^l \land \sigma \equiv l \land \forall \sigma' \equiv l(\sigma < \sigma' \to \sigma' \notin X_n^l).$$

Since φ is Π_1^b, we can show that $\forall n \in \mathbb{N} \forall l \in \mathbb{N} \exists! \sigma \varphi(l, n, \sigma)$ by Π_1^b-NI on l. Let $g(n) = \sigma$ such that $\varphi(n+2, n+2, \sigma)$. Then, for any $n \in \mathbb{N}$,

$$\forall d \in \mathbb{D}_{h(n+2)}(f_{n+2}(d) \leq g(n) + 2^{-n-2})$$

and

$$\exists d' \in \mathbb{D}_{h(n+2)}(f_{n+2}(d') \lceil (n+2) = g(n)).$$

Therefore, we can show that g is a real and that g is the least upper bound. □

Corollary 3.32 *The following is provable in* BTFA *plus* Σ_1^b-CA. *For each continuous function F on* $[0, 1] \times [0, 1]$ *with a polynomial modulus function, then there exists a continuous function* $G(x) = \sup_{0 \leq y \leq 1} F(x, y)$.

Proof. It is straightforward from the proof of the above lemma. □

Corollary 3.33 *The following is provable in* BTFA *plus* Σ_1^b-CA. *For each continuous function F on $[0,1]$ with a polynomial modulus function, then there exists a continuous function $G(x) = \sup_{0\leq y\leq x} F(y)$.*

Proof. We define a continuous function F' on $[0,1]\times[0,1]$ by

$$F'(x) = \begin{cases} F(0) & \text{if } x < y, \\ F(x-y) & \text{if } y \leq x. \end{cases}$$

Then F' has a polynomial modulus function. By the above lemma, we can obtain a continuous function $G(x) = \sup_{0\leq y\leq x} F(y)$. □

Theorem 3.34 *The following assertions are equivalent over* BTFA.

(1) Σ_1^b-CA.

(2) *For each continuous function F on $[0,1]$ with a polynomial modulus function, then there exists a continuous function $G(x) = \sup_{0\leq y\leq x} F(y)$.*

Proof. The implication from 1 to 2 is Corollary 3.33. It remains to prove that 2 implies 1. We reason in BTFA.

Let $\varphi(\sigma)$ be Σ_1^b. For simplicity, we assume $\varphi(\sigma)$ is of the form $\exists\tau \equiv t(\sigma)\psi(\sigma,\tau)$ where $\psi(\sigma,\tau)$ is a s.w.q. formula. (It is a routine to extend the following argument to the general case.)

For each $n \in \mathbb{N}$, let $a_n = 1 - 2^{-n} \in \mathbb{D}$. (Namely, $a_n = n$ in the sense of strings.) If σ is the length of n, then let $u_\sigma = a_n + 0^{n+1}s$ and $v_\sigma = u_\sigma + 2^{-2n-1}$. If τ is the length of $|t(n)|$, $y_{\sigma,\tau} = u_\sigma + 0^{2n+2}\tau$ and $z_{\sigma,\tau} = y_{\sigma,\tau} + 2^{-2n-2-|t(n)|}$.

Define a function $H : [0,1] \to [0,1]$ by

$$H(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2-2x & \text{if } 1/2 \leq x. \end{cases}$$

Now we define a continuous function $F = (\langle f_n : n \in \mathbb{N}\rangle, h)$. Let $h(n) = |t(n)|+2n+3$. Let $f_n(\sigma) = f_{n-1}(\sigma)$ for $\sigma \leq a_n$, $f(\sigma) = a_{n+1}$ for $\sigma \geq a_{n+1}$, and for $\sigma \in [a_n, a_{n+1}]$,

$$f_n(\sigma) = \begin{cases} 2\sigma - v_\sigma & \text{if } \sigma \in [(u_\sigma+v_\sigma)/2, v_\sigma], \\ u_\sigma & \text{if } \sigma \in [y_{\sigma,\tau}, z_{\sigma,\tau}] \text{ and } \neg\psi(\sigma,\tau), \\ u_\sigma + 2^{-|t(n)|-2n-2}h(2^{|t(n)|+2n+2}\cdot(\sigma - y_{\sigma,\tau})) & \text{if } \sigma \in [y_{\sigma,\tau}, z_{\sigma,\tau}] \text{ and } \psi(\sigma,\tau). \end{cases}$$

If $G(x) = \sup_{0 \le y \le x} F(y)$, then it is easy to see that $\exists \tau \le t(\sigma)\psi(\sigma, \tau)$ iff $G((u_\sigma + v_\sigma)/2) - 2^{-|t(\sigma)|-2\cdot|\sigma|-3} > u_\sigma$ iff $g_{|t(\sigma)|+2\cdot|\sigma|+5}((u_\sigma + v_\sigma)/2) > u_\sigma$, where $G = (\langle g_n : n \in \mathbb{N} \rangle, h')$. Therefore, $X = \{\sigma : \varphi(\sigma)\}$ exists. $\square$

Notes. The above theorem can be viewed as a formalized version of theorem 3.7 in Ko's book [18].

References

[1] J. Avigad, *Formalizing forcing arguments in subsystems of second order arithmetic*, Annals of Pure and Applied Logic **82** (1996), 165–191.

[2] P. Bernays and D. Hilbert, *Grundlagen der Mathematik*, volume I and volume II, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, second edition, 1968–1970.

[3] E. Bishop, *Foundations of constructive analysis*, McGraw-Hill, 1967.

[4] D. K. Brown and S. G. Simpson, *The baire category theorem in weak subsystems of second-oder arithmetic*, Annals of Pure and Applied Logic **58** (1993), 557–578.

[5] A. M. Fernandes, *A new conservation result of* WKL_0 *over* RCA_0, submitted.

[6] F. Ferreira, *Polynomial time computable arithmetic and conservative extensions*, Ph.D. thesis, Pennsylvania State University, 1988.

[7] F. Ferreira, *Polynomial time computable arithmetic*, Contemporary Mathematics **106** (1990), 137–156.

[8] F. Ferreira, *A feasible theory for analysis*, Journal of Symbolic Logic **59** (1994), 1001–1011.

[9] F. Ferreira, *Some notes on subword quantification and induction thereof*, Logic and algebra, Lecture Notes in Pure and Applied Mathematics **180** (1996), 477–489.

[10] D. Gabbay, *Semantical investigations in heyting's intuitionistic logic*, D. Reidel, Dordrecht, 1981.

[11] M. J. Groszek, M. Mytilinaios, and T. A. Slaman, *The sacks density theorem and* Σ_2*-bounding*, Journal of Symbolic Logic **61** (1996), 450–467.

[12] P. Hájek and P. Pudlák, *Metamathematics of First-Order Arithmetic*, Springer-Verlag, 1991.

[13] K. Hatzikiriakou, *Algebraic disguises of Σ^0_1 induction*, Archive for Mathematical Logic **29** (1989), 47–51.

[14] M. Hausner, *On a non-standard construction of haar measure*, Communications on Pure and Applied Mathematics **25** (1972), 403–405.

[15] H.Ishihara, B. Khoussainov and A. Nerode, *Decidable kripke models of intuitionistic theories*, Annals of Pure and Applied Logic **93** (1998), 115–123.

[16] C. G. Jockusch, Jr. and R. I. Soare, Π^0_1 *classes and degrees of theories*, Transactions of the American Mathematical Society **173** (1972), 33–56.

[17] R. Kaye, *Models of Peano Arithmetic*, Oxford University Press, 1991.

[18] K. Ko, *Complexity Theory of Real Functions*, Birkhäuser, 1991.

[19] D. Marker, *Enumerations of turing ideals with applications*, Notre Dame Journal of Formal Logic **30** (1990), 509–514.

[20] M. Mytilinaios, *Finite injury and* Σ_1*-induction*, Journal of Symbolic Logic **54** (1989), 38–49.

[21] L. S. Pontryagin, *Topological groups*, Gordon and Breach science, second edition, translated from the Russian by A. Brown, 1966.

[22] S. G. Simpson, *Partial realizations of hilbert's program*, Journal of Symbolic Logic **53** (1988), 349–363.

[23] S. G. Simpson, *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1998.

[24] S. G. Simpson and R. Smith, *Factorization of polynomials and* Σ^0_1 *induction*, Annals of Pure and Applied Logic **31** (1986), 289–306.

[25] K. Tanaka, *More on models of* WKL_0, hand-written notes, 1995.

[26] K. Tanaka, *The self-embedding theorem of* WKL_0 *and a non-standard method*, Annals of Pure and Applied Logic **84** (1997), 41–49.

[27] K. Tanaka and T. Yamazaki, *A non-standard construction of Haar measure and* WKL_0, Journal of Symbolic Logic **65** (2000), 173–186.

[28] A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics*, North-Holland, 1988.

[29] X. Yu and S. G. Simpson, *Measure theory and weak könig's lemma*, Archive for Mathematical Logic **30** (1990), 171–180.

[30] T. Yamazaki, *Some more conservation results on the Baire category theorem*, Mathematical Logic Quarterly **46**(2000), 105–110

Tohoku Mathematical Publications

No.1 Hitoshi Furuhata: *Isometric pluriharmonic immersions of Kähler manifolds into semi-Euclidean spaces*, 1995.

No.2 Tomokuni Takahashi: *Certain algebraic surfaces of general type with irregularity one and their canonical mappings*, 1996.

No.3 Takeshi Ikeda: *Coset constructions of conformal blocks*, 1996.

No.4 Masami Fujimori: *Integral and rational points on algebraic curves of certain types and their Jacobian varieties over number fields*, 1997.

No.5 Hisatoshi Ikai: *Some prehomogeneous representations defined by cubic forms*, 1997.

No.6 Setsuro Fujiié: *Solutions ramifiées des problèmes de Cauchy caractéristiques et fonctions hypergéométriques à deux variables*, 1997.

No.7 Miho Tanigaki: *Saturation of the approximation by spectral decompositions associated with the Schrödinger operator*, 1998.

No.8 Y. Nishiura, I. Takagi and E. Yanagida: *Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems — towards the Understanding of Singularities in Dissipative Structures —*, 1998.

No.9 Hideaki Izumi: *Non-commutative L^p-spaces constructed by the complex interpolation method*, 1998.

No.10 Youngho Jang: *Non-Archimedean quantum mechanics*, 1998.

No.11 Kazuhiro Horihata: *The evolution of harmonic maps*, 1999.

No.12 Tatsuya Tate: *Asymptotic behavior of eigenfunctions and eigenvalues for ergodic and periodic systems*, 1999.

No.13 Kazuya Matsumi: *Arithmetic of three-dimensional complete regular local rings of positive characteristics*, 1999.

No.14 Tetsuya Taniguchi: *Non-isotropic harmonic tori in complex projective spaces and configurations of points on Riemann surfaces*, 1999.

No.15 Taishi Shimoda: *Hypoellipticity of second order differential operators with sign-changing principal symbols*, 2000.

No.16 Tatsuo Konno: *On the infinitesimal isometries of fiber bundles*, 2000.

No.17 Takeshi Yamazaki: *Model-theoretic studies on subsystems of second order arithmetic*, 2000.