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Abstract

In this thesis, we present several effective methods of constructing harmonic maps between Hadamard manifolds. In particular, in the case of real hyperbolic spaces, we reduce the harmonic map equation to an ordinary differential equation defined on the nonnegative real line, and by showing the existence of solutions for a boundary value problem of this equation, we construct a variety of new harmonic maps between real hyperbolic spaces.

Furthermore, following the idea due to Donnelly, we establish the existence and uniqueness result for proper harmonic maps between Damek-Ricci spaces, which are a generalization of rank one symmetric spaces of noncompact type. A non-existence result for proper harmonic maps from complex hyperbolic spaces to real hyperbolic spaces is also proved.

Contents

Introduction

Let (M, g) and (M', g') be complete Riemannian manifolds, and $u : (M, g) \to (M', g')$ a C^2-map from M to M'. For a relatively compact domain $D \subset M$, we define the energy $E_D(u)$ of u over D by

$$E_D(u) := \frac{1}{2} \int_D |d_x u|^2 dv_g,$$

where $|d_x u|$ is the Hilbert-Schmidt norm of the differential $d_x u : T_x M \to T_{u(x)} M'$ of u at $x \in M$, and dv_g denotes the volume measure of (M, g). We call u a harmonic map if it is a critical point of E_D, considered as a functional defined on the space $C^2(M, M')$ of C^2-maps from M to M', for all variations with compact support for any D. In other words, u is a harmonic map if and only if it satisfies the Euler-Lagrange equation for the energy functional E_D for any D, that is,

$$\tau(u)(x) := \sum_{i=1}^{n} (\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i))(x) = 0, \quad x \in M,$$

where $\{e_i\}_{i=1}^n$ is an orthonormal frame field of M, ∇ is the Levi-Civita connection on the tangent bundle TM of M, and $\tilde{\nabla}$ is the induced connection on the pull-back bundle $u^{-1}TM'$ of the tangent bundle TM' of M' by u. Geometrically, $\tau(u)$ defines a section, called the tension field of u, of $u^{-1}TM'$. It should be remarked that any C^2 harmonic map u is smooth, since $\tau(u) = 0$ is in fact a system of semilinear elliptic partial differential equations of second order.

In the case where (M, g) and (M', g') are compact and without boundaries, a remarkable existence result of harmonic maps was established in 1964 by Eells-Sampson [17]. They proved that if the target manifold (M', g') has nonpositive sectional curvature everywhere, then there exists an energy minimizing harmonic map in each homotopy class of smooth maps from (M, g) to (M', g').

On the other hand, when (M, g) and (M', g') are noncompact, complete manifolds, not much has been known for the existence of harmonic maps between them. However, in the last decade, substantial progress has been made on the existence, the uniqueness and other fundamental properties of harmonic maps between Hadamard manifolds, that is, simply connected, connected, complete Riemannian manifolds of nonpositive sectional curvature, which are most typical examples of noncompact, complete Riemannian manifolds.

The primary object of the present thesis is to present several effective methods of constructing harmonic maps between these Hadamard manifolds.

Now we shall give a brief outline of the problems investigated in this thesis and summarize our results.

1) Since the existence theorem due to Eells and Sampson requires the nonpositivity of sectional curvatures on the target manifold, we can not apply it to the existence of harmonic maps between standard unit spheres, although they are among the simplest examples of Riemannian manifolds. However, by modifying the harmonic map equation, we see that several methods are available for constructing harmonic maps between standard unit spheres.

It follows from a well-known theorem of Takahashi that the harmonic map equation for a map $\varphi : (S^m, g_{S^m}) \to (S^n, g_{S^n})$ between standard unit spheres is equivalent to the equation

$$\Delta_{g_{S^m}} \Phi = 2e(\varphi)\Phi,$$

where $\Delta_{g_{S^m}}$ denotes the Laplace-Beltrami operator of (S^m, g_{S^m}), $e(\varphi) = |d\varphi|^2/2$ is the energy density function of φ, and $\Phi = i \circ \varphi$ for the inclusion map $i : S^n \to \mathbf{R}^{n+1}$. Therefore, when $e(\varphi)$ is constant, φ is a harmonic map if each component of φ is an eigenfunction of $\Delta_{g_{S^m}}$. Such harmonic maps φ are called eigenmaps, for which a typical example is provided with the Hopf fibration $\varphi : S^3 \to S^2$. Not only these are important objects in its own right, but also, using eigenmaps, we can obtain various harmonic maps between standard unit spheres, and between real hyperbolic spaces as well. In this regard, in Chapter 2, we shall present a new method of constructing a wealth of eigenmaps.

In order to construct a harmonic map between standard unit spheres, the notion of a join of two eigenmaps was introduced by Smith ([34]). More precisely, let $u : S^{m-1} \to S^{p-1}$ and $v : S^{n-1} \to S^{q-1}$ be eigenmaps, and $r : [0, \pi/2] \to [0, \pi/2]$ a smooth function. Noting that for any point $z \in S^{m+n-1}$, there exist $x \in S^{m-1}, y \in S^{n-1}$ and $t \in [0, \pi/2]$ such that z is expressed as

$$z = ((\cos t)x, (\sin t)y),$$

the join $u *_r v$ of these maps is defined by

$$u *_r v : S^{m+n-1} \ni ((\cos t)x, (\sin t)y) \mapsto ((\cos r(t))u(x), (\sin r(t))v(y)) \in S^{p+q-1}.$$

Consequently, the harmonic map equation for the join map is reduced to an ordinary differential equation in $r = r(t)$ with a suitable boundary condition. By solving this boundary value problem, Smith proved that there exists a harmonic representative in each homotopy class $\pi_n(S^n) \simeq \mathbf{Z}$ for $n \leq 7$. Subsequently, his method was extended by several authors. In particular, Ding([11]) utilized a variational method to clarify the meaning of the dumping condition, Eells and Ratto([15]) generalized the method to the case of ellipsoids, and Xin([45],[46]) reformulated Smith's method from the view point of Riemannian submersions (see also [16]).

2) We can generalize the notion of a join map to the case of real hyperbolic spaces $\mathbf{RH}^m$ as follows. First, note that for any point $z \in \mathbf{RH}^{m+n-1}$, there exist $x \in \mathbf{RH}^{m-1}, y \in S^{n-1}$ and $t \in [0, \infty)$ such that

$$z = ((\cosh t)x, (\sinh t)y).$$

Let $u : \mathbf{RH}^{m-1} \to \mathbf{RH}^{p-1}$ and $v : S^{n-1} \to S^{q-1}$ be eigenmaps, and $r : [0, \infty) \to [0, \infty)$ a smooth function. Then the join map $u *_r v$ of u, v and r is defined to be

$$u *_r v : \mathbf{RH}^{m+n-1} \ni ((\cosh t)x, (\sinh t)y) \mapsto ((\cosh r(t))u(x), (\sinh r(t))v(y)) \in \mathbf{RH}^{p+q-1}.$$

Then we see that $u *_r v$ is a harmonic map if and only if the function $r = r(t)$ satisfies the following ordinary differential equation defined on $[0, \infty)$:

$$\ddot{r}(t) + \left\{ p \, \frac{\sinh t}{\cosh t} + m \, \frac{\cosh t}{\sinh t} \right\} \dot{r}(t) - \left\{ \frac{\mu^2}{\cosh^2 t} + \frac{\nu^2}{\sinh^2 t} \right\} \sinh r(t) \cosh r(t) = 0,$$

where $e(u) = \mu^2/2$ and $e(v) = \nu^2/2$ denote the energy density functions of u and v, respectively.

On the other hand, the $(m+1)$-dimensional real hyperbolic space $\mathbf{RH}^{m+1}$ is represented as the warped product manifold

$$([0,\infty)\times S^m, dt^2+(\sinh^2 t)g_{S^m}).$$

Thus, for an eigenmap $\varphi : S^m \to S^n$, we may consider an equivariant map

$$\begin{aligned} u : ([0,\infty)\times S^m, dt^2+(\sinh^2 t)g_{S^m}) &\ni (t,\theta) \\ &\mapsto (r(t),\varphi(\theta)) \in ([0,\infty)\times S^n, dr^2+(\sinh^2 r)g_{S^n}). \end{aligned}$$

Then the harmonicity of u is also reduced to the following ordinary differential equation in $r=r(t)$ defined on $[0,\infty)$:

$$\ddot{r}(t)+m\,\frac{\cosh t}{\sinh t}\dot{r}(t)-\mu^2\frac{\sinh r(t)\cosh r(t)}{\sinh^2 t}=0,$$

where $e(\varphi)=\mu^2/2$ denotes the energy density function of φ.

In Chapter 1, we shall investigate the following ordinary differential equation defined on $[0,\infty)$, which stems from the study of such harmonic maps:

$$\ddot{r}(t)+\left\{p\frac{\dot{f}_1(t)}{f_1(t)}+q\frac{\dot{f}_2(t)}{f_2(t)}\right\}\dot{r}(t)-\left\{\mu^2\frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2}+\nu^2\frac{h_2(r(t))h_2'(r(t))}{f_2(t)^2}\right\}=0,$$

where p,q,μ and ν are some constants.

We first prove the short time existence of solutions following the method due to Baird ([3]). However, it seems to the present author that his proof needs some refinements, in particular, on the life span of solutions. In fact, under appropriate conditions for the given functions $f_i(t)$ and $h_i(r)$, we can elaborate his method and obtain a complete proof of the short time existence. Moreover, if we pose a condition on the growth rate of the function $f_i(t)$ as t goes to infinity, then such a short time solution can be extended to a global solution, which ensures the existence of an equivariant harmonic map as well as a harmonic map defined by the join. Applying these results, we can prove that there exists a family of harmonic maps from the real hyperbolic space $\mathbf{RH}^m$ onto itself for $m\geq 3$, which is parameterized by $\mathbf{Z}$.

Equivariant harmonic maps have been studied by many authors. For example, Kasue and Washio ([22]) investigated equivariant harmonic maps of the form

$$u : ([0,\infty) \times S^m, dt^2 + f(t)^2 g_{S^m}) \ni (t,\theta)$$
$$\mapsto (r(t), \varphi(\theta)) \in ([0,\infty) \times S^n, dr^2 + h(r)^2 g_{S^n}).$$

Here $\varphi : S^m \to S^n$ is an eigenmap, and $f = f(t)$ and $h = h(r)$ are warping functions. Since they constructed a comparison function to show the global existence of a solution for the original equation, the growth order of f and h are rather restricted. Thus one can not apply their argument to the case of real hyperbolic spaces.

3) For Hadamard manifolds (M, g) and (M', g'), one can consider the Eberlein-O'Neill compactifications $\overline{M}$ and $\overline{M'}$ of M and M', respectively, by adding their ideal boundaries, which are defined to be the spheres at infinity given by the asymptotic classes of geodesic rays. Then one can set up the Dirichlet problem at infinity for harmonic maps, which, for a given boundary map $f : \partial M \to \partial M'$, consists of the existence of a harmonic map $u : M \to M'$ which assumes the boundary value f continuously.

This problem can be regarded as a generalization of Hamilton's work [19] on the Dirichlet problem for harmonic maps between compact Riemannian manifolds with boundary to these noncompact Riemannian manifolds. However, since the Riemannian metrics under consideration blow up at the ideal boundaries, there appear much difficulties in analyzing the boundary behavior of solutions of the harmonic map equation.

The first progress to the above problem was accomplished around 1990's by Li-Tam [25] [26] [27] and Akutagawa [1]. Recall that most typical examples of Hadamard manifolds are the rank one symmetric spaces of noncompact type, that is, the real, the complex and the quaternion hyperbolic spaces and the Cayley hyperbolic plane. In their work, Li and Tam [25] [26] [27] solved the Dirichlet problem at infinity for harmonic maps between real hyperbolic spaces. In particular, exploiting the heat equation method, they established a general theory for the existence and uniqueness of solutions to this problem. For example, they proved the following: Suppose that Hadamard manifolds under consideration satisfy appropriate geometric conditions on the curvatures, the volume of unit balls and the bottom spectrum of the Laplace-Beltrami operator. Then, in order to obtain a global solution to the equation which converges to the desired harmonic map, it suffices to construct a suitable initial map for the parabolic harmonic map equation. On the other hand, by applying the elliptic method, Akutagawa [1] proved the existence of solutions to this problem in the case of the real hyperbolic plane.

In 1994, Donnelly [12] discussed in a general setting the problem for harmonic maps between rank one symmetric spaces of noncompact type. He investigated the unbounded models of these spaces realized as upper half space models, and made use of their global coordinates near the ideal boundaries. With these models and coordinates, he constructed good initial maps which enable one to apply Li-Tam's existence result, and solved the problem for boundary maps having sufficient regularity. A significant step involved is, by investigating the boundary behavior of proper harmonic maps, to deduce necessary conditions for the existence of solutions, which are expressed in terms of the relation between geometric structures around the ideal boundaries. Indeed, a given boundary map is related to the geometric structure on the boundary, for instance, to the conformal structure in the real case, and to the contact structure in the complex case.

Since all manifolds investigated so far are symmetric spaces and have strictly negative sectional curvature everywhere, the following question arises naturally: For non-symmetric or nonpositively curved manifolds, can one solve the Dirichlet problem at infinity for harmonic maps? As an answer to this question, in Chapter 3, we shall solve the Dirichlet problem for harmonic maps between Damek-Ricci spaces.

Damek-Ricci spaces were first introduced by Damek [7] as a semidirect extension of the generalized Heisenberg groups discovered by Kaplan [21]. These spaces may be regarded as a generalization of rank one symmetric spaces of noncompact type, since their geometries are quite similar to each other. However, Damek-Ricci spaces are not symmetric in general and have nonpositive sectional curvature. In fact, non-symmetric Damek-Ricci spaces admit vanishing sectional curvatures for certain 2-planes (see Theorem 3.1.2). We shall prove that Donnelly's existence and uniqueness results can be extended to the case of Damek-Ricci spaces. Thus the geometric conditions such as symmetry and strict negativity of sectional curvatures are proved to be not essential, and it is worthwhile studying the problem for Hadamard manifolds in further generality.

In the last section, we shall prove a non-existence result for proper harmonic maps from complex hyperbolic spaces to real hyperbolic spaces. To be precise, let $\mathbf{B}^m$ and $\mathbf{D}^n$ be the ball models of the m-dimensional complex hyperbolic space and the n-dimensional real hyperbolic space, respectively. Then we prove that there exists no proper harmonic map $u \in C^2(\mathbf{B}^m, \mathbf{D}^n)$ which has C^1-regularity up to the ideal boundary, where $m, n \geq 2$. Recently, Li and Ni [24] obtained the same result in the case of $n = 2$. Furthermore, we show that there exists a counter example to this result if we relax the regularity condition up to the ideal boundary.

It should be remarked that Donnelly [12] proved that, for a suitable boundary map, there exists a solution to the Dirichlet problem for harmonic maps from complex hyperbolic spaces to real hyperbolic spaces, which has sufficiently high regularity up to the ideal boundary. Hence our theorem appears to contradict his result. However, this is caused by the different choice of the models of complex hyperbolic spaces. For the upper half space model, Donnelly used the one defined by

$$M_1 = \left(\mathbf{R}_+ \times \mathbf{R}^{2m-1}, h_1 = \frac{dy^2}{y^2} + \frac{1}{y^2}g_1 + \frac{1}{y^4}g_2\right),$$

where $y \in \mathbf{R}_+$ and $g_1 + g_2$ is a Riemannian metric on $\mathbf{R}^{2m-1}$. On the other hand, our model is given by

$$M_2 = \left(\mathbf{R}_+ \times \mathbf{R}^{2m-1}, h_2 = \frac{d\eta^2}{4\eta^2} + \frac{1}{\eta}g_1 + \frac{1}{\eta^2}g_2\right),$$

where $\eta \in \mathbf{R}_+$. If one define a map $f : M_2 \to M_1$ by

$$f(\eta, x_i) = (\sqrt{\eta}, x_i) \qquad \text{for} \quad (\eta, x_i) \in \mathbf{R}_+ \times \mathbf{R}^{2m-1},$$

then it is easily verified that $f^*h_1 = h_2$, that is, $f : M_2 \to M_1$ is an isometry, in particular, a smooth map from M_2 to M_1. However, a function $y = \sqrt{\eta}$ is not smooth at $\eta = 0$, which implies that the differentiable structures around the ideal boundaries of M_1 and M_2 are different each other. Thus, even if $u : \overline{\mathbf{B}^m} \to \overline{\mathbf{D}^n}$ is smooth up to the ideal boundary with respect to the differentiable structure for Donnelly's model, it may not be the case with respect to ours.

When a point $p \in \mathbf{R}_+ \times \mathbf{R}^{2m-1}$ tends to the ideal boundary, each component of the metric h_1 blows up at least with the order of y^2, and likewise there exists a component of the metric h_2 which blows up with the order of η. Thus, we conjecture that, for other general Hadamard manifolds, the existence or non-existence of proper harmonic maps is also closely related to the growth rate of Riemannian metrics and the growth order of a proper harmonic map near ideal boundaries.

CHAPTER 1

Ordinary differential equations associated to equivariant harmonic maps

In this chapter, we study the existence of a positive solution $r = r(t)$ to the following equation (1.1.1) with the boundary condition (1.1.2):

$$\begin{aligned} \ddot{r}(t) &+ \left\{ p \, \frac{\dot{f}_1(t)}{f_1(t)} + q \, \frac{\dot{f}_2(t)}{f_2(t)} \right\} \dot{r}(t) \\ &- \left\{ \mu^2 \, \frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2} + \nu^2 \, \frac{h_2(r(t))h_2'(r(t))}{f_2(t)^2} \right\} = 0 \quad \text{on } [0, \infty), \end{aligned} \tag{1.1.1}$$

$$\lim_{t \to 0} r(t) = 0, \tag{1.1.2}$$

where $\dot{r}(t)$ (resp. $h'(r)$) means $(dr/dt)(t)$ (resp. $(dh/ds)(r)$), μ and ν are nonnegative numbers, and p and q are integers. Here we assume that

$$\mu^2 > \nu^2 \geq 0 \quad \text{and} \quad p > q \geq 0.$$

The equation (1.1.1) stems from the study of equivariant harmonic maps between complete noncompact Riemannian manifolds, for instance, the real (resp. complex) Euclidean space $(\mathbf{R}^m, g_{\mathbf{R}^m})$ (resp. $(\mathbf{C}^m, g_{\mathbf{C}^m})$), the real hyperbolic space $(\mathbf{RH}^m, g_{\mathbf{RH}^m})$ of constant negative sectional curvature -1 and the complex hyperbolic space $(\mathbf{CH}^m, g_{\mathbf{CH}^m})$ of constant holomorphic sectional curvature -1.

Throughout this chapter, we assume that C^∞-functions $f_i = f_i(t)$ $(i = 1, 2)$ defined on

$[0, \infty)$ satisfy the following conditions:

$$\begin{cases} \text{(F-1)} \quad f_i(t) > 0 \text{ on } (0, \infty), \text{ and } \dot{f}_i(t) \geq 0 \text{ on } [0, \infty), \\ \text{(F-2)} \quad \displaystyle\lim_{t \to 0} \frac{t}{f_i(t)} = a_i, \text{ where } a_1 > 0 \text{ and } a_2 \geq 0, \\ \text{(F-3)} \quad \text{for any } t \geq 0 \text{ the following holds:} \\ \qquad\quad \displaystyle 0 \leq p\, t\, \frac{\dot{f}_1(t)}{f_1(t)} + q\, t\, \frac{\dot{f}_2(t)}{f_2(t)} - 1, \text{ and} \\ \text{(F-4)} \quad \displaystyle\int^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau < \infty. \end{cases}$$

Moreover, we assume the following conditions for C^{∞}-functions $h_i = h_i(s)$ $(i = 1, 2)$ defined on $\mathbf{R}$:

$$\begin{cases} \text{(H-1)} \quad h_i(s) > 0 \text{ on } (0, \infty), \\ \text{(H-2)} \quad h_i(0) = 0, \quad h_i'(0) > 0, \quad \text{and} \\ \text{(H-3)} \quad \{h_i h_i{}'\}'(s) \geq 0 \text{ on } \mathbf{R}. \end{cases}$$

Our main purpose of this chapter is to prove the following theorems.

Theorem A. *If* $\displaystyle\int^{\infty} \frac{ds}{h(s)} < \infty$, *then there exists a global, strictly monotone increasing solution to the equation* (1.1.1) *with the boundary condition* (1.1.2). *Moreover, at least one solution is unbounded.*

Theorem B. *If* $\displaystyle\int^{\infty} \frac{ds}{h(s)} = \infty$, *then every solution to the equation* (1.1.1) *with the boundary condition* (1.1.2) *is a global, strictly monotone increasing, bounded solution.*

As a consequence, when $\displaystyle\int^{\infty} \frac{ds}{h(s)} = \infty$, the limit value $\displaystyle\lim_{t \to \infty} r(t)$ of a solution $r = r(t)$ exists. Conversely, for any $l \geq 0$, we can construct a global, strictly monotone increasing solution $r = r(t)$ which converges to l as $t \to \infty$ (see Theorem 1.2.6).

1.1 Local existence of solutions to (1.1.1) with (1.1.2)

In this section, we consider the following ordinary differential equation which includes the equation (1.1.1) as a special case:

$$\ddot{r}(t) + F(t)\dot{r}(t) - G(t, r(t)) = 0. \tag{1.1.3}$$

Here we assume that F and G satisfy the following conditions:

$$\begin{cases} \text{(C-1)} \quad F \in C^\infty((0,\infty)), \\ \text{(C-2)} \quad tF \in C([0,\infty)),\ tF(t) - 1 \geq 0 \quad \text{on } [0,\infty), \\ \text{(C-3)} \quad G \in C^\infty((0,\infty) \times \mathbf{R}),\ G(t,0) = 0 \quad \text{for} \ \ t > 0, \\ \text{(C-4)} \quad G(t,s) > 0 \quad \text{for } (t,s) \in (0,\infty) \times (0,\infty), \\ \qquad\qquad \lim_{t\to 0} t^2 G(t,s) > 0 \quad \text{for} \ \ s > 0, \qquad \lim_{t\to 0, s\to 0} t^2 G(t,s) = 0,\ \text{ and} \\ \text{(C-5)} \quad s_1 \leq s_2 \Longrightarrow G(t,s_1) \leq G(t,s_2) \quad \text{for any } t \in (0,\infty). \end{cases}$$

Let f_i and h_i be functions satisfying the conditions (F-1) through (F-4) and (H-1) through (H-3), respectively. Define $F(t)$ and $G(t,s)$ by

$$F(t) = p\frac{\dot{f}_1(t)}{f_1(t)} + q\frac{\dot{f}_2(t)}{f_2(t)},$$

$$G(t,s) = \mu^2 \frac{h_1(s)h_1'(s)}{f_1(t)^2} + \nu^2 \frac{h_2(s)h_2'(s)}{f_2(t)^2}.$$

Then it is easily verified that these functions, F and G, satisfy the conditions (C-1) through (C-5).

We are going to prove the short time existence of a solution to the equation (1.1.3). Our goal is to show the following

Theorem 1.1.1. *Under the conditions* (C-1) *through* (C-5)*, for any* $t_0 > 0$ *and* $r_0 > 0$*, there exists a unique positive solution* $r = r(t)$ *to the equation* (1.1.3) *on* $[0, t_0]$ *which satisfies the* *following conditions*:

(1) $\lim_{t\to 0} r(t) = 0$ *and* $r(t_0) = r_0$.

(2) $\dot{r}(t) > 0$ *on* $(0, t_0]$.

In the remainder of this section, we choose $t_0 > 0$ arbitrarily and fix it.

Lemma 1.1.2. *Assume that* $r = r(t)$ *and* $\rho = \rho(t)$ *are defined on* $[0, T]$ *and satisfy the following conditions on* $(0, T)$:

$$\begin{cases} \ddot{r}(t) + F(t)\dot{r}(t) = G(t, r(t)), \\ \ddot{\rho}(t) + F(t)\dot{\rho}(t) \leq G(t, \rho(t)). \end{cases}$$

If $r(T) = \rho(T)$ *and* $r(0) = \rho(0)$, *then* $r \leq \rho$ *on* $[0, T]$.

Proof. Set $w(t) := \rho(t) - r(t)$. Then we have

$$\ddot{w}(t) + F(t)\dot{w}(t) \leq G(t, \rho(t)) - G(t, r(t)).$$

Assume that there exists $t_1 \in (0, T)$ such that $w(t_1) < 0$. Then, since $w(0) = w(T) = 0$, we may suppose $\dot{w}(t_1) = 0$ and $\ddot{w}(t_1) > 0$. On the other hand, since $\rho(t_1) < r(t_1)$, it follows from (C-5) that

$$\begin{aligned} \ddot{w}(t_1) &\leq -F(t_1)\dot{w}(t_1) + G(t_1, \rho(t_1)) - G(t_1, r(t_1)) \\ &= G(t_1, \rho(t_1)) - G(t_1, r(t_1)) \\ &\leq 0, \end{aligned}$$

which is a contradiction. Thus $w \geq 0$ on $[0, T]$. $\square$

As an application of this Lemma, we can easily prove the uniqueness part of Theorem 1.1.1 as follows.

Corollary 1.1.3. *Let $r = r(t)$ and $\rho = \rho(t)$ be solutions to the equation* (1.1.3) *with the boundary condition* (1.1.2)*. If $r(t_0) = \rho(t_0)$ holds for some $t_0 \in (0,T)$, then we have*

$$r(t) = \rho(t) \qquad \textit{on} \quad [0,T),$$

where $[0,T)$ is the common life span of $r(t)$ and $\rho(t)$. In particular, if $r(t_0) = 0$, then $r \equiv 0$.

To prove the existence part of Theorem 1.1.1, we employ a method due to Baird ([3, Chapter 6]).

We replace the variable t in the equation (1.1.3) with $\tau = \log t$ to remove the singularity at $t = 0$. Then the equation (1.1.3) becomes

$$\frac{d^2 r}{d\tau^2}(\tau) + P(\tau)\frac{dr}{d\tau}(\tau) - Q(\tau, r(\tau)) = 0, \tag{1.1.4}$$

where

$$P(\tau) = e^{\tau} F(e^{\tau}) - 1, \qquad Q(\tau, s) = e^{2\tau} G(e^{\tau}, s).$$

Note that P and Q satisfy the following conditions:

$$\begin{cases} P \in C^{\infty}(\mathbf{R}),\ P(\tau) \geq 0 \quad \text{on } \mathbf{R}, \\ Q \in C^{\infty}(\mathbf{R} \times \mathbf{R}), \quad Q(\tau, s) > 0 \quad \text{for } (\tau, s) \in \mathbf{R} \times (0, \infty) \\ \lim\limits_{\tau \to -\infty} Q(\tau, s) > 0 \quad \text{for any } s > 0, \qquad \lim\limits_{\tau \to -\infty, s \to 0} Q(\tau, s) = 0, \\ Q(\tau, 0) = 0 \quad \text{for any } \tau \in \mathbf{R},\ \text{and,} \\ s_1 \leq s_2 \Longrightarrow Q(\tau, s_1) \leq Q(\tau, s_2) \quad \text{for any } \tau \in \mathbf{R}. \end{cases}$$

We also set $\tau_0 = \log t_0$.

Under these conditions, we prove the following

Theorem 1.1.4. *For any $\tau_0 \in \mathbf{R}$ and $r_0 > 0$, there exists a solution $r = r(\tau)$ to the equation* (1.1.4) *on $(-\infty, \tau_0]$ satisfying the following conditions*:

(1) $r(\tau) > 0,\ \dfrac{dr}{d\tau}(\tau) > 0$ *on $(-\infty, \tau_0]$, and*

(2) $\lim\limits_{\tau \to -\infty} r(\tau) = 0,\ \lim\limits_{\tau \to -\infty} \dfrac{dr}{d\tau}(\tau) = 0$ *and* $\lim\limits_{\tau \to -\infty} \dfrac{d^2 r}{d\tau^2}(\tau) = 0.$

It is easy to see that Corollary 1.1.3 together with Theorem 1.1.4 implies Theorem 1.1.1.

To prove Theorem 1.1.4, we begin with the following lemma concerning the life span of $r = r(\tau)$.

Lemma 1.1.5. *Let $r = r(\tau)$ be a solution to the equation* (1.1.4). *Then $(dr/d\tau)(\tau)$ is bounded if so is $r(\tau)$. Namely, if $(\bar{\tau}, \tau_0]$ is the life span of $r = r(\tau)$ and $-\infty < \bar{\tau}$, then $\lim_{\tau \to \bar{\tau}+0} |r(\tau)| = \infty$.*

Proof. Let $\tilde{P}(\tau) = \exp \int^\tau P(\rho)d\rho$. Then, from the equation (1.1.4), we have

$$\frac{d}{d\tau}\left\{\tilde{P}(\tau)\frac{dr}{d\tau}(\tau)\right\} = \tilde{P}(\tau)Q(\tau, r(\tau)).$$

Integrating both sides from $\tau(< \tau_0)$ to τ_0, we obtain

$$\tilde{P}(\tau_0)\frac{dr}{d\tau}(\tau_0) - \tilde{P}(\tau)\frac{dr}{d\tau}(\tau) = \int_\tau^{\tau_0} \tilde{P}(\rho)Q(\rho, r(\rho))d\rho.$$

Hence

$$\tilde{P}(\tau)\left|\frac{dr}{d\tau}(\tau)\right| \leq \tilde{P}(\tau_0)\left|\frac{dr}{d\tau}(\tau_0)\right| + \int_\tau^{\tau_0} \tilde{P}(\rho)|Q(\rho, r(\rho))|d\rho.$$

Since $Q \in C^\infty(\mathbf{R} \times \mathbf{R})$, the assertion holds. $\square$

Lemma 1.1.6. *Let $r_1 = r_1(\tau)$ and $r_2 = r_2(\tau)$ be solutions to the equation* (1.1.4) *satisfying*

$$r_1(\tau_0) \leq r_2(\tau_0) \quad \textit{and} \quad \frac{dr_1}{d\tau}(\tau_0) > \frac{dr_2}{d\tau}(\tau_0).$$

Then it holds that

$$r_1 < r_2 \quad \textit{and} \quad \frac{dr_1}{d\tau} > \frac{dr_2}{d\tau} \qquad \textit{on } (\bar{\tau}, \tau_0),$$

where $(\bar{\tau}, \tau_0)$ is the common life span of r_1 and r_2.

Proof. Let $w(\tau) = r_1(\tau) - r_2(\tau)$. Note that $w(\tau_0) \leq 0$ and $(dw/d\tau)(\tau_0) > 0$. Assume that there exists $\tau_1 \in (\bar{\tau}, \tau_0)$ such that $w(\tau_1) \geq 0$. Then there exists a point $\tau_2 \in (\tau_1, \tau_0)$ so that

$$w(\tau_2) < 0, \quad \frac{dw}{d\tau}(\tau_2) = 0 \quad \text{and} \quad \frac{d^2w}{d\tau^2}(\tau_2) > 0.$$

On the other hand, since r_1 and r_2 are solutions to the equation (1.1.4) and the function $s \mapsto Q(\cdot, s)$ is monotone, it holds that

$$\begin{aligned}\frac{d^2 w}{d\tau^2}(\tau_2) &= -P(\tau_2)\frac{dw}{d\tau}(\tau_2) + Q(\tau_2, r_1(\tau_2)) - Q(\tau_2, r_2(\tau_2)) \\ &= Q(\tau_2, r_1(\tau_2)) - Q(\tau_2, r_2(\tau_2)) \\ &\leq 0,\end{aligned}$$

which yields a contradiction. Hence $w(\tau) < 0$, that is, $r_1(\tau) < r_2(\tau)$ for $\tau \in (\bar{\tau}, \tau_0)$.

Since r_1 and r_2 are solutions to the equation (1.1.4), we have

$$\frac{d}{d\tau}\left\{\tilde{P}(\tau)\frac{d}{d\tau}(r_1(\tau) - r_2(\tau))\right\} = \tilde{P}(\tau)\{Q(\tau, r_1(\tau)) - Q(\tau, r_2(\tau))\}.$$

Multiplying both sides by $r_1 - r_2$ then yields

$$\begin{aligned}&(r_1(\tau) - r_2(\tau))\frac{d}{d\tau}\left\{\tilde{P}(\tau)\frac{d}{d\tau}\{r_1(\tau) - r_2(\tau)\}\right\} \\ &= (r_1(\tau) - r_2(\tau))\tilde{P}(\tau)\{Q(\tau, r_1(\tau)) - Q(\tau, r_2(\tau))\} \\ &\geq 0.\end{aligned}$$

Therefore, integrating both sides of this inequality from $\tau(> \bar{\tau})$ to τ_0, we obtain

$$\begin{aligned}&\tilde{P}(\tau_0)(r_1(\tau_0) - r_2(\tau_0))\frac{d}{d\tau}(r_1(\tau_0) - r_2(\tau_0)) \\ &\geq \tilde{P}(\tau)(r_1(\tau) - r_2(\tau))\frac{d}{d\tau}(r_1(\tau) - r_2(\tau)) + \int_\tau^{\tau_0} \tilde{P}(\tau)\left\{\frac{d}{d\tau}(r_1(\tau) - r_2(\tau))\right\}^2 d\tau \\ &> \tilde{P}(\tau)(r_1(\tau) - r_2(\tau))\frac{d}{d\tau}(r_1(\tau) - r_2(\tau)),\end{aligned}$$

from which it follows that

$$\begin{aligned}&\tilde{P}(\tau)(r_1(\tau) - r_2(\tau))\frac{d}{d\tau}(r_1(\tau) - r_2(\tau)) \\ &< \tilde{P}(\tau_0)(r_1(\tau_0) - r_2(\tau_0))\frac{d}{d\tau}(r_1(\tau_0) - r_2(\tau_0)) \\ &\leq 0.\end{aligned}$$

Since $r_1(\tau) < r_2(\tau)$, we have

$$\frac{dr_1}{d\tau}(\tau) > \frac{dr_2}{d\tau}(\tau)$$

for $\tau \in (\bar{\tau}, \tau_0)$. □

As a corollary of this lemma, we obtain the following

Corollary 1.1.7. *There is at most one solution to the two point boundary value problem of the equation* (1.1.4).

For each $r_0 > 0$ and $s_0 \in \mathbf{R}$ we shall solve the ordinary differential equation (1.1.4) backward with the initial condition

$$r(\tau_0) = r_0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau_0) = s_0. \tag{1.1.5}$$

Our aim is to show that for any $r_0 > 0$ there exists an $s_0 > 0$ such that a solution $r = r(\tau)$ exists on $(-\infty, \tau_0]$ and satisfies the condition (1) of Theorem 1.1.4.

Define a set $\mathcal{A}(r_0)$ by

$$\mathcal{A}(r_0) := \left\{ s_0 \in \mathbf{R} \;\middle|\; \begin{array}{l} \text{a solution } r(\tau) \text{ to (1.1.4) satisfying (1.1.5) exists, which decreases} \\ \text{monotonically to zero within finite time as } \tau \text{ decreases from } \tau_0 \end{array} \right\}.$$

Then we have the following

Proposition 1.1.8. *Let $r_0 > 0$. Then one of the following two cases occurs:*

(1) *The set $\mathcal{A}(r_0)$ is an empty set. In this case, for any $s_0 > 0$ there exists a positive solution $r = r(\tau)$ on $(-\infty, \tau_0]$ to the equation* (1.1.4) *with the initial condition* (1.1.5), *which satisfies* $\dfrac{dr}{d\tau}(\tau) > 0$.

(2) *The set $\mathcal{A}(r_0)$ is not empty. In this case, $\mathcal{A}(r_0)$ is an open set, and* $\inf \mathcal{A}(r_0) > 0$.

Proof. (1) For any fixed $r_0 > 0$, we take $s_0 > 0$ arbitrarily. Let $r = r(\tau)$ be a solution to the equation (1.1.4) with the initial condition (1.1.5), and let $(\bar{\tau}, \tau_0]$ be its life span. Then we have

$$\frac{dr}{d\tau}(\tau) > 0 \qquad \text{on } (\bar{\tau}, \tau_0].$$

Indeed, suppose that there exists $\tau_1 \in (\bar{\tau}, \tau_0)$ such that

$$\frac{dr}{d\tau}(\tau_1) = 0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau) > 0 \quad \text{on } (\tau_1, \tau_0].$$

Since $\mathcal{A}(r_0)$ is an empty set, $r(\tau) > 0$ for $\tau \in (\tau_1, \tau_0]$. Choose $\bar{s_0}$ so that

$$\bar{s_0} > \max\left\{s_0, C_1(\tau_0 - \tau) + \frac{r_0}{\tau_0 - \tau_1}\right\},$$

where $C_1 := \max_{\tau\in[\tau_1,\tau_0]} Q(\tau, r_0)$. We shall show that $\bar{s_0} \in \mathcal{A}(r_0)$.

Let $\rho = \rho(\tau)$ be a solution to the equation (1.1.4) satisfying

$$\rho(\tau_0) = r(\tau_0) \quad \text{and} \quad \frac{d\rho}{d\tau}(\tau_0) = \bar{s_0} > \frac{dr}{d\tau}(\tau_0),$$

and $(\tau_2, \tau_0]$ the life span of ρ. Then from Lemma 1.1.6 it holds that

$$\rho(\tau) < r(\tau) \quad \text{and} \quad \frac{d\rho}{d\tau}(\tau) > \frac{dr}{d\tau}(\tau)$$

as long as both r and ρ exist.

When $\tau_1 \leq \tau_2$, we have $\rho(\tau) \to -\infty$ as $\tau \to \tau_2 + 0$. Hence there exists $\tau_3 \in (\tau_2, \tau_0]$ such that

$$\rho(\tau_3) = 0 \quad \text{and} \quad \frac{d\rho}{d\tau}(\tau) > \frac{dr}{d\tau}(\tau) > 0 \quad \text{on } (\tau_3, \tau_0].$$

Therefore $\bar{s_0} \in \mathcal{A}(r_0)$, contradicting the fact $\mathcal{A}(r_0) = \emptyset$.

On the other hand, when $\tau_1 > \tau_2$, ρ satisfies $(d\rho/d\tau)(\tau) > (dr/d\tau)(\tau) > 0$ on $(\tau_1, \tau_0]$. Let

$$C_2 := \min_{\tau\in[\tau_1,\tau_0]} P(\tau).$$

Then $C_2 \geq 0$, and it holds that

$$\begin{aligned}
\frac{d^2\rho}{d\tau^2}(\tau) &= -P(\tau)\frac{d\rho}{d\tau}(\tau) + Q(\tau, \rho(\tau)) \\
&\leq -P(\tau)\frac{d\rho}{d\tau}(\tau) + Q(\tau, \rho(\tau_0)) \\
&\leq -C_2\frac{d\rho}{d\tau}(\tau) + C_1
\end{aligned}$$

on $[\tau_1, \tau_0]$. Integrating this inequality from $\tau \in [\tau_1, \tau_0]$ to τ_0, we have

$$\begin{aligned}
-\frac{d\rho}{d\tau}(\tau) &\leq -\frac{d\rho}{d\tau}(\tau_0) - C_2\{\rho(\tau_0) - \rho(\tau)\} + C_1(\tau_0 - \tau) \\
&\leq -\bar{s_0} + C_1(\tau_0 - \tau) \\
&< -\frac{r_0}{\tau_0 - \tau_1}.
\end{aligned}$$

Integrating again both sides from τ_1 to τ_0, we then obtain

$$\rho(\tau_1) < 0.$$

Therefore, $\bar{s_0} \in \mathcal{A}(r_0)$, which also contradicts the fact $\mathcal{A}(r_0) = \emptyset$. Hence

$$\frac{dr}{d\tau}(\tau) > 0 \qquad \text{on } (\bar{\tau}, \tau_0].$$

Assume that $\bar{\tau} > -\infty$. Then, it follows from Lemma 1.1.5 that $|r(\tau)| \to \infty$ as $\tau \to \bar{\tau} + 0$. If $r(\tau) \to \infty$, then, by the mean value theorem, there exists a point $\tau_1 \in (\bar{\tau}, \tau_0]$ such that

$$\frac{dr}{d\tau}(\tau_1) = 0,$$

which implies that $s_0 \in \mathcal{A}(r_0)$, a contradiction.

Similarly, if $r(\tau) \to -\infty$, then there exists a point $\tau_2 \in (\bar{\tau}, \tau_0]$ such that

$$r(\tau_2) = 0,$$

implying that $s_0 \in \mathcal{A}(r_0)$, a contradiction.

As a consequence, $\bar{\tau} = -\infty$, and

$$r(\tau) > 0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau) > 0 \quad \text{on } (-\infty, \tau_0].$$

(2) We first prove that $\mathcal{A}(r_0)$ is an open set. Fix $r_0 > 0$, and take $s_0 \in \mathcal{A}(r_0)$ arbitrarily. Let $r = r(\tau)$ be a solution to the equation (1.1.4) with the initial condition (1.1.5). Then we can find $\tau_1 < \tau_0$ such that

$$r(\tau_1) = 0 \quad \text{and} \quad r(\tau) > 0 \quad \text{on } (\tau_1, \tau_0).$$

Moreover, there exists an $\varepsilon > 0$ such that

$$\frac{dr}{d\tau}(\tau) > 0 \quad \text{on } I := [\tau_1 - \varepsilon, \tau_0].$$

Set

$$\eta := \frac{1}{2} \min_{\tau \in I} \frac{dr}{d\tau}(\tau), \quad \xi := -\frac{1}{2} r(\tau_1 - \varepsilon) \quad \text{and} \quad \delta := \min\{\eta, \xi\}.$$

Since the solutions depend continuously on initial values, it follows that for the above $\delta > 0$, there exists an $\bar{\varepsilon} > 0$ such that if $|\bar{s_0} - s_0| < \bar{\varepsilon}$, then

$$\sup_{\tau \in I} |\rho(\tau) - r(\tau)| + \sup_{\tau \in I} \left| \frac{d\rho}{d\tau}(\tau) - \frac{dr}{d\tau}(\tau) \right| < \delta,$$

where ρ is a solution to the equation (1.1.4) satisfying $\rho(\tau_0) = r(\tau_0) = r_0$ and $(d\rho/d\tau)(\tau_0) = \bar{s_0}$. Then we obtain

$$\frac{d\rho}{d\tau}(\tau) > \frac{dr}{d\tau}(\tau) - \delta > \eta > 0 \quad \text{on } [\tau_1 - \varepsilon, \tau_0],$$

and

$$\rho(\tau_1 - \varepsilon) < r(\tau_1 - \varepsilon) + \delta < \frac{1}{2} r(\tau_1 - \varepsilon) < 0.$$

Hence $\bar{s_0} \in \mathcal{A}(r_0)$. Thus $\mathcal{A}(r_0)$ is an open set.

Next we show that $\inf \mathcal{A}(r_0) > 0$. Suppose that $\inf \mathcal{A}(r_0) = 0$. For $s_0 \in \mathcal{A}(r_0)$, let $r = r(\tau)$ be a solution to the equation (1.1.4) with the initial condition (1.1.5). Then

$$\frac{d^2 r}{d\tau^2}(\tau_0) = -P(\tau_0) s_0 + Q(\tau_0, r_0).$$

Since $Q(\tau_0, r_0) > 0$, for sufficiently small $\delta > 0$, one can take $s_0 > 0$ so that

$$\frac{d^2 r}{d\tau^2}(\tau_0) > 2\delta.$$

Hence there exists an $\varepsilon = \varepsilon(s_0) > 0$ such that

$$\frac{d^2 r}{d\tau^2}(\tau) > \delta, \quad \frac{dr}{d\tau}(\tau) > 0 \quad \text{and} \quad r(\tau) > 0 \quad \text{on } (\tau_0 - 2\varepsilon, \tau_0 + 2\varepsilon).$$

For $\bar{s_0} \in \mathcal{A}(r_0)$ such that $\bar{s_0} < s_0$, let ρ be a solution to the equation (1.1.4) with

$$\rho(\tau_0) = r(\tau_0) = r_0 \quad \text{and} \quad \frac{d\rho}{d\tau}(\tau_0) = \bar{s_0}.$$

It follows from Lemma 1.1.6 that ρ exists on $(\tau_0 - 2\varepsilon, \tau_0 + 2\varepsilon)$ and satisfies

$$\rho(\tau) > r(\tau) \quad \text{and} \quad \frac{d\rho}{d\tau}(\tau) < \frac{dr}{d\tau}(\tau). \tag{1.1.6}$$

Thus we obtain

$$\begin{aligned} \frac{d^2 \rho}{d\tau^2}(\tau) &= -P(\tau) \frac{d\rho}{d\tau}(\tau) + Q(\tau, \rho(\tau)) \\ &\geq -P(\tau) \frac{dr}{d\tau}(\tau) + Q(\tau, r(\tau)) = \frac{d^2 r}{d\tau^2}(\tau) > \delta \end{aligned}$$

on $(\tau_0 - 2\varepsilon, \tau_0 + 2\varepsilon)$. Note that ε does not depend on the choice of $\bar{s_0}$, if $\bar{s_0} < s_0$. Integrating both sides of this inequality from $\tau_0 - \varepsilon$ to τ_0, we have

$$\frac{d\rho}{d\tau}(\tau_0 - \varepsilon) < \bar{s_0} - \delta\varepsilon.$$

Since $\inf \mathcal{A}(r_0) = 0$, we can take $\bar{s_0} \in \mathcal{A}(r_0)$ such that $\bar{s_0} < \delta\varepsilon$. Hence $(d\rho/d\tau)(\tau_0 - \varepsilon) < 0$. On the other hand, (1.1.6) implies that

$$\rho(\tau) > 0 \quad \text{on } (\tau_0 - \varepsilon, \tau_0].$$

These contradict the assumption that $\bar{s}_0 \in \mathcal{A}(r_0)$. Therefore, $\inf \mathcal{A}(r_0) > 0$. $\square$

Now, we are in a position to prove Theorem 1.1.4.

Proof of Theorem 1.1.4.

(1) Take $r_0 > 0$ arbitrarily. If $\mathcal{A}(r_0)$ is an empty set, we have already shown that there exists a solution $r = r(\tau)$ on $(-\infty, \tau_0]$ to the equation (1.1.4) satisfying

$$r(\tau) > 0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau) > 0.$$

Thus we assume that $\mathcal{A}(r_0)$ is not empty. Let $r_0' = \inf \mathcal{A}(r_0)$, and let $r = r(\tau)$ be a solution to the equation (1.1.4) satisfying

$$r(\tau_0) = r_0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau_0) = r_0'.$$

Note that $r_0' \notin \mathcal{A}(r_0)$ because $\mathcal{A}(r_0)$ is an open set.

Suppose that $(\tau_*, \tau_0]$ is the life span of $r = r(\tau)$. We first show that

$$r(\tau) > 0 \quad \text{on } (\tau_*, \tau_0].$$

We assume that there exists $\tau_1 \in (\tau_*, \tau_0)$ such that

$$r(\tau_1) = 0 \quad \text{and} \quad r(\tau) > 0 \quad \text{on } (\tau_1, \tau_0].$$

It is easily verified that if $(dr/d\tau)(\tau_1) = 0$, then $r = r(\tau)$ must be the zero solution. Thus, since $r_0' \notin \mathcal{A}(r_0)$, there exists $\tau_2 \in (\tau_1, \tau_0)$ such that

$$r(\tau_2) > 0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau_2) = 0.$$

Thus

$$\begin{aligned}\frac{d^2r}{d\tau^2}(\tau_2) &= -P(\tau_2)\frac{dr}{d\tau}(\tau_2) + Q(\tau_2, r(\tau_2))\\ &= Q(\tau_2, r(\tau_2)) > 0.\end{aligned}$$

Therefore, $r(\tau_2)$ is locally a minimum value. On the other hand, we can show that

$$\frac{dr}{d\tau}(\tau) < 0$$

on (τ_1, τ_2). Indeed, suppose that there exists $\tau_3 \in (\tau_1, \tau_2)$ such that

$$\frac{dr}{d\tau}(\tau_3) = 0 \quad \text{and} \quad \frac{dr}{d\tau}(\tau) < 0 \quad \text{on } (\tau_3, \tau_2).$$

Then we have

$$\frac{d^2r}{d\tau^2}(\tau_3) \leq 0.$$

However, from the equation (1.1.4), it holds that

$$\frac{d^2r}{d\tau^2}(\tau_3) = Q(\tau_3, r(\tau_3)) > 0,$$

which is a contradiction. Hence, $r(\tau_2)$ is positive and is the minimal value of r on $[\tau_1, \tau_0]$, which contradicts the assumption $r(\tau_1) = 0$. Thus $r > 0$ on $(\tau_*, \tau_0]$.

Second, we show that

$$\frac{dr}{d\tau}(\tau) > 0 \ \text{ on } \ (\tau_*, \tau_0].$$

Assume that there exists $\tau_1 \in (\tau_*, \tau_0)$ such that

$$\frac{dr}{d\tau}(\tau_1) = 0.$$

Then, by the same argument used for proving $r > 0$, we have

$$\frac{dr}{d\tau}(\tau) < 0 \quad \text{on } (\tau_*, \tau_1) \quad \text{and} \quad \frac{dr}{d\tau}(\tau) > 0 \quad \text{on } (\tau_1, \tau_0],$$

which implies that $r(\tau_1)$ is the minimal value of r on $(\tau_*, \tau_0]$. Take $\tau_2 \in (\tau_*, \tau_1)$ and $\tau_3 \in (\tau_1, \tau_0)$ so that $r(\tau_2) = r(\tau_3)$ and $I = [\tau_2, \tau_3]$. It then follows from the continuous dependence of solutions on initial values that for any positive number $\varepsilon < (1/2)\min\{r(\tau_1), r(\tau_2) - r(\tau_1)\}$, there exists $\delta > 0$ such that

$$\sup_{\tau \in I} |\rho(\tau) - r(\tau)| + \sup_{\tau \in I} \left| \frac{d\rho}{d\tau}(\tau) - \frac{dr}{d\tau}(\tau) \right| < \varepsilon$$

provided that $|r_0' - s_0| < \delta$. Here, ρ is a solution to the equation (1.1.4) satisfying $\rho(\tau_0) = r(\tau_0)$ and $(d\rho/d\tau)(\tau_0) = s_0$. In particular, we may assume that $s_0 \in \mathcal{A}(r_0)$. Since we have

$$\sup_{\tau \in I} |\rho(\tau) - r(\tau)| < \varepsilon,$$

it follows that

$$\rho(\tau) > r(\tau) - \varepsilon > r(\tau) - r(\tau_1) > 0 \quad \text{on } [\tau_2, \tau_0].$$

On the other hand, we have

$$\rho(\tau_2) > r(\tau_2) - \varepsilon > r(\tau_1) > \rho(\tau_1) > 0,$$
$$\rho(\tau_3) > r(\tau_3) - \varepsilon > r(\tau_1) > \rho(\tau_1) > 0.$$

Hence there exists $\tau_4 \in (\tau_2, \tau_3)$ such that $(d\rho/d\tau)(\tau_4) = 0$. As a consequence, $\rho = \rho(\tau)$ satisfies

$$\rho > 0 \quad \text{on } [\tau_2, \tau_0] \quad \text{and} \quad \frac{d\rho}{d\tau}(\tau_4) = 0 \quad \text{for} \quad \tau_4 \in (\tau_2, \tau_0),$$

which contradict the choice of s_0. Therefore, we have

$$\frac{dr}{d\tau}(\tau) > 0 \quad \text{on } (\tau_*, \tau_0].$$

Finally, we shall prove that $\tau_* = -\infty$. If $\tau_* > -\infty$, then $r = r(\tau)$ blows up at τ_*. Then it follows from Lemma 1.1.5 that

$$\lim_{\tau \to \tau_* + 0} r(\tau) = +\infty.$$

In this case, there exists $\bar{\tau} \in (\tau_*, \tau_0)$ such that $(dr/d\tau)(\bar{\tau}) = 0$, which contradicts the fact that $(dr/d\tau)(\tau) > 0$ on $(\tau_*, \tau_0]$. Thus $\tau_* = -\infty$.

(2) Although it can be proved, by using the same argument as that in [22], that

$$\lim_{\tau \to -\infty} \frac{dr}{d\tau}(\tau) = 0,$$

we present here a proof in our context.

Take $\tau_1 \in (-\infty, \tau_0)$ arbitrarily, and let A be a positive constant such that $P(\tau) \leq A$ for any $\tau \in (-\infty, \tau_1]$. Fix $\tau_2 \in (-\infty, \tau_1 - 1]$ arbitrarily, and let ρ be a unique solution to the following ordinary differential equation with the initial values:

$$\begin{cases} \dfrac{d^2\rho}{d\tau^2}(\tau) + A\dfrac{d\rho}{d\tau}(\tau) = 0, \\ \rho(\tau_2) = r(\tau_2) \quad \text{and} \quad \dfrac{d\rho}{d\tau}(\tau_2) = \dfrac{dr}{d\tau}(\tau_2). \end{cases}$$

Then it is easy to see that ρ is given by

$$\rho(\tau) = \frac{1}{A}(1 - e^{-A(\tau-\tau_2)})\frac{dr}{d\tau}(\tau_2) + r(\tau_2).$$

Let

$$R(\tau) = \rho(\tau)\frac{dr}{d\tau}(\tau) - r(\tau)\frac{d\rho}{d\tau}(\tau).$$

Then $R(\tau_2) = 0$ and $(dR/d\tau)(\tau_2) > 0$. Moreover, $R \geq 0$ on $(\tau_2, \tau_1]$. Indeed, if this is not the case, then there exists a point $\tau_* \in (\tau_2, \tau_1]$ such that

$$R(\tau_*) = 0, \ \frac{dR}{d\tau}(\tau_*) < 0, \quad \text{and} \quad R > 0 \quad \text{on } (\tau_2, \tau_*).$$

On the other hand, we have

$$\begin{aligned} \frac{dR}{d\tau}(\tau_*) &= \rho(\tau_*)\frac{d^2r}{d\tau^2}(\tau_*) - r(\tau_*)\frac{d^2\rho}{d\tau^2}(\tau_*) \\ &= \rho(\tau_*)\{-P(\tau_*)\frac{dr}{d\tau}(\tau_*) + Q(\tau_*, r(\tau_*))\} + Ar(\tau_*)\frac{d\rho}{d\tau}(\tau_*) \\ &\geq A\{r(\tau_*)\frac{d\rho}{d\tau}(\tau_*) - \rho(\tau_*)\frac{dr}{d\tau}(\tau_*)\} + Q(\tau_*, r(\tau_*))\rho(\tau_*) \\ &= Q(\tau_*, r(\tau_*))\rho(\tau_*) > 0, \end{aligned}$$

which leads to a contradiction. Thus $R \geq 0$ on $[\tau_2, \tau_1]$. Since $r > 0$ and $\rho > 0$, it holds that $r \geq \rho$ on $[\tau_2, \tau_1]$.

As a consequence, we obtain

$$r(\tau_2 + 1) - r(\tau_2) \geq \rho(\tau_2 + 1) - \rho(\tau_2) = \frac{1}{A}\frac{dr}{d\tau}(\tau_2)(1 - e^{-A}) \tag{1.1.7}$$

for any $\tau_2 \in (-\infty, \tau_1 - 1]$.

Now suppose that

$$\limsup_{\tau \to -\infty} \frac{dr}{d\tau}(\tau) > 0.$$

Then there exist a positive constant δ and a sequence $\{\tau_j\}$ satisfying

$$\tau_{j+1} \leq \tau_j - 1 \quad \text{and} \quad r(\tau_j) \geq \delta.$$

Then, from the inequality (1.1.7), we have

$$r(\tau_j + 1) - r(\tau_j) \geq \frac{1}{A}\frac{dr}{d\tau}(\tau_j)(1 - e^{-A}) > \frac{\delta}{A}(1 - e^{-A}).$$

Since r is a monotone increasing function, it holds that

$$r(\tau_j) < r(\tau_j + 1) - \frac{\delta}{A}(1 - e^{-A}) < r(\tau_{j-1}) - \frac{\delta}{A}(1 - e^{-A}).$$

This inequality implies that $r(\tau_j) < 0$ for sufficiently large j, which contradicts the fact $r > 0$. Hence it follows that

$$\lim_{\tau \to -\infty} \frac{dr}{d\tau}(\tau) = 0.$$

We shall show that $\lim\limits_{\tau \to -\infty} r(\tau) = 0$. Since r is monotone increasing, there exists $\eta \geq 0$ such that

$$\lim_{\tau \to -\infty} r(\tau) = \eta.$$

Assume that $\eta > 0$. Noting that $P(\tau)$ is bounded on $(-\infty, \tau_0]$, we have

$$\lim_{\tau \to -\infty} \frac{d^2 r}{d\tau^2}(\tau) = \lim_{\tau \to -\infty} Q(\tau, \eta) > 0.$$

Thus there exists $\delta > 0$ such that

$$\frac{d^2 r}{d\tau^2}(\tau) \geq \delta \qquad \text{on } (-\infty, \tau_1)$$

for τ_1 sufficiently small. Hence we obtain

$$\frac{dr}{d\tau}(\tau) < 0$$

for sufficiently small τ, which is a contradiction. Therefore

$$\lim_{\tau\to-\infty} r(\tau) = 0.$$

Finally, the equation (1.1.4) implies that

$$\frac{d^2r}{d\tau^2}(\tau) = -P(\tau)\frac{dr}{d\tau}(\tau) + Q(\tau, r(\tau)).$$

It follows from the condition (C-1) that

$$\lim_{\tau\to-\infty} P(\tau) \text{ exists and } \lim_{\tau\to-\infty} Q(\tau, r(\tau)) = 0,$$

which implies that

$$\lim_{\tau\to-\infty} \frac{d^2r}{d\tau^2}(\tau) = 0.$$

Thus the proof is completed. □

In the remainder of this section, by making use of a comparison function, we shall investigate the regularity of the solution $r = r(t)$ at $t = 0$.

We assume that there exist $k, l \geq 1$ such that F and G satisfy the following conditions.

$$\lim_{t\to 0} tF(t) = k, \qquad \lim_{t\to 0, s\to 0} t^2\frac{G(t,s)}{s} = l.$$

Then we can prove the following

Proposition 1.1.9. *Let $r = r(t)$ be a solution to the equation* (1.1.3) *with the boundary condition* (1.1.2)*. Then there exist $a > 0$ and $t_1 \in (0, t_0)$ such that*

$$0 < r(t) \leq \frac{r(t_1)}{t_1^a}t^a$$

for $t \in (0, t_1)$.

Proof. Set $\rho(t) := C_0t^a$, where positive constants a and C_0 will be determined later. Then

$$\ddot{\rho}(t) + F(t)\dot{\rho}(t) - G(t, \rho(t)) = C_0t^{a-2}\left\{a(a-1) + atF(t) - t^2\frac{G(t,\rho)}{\rho}\right\}.$$

For any $\varepsilon > 0$ we can find $t_1 \in (0, t_0)$ such that

$$|tF(t) - k| < \varepsilon \quad \text{and} \quad \left| t^2 \frac{G(t,\rho)}{\rho} - l \right| < \varepsilon$$

holds for $t \in (0, t_1)$. Hence we have

$$\ddot{\rho}(t) + F(t)\dot{\rho}(t) - G(t, \rho(t)) \leq C_0 t^{a-2}\{a^2 + (k - 1 + \varepsilon)a - (l - \varepsilon)\}.$$

Consequently, if we take a constant a to be

$$0 < a \leq \frac{-(k - 1 + \varepsilon) + \sqrt{(k - 1 + \varepsilon)^2 + 4(l - \varepsilon)}}{2},$$

then $\rho(t) = C_0 t^a$ satisfies

$$\ddot{\rho}(t) + F(t)\dot{\rho}(t) - G(t, \rho(t)) \leq 0 \quad \text{and} \quad \rho(0) = 0.$$

Choose the above constant C_0 so that $\rho(t_1) = r(t_1)$. Then, from Lemma 1.1.2, we have

$$r(t) \leq \rho(t) = C_0 t^a = \frac{r(t_1)}{t_1^a} t^a. \qquad \square$$

1.2 Global properties of solutions

In the previous section, we prove that for any $t_0 > 0$ and $r_0 > 0$, there exists $s_0 > 0$ such that a solution $r = r(t)$ to the equation (1.1.3) satisfies

$$\begin{cases} r(t_0) = r_0, \quad \dot{r}(t_0) = s_0, \\ \lim_{t \to 0} r(t) = 0, \quad \text{and} \quad \dot{r}(t) > 0 \ \text{ on } \ (0, t_0]. \end{cases}$$

We shall show that for a suitable $r_0 > 0$, there exists a global solution to the equation (1.1.1) with the initial condition $r(t_0) = r_0$.

The following lemma is an analogue of Lemma 1.1.5.

Lemma 1.2.1. *Let $r = r(t)$ be a solution to the equation* (1.1.3) *with the boundary condition* (1.1.2)*, and let $[0, T)$ be its life span.*

(1) *Assume that* $r(t_0) > 0$ *and* $\dot{r}(t_0) > 0$ *for some* $t_0 \in (0, T)$. *Then it holds that*

$$\dot{r}(t) > 0$$

on $(0, T)$. *Hence the solution constructed in the previous section is strictly monotone increasing as long as it exists.*

(2) *If* $T < \infty$, *then*

$$\lim_{t \uparrow T} r(t) = \infty.$$

Proof. (1) From Corollary 1.1.3, the solution $r = r(t)$ coincides with the one constructed in the previous section. Thus $\dot{r}(t) > 0$ on $(0, t_0]$. We shall show that $\dot{r}(t) > 0$ on (t_0, T). If this is not the case, we have a point $t_1 \in (t_0, T)$ such that

$$r(t_1) > 0, \quad \dot{r}(t_1) = 0 \quad \text{and} \quad \ddot{r}(t_1) \leq 0.$$

On the other hand, the equation (1.1.3) asserts that

$$\ddot{r}(t_1) = G(t_1, r(t_1)) > 0,$$

which is a contradiction. Thus $\dot{r}(t) > 0$ on $[t_0, T)$.

(2) Let $\tilde{F}(t) = \exp \int^t F(s)ds$. Then, from the equation (1.1.3), we have

$$\frac{d}{dt}\left\{\tilde{F}(t)\dot{r}(t)\right\} = \tilde{F}(t)G(t, r(t)).$$

Integrating both sides from t_0 to $t(> t_0)$, we obtain

$$\tilde{F}(t)\left|\dot{r}(t)\right| \leq \tilde{F}(t_0)\left|\dot{r}(t_0)\right| + \int_{t_0}^{t} \tilde{F}(t)|G(t, r(t))|dt.$$

Since $G \in C^\infty([0, \infty) \times \mathbf{R})$, if $r(t)$ is bounded when t tends to T, then so is $\dot{r}(t)$. Thus the assertion holds. $\square$

From now on, we shall consider the original ordinary differential equation (1.1.1).

Lemma 1.2.2. *If* $r = r(t)$ *is a solution to the equation* (1.1.1) *with the boundary condition* (1.1.2), *then it satisfies*

$$\dot{r}(t)^2 \leq \frac{\mu^2}{f_1(t)^2} h_1(r(t))^2 + \frac{\nu^2}{f_2(t)^2} h_2(r(t))^2 \tag{1.2.1}$$

on $[t_0, T)$.

Proof. It follows from the equation (1.1.1) that

$$\begin{aligned}
&\frac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{r}(t)\} \\
&\quad = f_1(t)^{p-2} f_2(t)^{q-2}\{\mu^2 f_2(t)^2 h_1(r(t)) h_1'(r(t)) + \nu^2 f_1(t)^2 h_2(r(t)) h_2'(r(t))\}.
\end{aligned}$$

Multiplying both sides by $f_1(t)^p f_2(t)^q \dot{r}(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 \\
&\quad = \mu^2 f_1(t)^{2p-2} f_2(t)^{2q} \frac{d}{dt} h_1(r(t))^2 + \nu^2 f_1(t)^{2p} f_2(t)^{2q-2} \frac{d}{dt} h_2(r(t))^2.
\end{aligned}$$

Integrating both sides of this equation from t_1 to t, we have

$$\begin{aligned}
&\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 \\
&= \{f_1(t_1)^p f_2(t_1)^q \dot{r}(t_1)\}^2 \\
&\quad + \mu^2\{f_1(t)^{2p-2} f_2(t)^{2q} h_1(r(t))^2 - f_1(t_1)^{2p-2} f_2(t_1)^{2q} h_1(r(t_1))^2\} \\
&\quad + \nu^2\{f_1(t)^{2p} f_2(t)^{2q-2} h_2(r(t))^2 - f_1(t_1)^{2p} f_2(t_1)^{2q-2} h_2(r(t_1))^2\} \\
&\quad - \int_{t_1}^{t} \mu^2 h_1(r(\tau))^2 \frac{d}{d\tau}\{f_1(\tau)^{2p-2} f_2(\tau)^{2q}\} d\tau \\
&\quad - \int_{t_1}^{t} \nu^2 h_2(r(\tau))^2 \frac{d}{d\tau}\{f_1(\tau)^{2p} f_2(\tau)^{2q-2}\} d\tau.
\end{aligned} \tag{1.2.2}$$

The condition (F-1) implies

$$\frac{d}{d\tau}\{f_1(\tau)^m f_2(\tau)^n\} \geq 0$$

for any nonnegative integers m and n. As a consequence, we have

$$\begin{aligned}\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 \leq \{f_1(t_1)^p f_2(t_1)^q \dot{r}(t_1)\}^2 \\ + \mu^2\{f_1(t)^{2p-2} f_2(t)^{2q} h_1(r(t))^2 - f_1(t_1)^{2p-2} f_2(t_1)^{2q} h_1(r(t_1))^2\} \\ + \nu^2\{f_1(t)^{2p} f_2(t)^{2q-2} h_2(r(t))^2 - f_1(t_1)^{2p} f_2(t_1)^{2q-2} h_2(r(t_1))^2\}.\end{aligned}$$

Letting t_1 tend to 0, we have

$$\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 \leq \mu^2 f_1(t)^{2p-2} f_2(t)^{2q} h_1(r(t))^2 + \nu^2 f_1(t)^{2p} f_2(t)^{2q-2} h_2(r(t))^2.$$

Dividing both sides by $f_1(t)^{2p} f_2(t)^{2q}$, we obtain the conclusion. □

Using this lemma, we present a sufficient condition for the existence of a global solution in terms of a relation between t_0 and r_0.

If $r(t) = 0$ for some $t > 0$, then $r(t) \equiv 0$ by Corollary 1.1.3. Hence it is a global solution. In the sequel of this section we assume that $r(t) > 0$ for all $t > 0$.

Let

$$h(r) = \begin{cases} \max\{h_1(r), h_2(r)\} & (\nu > 0), \\ h_1(r) & (\nu = 0). \end{cases}$$

Since $\dot{r}(t) > 0$, we have

$$\frac{\dot{r}(t)}{h(r(t))} \leq \gamma \left\{ \frac{1}{f_1(t)} + \frac{1}{f_2(t)} \right\},$$

where $\gamma = \max\{\mu, \nu\}$. Integrating both sides from t_0 to $t (\in [t_0, T])$, we obtain

$$\int_{r_0}^{r(t)} \frac{ds}{h(s)} \leq \gamma \int_{t_0}^{t} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau. \tag{1.2.3}$$

Theorem 1.2.3. *If r_0 satisfies*

$$\int_{r_0}^{\infty} \frac{dr}{h(r)} > \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau, \tag{1.2.4}$$

then the solution to the equation (1.1.1) *with the boundary condition* (1.1.2) *exists globally and is bounded.*

Proof. Let $[0, T)$ be the life span of r, and assume that $T < \infty$. Then, by virtue of Lemma 1.2.1 and (1.2.3), we have

$$\int_{r_0}^{\infty} \frac{ds}{h(s)} = \lim_{t \uparrow T} \int_{r_0}^{r(t)} \frac{ds}{h(s)} \leq \gamma \lim_{t \uparrow T} \int_{t_0}^{t} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau \leq \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau,$$

which contradicts the assumption. Hence $r = r(t)$ is a global solution. If it is unbounded, then from (1.2.3), we obtain

$$\int_{r_0}^{\infty} \frac{ds}{h(s)} = \lim_{t \to \infty} \int_{r_0}^{r(t)} \frac{ds}{h(s)} \leq \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau,$$

which also contradicts our assumption. □

Corollary 1.2.4. *If*

$$\int^{\infty} \frac{ds}{h(s)} = \infty,$$

then any solution $r = r(t)$ to the equation (1.1.1) *with the boundary condition* (1.1.2) *can be extended globally in t and is bounded.*

Proof. The left-hand side of (1.2.4) is infinite for any r_0, and hence the assumption of Theorem 1.2.3 is always satisfied. □

Now, we study the distribution of the limit value $\lim_{t \to \infty} r(t)$ for our global solution $r = r(t)$.

Proposition 1.2.5. *The set*

$$L = \left\{ \lim_{t \to \infty} r(t) \;\middle|\; r \text{ is a global solution to (1.1.1) with (1.1.2)} \right\}$$

is dense in $[0, \infty)$.

Proof. First note that we have the zero solution, and hence $0 \in L$. Let $l > 0$. We shall show that

$$(l - \varepsilon, l + \varepsilon) \cap L \neq \emptyset$$

for any $\varepsilon \in (0, l)$. Note that we may choose $T_0 > 0$ so that

$$\int_{l-\varepsilon}^{\infty} \frac{ds}{h(s)} > \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau$$

for any $t_0 > T_0$. Then, by Theorem 1.2.3, there exists globally a solution $r = r(t)$ to the equation (1.1.1) with the boundary condition (1.1.2) satisfying

$$r(t_0) = l - \varepsilon > 0.$$

Let $r(t; t_0)$ denote this solution. Since it is bounded and increasing, the limit $r(\infty; t_0)$ exists and

$$0 < l - \varepsilon < r(\infty; t_0) < \infty.$$

By the inequality (1.2.3) with $t = \infty$, we have

$$\int_{l-\varepsilon}^{r(\infty;t_0)} \frac{ds}{h(s)} \leq \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau.$$

Since the right hand side tends to 0 as $t_0 \to \infty$, we obtain

$$\lim_{t_0 \to \infty} r(\infty; t_0) = l - \varepsilon.$$

Therefore, for sufficiently large t_0, it holds that

$$r(\infty; t_0) \in (l - \varepsilon, l + \varepsilon).$$ □

We are now in a position to prove our existence result.

Theorem 1.2.6. *For any $l \geq 0$, there exists a global solution $r = r(t)$ to the equation* (1.1.1) *with the boundary condition* (1.1.2) *satisfying $\lim_{t\to\infty} r(t) = l$. Therefore, $L = [0, \infty)$.*

Proof. If $l = 0$, then $r(t) \equiv 0$ is a desired solution. Hence we assume $l > 0$. By virtue of Proposition 1.2.5, there exist sequences $\{\underline{l}_i\}$, $\{\bar{l}_j\}$ and solutions $\underline{r}_i$, $\bar{r}_j$ to the equation (1.1.1) with the boundary condition (1.1.2) such that

$$\underline{l}_1 < \underline{l}_2 < \cdots \to l, \qquad \lim_{t\to\infty} \underline{r}_i(t) = \underline{l}_i,$$

and

$$\bar{l}_1 > \bar{l}_2 > \cdots \to l, \qquad \lim_{t\to\infty} \bar{r}_j(t) = \bar{l}_j.$$

It follows from Corollary 1.1.3 that

$$\underline{r}_1(t) < \underline{r}_2(t) < \cdots < \bar{r}_2(t) < \bar{r}_1(t).$$

Fix $t_0 > 0$, and define two positive numbers $\underline{\alpha}$ and $\bar{\alpha}$ by

$$\underline{\alpha} = \lim_{i\to\infty} \underline{r}_i(t_0) \quad \text{and} \quad \bar{\alpha} = \lim_{j\to\infty} \bar{r}_j(t_0).$$

Choose α such that $\underline{\alpha} \le \alpha \le \bar{\alpha}$, and let r_α be a unique solution to the equation (1.1.1) with the boundary condition (1.1.2) satisfying $r_\alpha(t_0) = \alpha$. Making use of Corollary 1.1.3 again, we get

$$\underline{r}_i(t) < r_\alpha(t) < \bar{r}_j(t)$$

as long as r_α exists. Since $\underline{r}_i(t)$ and $\bar{r}_j(t)$ exist on $[0, \infty)$, so does r_α. Letting $t \to \infty$ first, and then $i \to \infty, j \to \infty$, we obtain

$$\lim_{t\to\infty} r_\alpha(t) = l. \qquad \square$$

Let us now turn to the blow-up time problem. Namely, we shall show that for any $T > 0$ there exists a solution to the equation (1.1.1) with the boundary condition (1.1.2) satisfying $r(t) \to +\infty$ as $t \to T$.

We suppose that $h(s) = \max\{h_1(s), h_2(s)\}$ satisfies

$$\int^\infty \frac{ds}{h(s)} < \infty. \tag{1.2.5}$$

Note that if the assumption (1.2.5) is false, then any solution is bounded (Corollary 1.2.4).

For $t_0 > 0$ and $r_0 > 0$, let $r = r(t)$ be a solution to the equation (1.1.1) with the boundary condition (1.1.2) satisfying $r(t_0) = r_0$. Such a solution exists uniquely. We denote $\dot{r}(t_0)$ by $\beta(r_0, t_0)$, which is uniquely determined by r_0 and t_0. We define

$$\phi(r_0, t_0) = \int_{r_0}^\infty [\gamma_1(t_0)^2\{h_1(s)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2\{h_2(s)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2]^{-1/2} ds,$$

where

$$\gamma_1(t_0) = \frac{\mu}{f_1(t_0)} \quad \text{and} \quad \gamma_2(t_0) = \frac{\nu}{f_2(t_0)}.$$

Lemma 1.2.7. *The function $\phi(r_0, t_0)$ is well-defined for $r_0 > 0$ and $t_0 > 0$, and*

$$\lim_{r_0 \to \infty} \phi(r_0, t_0) = 0.$$

Proof. It follows from (H-3) that $(h_i^2)'(s) \geq (h_i^2)'(s - r_0)$ for $s > r_0$. Integrating both sides with respect to s from r_0 to s, we get

$$h_i(s)^2 - h_i(r_0)^2 \geq h_i(s - r_0)^2.$$

Therefore, it holds that for $K > 0$

$$\begin{aligned}
&\int_{r_0+K}^{\infty} [\gamma_1(t_0)^2\{h_1(s)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2\{h_2(s)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2]^{-1/2} ds \\
&\qquad \leq C(t_0) \int_{r_0+K}^{\infty} [h_1(s - r_0)^2 + h_2(s - r_0)^2]^{-1/2} ds \\
&\qquad \leq C(t_0) \int_{K}^{\infty} \frac{ds}{h(s)}.
\end{aligned}$$

Note that it follows from the condition (1.2.5) that for any $\varepsilon > 0$

$$\int_K^\infty \frac{ds}{h(s)} < \varepsilon$$

for sufficiently large K.

Using the Taylor expansion formula, we have for $s > r_0$

$$h_i(s)^2 - h_i(r_0)^2 = 2h_i(\rho_i)h_i'(\rho_i)(s - r_0),$$

where $\rho_i \in (r_0, s)$. It follows from (H-3) that

$$h_i(s)^2 - h_i(r_0)^2 \geq 2h_i(r_0)h_i'(r_0)(s - r_0) > 0.$$

Therefore it holds that

$$\begin{aligned}
&\int_{r_0}^{r_0+K} [\gamma_1(t_0)^2\{h_1(s)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2\{h_2(s)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2]^{-1/2} ds \\
&\qquad \leq [2(\gamma_1(t_0)^2 h_1(r_0)h_1'(r_0) + \gamma_2(t_0)^2 h_2(r_0)h_2'(r_0))]^{-1/2} \int_{r_0}^{r_0+K} \frac{ds}{\sqrt{s - r_0}} \\
&\qquad \leq C(t_0)\sqrt{K} \min\left\{[h_1(r_0)h_1'(r_0)]^{-1/2}, [h_2(r_0)h_2'(r_0)]^{-1/2}\right\}.
\end{aligned}$$

We shall show that the right hand side converges to zero as $r_0 \to \infty$. Assume that $h_i(s)h_i'(s)$'s are bounded functions. Then, by virtue of (H-3), we get

$$h_i(s)^2 \leq Cs$$

for some positive constant C. As a result, we have

$$\int^{\infty} \frac{ds}{h(s)} \geq C \int^{\infty} \frac{ds}{\sqrt{s}} = \infty,$$

which contradicts (1.2.5). Hence $h_i(s)h_i'(s)$'s are unbounded. Taking (H-3) into consideration, we get

$$\lim_{r_0 \to \infty} \max \left\{ \sqrt{h_1(r_0)h_1'(r_0)}, \sqrt{h_2(r_0)h_2'(r_0)} \right\} = \infty,$$

and therefore

$$\lim_{r_0 \to \infty} \int_{r_0}^{r_0+K} [\gamma_1(t_0)^2\{h_1(s)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2\{h_2(s)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2]^{-1/2} ds = 0.$$

Summing up these estimates for integrations over $[r_0, r_0 + K]$ and $[r_0 + K, \infty)$, we obtain the assertion. $\square$

Proposition 1.2.8. *The set*

$$B = \left\{ T \in (0, \infty) \;\middle|\; \lim_{t \uparrow T} r(t) = \infty, \text{ where } r \text{ is a solution to } (1.1.1) \text{ with } (1.1.2) \right\}$$

is dense in $(0, \infty)$.

Proof. Let $T > 0$. We shall show that

$$(T - \varepsilon, T + \varepsilon) \cap B \neq \emptyset$$

for any $\varepsilon \in (0, T)$. Let r_α be a solution to the equation (1.1.1) with the boundary condition (1.1.2) satisfying $r_\alpha(T) = \alpha$, and $[0, T_\alpha)$ its life-span. By Lemma 1.2.9 below, we have

$$\phi(\alpha, T) \geq f_1(T)^p f_2(T)^q \int_T^{T_\alpha} \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}.$$

Letting $\alpha \to \infty$, we obtain from Lemma 1.2.7 that

$$\lim_{\alpha\to\infty} T_\alpha = T.$$

Hence $T_\alpha \in (T-\varepsilon, T+\varepsilon)$ for sufficiently large α. □

Lemma 1.2.9. *Let $[0,T)$ be the life span of a solution r. Then we obtain*

$$\phi(\alpha, t_0) \geq f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{T} \frac{dt}{f_1(t)^p f_2(t)^q}.$$

Proof. Since $\dot{f}_i(t) \geq 0$ and $h_i(s)^2 \leq h_i(t)^2$ for all $s \leq t$, we have

$$\int_{t_0}^{t} h_1(r(\tau))^2 \frac{d}{d\tau}\{f_1(\tau)^{2p-2} f_2(\tau)^{2q}\} d\tau \leq h_1(r(t))^2 \{f_1(t)^{2p-2} f_2(t)^{2q} - f_1(t_0)^{2p-2} f_2(t_0)^{2q}\},$$

and

$$\int_{t_0}^{t} h_2(r(\tau))^2 \frac{d}{d\tau}\{f_1(\tau)^{2p} f_2(\tau)^{2q-2}\} d\tau \leq h_2(r(t))^2 \{f_1(t)^{2p} f_2(t)^{2q-2} - f_1(t_0)^{2p} f_2(t_0)^{2q-2}\}.$$

Making use of the equation (1.2.2) and these inequalities, we get

$$\begin{aligned}
\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 \geq{}& \{f_1(t_0)^p f_2(t_0)^q \dot{r}(t_0)\}^2 \\
&+ \mu^2 f_1(t_0)^{2p-2} f_2(t_0)^{2q} \{h_1(r(t))^2 - h_1(r(t_0))^2\} \\
&+ \nu^2 f_1(t_0)^{2p} f_2(t_0)^{2q-2} \{h_2(r(t))^2 - h_2(r(t_0))^2\},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\dot{r}(t)[\gamma_1(t_0)^2\{h_1(r(t))^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2\{h_2(r(t))^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2]^{-1/2} \\
&\geq f_1(t_0)^p f_2(t_0)^q \frac{1}{f_1(t)^p f_2(t)^q}.
\end{aligned}$$

Integrating both sides from t_0 to T, we obtain

$$\begin{aligned}
&\int_{r_0}^{r(T)} [\gamma_1(t_0)^2\{h_1(s)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2\{h_2(s)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2]^{-1/2} ds \\
&\geq f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{T} \frac{dt}{f_1(t)^p f_2(t)^q}.
\end{aligned}$$

This implies that

$$\phi(r_0, t_0) \geq f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{T} \frac{dt}{f_1(t)^p f_2(t)^q}. \qquad \square$$

Note. For any $t \in (0, T)$, we can prove that

$$\phi(r(t), t) \geq f_1(t)^p f_2(t)^q \int_{t}^{T} \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}.$$

Theorem 1.2.10. *For any given* $T > 0$, *there exists a solution to the equation* (1.1.1) *with the boundary condition* (1.1.2) *satisfying* $\lim_{t \to T} r(t) = +\infty$.

Proof. By virtue of Proposition 1.2.8, there exist sequences $\{\underline{T}_i\}$, $\{\bar{T}_j\}$ and solutions $\underline{r}_i$, $\bar{r}_j$ to the equation (1.1.1) with the boundary condition (1.1.2) such that

$$0 < t_0 < \underline{T}_1 < \underline{T}_2 < \cdots \to T, \qquad \lim_{t \uparrow \underline{T}_i} \underline{r}_i(t) = \infty,$$

and

$$\bar{T}_1 > \bar{T}_2 > \cdots \to T, \qquad \lim_{t \uparrow \bar{T}_j} \bar{r}_j(t) = \infty.$$

It follows from Corollary 1.1.3 that

$$\overline{r}_1(t) < \overline{r}_2(t) < \cdots < \underline{r}_2(t) < \underline{r}_1(t)$$

as long as they exist. Fix $t_0 > 0$, and define two positive numbers $\underline{\alpha}$ and $\bar{\alpha}$ by

$$\underline{\alpha} = \lim_{i \to \infty} \underline{r}_i(t_0) \quad \text{and} \quad \bar{\alpha} = \lim_{j \to \infty} \bar{r}_j(t_0).$$

Choose α such that $\overline{\alpha} \leq \alpha \leq \underline{\alpha}$, and let r_α be the solution to the equation (1.1.1) with the boundary condition (1.1.2) and $r_\alpha(t_0) = \alpha$. Using Corollary 1.1.3 again, we get

$$\overline{r}_i(t) < r_\alpha(t) < \underline{r}_j(t)$$

as long as they exist. This estimate asserts that

$$\underline{T}_i < T_\alpha < \bar{T}_j,$$

where $[0, T_\alpha)$ is the life-span of r_α. Letting $i \to \infty$ and $j \to \infty$, we obtain $T_\alpha = T$. $\square$

Now we prove the following result, which shows that the conclusion of Theorem 1.2.10 is true even for $T = \infty$.

Theorem 1.2.11. *There exists a global, positive and strictly monotone increasing solution $r = r(t)$ satisfying*

$$\lim_{t\to\infty} r(t) = \infty.$$

Proof. Fix $t_0 > 0$ arbitrarily. Then, it follows from Theorem 1.2.8 and Theorem 1.2.10 that there exist sequences of positive numbers $\{T_i\}_{i=1}^{\infty}$, $\{l_j\}_{j=1}^{\infty}$ and sequences of solutions $\{\overline{r}_i\}_{i=1}^{\infty}$, $\{\underline{r}_j\}_{j=1}^{\infty}$ to the equation (1.1.1) with the boundary condition (1.1.2) so that

$$0 < t_0 < T_1 < T_2 < \cdots \to \infty, \qquad \lim_{t\uparrow T_i} \overline{r}_i(t) = \infty,$$

and

$$l_1 < l_2 < \cdots \to \infty, \qquad \lim_{t\to\infty} \underline{r}_j(t) = l_j.$$

Define

$$\overline{\alpha} = \lim_{i\to\infty} \overline{r}_i(t_0), \qquad \underline{\alpha} = \lim_{j\to\infty} \underline{r}_j(t_0).$$

Then $0 < \underline{\alpha} \le \overline{\alpha} < \infty$. Let $r_\alpha = r_\alpha(t)$ be a solution to the equation (1.1.1) with the boundary condition (1.1.2) and $r_\alpha(t_0) = \alpha$, where $\underline{\alpha} \le \alpha \le \overline{\alpha}$. Then $r_\alpha(t)$ satisfies

$$\underline{r}_j(t) \le r_\alpha(t) \le \overline{r}_i(t)$$

as long as it exists. Letting $i \to \infty$ first, we have $T_\alpha = \infty$, and then $j \to \infty$ we get

$$\lim_{t\to\infty} r_\alpha(t) = \infty. \qquad \square$$

1.3 Addendum

In the case that $p = 1$ and $\nu = 0$, the equation (1.1.1) becomes

$$\ddot{r}(t) + \frac{\dot{f}(t)}{f(t)}\dot{r}(t) - \mu^2\, \frac{h(r(t))h'(r(t))}{f(t)^2} = 0, \tag{1.3.1}$$

which can be solved explicitly.

Indeed, multiplying both sides of the equation (1.3.1) by $f(t)^2\dot{r}(t)$ and integrating from t_0 to t, we obtain

$$\{f(t)\dot{r}(t)\}^2 - \{f(t_0)\dot{r}(t_0)\}^2 = \mu^2\{h(r(t))^2 - h(r(t_0))^2\}. \tag{1.3.2}$$

Under the condition

$$r(0) = 0 \quad \text{and} \quad \dot{r}(0) = 0,$$

the identity (1.3.2) becomes

$$\{f(t)\dot{r}(t)\}^2 = \mu^2 h(r(t))^2.$$

Thus

$$f(t)\dot{r}(t) = \mu h(r(t)).$$

As a consequence, we can solve the ordinary differential equation (1.3.1) by making use of the method of separation of variables.

CHAPTER 2

Equivariant harmonic maps

In this chapter, we shall construct equivariant harmonic maps between noncompact, complete Riemannian manifolds by making use of the results established in the previous chapter.

We begin with fixing our notation and a brief review of relevant theorems for eigenmaps.

2.1 Eigenmaps

Let (S^m, g_{S^m}) be the m-dimensional standard unit sphere. A map $\varphi : (S^m, g_{S^m}) \to (S^n, g_{S^n})$ is called an *eigenmap* if it is a harmonic map with constant energy density. It is a well-known result that from Takahashi's theorem, we have the following

Lemma 2.1.1. *Let $\varphi : (S^m, g_{S^m}) \to (S^n, g_{S^n})$ be an eigenmap. Then all components of the map $\Phi = i \circ \varphi : S^m \to \mathbf{R}^{n+1}$ are harmonic homogeneous polynomials on $\mathbf{R}^{m+1}$ of the same polynomial degree, where $i : S^n \to \mathbf{R}^{n+1}$ is the inclusion map.*

Thus, if we can find a family of harmonic homogeneous polynomials of the same degree, say $\{\Phi_i\}_{i=1}^{n+1}$, satisfying

$$\sum_{i=1}^{n+1} \Phi_i^2(x) = 1 \qquad \text{for} \quad x \in S^m,$$

then we obtain an eigenmap $\varphi : S^m \to S^n$. An eigenmap $\varphi : S^m \to S^n$ is said to be *of degree k* if all components of φ are harmonic homogeneous polynomial of degree k.

There is no general theory of constructing an eigenmap $\varphi : S^m \to S^n$ for a given degree. It should be remarked that almost all known results have been for the case of degree two. We shall give an algorithm of constructing a new eigenmap of degree two from the old ones of the same degree.

A bilinear map $F : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^r$ is called an *orthogonal multiplication* if for any $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$ it holds that

$$\|F(x, y)\| = \|x\|\|y\|.$$

An orthogonal multiplication $F : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^r$ is said to be *full* if its image is not contained in any hyperplane of $\mathbf{R}^r$.

In the case of $m = n$, associated with an orthogonal multiplication $F : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^r$, we obtain an eigenmap $\varphi : S^{2m-1} \to S^r$ defined by

$$\varphi(x, y) = (\|x\|^2 - \|y\|^2, 2F(x, y)),$$

which is called the Hopf construction. Moreover if F is full, then so is φ, that is, the image of φ is not contained in any totally geodesic submanifold of dimension $r - 1$ in S^r.

Typical examples of a full orthogonal multiplication are given by the complex multiplication $F : \mathbf{C} \times \mathbf{C} \ni (x, y) \mapsto xy \in \mathbf{C}$ and the quaternion multiplication $F : \mathbf{H} \times \mathbf{H} \ni (p, q) \mapsto pq \in \mathbf{H}$. The former induces the Hopf fibration $\varphi : S^3 \to S^2$, and the latter induces an eigenmap $\varphi : S^7 \to S^4$.

We shall now deduce a necessary condition for the existence of orthogonal multiplications. Assume $m \leq n$. Since each orthogonal multiplication $F : \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^r$ is a bilinear map, we can express it as

$$F(x, y) = \sum a_{ij} x_i y_j,$$

where $x = (x_1, \ldots, x_m) \in \mathbf{R}^m,\ y = (y_1, \ldots, y_n) \in \mathbf{R}^n$ and $a_{ij} \in \mathbf{R}^r$. Since $\|F(x, y)\| = \|x\|\|y\|$, we have

$$\begin{cases} \langle a_{ik}, a_{il} \rangle & = \delta_{kl}, \\ \langle a_{ik}, a_{jk} \rangle & = \delta_{ij}, \\ \langle a_{ik}, a_{jl} \rangle & + \langle a_{il}, a_{jk} \rangle = 0 \qquad (i \neq j, \quad k \neq l), \end{cases} \tag{2.1.1}$$

which imply that for each k_0 $(1 \leq k_0 \leq n)$ and i_0 $(1 \leq i_0 \leq m)$, both $\{a_{ik_0}\}_{i=1}^m$ and $\{a_{i_0 k}\}_{k=1}^n$ are orthonormal systems in $\mathbf{R}^r$, and hence $\max\{m, n\} \leq r$. Moreover, if F is full, then $r \leq mn$.

Following a method due to M. Parker [30], who applied it to the case $m = n$, we now define an $mn \times mn$-matrix $G(F)$ by

$$G(F) := \begin{bmatrix} I_n & A_{12} & \cdots & A_{1m} \\ A_{21} & I_n & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & I_n \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix and A_{ij} are $n \times n$-matrix whose entries are

$$(A_{ij})_{kl} = \langle a_{ik}, a_{jl} \rangle, \quad 1 \leq k, l \leq n.$$

By virtue of (2.1.1), each A_{ij} is a skew-symmetric matrix and $A_{ji} = -A_{ij}$. Note that the determinant of $G(F)$ coincides with Gram's determinant with respect to the system of vectors $\{a_{ij}\}$. Hence rank $G(F) = r$.

We consider only the case of $m = 2$, and prove the following existence result of orthogonal multiplications.

Proposition 2.1.2. *There exists a full orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^n \to \mathbf{R}^r$ if and only if r is even, where $n \leq r \leq 2n$.*

Proof. We first prove that rank $G(F)(= r)$ must be even whenever a full orthogonal multiplication exists. Recall that the characteristic polynomial of $G(F)$ is

$$\det(G(F) - \mu I_{2n}) = \det \begin{bmatrix} (1-\mu)I_n & -A \\ A & (1-\mu)I_n \end{bmatrix},$$

where $A = A_{21}$. Since

$$\det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \left|\det[A + \sqrt{-1}B]\right|^2,$$

A and B being real matrices and $|\cdot|$ denoting the absolute value, we have

$$\det(G(F) - \mu I_{2n}) = \left|\det[(1-\mu)I_n + \sqrt{-1}A]\right|^2.$$

Consequently, the multiplicity of the zero eigenvalue of the matrix $G(F)$ is even. Hence r must be even.

Define orthogonal multiplications $F_1 : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ and $F_2 : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$ by

$$\begin{cases} F_1(x, y) = (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1), \\ F_2(x, y) = (x_1y_1, x_1y_2, x_2y_1, x_2y_2), \end{cases}$$

respectively, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, they are full and satisfy

$$\|F_1(x, y)\| = \|F_2(x, y)\| = \|x\|\|y\|.$$

In the case of $n = 2k$, we decompose $\mathbf{R}^n$ as the direct sum of k-copies of $\mathbf{R}^2$, that is, $\mathbf{R}^n = \mathbf{R}^2 \oplus \cdots \oplus \mathbf{R}^2$. Hence $\mathbf{R}^2 \times \mathbf{R}^n = (\mathbf{R}^2 \times \mathbf{R}^2) \oplus \cdots \oplus (\mathbf{R}^2 \times \mathbf{R}^2)$ (direct sum of k-copies of $\mathbf{R}^2 \times \mathbf{R}^2$). Then, for i $(0 \le i \le k)$,

$$F = \overbrace{F_1 \oplus \cdots \oplus F_1}^{i\text{-copies}} \oplus \overbrace{F_2 \oplus \cdots \oplus F_2}^{(k-i)\text{-copies}}$$

defines an orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^n \to \mathbf{R}^{2(n-i)}$. Thus for even r, where $n \le r \le 2n$, there exists an orthogonal multiplication $\mathbf{R}^2 \times \mathbf{R}^{2k} \to \mathbf{R}^r$.

In the case of $n = 2k + 1$, we have the direct sum $\mathbf{R}^n = \mathbf{R}^{2k} \oplus \mathbf{R}$. Associated with the orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^{2k} \to \mathbf{R}^r$, we obtain an orthogonal multiplication $\tilde{F} : \mathbf{R}^2 \times \mathbf{R}^{2k+1} \to \mathbf{R}^{r+2}$ defined by

$$\tilde{F}((x_1, x_2), (y_1, \dots, y_{2k}, y_{2k+1})) = (F((x_1, x_2), (y_1, \dots, y_{2k})), (x_1y_{2k+1}, x_2y_{2k+1})),$$

where $(x_1, x_2) \in \mathbf{R}^2$ and $(y_1, \dots, y_{2k}, y_{2k+1}) \in \mathbf{R}^{2k+1}$. Hence for even r, where $n+1 \le r \le 2n$, there exists an orthogonal multiplication $\mathbf{R}^2 \times \mathbf{R}^{2k+1} \to \mathbf{R}^r$. $\square$

Now we introduce a method of constructing a new eigenmap of degree two from old known ones.

Let $f : S^m \to S^p$ and $g : S^n \to S^q$ be eigenmaps of degree two, and $F : \mathbf{R}^{m+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^r$ an orthogonal multiplication. Define a map $\varphi : \mathbf{R}^{m+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{p+q+r+2}$ by

$$\varphi(x, y) := (f(x), g(y), \sqrt{2}F(x, y)). \tag{2.1.2}$$

Then the restriction $\varphi_{|S^{m+n+1}}$ of φ to S^{m+n+1} gives rise to an eigenmap of degree two from S^{m+n+1} into $S^{p+q+r+1}$. Moreover if f, g and F are full, then so is φ.

Most significant case is that of $m = p = 1$. In this case, $f : S^1 \to S^1$ is given by the Hopf map, that is, $f(x_1, x_2) = (|x_1|^2 - |x_2|^2, 2x_1x_2)$, and Proposition 2.1.2 ensures the existence of a full orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^{n+1} \to \mathbf{R}^r$ for even r. Thus we obtain the following

Proposition 2.1.3. *Let $g : S^n \to S^q$ be a full eigenmap of degree two. Then we have a new full eigenmap $\varphi : S^{n+2} \to S^{q+r+2}$ of degree two, where r is even and $n + 1 \leq r \leq 2n + 2$.*

By making use of this proposition, we can prove the following

Theorem 2.1.4. (1) (i) *A full eigenmap $\varphi : S^5 \to S^n$ of degree two exists for $n = 4$ or $7 \leq n \leq 19$.*

(ii) *A full eigenmap $\varphi : S^6 \to S^n$ of degree two exists for $11 \leq n \leq 26$.*

(2) *Let $k \geq 3$.*

(i) *A full eigenmap $\varphi : S^{2k+1} \to S^n$ of degree two exists for $k^2 + 3k - 10 \leq n \leq 2k^2 + 5k + 1$.*

(ii) *A full eigenmap $\varphi : S^{2k+2} \to S^n$ of degree two exists for $k^2 + 5k - 7 \leq n \leq 2k^2 + 7k + 4$.*

Note. The space of harmonic polynomials of degree two on $\mathbf{R}^{m+1}$ has the dimension $m(m + 3)/2$. Thus, in Theorem 2.1.4, the upper bound of the dimension of target manifold is best possible to assure the existence of a full eigenmap of degree two.

Note. Gauchman and Toth ([18]) proved that a full eigenmap of degree two $\varphi : S^4 \to S^n$ exists for $n = 4, 7$ or $9 \leq n \leq 13$.

In order to prove Theorem 2.1.4, we use the following result.

Lemma 2.1.5 (Gauchman and Toth [18]). *From a full eigenmap $g : S^m \to S^n$ of degree two, one can construct a full eigenmap $\tilde{g} : S^{m+1} \to S^{m+n+2}$ of the same degree.*

In fact, they proved an explicit formula for constructing $\tilde{g}$ from g, but we do not need it here.

Proof of Theorem 2.1.4. (1) (i) In [18], it is proved that there exist full eigenmaps $\varphi : S^5 \to S^n$ of degree two for $n = 4, 7, 8, 9, 13, 15, 16, 17, 18, 19$. On the other hand, it is well-known that a full eigenmap $g : S^3 \to S^m$ exists if and only if $m = 2$ or $4 \leq m \leq 8$. Since we have an orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^4 \to \mathbf{R}^4$, it follows from Proposition 2.1.3 that there exists a full eigenmap $\varphi : S^5 \to S^n$ of degree two for $10 \leq n \leq 14$.

(ii) From Lemma 2.1.5 and (i) we have a full eigenmap $\varphi : S^6 \to S^n$ of degree two for $n = 11$ or $14 \leq n \leq 26$. Applying Proposition 2.1.3 to a full eigenmap $g : S^4 \to S^4$ of degree two together with a full orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^5 \to \mathbf{R}^6$, we can construct a new eigenmap $\varphi : S^6 \to S^{12}$ of degree two. Moreover, by making use of the result due to Wood ([44]), we can find a full orthogonal multiplication $F : \mathbf{R}^3 \times \mathbf{R}^4 \to \mathbf{R}^4$. Hence there exists a full eigenmap $\varphi : S^6 \to S^{13}$ of degree two defined by $\varphi(x, y) = (f(x), g(x), \sqrt{2}F(x, y))$, where $f : S^3 \to S^4$ is a full eigenmap of degree two and $g : S^2 \to S^4$ is the Veronese map.

(2) We shall proved by an induction argument.

Step 1. In the case of $k = 3$.

(i) Since there exists a full eigenmap $\varphi : S^6 \to S^n$ of degree two for $11 \leq n \leq 26$, it follows from Lemma 2.1.5 that there exists a full eigenmap $\varphi : S^7 \to S^n$ of degree two for $19 \leq n \leq 34$. On the other hand, using Proposition 2.1.3, we have a full eigenmap $\varphi : S^7 \to S^{q+r+2}$ of degree two constructed from a full eigenmap $g : S^5 \to S^q$ of degree two together with a full orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^6 \to \mathbf{R}^r$. Then Proposition 2.1.2 and (1) imply that a full eigenmap $\varphi : S^7 \to S^n$ of degree two exists for $n = 12$ or $14 \leq n \leq 33$.

Let $F : \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$ be the orthogonal multiplication given by the quaternion multiplication, and let $f : S^3 \to S^p$ and $g : S^3 \to S^q$ be full eigenmaps of degree two. Then Proposition 2.1.3 ensures the existence of a full eigenmap $\varphi : S^7 \to S^{p+q+5}$. When $p = q = 2$, we have $\varphi : S^7 \to S^9$, and in the case of $p = 2, q = 4$, we have $\varphi : S^7 \to S^{11}$. Finally, there exists a full orthogonal multiplication $F : \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^r$ for $r = 8, 11$. Therefore the Hopf construction

$$\varphi(x, y) = (\|x\|^2 - \|y\|^2, 2F(x, y)) \qquad (x, y \in \mathbf{R}^4)$$

gives rise to a full eigenmap $\varphi : S^7 \to S^n$ of degree two for $n = 8, 11$. As a consequence, we have a full eigenmap $\varphi : S^7 \to S^n$ of degree two for $8 \leq n \leq 34$. Thus (i) is true for $k = 3$.

(ii) Using Lemma 2.1.5 and the above result, we have a full eigenmap $\varphi : S^8 \to S^n$ of degree two for $17 \leq n \leq 43$. Thus (ii) is also true for $k = 3$.

Step 2. (i) We assume that the statements (i) and (ii) are true up to $k = l(\geq 3)$. Since there exists an eigenmap $g : S^{2l+2} \to S^q$ of degree two for $l^2 + 5l - 7 \leq q \leq 2l^2 + 7l + 4$, it follows from Lemma 2.1.5 that we have a new eigenmap

$$\varphi : S^{2l+3} \to S^n \quad \text{for} \quad l^2 + 7l - 3 \leq n \leq 2l^2 + 9l + 8.$$

On the other hand, from our assumption, there exist a full orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^{2l+2} \to \mathbf{R}^r$ for $2l + 2 \leq r \leq 4l + 4$ and a full eigenmap $g : S^{2l+1} \to S^q$ for $l^2 + 3l - 10 \leq q \leq 2l^2 + 5l + 1$. Thus, from Proposition 2.1.3, we have a new eigenmap

$$\varphi : S^{2l+3} \to S^n \quad \text{for} \quad l^2 + 5l - 6 \leq n \leq 2l^2 + 6l + 10.$$

Since

$$l^2 + 5l - 6 < l^2 + 7l - 3 < 2l^2 + 6l + 10 < 2l^2 + 9l + 8,$$

it follows that a full eigenmap $\varphi : S^{2l+3} \to S^n$ exists for $l^2 + 5l - 6 \leq n \leq 2l^2 + 9l + 8$, that is, $(l + 1)^2 + 3(l + 1) - 10 \leq n \leq 2(l + 1)^2 + 5(l + 1) + 1$.

Hence the statement (i) is true for $k = l + 1$.

(ii) Since there exists an eigenmap $g : S^{2l+2} \to S^q$ of degree two for $l^2 + 5l - 7 \leq q \leq 2l^2 + 7l + 4$, by making use of Proposition 2.1.2 and 2.1.3, we have a new eigenmap

$$\varphi : S^{2l+4} \to S^n \quad \text{for} \quad l^2 + 7l - 1 \leq n \leq 2l^2 + 10l + 15.$$

On the other hand, it follows from Lemma 2.1.5 applied to the full eigenmap $g : S^{2l+3} \to S^q$ that there exists a new eigenmap

$$\varphi : S^{2l+4} \to S^n \quad \text{for} \quad l^2 + 7l - 1 \leq q \leq 2l^2 + 11l + 13.$$

Noting that

$$2l^2 + 10l + 15 < 2l^2 + 11l + 13,$$

we conclude that a full eigenmap $\varphi : S^{2l+4} \to S^n$ exists for $l^2 + 7l - 1 \le n \le 2l^2 + 11l + 13$, that is, $(l+1)^2 + 5(l+1) - 7 \le n \le 2(l+1)^2 + 7(l+1) + 4$.

Therefore the statement (ii) is also true for $k = l + 1$. □

Note. Let $\varphi_1 : S^m \to S^n$ and $\varphi_2 : S^n \to S^p$ be eigenmaps of degree two. Then the composite $\varphi_2 \circ \varphi_1$ of φ_1 and φ_2 yields an eigenmap of degree four from S^m to S^p. Thus, by an induction argument, we can obtain an eigenmap of degree 2^k.

2.2 Reduction of harmonic map equations

We begin with the general situation. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, and $I(\subset \mathbf{R})$ an interval. For positive smooth functions f_1 and f_2 defined on I, we consider the doubly warped product Riemannian manifold defined by

$$(M, g) = (I \times M_1 \times M_2, dt^2 + f_1(t)^2 g_1 + f_2(t)^2 g_2),$$

where $t \in I$.

Lemma 2.2.1. *Let $\Gamma_j{}^i{}_k$ be the Christoffel symbols and Δ_g the Laplace-Beltrami operator of (M, g). Then we have*

$$\begin{cases} \Gamma_j{}^1{}_k = -\dot{f}_1(t) f_1(t) (g_1)_{j-1,k-1} & (2 \le j, k \le m_1 + 1), \\ \Gamma_j{}^1{}_k = -\dot{f}_2(t) f_2(t) (g_2)_{j-m_1-1,k-m_1-1} & (m_1 + 2 \le j, k \le m_1 + m_2 + 1), \\ \Gamma_j{}^1{}_k = 0 & (\text{otherwise}), \end{cases}$$

$$\begin{cases} \Gamma_j{}^i{}_1 = \Gamma_1{}^i{}_j = \dfrac{\dot{f}_1(t)}{f_1(t)} \delta_{ij} & (2 \le j \le m_1 + 1), \\ \Gamma_j{}^i{}_k = {}^1\Gamma_j{}^i{}_k & (2 \le j, k \le m_1 + 1), \\ \Gamma_j{}^i{}_k = 0 & (\text{otherwise}), \end{cases}$$

$$\begin{cases} \Gamma_{j\ 1}^{\ i} = \Gamma_{1\ j}^{\ i} = \dfrac{\dot f_2(t)}{f_2(t)}\delta_{ij} & (m_1+2 \le j \le m_1+m_2+1), \\ \Gamma_{j\ k}^{\ i} = {}^2\Gamma_{j\ k}^{\ i} & (m_1+2 \le j,k \le m_1+m_2+1), \ \textit{and} \\ \Gamma_{j\ k}^{\ i} = 0 & (\text{otherwise}), \end{cases}$$

$$\Delta_g = \frac{\partial^2}{\partial t^2} + \left(m_1 \frac{\dot f_1(t)}{f_1(t)} + m_2 \frac{\dot f_2(t)}{f_2(t)} \right) \frac{\partial}{\partial t} + \frac{1}{f_1(t)^2}\Delta_1 + \frac{1}{f_2(t)^2}\Delta_2.$$

Here $x_1 = t \in I, (x_2, \ldots, x_{m_1+1}) \in M_1, (x_{m_1+2}, \ldots, x_{m_1+m_2+1}) \in M_2,\ \ m_1 = \mathrm{dim} M_1,$ *and* $m_2 = \mathrm{dim} M_2$. *Also, we denote by* ${}^p\Gamma_{j\ k}^{\ i}$ (resp. Δ_p) *the Christoffel symbols* (resp. *the Laplace-Beltrami operator*) *of* (M_p, g_p).

Let $(\tilde M_1, \tilde g_1)$ and $(\tilde M_2, \tilde g_2)$ be Riemannian manifolds, $\tilde I (\subset \mathbf{R})$ an interval, and h_1 and h_2 nonnegative smooth functions on $\tilde I$. We consider a product map u defined by

$$\begin{aligned} u : (I \times M_1 \times M_2,\ dt^2 + f_1(t)^2 g_1 + f_2(t)^2 g_2) \ni (t, x, y) \\ \mapsto (r(t), \varphi(x), \psi(y)) \in (\tilde I \times \tilde M_1 \times \tilde M_2,\ dr^2 + h_1(r)^2 \tilde g_1 + h_2(r)^2 \tilde g_2), \end{aligned} \tag{2.2.1}$$

where $r : I \to \tilde I,\ \varphi : M_1 \to \tilde M_1$ and $\psi : M_2 \to \tilde M_2$ are smooth maps. The tension field $\tau(u)$ of the map u is computed as follows.

In local coordinates, $x = (x_i) \in M, u(x) = (u^\alpha(x)) \in \tilde I \times \tilde M_1 \times \tilde M_2$, the components of the tension field $\tau(u)$ is given by

$$\tau(u)^\alpha = \Delta_g u^\alpha + \sum_{i,j=1}^{m_1+m_2+1} \sum_{\beta,\gamma=1}^{n_1+n_2+1} g^{ij} \tilde\Gamma_{\beta\ \gamma}^{\ \alpha}(u(x)) \frac{\partial u^\beta}{\partial x_i} \frac{\partial u^\gamma}{\partial x_j}.$$

Then, from Lemma 2.2.1, we have

Lemma 2.2.2.

$$\begin{cases} \tau(u)^1 &= \ddot{r}(t) + \left(m_1 \dfrac{\dot{f}_1(t)}{f_1(t)} + m_2 \dfrac{\dot{f}_2(t)}{f_2(t)} \right) \dot{r}(t) \\ & \qquad -2e(\varphi) \dfrac{h_1'(r(t))h_1(r(t))}{f_1(t)^2} - 2e(\psi) \dfrac{h_2'(r(t))h_2(r(t))}{f_2(t)^2}, \\ \tau(u)^\alpha &= \dfrac{1}{f_1(t)^2} \tau(\varphi)^{\alpha-1} \qquad\qquad (2 \leq \alpha \leq n_1+1), \\ \tau(u)^\alpha &= \dfrac{1}{f_2(t)^2} \tau(\psi)^{\alpha-n_1-1} \qquad (n_1+2 \leq \alpha \leq n_1+n_2+1), \end{cases}$$

where $e(\varphi)$ and $\tau(\varphi)$ denote the energy density function and the tension field of $\varphi : (M_1, g_1) \to (\tilde{M}_1, \tilde{g}_1)$, and $e(\psi)$ and $\tau(\psi)$ denote the energy density function and the tension field of $\psi : (M_2, g_2) \to (\tilde{M}_2, \tilde{g}_2)$, respectively.

Proposition 2.2.3. *Let $u : (I \times M_1 \times M_2,\ dt^2 + f_1(t)^2 g_1 + f_2(t)^2 g_2) \to (\tilde{I} \times \tilde{M}_1 \times \tilde{M}_2,\ dr^2 + h_1(r)^2 \tilde{g}_1 + h_2(r)^2 \tilde{g}_2)$ be a map as in* (2.2.1)*. Then u is a harmonic map if and only if the following two conditions hold*:

(1) *$r = r(t)$ is a solution to the following ordinary differential equation*

$$\begin{aligned} \ddot{r}(t) + \left(m_1 \frac{\dot{f}_1(t)}{f_1(t)} + m_2 \frac{\dot{f}_2(t)}{f_2(t)} \right) \dot{r}(t) & \\ -2e(\varphi) \frac{h_1'(r(t))h_1(r(t))}{f_1(t)^2} - 2e(\psi) \frac{h_2'(r(t))h_2(r(t))}{f_2(t)^2} &= 0. \end{aligned} \tag{2.2.2}$$

(2) *$\varphi : (M_1, g_1) \to (\tilde{M}_1, \tilde{g}_1)$ and $\psi : (M_2, g_2) \to (\tilde{M}_2, \tilde{g}_2)$ are harmonic maps with constant energy density, that is, φ and ψ are eigenmaps.*

Now, making use of Proposition 2.2.3, we shall construct equivariant harmonic maps between noncompact space forms.

2.3 Constructions of equivariant harmonic maps

2.3.1 Join parameterization

We introduce the join parameterization of real hyperbolic spaces, which is an analogue of that for standard unit spheres due to Smith([34]).

We consider the hyperboloid model of the real hyperbolic space $\mathbf{RH}^{m+p+1}$, that is,

$$\mathbf{RH}^{m+p+1} = \{(x_0, x_1, \cdots, x_{m+p+1}) \in \mathbf{R}^{m+p+2} \mid -x_0^2 + x_1^2 + \cdots + x_{m+p+1}^2 = -1,\ x_0 > 0\}.$$

For any $z \in \mathbf{RH}^{m+p+1}$ there exist $x \in \mathbf{RH}^p, y \in S^m$ and $t \in [0, \infty)$ such that

$$z = ((\cosh t)x, (\sinh t)y).$$

Note that x and y are uniquely determined for $t > 0$. Define a map $f : [0, \infty) \times \mathbf{RH}^p \times S^m \to \mathbf{RH}^{m+p+1}$ by

$$f(t, x, y) = ((\cosh t)x, (\sinh t)y).$$

Then the pull-back metric of the standard metric $\tilde{g}$ on $\mathbf{RH}^{m+p+1}$ via f is given by

$$f^*\tilde{g} = dt^2 + (\cosh^2 t)g_1 + (\sinh^2 t)g_2,$$

where g_1 (resp. g_2) is the standard metric on $\mathbf{RH}^p$ (resp. S^m).

Let $\psi : \mathbf{RH}^p \to \mathbf{RH}^q, \varphi : S^m \to S^n$ be eigenmaps, and $2e(\psi) = \mu^2,\ 2e(\varphi) = \nu^2$. Then the map

$$u : \mathbf{RH}^{m+p+1} \ni (t, x, y) \to (r(t), \psi(x), \varphi(y)) \in \mathbf{RH}^{n+q+1}$$

is a harmonic map if and only if $r = r(t)$ is a solution to the following ordinary differential equation:

$$\begin{aligned} \ddot{r}(t) + \left\{ p\, \frac{\sinh t}{\cosh t} + m\, \frac{\cosh t}{\sinh t} \right\} \dot{r}(t) \\ - \left\{ \frac{\mu^2}{\cosh^2 t} + \frac{\nu^2}{\sinh^2 t} \right\} \sinh r(t) \cosh r(t) = 0. \end{aligned} \tag{2.2.3}$$

In order for u to be continuous, we require that $r = r(t)$ satisfies

$$\lim_{t \to 0} r(t) = 0.$$

The equation (2.2.3) is a special case of the equation (1.1.1), where $q = m$, $f_1(t) = \cosh t$, $f_2(t) = \sinh t$, and $h_1(r) = h_2(r) = \sinh r$. Since these functions satisfy the conditions (F-1) through (F-4) and (H-1) through (H-3), respectively, a solution $r = r(t)$ to the equation (2.2.3) exists globally. Moreover, since

$$\int^{\infty} \frac{dr}{\sinh r} < \infty,$$

we have a global, strictly monotone increasing, and unbounded solution $r = r(t)$ to the equation (2.2.3) (see Theorem A in Chapter 1).

In particular, let $p = q$ and $\psi : \mathbf{RH}^p \to \mathbf{RH}^p$ the identity map. Then, from Theorem 2.1.4, we have the following

Theorem 2.3.1. *There exists an equivariant harmonic map $u : \mathbf{RH}^{m+p+1} \to \mathbf{RH}^{n(m)+p+1}$. Here $n(m)$ is the integer for which an eigenmap $\varphi : S^m \to S^{n(m)}$ exists.*

On the other hand, when $m = 1$, there exists a family of eigenmaps $S^1 \ni z \mapsto z^k \in S^1$, where $k \in \mathbf{Z}$ and $z \in \mathbf{C}$. Thus we obtain

Theorem 2.3.2. *Let $m \geq 3$. Then there exists a family of harmonic maps from $\mathbf{RH}^m$ onto itself, which is parameterized by $\mathbf{Z}$.*

Proof. Since the equation (2.2.3) has a global, strictly monotone increasing, and unbounded solution $r = r(t)$, if we define $u : \mathbf{RH}^{p+2} \to \mathbf{RH}^{p+2}$ by

$$u : \mathbf{RH}^{p+2} \ni (t, x, z) \mapsto (r(t), x, z^k) \in \mathbf{RH}^{p+2} \qquad (x \in \mathbf{RH}^p, z \in S^1),$$

then u is a surjective harmonic map. $\square$

2.3.2 Warped product manifolds

Let $M = ([0, \infty) \times S^m, dt^2 + f(t)^2 g_{S^m})$ and $N = ([0, \infty) \times S^n, dr^2 + h(r)^2 g_{S^n})$, where $t, r \in [0, \infty)$. Let $f = f(t)$ and $h = h(r)$ be smooth functions on $[0, \infty)$ satisfying $f(t) > 0$ and $h(r) > 0$ for $t > 0$ and $r > 0$. If f and h satisfy

$$f(0) = 0, \quad \dot{f}(0) = 1, \quad h(0) = 0 \quad \text{and} \quad h'(0) = 1,$$

then M and N are smooth Riemannian manifolds diffeomorphic to Euclidean space.

Let $\varphi : S^m \to S^n$ be an eigenmap with $2e(\varphi) = \mu^2$. Then a product map $u(t,x) = (r(t), \varphi(x))$ is a harmonic map if and only if $r = r(t)$ is a solution to the following ordinary differential equation:

$$\ddot{r}(t) + m\frac{\dot{f}(t)}{f(t)}\dot{r}(t) - \mu^2\frac{h(r(t))h'(r(t))}{f(t)^2} = 0. \tag{2.2.4}$$

In order for u to be continuous, we require that $r = r(t)$ satisfies

$$\lim_{t\to 0} r(t) = 0.$$

Then Theorem A and Theorem B in Chapter 1 imply the following result.

Theorem 2.3.3. *Let $f = f(t)$ and $h = h(r)$ be smooth functions on $[0,\infty)$ and $\mathbf{R}$, respectively. Assume that f and h satisfy the following conditions*:

$$\begin{cases} f(0) = 0,\ \dot{f}(0) = 1,\ f(t) > 0 \text{ and } \dot{f}(t) \geq 0 \quad \text{for } \ t > 0, \\[2ex] 0 \leq m\dfrac{\dot{f}(t)}{f(t)} - 1 \ \text{ for } t \geq 0, \qquad \displaystyle\int^\infty \frac{dt}{f(t)} < \infty, \\[2ex] h(0) = 0,\ h'(0) = 1,\ h(r) > 0 \ \text{ for } r > 0 \ \text{ and } \ (h^2)''(r) \geq 0 \quad \text{for } r \in \mathbf{R}. \end{cases}$$

Then, if there exists an eigenmap $\varphi : S^m \to S^n$, then we can construct an equivariant harmonic map

$$u : ([0,\infty) \times S^m, dt^2 + f(t)^2 g_{S^m}) \ni (t,x) \mapsto (r(t), \varphi(x)) \in ([0,\infty) \times S^n, dr^2 + h(r)^2 g_{S^n}).$$

As applications of this theorem, we now illustrate few examples of equivariant harmonic maps.

Case 1. Let $f(t) = \sinh t$ and $h(r) = \sinh r$. Then M and N are isometric to the real hyperbolic spaces of dimension $m+1$ and $n+1$, respectively. Since these f and h satisfy the conditions in Theorem 2.3.3 and

$$\int^\infty \frac{dr}{h(r)} < \infty,$$

there exists a global, strictly monotone increasing, and unbounded solution $r = r(t)$ to the equation (2.2.4). Therefore, from Theorem 2.1.4, we obtain the following

Theorem 2.3.4. (1) (i) *A full equivariant harmonic map* $u : \mathbf{RH}^6 \to \mathbf{RH}^n$ *exists for* $n = 5$ *or* $8 \leq n \leq 20$.

(ii) *A full equivariant harmonic map* $u : \mathbf{RH}^7 \to \mathbf{RH}^n$ *exists for* $12 \leq n \leq 27$.

(2) *Let* $k \geq 4$.

(i) *A full equivariant harmonic map* $u : \mathbf{RH}^{2k} \to \mathbf{RH}^n$ *exists for* $k^2+k-12 \leq n \leq 2k^2+k-2$.

(ii) *A full equivariant harmonic map* $u : \mathbf{RH}^{2k+1} \to \mathbf{RH}^n$ *exists for* $k^2 + 3k - 11 \leq n \leq 2k^2 + 3k - 1$.

Note. From the eigenmap $S^1 \ni z \mapsto z^k \in S^1$ $(k \in \mathbf{Z})$, we obtain an equivariant harmonic map $u : \mathbf{RH}^2 \to \mathbf{RH}^2$. However, u is holomorphic or anti-holomorphic. Indeed, using the Poincaré disc model $\mathbf{D}^2$, it is given by $\mathbf{D}^2 \ni z \mapsto z^k \in \mathbf{D}^2$.

Case 2. Let $f(t) = \sinh t$ and $h(r) = r$. Then M and N are isometric to the real hyperbolic space and the real Euclidean space, respectively. Since these f and h satisfy the conditions in Theorem 2.3.3, there exists a global solution $r = r(t)$ to the equation (2.2.4). Moreover, it holds that

$$\int^{\infty} \frac{dr}{h(r)} = \infty.$$

Thus, from Theorem 2.1.4, we obtain the following

Theorem 2.3.5. *There exists a family of equivariant harmonic maps from* $\mathbf{RH}^{m+1}$ *into* $\mathbf{R}^{n(m)+1}$, *each of which has a bounded image.*

Recently, Nagasawa and Tachikawa studied the case where

$$\int^{\infty} \sqrt{\frac{\mu^2}{f_1(t)^2} + \frac{\nu^2}{f_2(t)^2}}\, dt = \infty,$$

and proved, for example, the following

Theorem 2.3.6 ([28]). *Let* $M = (\mathbf{R}_+ \times S^m, dt^2 + f(t)^2 d\theta^2)$ *and* $N = (\mathbf{R}_+ \times S^n, dr^2 + h(r)^2 d\varphi^2)$. *Assume that, in addition to the conditions* (F-1) *through* (F-3) *and* (H-1) *through* (H-3), f *and* h *satisfy*

$$\int^{\infty} \frac{dt}{f(t)} dt = \infty \quad and \quad \int^{\infty} \frac{dr}{h(r)} dr < \infty.$$

Then there exists no equivariant harmonic map from M *to* N *except constant maps.*

As a corollary, there is no equivariant harmonic map from $\mathbf{R}^m$ to $\mathbf{RH}^n$, which was independently proved by Tachikawa([35]). See also [2] and [36].

In consequence, it is observed that the existence or non-existence of equivariant harmonic maps is closely related to the growth orders of warping function, f and h, when t and r tend to ∞, respectively. We shall investigate a related non-existence result in the last chapter.

CHAPTER 3

Dirichlet problem at infinity for harmonic maps between Damek-Ricci spaces

3.1 Damek-Ricci spaces

3.1.1 Generalized Heisenberg algebra

We start this section with a brief review of generalized Heisenberg algebras due to Kaplan [21].

Let $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$ be a 2-step nilpotent Lie algebra with inner product $\langle\ ,\ \rangle_{\mathfrak{n}}$, $\mathfrak{z}$ its center and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle_{\mathfrak{n}}$. Since $\mathfrak{n}$ is 2-step nilpotent, $\operatorname{ad} v|_{\mathfrak{v}}$ is a map from $\mathfrak{v}$ to $\mathfrak{z}$ for any $v \in \mathfrak{v}$. We set $\mathfrak{k}_v := \{u \in \mathfrak{v} \mid \operatorname{ad} v(u) = [v, u] = 0\}$ and consider the orthogonal decomposition $\mathfrak{v} = \mathfrak{k}_v \oplus \mathfrak{v}_v$ with respect to $\langle\ ,\ \rangle_{\mathfrak{n}}$. We call $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$ a *generalized Heisenberg algebra* if $\operatorname{ad} v|_{\mathfrak{v}_v} : \mathfrak{v}_v \to \mathfrak{z}$ is a surjective isometry for any unit vector $v \in \mathfrak{v}$.

Generalized Heisenberg algebras can be constructed systematically in the following fashion: Let $(\mathfrak{u}, \langle\ ,\ \rangle_{\mathfrak{u}})$ and $(\mathfrak{v}, \langle\ ,\ \rangle_{\mathfrak{v}})$ be real vector spaces equipped with inner product. Let $\mu : \mathfrak{u} \times \mathfrak{v} \to \mathfrak{v}$ be a composition of quadratic forms, that is, μ is a bilinear map satisfying $|\mu(u, v)|_{\mathfrak{v}} = |u|_{\mathfrak{u}}|v|_{\mathfrak{v}}$ for all $u \in \mathfrak{u}$ and $v \in \mathfrak{v}$. Note that $\mu : \mathfrak{u} \times \mathfrak{v} \to \mathfrak{v}$ satisfies $\mu(u_0, v) = v$ for some $u_0 \in \mathfrak{u}$. Indeed, for any given $u_0 \in \mathfrak{u}$, set $T : \mathfrak{v} \to \mathfrak{v}$ by $T(v) := \mu(u_0, v)$. Then $\mu'(u, v) := \mu(u, T^{-1}(v))$ satisfies $\mu'(u_0, v) = v$. Hence we may suppose $\mu(u_0, v) = v$. Define $\phi : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{u}$ by

$$\langle u, \phi(v, v')\rangle_{\mathfrak{u}} = \langle \mu(u, v), v'\rangle_{\mathfrak{v}}.$$

Let $\mathfrak{z}$ be the orthogonal complement of $\mathbf{R}u_0 = \{ru_0 \mid r \in \mathbf{R}\}$ in $\mathfrak{u}$, $\pi : \mathfrak{u} \to \mathfrak{z}$ the orthogonal projection and $\mathfrak{n}$ the direct sum $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$ of $\mathfrak{v}$ and $\mathfrak{z}$ with the natural inner product. Define a Lie bracket on $\mathfrak{n}$ by

$$[v+z, v'+z'] := \pi \circ \phi(v, v'). \tag{3.1.1}$$

Theorem 3.1.1 ([21]). $\mathfrak{n}$, *equipped with the above inner product and a Lie bracket, is a generalized Heisenberg algebra. Conversely, any generalized Heisenberg algebra arises in this manner.*

Regarding the existence of a composition of quadratic forms, we remark the following. Let ρ be a function defined on positive integers by

$$\rho(n) := 8p + 2^q \ \text{ if } \ n = (\text{odd number}) \times 2^{4p+q}, \ 0 \le q \le 3.$$

Then it is known by Hurwitz, Radon and Eckmann that a composition of quadratic forms $\mu : \mathfrak{u} \times \mathfrak{v} \to \mathfrak{v}$ exists if and only if $0 < \dim \mathfrak{u} \le \rho(\dim \mathfrak{v})$. Therefore we may conclude (see [21]):

Fact. *Given integers $m, n > 0$, there exists an $(n+m)$-dimensional generalized Heisenberg algebra with m-dimensional center if and only if* $0 < m < \rho(n)$.

As a consequence, we can construct a rich variety of generalized Heisenberg algebras and Damek-Ricci spaces as well.

3.1.2 Damek-Ricci space

Let $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$ be a generalized Heisenberg algebra, $\mathfrak{a}$ a one-dimensional real vector space and h a unit vector in $\mathfrak{a}$. Since $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, any element of $\mathfrak{n}$ is written uniquely as $v + z$ with $v \in \mathfrak{v}, z \in \mathfrak{z}$. Now we define on the linear space $\mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n}$ a canonical inner product and a Lie bracket by

$$\langle th + v + z, t'h + v' + z' \rangle := tt' + \langle v + z, v' + z' \rangle_{\mathfrak{n}},$$
$$[th + v + z, t'h + v' + z'] := tv' - t'v + 2tz' - 2t'z + 2[v + z, v' + z']_{\mathfrak{n}},$$

where $[\ ,\ ]_{\mathfrak{n}}$ is the Lie bracket on $\mathfrak{n}$ defined by (3.1.1). In this way, $\mathfrak{s}$ becomes a Lie algebra with an inner product. The simply connected Lie group S associated with $\mathfrak{s}$, equipped with the left invariant metric g_S, is called a *Damek-Ricci space*.

Example. Let $\mathfrak{v} = \mathbf{R}^{2k}, \mathfrak{u} = \mathbf{R}^2$, and $\mu(z,v) = (-y,x), \mu(w,v) = (x,y)$, where $\{z,w\}$ is an orthonormal basis of $\mathbf{R}^2$ and $v = (x,y) \in \mathbf{R}^k \oplus \mathbf{R}^k$. Then $\mathfrak{n}$ is a (classical) Heisenberg algebra and (S, g_S) is isometric to the complex hyperbolic space whose sectional curvature K satisfies $-4 \leq K \leq -1$. The quaternion hyperbolic space and the Cayley hyperbolic plane are also Damek-Ricci spaces. These three spaces are called *classical* (cf. [8]). Thus Damek-Ricci spaces are generalizations of rank one symmetric spaces of noncompact type.

Let ∇ be the Levi-Civita connection on (S, g_S). Using the formula

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle$$

for $X, Y, Z \in \mathfrak{s}$, one obtains

$$\begin{cases} \nabla_h X = 0,\ \nabla_X h = -[h,X], \\ \nabla_v v' = \langle v, v'\rangle h + \dfrac{1}{2}[v,v'], \\ \nabla_v z = \nabla_z v = -\mu(z,v), \\ \nabla_z z' = 2\langle z, z'\rangle h, \end{cases}$$

where $X \in \mathfrak{s}, v, v' \in \mathfrak{v}, z, z' \in \mathfrak{z}$ and μ is the composition of quadratic forms in the definition of $\mathfrak{n}$. Then the following results have been proved.

Theorem 3.1.2. (1) *A Damek-Ricci space is a symmetric space if and only if it is classical* ([7], [8]).

(2) *A Damek-Ricci space has strictly negative sectional curvature if and only if it is classical* ([13]).

In fact, Damek constructed a simple example of Damek-Ricci space whose sectional curvature attains zero for some two-dimensional plane. Other examples have been given in [5]. The second assertion was first proved by Boggino [4]. However, his proof needed a refinement which had been accomplished by Dotti([13]), see also [23]. The above theorem claims that every non-symmetric Damek-Ricci space has two-dimensional planes for which the sectional curvature vanishes.

We refer to [5] for further information about the geometry, in particular about the sectional curvature, of the Damek-Ricci spaces.

Finally, we estimate the bottom spectrum $\lambda_1(S)$ of the Laplace-Beltrami operator for S with the left invariant metric.

Lemma 3.1.3 ([9]). *Let (r, ω) be the polar coordinate on (S, g_S) around an origin. Then the volume form dv_S of S is given by*

$$dv_S = 2^{-m}(\sinh r)^n(\sinh 2r)^m dr d\sigma(\omega),$$

where $n = \dim \mathfrak{v}, m = \dim \mathfrak{z}$ and $d\sigma(\omega)$ is the surface element of the standard unit sphere S^{n+m}.

From this lemma and the fact $\lambda_1(S) \geq \inf(\Delta_S r)^2/4$, where Δ_S denotes the positive Laplace-Beltrami operator, we obtain the following

Corollary 3.1.4.

$$\lambda_1(S) \geq \frac{1}{4}(n + 2m)^2.$$

Note. Damek and Ricci [10] proved that the equality holds in the above inequality.

3.2 Harmonic maps between Damek-Ricci spaces

In this section, following Donnelly [12], we shall prove the existence and uniqueness of solutions of the Dirichlet problem at infinity for harmonic maps.

3.2.1 Harmonic map equation

We compute explicitly the tension field of a harmonic map between Damek-Ricci spaces. Let (S, g_S) be a Damek-Ricci space with left invariant metric g_S, and set $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$. Take the unit vector $h \in \mathfrak{a}$ such that $\operatorname{ad} h(v) = v$, $\operatorname{ad} h(z) = 2z$ for any $v \in \mathfrak{v}, z \in \mathfrak{z}$ and an orthonormal basis $\{h, v_1, \ldots, v_n, z_1, \ldots, z_m\}$ of $\mathfrak{s}$, $\{v_i\}$ (resp. $\{z_j\}$) being an orthonormal basis of $\mathfrak{v}$ (resp. $\mathfrak{z}$). We define a map $\varphi : \mathbf{R} \times N \to S$ by

$$\varphi(t, n) := n(\exp(th)),$$

where N is the simply connected Lie group associated with $\mathfrak{n}$ and exp is the exponential map on $\mathfrak{a}$. Then the induced metric $\varphi^* g_S$ is given by

$$\varphi^* g_S = dt^2 + e^{-2t} g_{\mathfrak{v}} + e^{-4t} g_{\mathfrak{z}},$$

where $g_{\mathfrak{v}} + g_{\mathfrak{z}}$ is a left invariant metric on N. Setting $y = e^t$, one can verify that (S, g_S) is isometric to $M := \mathbf{R}_+ \times N$ with the Riemannian metric

$$g_M := y^{-2} dy^2 + y^{-2} g_{\mathfrak{v}} + y^{-4} g_{\mathfrak{z}},$$

where $\mathbf{R}_+ = \{r \in \mathbf{R} \mid r > 0\}$. A straightforward calculation then yields that the Levi-Civita connection ∇ on (S, g_S) is given by

$$\begin{cases} \nabla_\eta \eta = -y^{-1}\eta, \quad \nabla_\eta v_i = -y^{-1} v_i, \quad \nabla_\eta z_k = -2y^{-1} z_k, \\ \nabla_{v_i} v_j = y^{-1}\delta_{ij}\eta + \dfrac{1}{2}\displaystyle\sum_{k=1}^{m} a_i{}^k{}_j z_k, \\ \nabla_{v_i} z_k = \dfrac{1}{2} y^{-2} \displaystyle\sum_{j=1}^{n} a_j{}^k{}_i v_j, \quad \nabla_{z_k} z_l = 2y^{-3}\delta_{kl}\eta, \end{cases} \tag{3.2.1}$$

where $a_i{}^k{}_j$ are the structure constants, $[v_i, v_j] = \displaystyle\sum_{k=1}^{m} a_i{}^k{}_j z_k$, $\eta = \partial/\partial y$ and, v_i and z_k are regarded as left invariant vector fields on S.

Now we compute the tension field. Let (S, g_S) and $(S', g_{S'})$ be Damek-Ricci spaces and $u \in C^2(S, S')$. We represent (S, g_S) (resp. $(S', g_{S'})$) as

$$(\mathbf{R}_+ \times N, y^{-2}dy^2 + y^{-2}g_{\mathfrak{v}} + y^{-4}g_{\mathfrak{z}}) \quad (\text{resp. } (\mathbf{R}_+ \times N', \bar{y}^{-2}d\bar{y}^2 + \bar{y}^{-2}g_{\mathfrak{v}'} + \bar{y}^{-4}g_{\mathfrak{z}'})),$$

where $y, \bar{y} \in \mathbf{R}_+$ and $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$ (resp. $\mathfrak{n}' = \mathfrak{v}' + \mathfrak{z}'$) with $\dim \mathfrak{v} = n$ (resp. $\dim \mathfrak{v}' = n'$), $\dim \mathfrak{z} = m$ (resp. $\dim \mathfrak{z}' = m'$). Take an orthonormal basis $\{v_1, \dots, v_n, z_1, \dots, z_m\}$ (resp. $\{\bar{v}_1, \dots, \bar{v}_{n'}, \bar{z}_1, \dots, \bar{z}_{m'}\}$) of $\mathfrak{n}$ (resp. $\mathfrak{n}'$) as above, and denote their left invariant extensions on S by the same letters. Let

$$\begin{cases} e_0 = \dfrac{\partial}{\partial y}, \\ e_i = v_i \quad (1 \le i \le n), \\ e_i = z_{i-n} \ (n+1 \le i \le n+m), \end{cases} \qquad \begin{cases} f_0 = \dfrac{\partial}{\partial \bar{y}}, \\ f_\alpha = \bar{v}_\alpha \ (1 \le \alpha \le n'), \\ f_\alpha = \bar{z}_{\alpha-n'} \ (n'+1 \le \alpha \le n'+m'), \end{cases}$$

and

$$u_j^\alpha = f_\alpha^*(du(e_i)), \quad u_{ij}^\alpha = e_j \cdot u_i^\alpha, \quad \tau^\alpha(u) = f_\alpha^*(\tau(u)),$$

where f_α^* is the dual frame of f_α. Then, since the tension field of u is given by

$$\tau(u) := \sum_{i=1}^{n} (\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)) = 0,$$

we get

$$(3.2.2) \qquad \begin{cases} \tau^0(u) = \displaystyle\sum_{i=0}^{n+m} g^{ii} u_{ii}^0 + (1-n-2m) u_0^0 y - (u^0)^{-1} \sum_{i=0}^{n+m} g^{ii} (u_i^0)^2 \\ \qquad\qquad + (u^0)^{-1} \displaystyle\sum_{i=0}^{n+m} g^{ii} \sum_{\beta=1}^{n'} (u_i^\beta)^2 + 2(u^0)^{-3} \sum_{i=0}^{n+m} g^{ii} \sum_{\beta=n'+1}^{n'+m'} (u_i^\beta)^2, \\ \tau^\alpha(u) = \displaystyle\sum_{i=0}^{n+m} g^{ii} u_{ii}^\alpha + (1-n-2m) u_0^\alpha y - 2(u^0)^{-1} \sum_{i=0}^{n+m} g^{ii} u_i^0 u_i^\alpha \\ \qquad\qquad + (u^0)^{-2} \displaystyle\sum_{i=0}^{n+m} g^{ii} \sum_{\beta=1}^{n'} \sum_{\gamma=n'+1}^{n'+m'} u_i^\gamma u_i^\beta \Gamma_{\alpha\ \beta}^{\ \gamma-n'} \qquad (1 \le \alpha \le n'), \\ \tau^\alpha(u) = \displaystyle\sum_{i=0}^{n+m} g^{ii} u_{ii}^\alpha + (1-n-2m) u_0^0 y - 4(u^0)^{-1} \sum_{i=0}^{n+m} g^{ii} u_i^0 u_i^\alpha \\ \qquad\qquad\qquad\qquad (n'+1 \le \alpha \le n'+m'), \end{cases}$$

where $u^0 = \bar{y}(u)$, and (g_{ij}) denotes the matrix component of the metric g_M, (g^{ij}) its inverse matrix. Note that $u_{ij}^\alpha \ne u_{ji}^\alpha$ because $[v_i, v_j] \ne 0$.

3.2.2 Uniqueness theorem

Let S and S' be Damek-Ricci spaces and assume that they are represented as in Section 3.2.1. We denote by $\bar{S}$ (resp. $\bar{S}'$) the Eberlein-O'Neill compactification of S (resp. S'). Then $\{y=0\} \times N$ (resp. $\{\bar{y}=0\} \times N'$) represents the ideal boundary, ∂S (resp. $\partial S'$), except a point in ∂S (resp. $\partial S'$) (cf. [6]).

Let $u \in C^2(S, S')$ be a proper harmonic map. First, we investigate a necessary condition for the existence of a C^2-extension $u : \bar{S} \to \bar{S}'$ of u, and then prove the following uniqueness theorem.

Theorem 3.2.1 (Uniqueness theorem). *Let u and w be proper harmonic maps between Damek-Ricci spaces S and S'. Suppose $u, w \in C^2(\bar{S}, \bar{S}')$ and $f := u_{|\partial S} = w_{|\partial S}$. If*

$$\sum_{j=n+1}^{n+m} \sum_{\gamma=n'+1}^{n'+m'} (f_j^\gamma)^2 > 0 \text{ on } \partial S,$$

then $u = w$ on $\bar{S}$.

In order to prove this theorem, the following lemma plays an important role.

Lemma 3.2.2 ([12], [26]). *Suppose that $\omega \in C^1\Lambda^1 S \cap C^0\Lambda^1\bar{S}$ is a 1-form defined on a neighborhood of $p \in \partial S$. Let $\omega = \sum_{i=0}^{n+m} \omega_i e_i^*$, where $\{e_i^*\}$ is the dual coframe of $\{e_i\}$. Then there exists a sequence of points $\{q_k\} \subset S$ such that $q_k \to p$ and*

$$\lim_{k\to\infty} \left(y^{-1} \sum_{j=0}^{n+m} g^{jj}\omega_{jj} \right)(q_k) = 0.$$

Using this lemma together with $\tau(u) = 0$, we obtain the following necessary condition.

Lemma 3.2.3. *Let $u \in C^2(\bar{S}, \bar{S}')$ be a proper harmonic map. Then at the ideal boundary we have the following:*

(1) $\displaystyle\sum_{j=0}^{n} \sum_{\beta=n'+1}^{n'+m'} (u_j^\beta)^2 = 0,$

(2) $(n+2m)(u_0^0)^4 - \sum_{j=0}^{n}\sum_{\beta=1}^{n'}(u_j^\beta)^2(u_0^0)^2 - 2\sum_{j=0}^{n}\sum_{\beta=n'+1}^{n'+m'}(u_{j0}^\beta)^2 - 2\sum_{j=n+1}^{n+m}\sum_{\beta=n'+1}^{n'+m'}(u_j^\beta)^2 = 0,$

(3) $(1+n+2m)u_0^\alpha(u_0^0)^2 - \sum_{j=0}^{n}\sum_{\beta=1}^{n'}\sum_{\gamma=n'+1}^{n'+m'}\Gamma_{\alpha\ \beta}^{\ \gamma-n'}u_j^\beta u_{j0}^\gamma = 0 \quad (1 \le \alpha \le n'),$

(4) $(2+n+2m)u_0^0 u_{00}^\alpha = 0 \quad (n'+1 \le \alpha \le n'+m').$

Proof. Since $u \in C^2(\bar{S}, \bar{S}')$ is proper, we have $u^0 = O(y)$ and $\lim_{y\to 0} u^0 y^{-1} = u_0^0$. Multiplying by $(u^0)^3 y^{-2}$ both sides of the first equation in (3.2.2), we let $y \to 0$. Then, by virtue of Lemma 3.2.2, the first term on the right hand side tends to 0. Hence we obtain (1).

The rest of the statement follows similarly if we multiply the first, second and third equations in (3.2.2) by $(u^0)^3y^{-4}, (u^0)^2y^{-3}$ and u^0y^{-3}, respectively. ☐

Since $u_{0j}^\gamma = 0$ $(\gamma \ge n'+1)$ at the boundary and $ddu = 0$, we have

$$u_{j0}^\gamma = -\sum_{\mu,\nu=1}^{n'}\Gamma_{\mu\ \nu}^{\ \gamma-n'}u_0^\mu u_j^\nu \quad (0 \le j \le n, n'+1 \le \gamma \le n'+m'). \tag{3.2.3}$$

Substituting this into the equation (3) of Lemma 3.2.3 and adding the result in α, we get

$$(1+n+2m)(u_0^0)^2\sum_{\alpha=1}^{n'}(u_0^\alpha)^2 + \sum_{j=0}^{n}\sum_{\gamma=n'+1}^{n'+m'}\left(\sum_{\alpha,\beta=1}^{n'}\Gamma_{\alpha\ \beta}^{\ \gamma-n'}u_0^\alpha u_j^\beta\right)^2 = 0. \tag{3.2.4}$$

Now we can deduce the following

Proposition 3.2.4. *Let $u \in C^2(\bar{S}, \bar{S}')$ be a proper harmonic map. If $f := u_{|\partial S}$ satisfies*

$$\sum_{j=n+1}^{n+m}\sum_{\gamma=n'+1}^{n'+m'}(f_j^\gamma)^2 > 0,$$

then at the ideal boundary u must satisfy $u_0^0 > 0, u_0^\alpha = 0$ $(1 \le \alpha \le n'+m')$ and $u_{k0}^\beta = u_{00}^\beta = 0$ $(1 \le k \le n, n'+1 \le \beta \le n'+m')$.

Proof. The assumption on f implies that the fourth term in the equation (2) of Lemma 3.2.3 never vanishes. Thus $u_0^0 > 0$, which implies

$$u_0^\alpha = 0, \quad u_{00}^\alpha = 0 \quad \text{and} \quad \sum_{\alpha,\beta=1}^{n'} \Gamma_{\alpha\ \beta}^{\ \gamma-n'} u_j^\alpha u_j^\beta = 0$$

from the equation (4) of Lemma 3.2.3 and (3.2.4). Finally, (3.2.3) implies $u_{k0}^\beta = 0$. □

When S and S' are real hyperbolic planes, this proposition means that a proper harmonic map must be conformal at the ideal boundary.

By Proposition 3.2.4 we get the following

Corollary 3.2.5. *We have*

$$\begin{cases} u_0^\alpha = O(y) & (1 \le \alpha \le n'), \\ u_0^\alpha = o(y) & (n'+1 \le \alpha \le n'+m'). \end{cases}$$

We shall now complete the proof of Theorem 3.2.1.

If we write

$$u(y,n) = (\bar{y}(u), \bar{n}(u)),\ \ w(y,n) = (\bar{y}(w), \bar{n}(w)),\ \ f(n) = \bar{n}(f),$$

then it holds that

$$\begin{aligned} d(u,w) \le\ & d((\bar{y}(u), \bar{n}(u)), (\bar{y}(u), \bar{n}(f))) \\ & + d((\bar{y}(u), \bar{n}(f)), (\bar{y}(w), \bar{n}(f))) + d((\bar{y}(w), \bar{n}(f)), (\bar{y}(w), \bar{n}(w))). \end{aligned} \tag{3.2.5}$$

From the explicit expression for the metrics and $\bar{n}(f) = \bar{n}(u(0,n))$, we see that the first term on the right hand side of (3.2.5) is

$$\int_0^y \left| \frac{\partial \bar{n}}{\partial t}(u(t,n)) \right| dt \le \int_0^y \left[t^{-1} \sum_{\alpha=1}^{n'} |u_0^\alpha| + t^{-2} \sum_{\alpha=n'+1}^{n'+m'} |u_0^\alpha| \right] dt = o(1).$$

The last estimate in the above follows from Corollary 3.2.5. Similarly, the third term is $O(y)$. On the other hand, the second term on the right hand side of (3.2.5) is

$$\left| \log \frac{\bar{y}(u)}{\bar{y}(w)} \right| = \left| \log \frac{u^0(y,n)}{w^0(y,n)} \right|.$$

Since $u_0^0((0,n))$ and $w_0^0(w(0,n))$ are uniquely determined by f and positive, we have

$$\left|\log\frac{\bar{y}(u)}{\bar{y}(w)}\right| = \left|\log\frac{u_0^0(0,n)y+o(y)}{w_0^0(0,n)y+o(y)}\right| = o(1).$$

Therefore $d(u,w)=0$ at the ideal boundary. Hence the maximal principle ([33]) implies Theorem 3.2.1.

3.2.3 Existence theorem

To show the existence of a solution of the Dirichlet problem at infinity, we use the heat flow method due to Li and Tam [25]. As we see in the previous section, Damek-Ricci spaces are homogeneous spaces of nonpositive sectional curvature and have positive bottom spectrum of the Laplace-Beltrami operator. Therefore, we can apply the general existence theory [25, Theorem 5.2] if there exists a suitable initial map. Indeed, h in Proposition 3.2.7 can be taken as our initial map.

The decay order of the tension field of the initial map can be estimated by the following

Lemma 3.2.6. *Let* $h \in C^{2,\varepsilon}(\partial S, \partial S')$ $(0<\varepsilon<1)$. *If* h *satisfies* (1) *through* (4) (*with* u *replaced by* h) *in Lemma* 3.2.3 at the ideal boundary, then its tension field $\tau(h)$ has the following decay order as $y \to 0$:

$$\begin{cases} \tau^{\alpha}(h) = O(y^{1+\varepsilon}) \quad (0 \le \alpha \le n'), \\[2ex] \tau^{\alpha}(h) = O(y^{2+\varepsilon}) \quad (n'+1 \le \alpha \le n'+m'). \end{cases}$$

This lemma follows easily from the Taylor expansion of $\tau^{\alpha}(h)$ in (3.2.2) with respect to y.

Proposition 3.2.7. *Suppose that* $f \in C^{3,\varepsilon}(\partial S, \partial S')$ $(0<\varepsilon<1)$ *satisfies*

$$\begin{cases} f_j^{\gamma} = 0 \quad (1 \le j \le n, n'+1 \le \gamma \le n'+m'), \\[2ex] \displaystyle\sum_{j=n+1}^{n+m}\sum_{\gamma=n'+1}^{n'+m'} (f_j^{\gamma})^2 > 0. \end{cases}$$

Then there exists an $h \in C^{2,\varepsilon}(\bar{S}, \bar{S}')$ *such that* $h = f$ *at the ideal boundary and* $\|\tau(h)\| = O(y^{\varepsilon})$, *where* $\|\tau(h)\|$ *is the norm of the tension field in the Riemannian metric.*

Proof. Let $\phi > 0$ be a unique solution of

$$(n+2m)\phi^4 - \sum_{j=1}^{n}\sum_{\beta=1}^{n'}(f_j^{\beta})^2\phi^2 - 2\sum_{j=n+1}^{n+m}\sum_{\beta=n'+1}^{n'+m'}(f_j^{\beta})^2 = 0.$$

Since $f \in C^{3,\varepsilon}$, we have $\phi \in C^{2,\varepsilon}(\partial S)$. Set $h(y,n) := (y\phi(n), f(n))$. Then $h \in C^{2,\varepsilon}(\partial S, \partial S')$. Moreover, it is easy to verify that h satisfies

$$\begin{cases} h_0^0 = \phi, \\ h_0^{\alpha} = 0 \quad (1 \leq \alpha \leq n'+m'), \\ h_{00}^{\beta} = 0 \quad (n'+1 \leq \beta \leq n'+m'), \\ h_{j0}^{\gamma} = 0 \quad (1 \leq j \leq n, n'+1 \leq \gamma \leq n'+m'), \end{cases}$$

and $h = f$ at the ideal boundary. Lemma 3.2.6 and the explicit expression for the metric then imply $\|\tau(h)\| = O(y^{\varepsilon})$. $\square$

Next we construct a comparison function.

Lemma 3.2.8. *For sufficiently large* r_0 *and some constant* s, *define*

$$\psi(r) := \begin{cases} e^{-sr_0} & (r \leq r_0), \\ e^{-sr} & (r \geq r_0). \end{cases}$$

If $0 < s \leq n+2m$, *then* ψ *is a superharmonic function on* S. *Here* r *is the distance function as in Lemma* 3.1.3.

Proof. It follows from Lemma 3.1.3 that the Laplace-Beltrami operator on (S, g_S) has the form

$$\Delta_S = -\frac{\partial^2}{\partial r^2} - \left(n\frac{\cosh r}{\sinh r} + 2m\frac{\cosh 2r}{\sinh 2r}\right)\frac{\partial}{\partial r} + (\text{spherical part}).$$

A straightforward calculation then yields the lemma. $\square$

Now we prove the main theorem of this section.

Theorem 3.2.9 (Existence Theorem). *Suppose that* $f \in C^{3,\varepsilon}(\partial S, \partial S')$ $(0 < \varepsilon < 1)$ *satisfies*

$$\begin{cases} f_j^\gamma = 0 \quad (1 \leq j \leq n, n'+1 \leq \gamma \leq n'+m'), \\ \displaystyle\sum_{j=n+1}^{n+m} \sum_{\gamma=n'+1}^{n'+m'} (f_j^\gamma)^2 > 0. \end{cases}$$

Then there exists a harmonic map $u : S \to S'$*, which is continuous up to the ideal boundary and assumes* f *as the boundary value.*

Proof. For a map $v : S \times [0, \infty) \to S'$, consider the heat equation for harmonic maps

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = \tau(v)(x,t), & (x,t) \in S \times [0,\infty), \\ v(x,0) = h(x), & x \in S, \end{cases}$$

where h is the map constructed in Proposition 3.2.7. Since the initial map h satisfies the conditions in [25, Theorem 5.2], a solution v exists globally in time t and converges to a harmonic map v_∞ as $t \to \infty$.

We shall show that $u = v_\infty$ has f as the boundary value. Let ψ be a supersolution in Lemma 3.2.8, then (cf. [20])

$$\left(\frac{\partial}{\partial t} + \Delta_S \right) \left(\|v_t\|^2 - c\psi \right) \leq 0$$

and

$$\|v_t(\cdot, 0)\|^2 - c\psi = \|\tau(h)\|^2 - c\psi,$$

where $v_t = \partial v / \partial t$. Taking $s = 2\varepsilon$ and c sufficiently large, we may assume

$$\|v_t(\cdot, 0)\|^2 - c\psi = \|\tau(h)\|^2 - c\psi < 0.$$

Therefore the parabolic maximum principle implies

$$\|v_t(x,t)\| \leq c e^{-\varepsilon r}, \quad \text{for all } (x,t) \in S \times [0, \infty).$$

By virtue of [25, Lemma 5.1], we have

$$\|v_t(x,t)\| \leq c_1 e^{-c_2 t}$$

for some constants $c_1, c_2 > 0$. Then for any $T > 0$

$$\begin{aligned} d_{S'}(u(x), h(x)) &\leq \int_0^\infty \|v_t(x,t)\| dt \leq \int_0^T \|v_t(x,t)\| dt + \int_T^\infty \|v_t(x,t)\| dt \\ &\leq c e^{-\varepsilon r(x)} T + c_1 e^{-c_2 T}. \end{aligned}$$

Choosing $T = \varepsilon r$, we get

$$c e^{-\varepsilon r(x)} T \text{ and } c_1 e^{-c_2 T} \to 0 \quad \text{as } r \to \infty.$$

Therefore, $d_{S'}(u(x), h(x)) \to 0$ as $x \to \partial S$ and $u = h = f$ at the ideal boundary. $\square$

Note. Our result remains true, with necessary modifications, when the target manifold is replaced by the real hyperbolic space.

CHAPTER 4

Non-existence of proper harmonic maps from complex hyperbolic spaces into real hyperbolic spaces

In this chapter, we investigate proper harmonic maps from the m-dimensional complex hyperbolic space $\mathbf{B}^m$ to the n-dimensional real hyperbolic space $\mathbf{D}^n$. Our goal is to prove the following non-existence result.

Theorem 4.1.0. *Let $m, n \geq 2$. Then there is no proper harmonic map $u \in C^2(\mathbf{B}^m, \mathbf{D}^n)$ which has C^1-regularity up to the ideal boundary.*

The C^1-regularity assumption up to the ideal boundary, supposed for proper harmonic maps in Theorem 4.1.0, plays a crucial role in our proof. Indeed, in the last section, we show that there exists a counter example to this theorem if we relax its regularity condition up to the ideal boundary.

4.1 Proof of Theorem

Let $(\mathbf{B}^m, g)$ and $(\mathbf{D}^n, g')$ be the ball models of the m-dimensional complex hyperbolic space and the n-dimensional real hyperbolic space, respectively. Namely,

$$\mathbf{B}^m = \left(\{z \in \mathbf{C}^m \mid |z| < 1\},\ g = \frac{1}{(1-|z|^2)^2} \sum_{i,j=1}^{m} \left\{ (1-|z|^2)\delta_{ij} + \bar{z}^i z^j \right\} dz^i d\bar{z}^j \right),$$

$$\mathbf{D}^n = \left(\{x \in \mathbf{R}^n \mid |x| < 1\},\ g' = \frac{4}{(1-|x|^2)^2} \sum_{i=1}^{n} (dx^i)^2 \right),$$

where $z = (z^1, \ldots, z^m) \in \mathbf{B}^m$ and $x = (x^1, \ldots, x^n) \in \mathbf{D}^n$ are the canonical Euclidean coordinates. We endow the Eberlein-O'Neill compactification $\overline{\mathbf{B}^m}$ (resp. $\overline{\mathbf{D}^n}$) with the natural smooth structure inherited from $\mathbf{C}^m$ (resp. $\mathbf{R}^n$), and regard it as a submanifold with boundary in $\mathbf{C}^m$ (resp. $\mathbf{R}^n$). With these understood, we have the following

Lemma 4.1.1. (1) *Let $g_{i\bar{j}}$, $\Gamma_{j\ k}^{\ i}$ and $\Delta_{\mathbf{B}^m}$ be the components, the Christoffel symbols and the Laplace-Beltrami operator of Bergman metric g, respectively. Then we have*

$$\begin{cases} g_{i\bar{j}} = (1-|z|^2)^{-2}\{(1-|z|^2)\delta_{ij} + \bar{z}^i z^j\}, \\ \Gamma_{j\ k}^{\ i} = (1-|z|^2)^{-1}(\bar{z}^j\delta_{ik} + \bar{z}^k\delta_{ij}), \\ \Delta_{\mathbf{B}^m} = (1-|z|^2)\displaystyle\sum_{i,j=1}^{m}(\delta_{ij} - z^i\bar{z}^j)\frac{\partial^2}{\partial z^i\partial\bar{z}^j}. \end{cases}$$

(2) *Let $g'_{\alpha\beta}$, $\tilde{\Gamma}_{\beta\ \gamma}^{\ \alpha}$ and $\Delta_{\mathbf{D}^n}$ be the components, the Christoffel symbols and the Laplace-Beltrami operator of Poincaré metric g', respectively. Then we have*

$$\begin{cases} g'_{\alpha\beta} = \dfrac{1}{(1-|x|^2)^2}\delta_{\alpha\beta}, \\ \tilde{\Gamma}_{\beta\ \gamma}^{\ \alpha} = \dfrac{2}{(1-|x|^2)}(x^\beta\delta_{\alpha\gamma} + x^\gamma\delta_{\alpha\beta} - x^\alpha\delta_{\beta\gamma}), \\ \Delta_{\mathbf{D}^n} = (1-|x|^2)^2\displaystyle\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial x^\alpha\partial x^\alpha} + 2(n-2)(1-|x|^2)\sum_{\alpha=1}^{n}x^\alpha\frac{\partial}{\partial x^\alpha}. \end{cases}$$

Following [24], we define a family of global vector fields N and X_j, $1 \le j \le m$, on $\mathbf{C}^m$ in the following manner:

$$\begin{cases} N = \displaystyle\sum_{i=1}^{m} z^i\frac{\partial}{\partial z^i}, \\ X_j = \displaystyle\sum_{i=1}^{m}(\delta_{ij} - z^i\bar{z}^j)\frac{\partial}{\partial z^i} = \frac{\partial}{\partial z^j} - \langle\frac{\partial}{\partial z^j}, N\rangle N, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian inner product on $\mathbf{C}^m$. Then we get

Lemma 4.1.2 ([24]). (1) *$N + \bar{N}$ is the outer normal vector field on $\partial\mathbf{B}^m$.*

(2) *$\{X_j + \bar{X}_j,\ \sqrt{-1}(X_j - \bar{X}_j),\ \sqrt{-1}(N - \bar{N})\}_{j=1}^m$ is a basis of the tangent space $T_p\partial\mathbf{B}^m$ at each $p \in \partial\mathbf{B}^m$.*

Moreover, we obtain the following

Lemma 4.1.3 ([24]). *Let* $L = \sum_{i,j=1}^{m} (\delta_{ij} - z^i \bar{z}^j) \frac{\partial^2}{\partial z^i \partial \bar{z}^j}$. *Then*

$$\begin{aligned} L &= \sum_{j=1}^{m} X_j \bar{X}_j + (m - |z|^2) \bar{N} + (1 - |z|^2) N \bar{N} \\ &= \sum_{j=1}^{m} \bar{X}_j X_j + (m - |z|^2) N + (1 - |z|^2) \bar{N} N. \end{aligned}$$

In terms of our frame fields, the tension field of u is expressed as follows.

Lemma 4.1.4. *For* $u \in C^2(\mathbf{B}^m, \mathbf{D}^n)$ *we have*

$$\begin{aligned} \tau(u)^\alpha = (1 - |z|^2) &\left[L u^\alpha + \frac{2}{a(u)(z)} \{ (N u^\alpha) \langle \bar{N} u, u \rangle + (\bar{N} u^\alpha) \langle N u, u \rangle - u^\alpha |N u|^2 \} \right] \\ &+ \frac{2}{a(u)(z)} \sum_{j=1}^{m} \{ (X_j u^\alpha) \langle \bar{X}_j u, u \rangle + (\bar{X}_j u^\alpha) \langle X_j u, u \rangle - u^\alpha |X_j u|^2 \}, \end{aligned}$$

where $a(u)(z) = (1 - |u(z)|^2)/(1 - |z|^2)$ *and* $u(z) = (u^1(z), \dots, u^n(z))$.

Proof. In our coordinates, the tension field of $u \in C^2(\mathbf{B}^m, \mathbf{D}^n)$ is given by

$$\tau(u)^\alpha = \Delta_{\mathbf{B}^m} u^\alpha + \sum_{i,j=1}^{m} \sum_{\beta,\gamma=1}^{n} g^{i\bar{j}} \tilde{\Gamma}_{\beta}{}^{\alpha}{}_{\gamma}(u(z)) u^\beta_i u^\gamma_{\bar{j}}.$$

Hence, from Lemma 4.1.1, we have

$$\begin{aligned} \tau(u)^\alpha &= (1 - |z|^2) \sum_{i,j=1}^{m} (\delta_{ij} - z^i \bar{z}^j) \frac{\partial^2 u^\alpha}{\partial z^i \partial \bar{z}^j} \\ &\quad + 2(1 - |z|^2)(1 - |u(z)|^2)^{-1} \sum_{i,j=1}^{m} \sum_{\beta,\gamma=1}^{n} (\delta_{ij} - z^i \bar{z}^j)(u^\beta \delta_{\alpha\gamma} + u^\gamma \delta_{\alpha\beta} + u^\alpha \delta_{\beta\gamma}) u^\gamma_i u^\beta_{\bar{j}} \\ &= (1 - |z|^2) L u^\alpha + 2a(u)(z)^{-1} \sum_{\beta=1}^{n} \sum_{j=1}^{m} \left[\left\{ \sum_{i=1}^{m} (\delta_{ij} - z^i \bar{z}^j) u^\alpha_i \right\} u^\beta u^\beta_{\bar{j}} \right. \\ &\qquad \left. + \left\{ \sum_{i=1}^{m} (\delta_{ij} - z^i \bar{z}^j) u^\beta_i \right\} u^\alpha_{\bar{j}} u^\beta + \left\{ \sum_{i=1}^{m} (\delta_{ij} - z^i \bar{z}^j) u^\beta_i \right\} u^\alpha u^\beta_{\bar{j}} \right] \\ &= (1 - |z|^2) L u^\alpha \\ &\quad + 2a(u)(z)^{-1} \sum_{\beta=1}^{n} \sum_{j=1}^{m} \{ (X_j u^\alpha) u^\beta u^\beta_{\bar{j}} + (X_j u^\beta) u^\alpha_{\bar{j}} u^\beta + (X_j u^\beta) u^\alpha u^\beta_{\bar{j}} \} = (*1). \end{aligned}$$

From the formula $\partial/\partial \bar{z}^j = \bar{X}_j + z^j \bar{N}$, we then have

$$\begin{aligned}
(*1) &= (1-|z|^2)Lu^\alpha \\
&\quad + 2a(u)(z)^{-1} \sum_{\beta=1}^{n} \sum_{j=1}^{m} \{(X_j u^\alpha) u^\beta (\bar{X}_j u^\beta + z^j \bar{N} u^\beta) \\
&\qquad\qquad + (X_j u^\beta) u^\beta (\bar{X}_j u^\alpha + z^j \bar{N} u^\alpha) + (X_j u^\beta) u^\alpha (\bar{X}_j u^\beta + z^j \bar{N} u^\beta)\} \\
&= (1-|z|^2)Lu^\alpha \\
&\quad + (1-|z|^2) \frac{2}{a(u)(z)} \sum_{\beta=1}^{n} \{(N u^\alpha)(u^\beta \bar{N} u^\beta) + (\bar{N} u^\alpha)(u^\beta N u^\beta) - u^\alpha (N u^\beta)(\bar{N} u^\beta)\} \\
&\quad + \frac{2}{a(u)(z)} \sum_{j=1}^{m} \sum_{\beta=1}^{n} \{(X_j u^\alpha)(u^\beta \bar{X}_j u^\beta) + (\bar{X}_j u^\alpha)(u^\beta X_j u^\beta) - u^\alpha (X_j u^\beta)(\bar{X}_j u^\beta)\}. \qquad \square
\end{aligned}$$

In order to investigate the asymptotic behavior of proper harmonic maps, Li and Ni proved the following

Lemma 4.1.5 ([24]). *Assume $f \in C^2(\mathbf{B}^m) \cap C^1(\overline{\mathbf{B}^m})$. Then we have*

$$\lim_{z \to \partial \mathbf{B}^m} \frac{1-|z|^2}{\varepsilon(z)^{2m}} \int_{B(z,\varepsilon(z))} (\bar{f} L f)(\zeta) d\zeta = 0,$$

where $\varepsilon(z) = (1-|z|)/2$ and $B(z, \varepsilon(z))$ is the open ball in $(\mathbf{C}^m, g_{\mathbf{C}^m})$ with the radius $\varepsilon(z)$ centered at z, and $d\zeta$ is the volume element of $(\mathbf{C}^m, g_{\mathbf{C}^m})$.

As a corollary of this lemma, we have the following result, which is in essential use in our proof.

Corollary 4.1.6. *Assume $u \in C^2(\mathbf{B}^m, \mathbf{C}^n) \cap C^1(\overline{\mathbf{B}^m}, \mathbf{C}^n)$. Then for any point $p \in \partial \mathbf{B}^m$ there exists a sequence $\{z_j\}_{j=1}^\infty \subset \mathbf{B}^m$ satisfying the following properties:*

(1) $z_j \to p \quad (j \to \infty)$.

(2) $\lim_{j\to\infty} (1-|z_j|^2)\langle Lu, u\rangle(z_j) = 0$.

We first prove the following result.

Proposition 4.1.7. *Let $m, n \geq 2$ and $u \in C^2(\mathbf{B}^m, \mathbf{D}^n) \cap C^1(\overline{\mathbf{B}^m}, \overline{\mathbf{D}^n})$. If u is a proper harmonic map, then the boundary value of u is a constant map.*

Proof. Since $u^\alpha \in \mathbf{R}$ and $\tau(u)^\alpha = 0$, we have

$$\begin{aligned} 0 = {} & a(u)(z)(1-|z|^2)\langle Lu, u\rangle + 2(1-|z|^2)(2|\langle Nu, u\rangle|^2 - |u|^2|Nu|^2) \\ & + 2\sum_{j=1}^{m}(2|\langle X_j u, u\rangle|^2 - |u|^2|X_j u|^2). \end{aligned}$$

For $p \in \partial\mathbf{B}^m$, let $\{z_j\}_{j=1}^{\infty}$ be a sequence satisfying the conditions in Corollary 4.1.6. Then, since $u \in C^1(\overline{\mathbf{B}^m}, \overline{\mathbf{D}^n})$, it follows from Corollary 4.1.6 that

$$\begin{aligned} &\lim_{j\to\infty} a(u)(z_j)(1-|z_j|^2)\langle Lu, u\rangle(z_j) = 0, \\ &\lim_{j\to\infty} 2(1-|z_j|^2)(2|\langle Nu, u\rangle|^2 - |u|^2|Nu|^2)(z_j) = 0. \end{aligned}$$

Thus we conclude that

$$2\sum_{j=1}^{m}|\langle X_j u, u\rangle|^2 = \sum_{j=1}^{m}|X_j u|^2 \quad \text{at } p,$$

and hence

$$2\sum_{j=1}^{m}|\langle X_j u, u\rangle|^2 = \sum_{j=1}^{m}|X_j u|^2 \quad \text{on } \partial\mathbf{B}^m.$$

On the other hand, on $\partial\mathbf{B}^m$, it holds that

$$0 = X_j|u|^2 = 2\langle X_j u, u\rangle.$$

Therefore

$$\sum_{j=1}^{m}|X_j u|^2 = 0,$$

which asserts that

$$X_j u^\alpha = 0 \quad \text{on } \partial\mathbf{B}^m \ \text{ for } \ 1 \leq j \leq m, \ 1 \leq \alpha \leq n.$$

Since $u^\alpha \in \mathbf{R}$, we obtain

$$\bar{X}_j u^\alpha = 0 \quad \text{on } \partial\mathbf{B}^m \ \text{ for } \ 1 \leq j \leq m, \ 1 \leq \alpha \leq n.$$

Moreover, from Lemma 4.1.3, we have

$$(m-1)(N-\bar{N}) = \sum_{j=1}^{m}(X_j\bar{X}_j - \bar{X}_jX_j) \quad \text{on } \partial\mathbf{B}^m.$$

Hence

$$(m-1)(N-\bar{N})u^\alpha = \sum_{j=1}^{m}(X_j\bar{X}_j - \bar{X}_jX_j)u^\alpha = 0 \quad \text{on } \partial\mathbf{B}^m.$$

Thus u is a constant map at the ideal boundary. □

Proof of Theorem. Let $\{r_j\}_{j=1}^{\infty}$ be a sequence such that $0 < r_j < 1$, $r_j \uparrow 1$ $(j \to \infty)$, and $B(r_j) = \{z \in \mathbf{C}^m \mid |z| < r_j\}$. Let D_j be the smallest geodesic ball containing $u(\partial B(r_j))$. Since u is a harmonic map and $\mathbf{D}^m$ has nonpositive sectional curvature, the function f defined by

$$f(z) = d_{\mathbf{D}^m}(u(z), p)^2$$

is a subharmonic function on $\overline{B(r_j)}$, where p is the center of D_j and $d_{\mathbf{D}^m}$ is the distance function with respect to the Poincaré metric. Hence the maximum principle implies that $u(\overline{B(r_j)}) \subset D_j$ for each j. On the other hand, since $u \in C^0(\overline{\mathbf{B}^m}, \overline{\mathbf{D}^n})$ and $u(\partial\mathbf{B}^m) = p_0 \in \partial\mathbf{D}^n$, it holds that $D_j \to p_0$ uniformly as $j \to \infty$. Thus we have $u(B(r_j)) \to p_0$ as $j \to \infty$. □

Note. The last argument is also true if the target manifold is replaced by any Hadamard manifold.

4.2 A counter example

In this section, for two-dimensional complex and real hyperbolic spaces $\mathbf{B}^2$ and $\mathbf{D}^2$, we use the following upper halfspace models, $\mathbf{CH}^2$ and $\mathbf{RH}^2$, respectively.

$$\mathbf{CH}^2 = \left(\mathbf{R}_+ \times \mathbf{R}^3, \frac{1}{4y^2}dy^2 + \frac{1}{y}(d\eta^2 + d\xi^2) + \frac{1}{y^2}|dx + \eta d\xi - \xi d\eta|^2\right),$$

$$\mathbf{RH}^2 = \left(\mathbf{R}_+ \times \mathbf{R}, \frac{dY^2 + dX^2}{Y^2}\right),$$

where $(y, \eta, \xi, x) \in \mathbf{R}_+ \times \mathbf{R}^3$ and $(Y, X) \in \mathbf{R}_+ \times \mathbf{R}$. Note that the vector field $\partial/\partial x$ is mapped to the vector field $\partial/\partial y$ via the complex structure of $\mathbf{CH}^2$. Let $\Phi : \mathbf{B}^2 \to \mathbf{CH}^2$ and $\Psi : \mathbf{D}^2 \to \mathbf{RH}^2$ be the Cayley transforms of $\mathbf{CH}^2$ and $\mathbf{RH}^2$, respectively. Assume that a map $\tilde{u} : \mathbf{CH}^2 \to \mathbf{RH}^2$ takes the following form:

$$\tilde{u}(y, \eta, \xi, x) = (f(y), g(x)).$$

Then the harmonic map equation for $\tilde{u}$ is reduced to the following system of ordinary differential equations:

$$\begin{cases} 4y^2 f_{yy}(y) - 4y f_y(y) - y^2 f(y)^{-1}\{4f_y(y)^2 - g_x(x)^2\} = 0, \\ \\ g_{xx}(x) = 0, \end{cases}$$

where $f_y = df/dy, f_{yy} = d^2f/dy^2, g_x = dg/dx, g_{xx} = d^2g/dx^2$.

For any $a, b \in \mathbf{R}$, we can easily verify that $f(y) = (|a|/2\sqrt{2})y$ and $g(x) = ax + b$ are solutions to this system. Therefore we obtain a family of harmonic maps

$$\tilde{u}_{(a,b)} : \mathbf{CH}^2 \ni (y, \eta, \xi, x) \mapsto \left(\frac{|a|}{2\sqrt{2}}y, ax + b\right) \in \mathbf{RH}^2.$$

Let

$$\begin{cases} \tilde{\sigma}_{(a,b)}(y, x) = \left(\dfrac{|a|}{2\sqrt{2}}y, \dfrac{a}{2\sqrt{2}}x + b\right) : \mathbf{RH}^2 \to \mathbf{RH}^2, \\ \\ \sigma_{(a,b)} = \Psi^{-1} \circ \tilde{\sigma}_{(a,b)} \circ \Psi : \mathbf{D}^2 \to \mathbf{D}^2, \\ \\ \tilde{u}(y, \eta, \xi, x) = \tilde{u}_{(2\sqrt{2},0)}(y, \eta, \xi, x) = (y, 2\sqrt{2}x) : \mathbf{CH}^2 \to \mathbf{RH}^2, \\ \\ u_{(a,b)} = \Psi^{-1} \circ \tilde{u}_{(a,b)} \circ \Phi : \mathbf{B}^2 \to \mathbf{D}^2. \end{cases}$$

Note that, if $a \neq 0$, each $u_{(a,b)}$ is smooth up to the boundary except for $(0, 1) \in \partial\mathbf{B}^2$. Thus we need to investigate the boundary behavior of $u_{(a,b)}$ near the point $(0, 1)$. To this end, since any $\sigma_{(a,b)}$ is an isometry of $\mathbf{D}^2$ which fixes the point $(0, 1)$ commonly, it suffices to consider the case where $a = 2\sqrt{2}$ and $b = 0$. Set $u = \Psi^{-1} \circ \tilde{u}_{(2\sqrt{2},0)} \circ \Phi$.

It is then easy to see that u is explicitly given by

$$u(z^1, z^2) = \frac{1}{(1-|z|^2+2|1-z^2|^2)^2+8(\Im(z^2))^2} \\ \times (-8\sqrt{2}|1-z^2|^2\Im(z^2), (1-|z|^2)^2-4|1-z^2|^4+8(\Im(z^2))^2),$$

where $z=(z^1,z^2)$. Now, by a long but straightforward calculation, we can verify that u has the following properties:

(1) $u(0,1)=(0,1)$.

(2) u is Lipschitz continuous at $(0,1)$.

(3) u is not C^1 at $(0,1)$.

(4) u is not a constant map at the ideal boundary.

Therefore the C^1-regularity up to the ideal boundary is an optimal condition for our theorem.

Now, we prove that u satisfies the properties mentioned above. Since the boundary map of $\tilde{u}:\mathbf{CH}^2\to\mathbf{RH}^2$ is given by $\tilde{u}(0,\eta,\xi,x)=(0,2\sqrt{2}x)$, it is clear that $u:\mathbf{B}^2\to\mathbf{D}^2$ is not a constant map at the ideal boundary. Also it is easily verified that

$$(1-|z|^2+2|1-z^2|^2)^2+8(\Im z^2)^2=0,\ |z|\le 1 \Longleftrightarrow z^1=0,\ z^2=1.$$

Claim 1. $\displaystyle\lim_{(z^1,z^2)\to(0,1)} u(z^1,z^2)=(0,1).$

Let

$$z^1=r(\theta^1+\sqrt{-1}\theta^2),\quad z^2=1+r(\theta^3+\sqrt{-1}\theta^4),$$

where $(\theta^1)^2+(\theta^2)^2+(\theta^3)^2+(\theta^4)^2=1$. Note that

$$(z^1,z^2)\ne(0,1)\Longleftrightarrow|\theta|^2:=(\theta^3)^2+(\theta^4)^2\ne 0.$$

Then it is easy to see that

$$1-|z|^2=-r(2\theta^3+r),\quad\text{and}\quad|1-z^2|^2=r^2|\theta|^2.$$

Hence we have

$$\begin{aligned}
&u(z^1, z^2)\\
&\quad = \frac{1}{(2\theta^3 + r - 2r|\theta|^2)^2 + 8(\theta^4)^2}(-8\sqrt{2}r|\theta|^2\theta^4, (2\theta^3 + r)^2 - 4r^2|\theta|^4 + 8(\theta^4)^2)\\
&\quad \longrightarrow (0,1) \quad (r \to 0).
\end{aligned}$$

Consequently, if we define $u(0,1) = (0,1)$, then $u \in C^0(\overline{\mathbf{B}^2}, \overline{\mathbf{D}^2})$.

Claim 2. u is Lipschitz continuous at $(0,1)$.

We shall prove that there exists a positive constant C such that

$$\sup_{z \in \overline{\mathbf{B}^2}, z \neq (0,1)} \frac{\|u(z^1, z^2) - (0,1)\|_{\mathbf{R}^2}}{\|(z^1, z^2) - (0,1)\|_{\mathbf{C}^2}} \leq C.$$

It is immediate from the definition that

$$\begin{aligned}
&u(z^1, z^2) - (0,1)\\
&\quad = \frac{4|1 - z^2|^2}{(1 - |z|^2 + 2|1 - z^2|^2)^2 + 8(\Im z^2)^2}(-2\sqrt{2}\Im z^2, -2|1 - z^2|^2 - (1 - |z|^2).
\end{aligned}$$

Hence we obtain

$$\|u(z^1, z^2) - (0,1)\|_{\mathbf{R}^2} = \frac{4|1 - z^2|^2}{\sqrt{(1 - |z|^2 + 2|1 - z^2|^2)^2 + 8(\Im z^2)^2}}.$$

Using the coordinate used in the proof of Claim1, we have

$$\begin{aligned}
\frac{\|u(z^1, z^2) - (0,1)\|}{\|(z^1, z^2) - (0,1)\|} &= \frac{4|\theta|^2}{\sqrt{(2\theta^3 + r - 2r|\theta|^2)^2 + 8(\theta^4)^2}}\\
&\longrightarrow \frac{2|\theta|^2}{\sqrt{(\theta^3)^2 + 2(\theta^4)^2}} \quad (r \to 0).
\end{aligned}$$

On the other hand, it holds that

$$\frac{2|\theta|^2}{\sqrt{(\theta^3)^2 + 2(\theta^4)^2}} = \sqrt{(\theta^3)^2 + 2(\theta^4)^2} + \frac{2|\theta^3|^2}{\sqrt{(\theta^3)^2 + 2(\theta^4)^2}} \leq \sqrt{(\theta^3)^2 + 2(\theta^4)^2} + |\theta^3|.$$

Thus, for some positive constant C, we have

$$\lim_{(z^1, z^2) \to (0,1)} \frac{\|u(z^1, z^2) - (0,1)\|}{\|(z^1, z^2) - (0,1)\|} \leq C.$$

Hence the function

$$\frac{\|u(z^1, z^2) - (0,1)\|}{\|(z^1, z^2) - (0,1)\|}$$

is bounded on $\overline{\mathbf{B}^2} - \{(0,1)\}$, which implies that u is Lipschitz continuous at $(0,1)$.

Claim 3. u is not C^1 at $(0,1)$.

Let v be the second component of u, that is,

$$v(z^1, z^2) = \frac{(1-|z|^2)^2 - 4|1-z^2|^4 + 8(\Im z^2)^2}{(1-|z|^2 + 2|1-z^2|^2)^2 + 8(\Im z^2)^2}.$$

Then

$$\frac{\partial v}{\partial z^2}(0,1) = \lim_{h \to +0} \frac{v(0,1) - v(0,1-h)}{h} = \lim_{h \to +0} \frac{4(2-h)}{(2+h)^2} = 2.$$

On the other hand, from a straightforward calculation, we obtain

$$\begin{aligned}
&v_{z^2} \times \{(1-|z|^2 + 2|1-z^2|^2)^2 + 8(\Im z^2)^2\}^2 \\
&= 16(\Im z^2)^2[\bar{z^2}\{2|1-z^2|^2 - (1-|z|^2) - (1-|z|^2)^2\} + 2(1-|z|^2)] \\
&\quad -32\sqrt{-1}\Im z^2|1-z^2|^2\{2|1-z^2|^2 + 1 - |z|^2\} \\
&\quad -2(1-|z|^2 + 2|1-z^2|^2)^2\{\bar{z^2}(1-|z|^2)^2 + 4|1-z^2|^2(1-\bar{z^2})\} \\
&\quad +2(2-\bar{z^2})\{(1-|z|^2)^2 - 4|1-z^2|^4\}(1-|z|^2 + 2|1-z^2|^2).
\end{aligned}$$

Here we have

$$\{(1-|z|^2 + 2|1-z^2|^2)^2 + 8(\Im z^2)^2\}^2 = \{(2\theta^3 + r - 2r\|\theta\|^2)^2 + 8(\theta^4)^2\}r^4.$$

Hence

$$\begin{aligned}
v_{z^2}(0, 1-r) &= \frac{-2r(2+r)\{(1-r)(2-r)^2 + 4r\} + 2(1+r)\{(2-r)^2 - 4r^2\}}{r^4(r+2)^4} \\
&\longrightarrow +\infty \quad (r \to 0).
\end{aligned}$$

Therefore, u is not C^1 at $(0,1)$.

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