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Stability and singularities of harmonic maps into spheres

by

Tôru Nakajima

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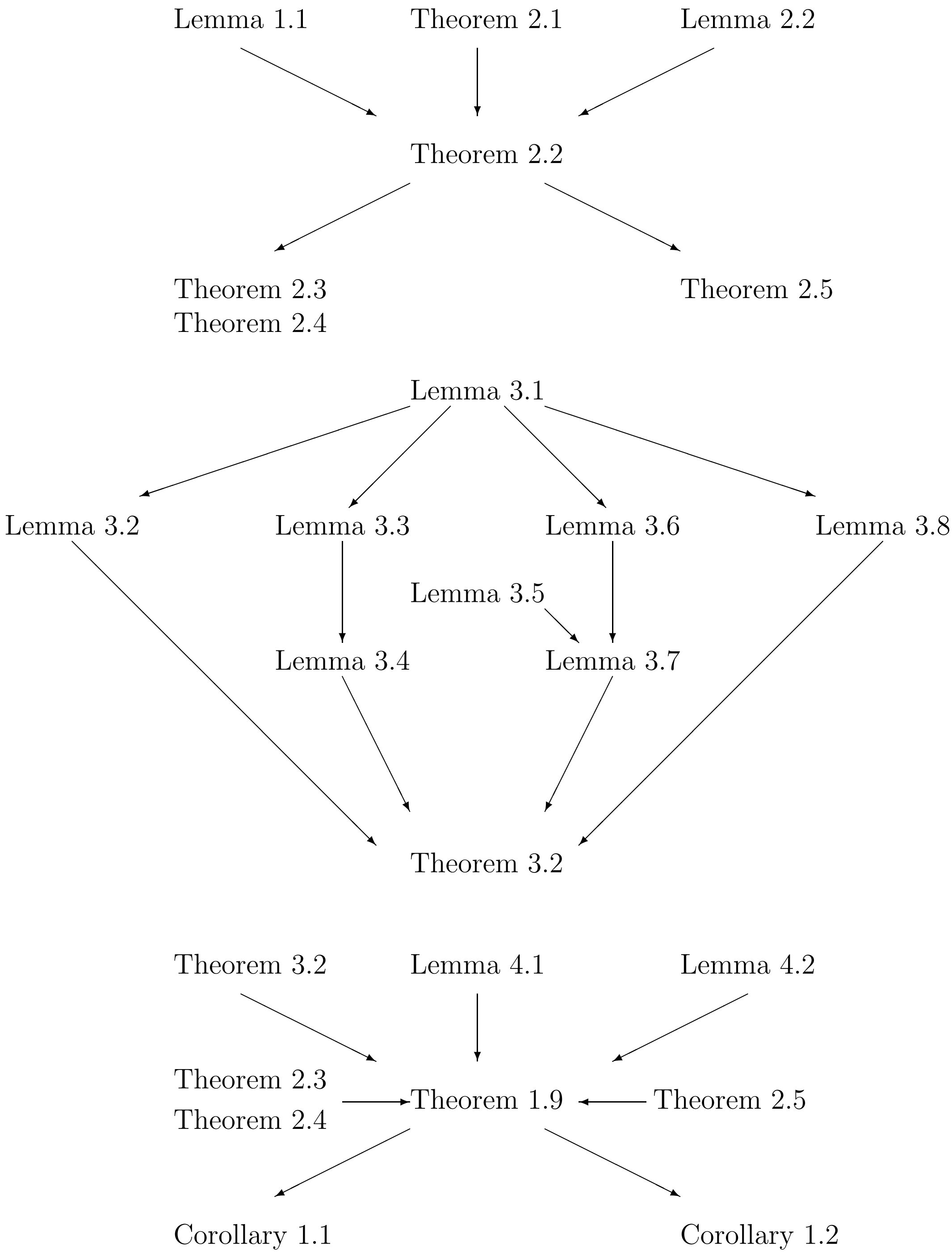

Preface

This monograph is based on the author's doctoral thesis [27] submitted to the Mathematical Institute, Tohoku University in 2002. In that thesis, the author studied the relation between singularities and stability of harmonic maps from domains in the 4-dimensional Euclidean space into 3-spheres. We shall give basic notation and review the history of study for harmonic maps briefly in the first two chapters, especially several known results closely related to ours. In Chapter 3, we shall prove a theorem on energies of harmonic maps between spheres following Ramanathan's paper [29]. This plays an important role in proving the main result in the thesis [27]. In Chpater 4, we shall prove the main theorem following the author's papers [25] and [26].

Harmonic maps are the critical points of the Dirichlet energy functional defined for maps between Riemannian manifolds, and they play very important roles in various context in differential geometry as well as in physics.

Let Ω be a bounded domain with smooth boundary in the m-dimensional Euclidean space $\mathbb{R}^m$ and $\mathbb{S}^n$ denote the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, where m and n are integers greater than or equal to 2. We define the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^{n+1})$ of maps from Ω to $\mathbb{R}^{n+1}$ to be

$$W^{1,2}(\Omega, \mathbb{R}^{n+1}) = \left\{ u = (u^i)_{1 \leq i \leq n+1} \,\middle|\, \begin{array}{l} u^i \in L^2(\Omega, \mathbb{R}),\ \dfrac{\partial u^i}{\partial x^\alpha} \in L^2(\Omega, \mathbb{R}) \\ \text{for } 1 \leq i \leq n+1,\ 1 \leq \alpha \leq m \end{array} \right\},$$

where $\partial u^i / \partial x^\alpha$ is the derivative in distribution sense. The inner product

$(\cdot,\cdot)_{W^{1,2}}$ of $W^{1,2}(\Omega,\mathbb{R}^{n+1})$ is defined by

$$(u,v)_{W^{1,2}} = \int_\Omega u\cdot v\,dx + \int_\Omega \langle \nabla u, \nabla v\rangle\,dx.$$

Here and in what follows, we use the notation:

$$u\cdot v = \sum_{i=1}^{n+1} u^i v^i,\quad |u|^2 = u\cdot u,\quad \nabla u = \left(\frac{\partial u^i}{\partial x^\alpha}\right)_{1\le i\le n+1,\ 1\le\alpha\le m},$$
$$\langle \nabla u,\nabla v\rangle = \sum_{i=1}^{n+1}\sum_{\alpha=1}^{m} \frac{\partial u^i}{\partial x^\alpha}\frac{\partial v^i}{\partial x^\alpha},\quad |\nabla u|^2 = \langle \nabla u,\nabla u\rangle.$$

Also, $W_0^{1,2}(\Omega,\mathbb{R}^{n+1})$ denotes the closure of $C_0^\infty(\Omega,\mathbb{R}^{n+1})$ in $W^{1,2}(\Omega,\mathbb{R}^{n+1})$. Furthermore we define the class $W^{1,2}(\Omega,\mathbb{S}^n)$ of Sobolev maps to be

$$W^{1,2}(\Omega,\mathbb{S}^n) = \{u\in W^{1,2}(\Omega,\mathbb{R}^{n+1})|\ |u(x)|=1 \text{ for almost every } x\in\Omega\}.$$

Also, for any $\zeta\in C^\infty(\partial\Omega,\mathbb{S}^n)$, we define the class $W_\zeta^{1,2}(\Omega,\mathbb{S}^n)$ to be

$$W_\zeta^{1,2}(\Omega,\mathbb{S}^n) = \{u\in W^{1,2}(\Omega,\mathbb{S}^n)\mid u=\zeta \text{ on } \partial\Omega\}.$$

To every element $u\in W^{1,2}(\Omega,\mathbb{S}^n)$, we associate a non-negative real number

$$\mathbf{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx,$$

which is called the *Dirichlet energy* of u, and $\mathbf{E}$ is referred to as the *Dirichlet energy functional*. In the present monograph, we investigate the variational problem of $\mathbf{E}$. More precisely, for a given $\zeta\in C^\infty(\partial\Omega,\mathbb{S}^n)$, we look for minimum points and critical points of $\mathbf{E}$ in the class $W_\zeta^{1,2}(\Omega,\mathbb{S}^n)$. Namely,

Problem 1. Find a map $u_{\min}\in W_\zeta^{1,2}(\Omega,\mathbb{S}^n)$ which satisfies

$$\mathbf{E}(u_{\min}) = \operatorname*{Inf}_{u\in W_\zeta^{1,2}(\Omega,\mathbb{S}^n)} \mathbf{E}(u),$$

and study its regularity.

We call the map $u_{\min}$ an *energy minimizing map*, which is a natural generalization of the notion of harmonic functions. When $W_\zeta^{1,2}(\Omega,\mathbb{S}^n)$ is non-empty, the existence of such maps can be proved by the direct method in the calculus of variations. Since elements of $W^{1,2}(\Omega, \mathbb{S}^n)$ are not necessarily continuous, it is important to study the regularity of energy minimizing maps. In accordance with custom, we use the word *regularity* when we discuss the differentiability of a Sobolev map. Also, the word *smooth* means being infinitely differentiable.

In contrast to harmonic functions, energy minimizing maps are not necessarily continuous. For an energy minimizing map u, a point of $\overline{\Omega}$ is said to be a *singular point* of u if u is discontinuous at a. A typical example of discontinuous energy minimizing maps is given by the map $x/|x| \in W^{1,2}_{\mathrm{id}_{\mathbb{S}^{m-1}}}(\mathbb{B}^m, \mathbb{S}^{m-1})$ ($m \geq 3$) ([21]), where $\mathbb{B}^m$ is an m-dimensional unit open ball in $\mathbb{R}^m$. Obviously, this is discontinuous at 0. On the regularity of energy minimizing maps, the following facts have been already established.

(1) There exists a neighborhood of $\partial\Omega$ in which $u_{\min}$ is smooth (see Schoen-Uhlenbeck [34]).

(2) If $m = 2$, then $u_{\min}$ is smooth in Ω (see Morrey [23]).

(3) If $m \geq 3$, there exists a closed set $\Sigma \subset \Omega$ with $\dim_{\mathcal{H}}(\Sigma) \leq m - 3$ such that $u_{\min}$ is smooth on $\Omega \backslash \Sigma$. Moreover, Σ is a discrete set if $m = 3$ (see Schoen-Uhlenbeck [33]). Here, $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension.

By the very definition, an energy minimizing map is a minimum point of the Dirichlet energy functional, and hence it is also a critical point, that is, a weak solution to the Euler-Lagrange equation of the Dirichlet energy functional. Here, two types of the Euler-Lagrange equations are to be considered. One is obtained by the variation in the target $\mathbb{S}^n$, and the other is obtained by the variation in the domain Ω. The Euler-Lagrange equation obtained by the former variation is given by

$$\int_\Omega \{\langle \nabla u, \nabla \phi \rangle - |\nabla u|^2 u \cdot \phi\} \, dx = 0 \quad \text{for all } \phi \in \mathrm{C}_0^\infty(\Omega, \mathbb{R}^{\mathrm{n}+1}), \tag{1}$$

and that for the latter variation is given by

$$\int_\Omega \left\{ |\nabla u|^2 \mathrm{div}(\eta) - 2\sum_{\alpha,\beta=1}^{m} \frac{\partial \eta^\beta}{\partial x^\alpha}\frac{\partial u}{\partial x^\alpha}\cdot\frac{\partial u}{\partial x^\beta} \right\} dx = 0 \quad \text{for all } \eta \in \mathrm{C}_0^\infty(\Omega, \mathbb{R}^{\mathrm{m}}). \tag{2}$$

We call $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ a *weakly harmonic map* if u satisfies (1); and u is said to be a *stationary harmonic map* if u satisfies both (1) and (2). If u is smooth and satisfies (1), we call u a *smooth harmonic map.* In the present monograph, we use the terminology "harmonic maps" to mean both in the regular sense and in the weak sense. Energy minimizing maps are always stationary harmonic maps, and it is immediate by the definition that stationary harmonic maps are always weakly harmonic maps. The converse, however, is not true.

Now we state a natural problem on harmonic maps.

Problem 2. Given any $\zeta \in C^\infty(\partial\Omega, \mathbb{S}^n)$, find harmonic maps $u \in W_\zeta^{1,2}(\Omega, \mathbb{S}^n)$ and study the regularity of them.

Contrary to the case of energy minimizing maps, weakly harmonic maps may be discontinuous on larger sets. Indeed, Rivière [30] constructed a weakly harmonic map from $\mathbb{B}^3$ into $\mathbb{S}^2$ which is discontinuous everywhere on $\overline{\mathbb{B}^3}$. On the other hand, Bethuel [3] and Evans [9] proved that any stationary harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ is smooth on Ω except for an $\mathcal{H}^{m-2}$-null set, where $\mathcal{H}^{m-2}$ is the $(m-2)$-dimensional Hausdorff measure.

Recall that the harmonicity is a condition on the first variation of the Dirichlet energy functional, and harmonic maps correspond to critical points of the Dirichlet energy functional. Thus we next consider the second variation of the Dirichlet energy functional at a harmonic map. Let $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a weakly harmonic map. Then the second variation of $\mathbf{E}$ at u is given by

$$\delta_u^2 \mathbf{E}(\psi) = \int_\Omega \{|\nabla\psi|^2 - |\nabla u|^2|\psi|^2\}\, dx$$

for $\psi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfying $\psi(x)\cdot u(x) = 0$ for almost every $x \in \Omega$. A harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ is said to be *weakly stable* if

$$\delta_u^2 \mathbf{E}(\psi) \geq 0$$

for all $\psi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^{n+1})$ satisfying $u(x) \cdot \psi(x) = 0$ for almost every $x \in \Omega$. Otherwise, we call u *unstable*. The orthogonality condition means that ψ is a vector field along u.

An energy minimizing map is weakly stable because it is a minimum point of the Dirichlet energy functional. Hong [17] and Hong-Wang [18] proved that a weakly stable stationary harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ is smooth on Ω if $n \geq 3$ except for a closed set Σ with $\dim_{\mathcal{H}}(\Sigma) \leq m - 3$.

There have been many other results on the regularity of harmonic maps. Most of them, however, are concerned with the estimate of the size of the singular set (the set of points of discontinuity).

In 1987, Brezis-Coron-Lieb [5] investigated the behavior of an energy minimizing map from a domain in $\mathbb{R}^3$ into $\mathbb{S}^2$, and showed that its mapping degree around a singular point is equal to $+1$ or -1. Indeed, they analyzed the precise behavior around the singular point. Subsequently, many people made use of the technique of Brezis-Coron-Lieb and developed a deep theory. Typical results are obtained on estimates of the number of singular points (due to Almgren-Lieb [1], Hardt-Lin [14]), and the existence of infinitely many weakly harmonic maps (due to Bethuel-Brezis-Coron [4]). Now we have a natural problem.

Problem 3. Analyze the behavior of an energy minimizing map around its singular points.

The work of Brezis-Coron-Lieb, however, depends heavily on the fact that their target manifold is $\mathbb{S}^2$, which is a Riemann surface. Therefore it seems rather difficult to apply their techniques in higher dimensional cases. Since spheres of different dimensions have distinct geometric properties, we must use different techniques. In the present monograph, we consider the weakly stable stationary harmonic maps from a domain in $\mathbb{R}^4$ into $\mathbb{S}^3$. In this case, due to Hong-Wang [18], Okayasu [28] and Schoen-Uhlenbeck [35], the set of interior singular points of a weakly stable stationary harmonic map is a discrete set.

Now, let us state the main theorem of the present monograph (Theorem 1.9 (1) in Chapter 1). This result was proved in [26].

Theorem 0.1 (Mapping degree around a singular point) *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary. If a weakly stable stationary harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ has a singular point $\xi \in \Omega$, then the mapping degree of u around ξ is equal to $+1$, -1 or 0.*

As a consequence, if a stationary harmonic map from a 4-dimensional domain in $\mathbb{R}^4$ into $\mathbb{S}^3$ has an isolated singular point $\xi \in \Omega$ and if the mapping degree of u at ξ is neither ± 1 nor 0, then u is revealed to be unstable.

Unfortunately, we do not know whether there exists a weakly stable stationary harmonic map having a singular point around which the mapping degree of u is equal to 0. Although we do not know such an example, we cannot exclude the possibility from our proof at present. It is the author's personal opinion that we may exclude the possibility by another consideration. This is one of our future problems.

The assumption of weak stability of u is essential for determining the mapping degree around a singular point in Theorem 0.1. Indeed, for any integer d, there exists a stationary harmonic map $u_d \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ having an isolated singular point at the origin, at which the mapping degree of u is equal to d (see Theorem 1.10 in Chapter 1). Therefore, in the case of maps from a 4-dimensional domain into $\mathbb{S}^3$, the weak stability effects the behavior of a stationary harmonic map around its isolated singular points.

In addition to the mapping degree, the exact behavior of a stable stationary harmonic map around a singular point can be determined if the mapping degree there is equal to $+1$ or -1. To state the results precisely, we now introduce a *rescaled map*. For a point $\xi \in \Omega$ and $0 < \rho < \operatorname{dist}(\xi, \partial\Omega)$ we define the map

$u_{\xi,\rho} \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^{n-1})$ to be

$$u_{\xi,\rho}(x) = u(\xi + \rho x),$$

where $\mathbb{B}^m$ is the unit ball in $\mathbb{R}^m$ with center at the origin. It can be thought that $u_{\xi,\rho}$ emphasizes the behavior of u around ξ if ρ is small. We have the following result (Theorem 1.9 (2) in Chapter 1) (see [26]).

Theorem 0.2 *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Suppose that a weakly stable stationary harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ has an isolated singular point $\xi \in \Omega$, and that its mapping degree around ξ is equal to $+1$ or -1. Then there exist a sequence $\{\rho_j\}_{j=1}^{\infty}$ of positive numbers tending to 0, and a 4×4 constant orthogonal matrix S such that for any multi-index (l_1, l_2, l_3, l_4), where each l_k is a non-negative integer,*

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\left(\frac{\partial}{\partial x^2}\right)^{l_2}\left(\frac{\partial}{\partial x^3}\right)^{l_3}\left(\frac{\partial}{\partial x^4}\right)^{l_4} u_{\xi,\rho_j}$$

converges to

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\left(\frac{\partial}{\partial x^2}\right)^{l_2}\left(\frac{\partial}{\partial x^3}\right)^{l_3}\left(\frac{\partial}{\partial x^4}\right)^{l_4} S\frac{x}{|x|}$$

uniformly on every compact subset of $\mathbb{B}^4 \setminus \{0\}$ as j tends to ∞.

Theorem 0.1 and Theorem 0.2 correspond to Theorem 1.9 in Chapter 1 and the proofs will be given in Chapter 4. The proofs are done by using the so-called *blow-up technique* and a precise analysis of the second variation. Beside these, we are obliged to set a stronger condition on the second variation, called *strict stability*, as follows. Suppose that the singular set of a weakly harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ consists of a finite number of interior points. Then u is said to be *strictly stable* if there exists a positive number $\lambda > 0$ such that

$$\delta_u^2 \mathbf{E}(\psi) \geq \lambda \int_\Omega d(x)^{-2} |\psi|^2 \, dx$$

for any $\psi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^{n+1})$ satisfying $u(x) \cdot \psi(x) = 0$ for almost every $x \in \Omega$. Here $d(x) = \mathrm{dist}(x, \mathrm{Sing}(u))$ denotes the distance from x to $\mathrm{Sing}(u)$, where $\mathrm{Sing}(u)$ is the set of points of discontinuity of u. The weight function $d(x)^{-2}$ reflects the behavior of u near $\mathrm{Sing}(u)$. We denote by $\lambda(u)$ the supremum of such λ. We use the following curious phenomenon (Theorem 1.9 (1)) (see [26]).

Theorem 0.3 *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary. Suppose that a weakly stable stationary harmonic map $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ has an isolated singular point ξ and that the mapping degree of u around ξ is equal to $+1$ or -1. Then u is not strictly stable.*

It is known that an energy minimizing map $x/|x| \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^{m-1})$ satisfies

$$\lambda\Big(\frac{x}{|x|}\Big) = \frac{(m-4)^2}{4} \quad \text{for } m \geq 3.$$

Therefore our method seems neither to be used for higher dimensional cases nor to give a new proof of Brezis-Coron-Lieb's result.

Study of the behavior of a harmonic map around a singular point is very interesting not only from analytical point of view, but also from geometrical point of view. Also, it is important to investigate what type of conditions on harmonic maps influences the behavior of harmonic maps around a singular point. Theorems mentioned above are the first result which clarifies the effect of stability on the behavior of a stationary harmonic map around an isolated singular point in a simple setting. Although our method seems not applicable to the case of maps from an m-dimensional domain into an $(m-1)$-sphere if $m \neq 4$, the author believes that the precise analysis of the second variation is an efficient method in the study of singularity of harmonic maps. Also, the non-strict stability appearing in Theorem 0.3 reflects some special structure of maps from a 4-dimensional domain into a 3-sphere.

The present monograph is organized as follows.

In Chapter 1, we collect several definitions used throughout the monograph and review relevant known results on harmonic maps. We state the main theorem (Theorem 1.9) in the author's thesis [27] and some corollaries derived from it.

In Chapter 2, we prove the energy identity for stationary harmonic maps and introduce the blow-up technique. By virtue of this technique, the proof of Theorem 1.9 is reduced to the study of smooth harmonic maps between 3-spheres. In the course of the discussion of the blow-up, we prove a simple inequality (Lemma 2.1) that relates the upper bound of the Dirichlet energy of a weakly stable stationary harmonic map.

In Chapter 3, we present some relevant results on conformal geometry and discuss the energy of harmonic maps between spheres. Theorems obtained in this Chapter are used to obtain the lower bound of the Dirichlet energy of weakly harmonic maps satisfying some additional conditions.

In Chapter 4, we prove Theorem 1.9 by applying the estimate of the Dirichlet energy and the stability condition. Also, we make several remarks and comments on some of our future problems.

List of Basic Notation

Let Ω be a bounded domain in $\mathbb{R}^m$ and M and N denote smooth compact Riemannian manifolds.

[1] For a point $\xi = (\xi^1, \xi^2, \cdots, \xi^m) \in \mathbb{R}^m$ and a constant $\rho > 0$, we set $\mathbb{B}^m_\rho(\xi) = \{x = (x^1, x^2, \cdots, x^m) \in \mathbb{R}^m | \sum_{\alpha=1}^m (x^\alpha - \xi^\alpha)^2 < \rho^2\}$. We simply write $\mathbb{B}^m$ for $\mathbb{B}^m_1(0)$.

[2] For vectors $x = (x^1, x^2, \cdots, x^m)$, $y = (y^1, y^2, \cdots, y^m) \in \mathbb{R}^m$ and $u = (u^1, u^2, \cdots, u^D)$, $v = (v^1, v^2, \cdots, v^D) \in \mathbb{R}^D$, the notation $\cdot$ stands for the standard Euclidean inner product of $\mathbb{R}^m$ and $\mathbb{R}^D$. That is, $x \cdot y = \sum_{\alpha=1}^m x^\alpha y^\alpha$, $u \cdot v = \sum_{i=1}^D u^i v^i$. Also, $|\ |$ denotes the standard norm $|x| = (x \cdot x)^{1/2}$, $|u| = (u \cdot u)^{1/2}$.

[3] For matrices $A = (A^i_\alpha)$, $B = (B^i_\alpha)$ $(1 \leq \alpha \leq m, 1 \leq i \leq D)$, we define $\langle A, B\rangle$ to be $\langle A, B\rangle = \mathrm{trace}(A^t B) = \sum_{\alpha=1}^m \sum_{i=1}^D A^i_\alpha B^i_\alpha$ and $|A|$ to be $|A| = \langle A, A\rangle^{1/2}$.

[4] For $s, t \in \mathbb{R}$, we denote $s \vee t = \mathrm{Max}\{s, t\}$ and $s \wedge t = \mathrm{Min}\{s, t\}$.

[5] For a map $v : \mathbb{R}^m \to \mathbb{R}^{n+1}$ we define the gradient ∇u of u to be

$$\nabla u = \left(\frac{\partial u^i}{\partial x^\alpha}\right)_{1\leq\alpha\leq m,\ 1\leq i\leq n+1}.$$

[6] $C^\infty(\Omega, \mathbb{R}^{n+1})$ is the space of $\mathbb{R}^{n+1}$-valued infinitely differentiable maps in Ω. $\mathcal{D}(\Omega) = C^\infty_0(\Omega, \mathbb{R}^{n+1})$ is the space of infinitely differentiable maps with compact support in Ω, and $\mathcal{D}'(\Omega)$ is the dual space of $\mathcal{D}(\Omega)$. For $T \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, $\langle T, \phi\rangle$ is the pairing of T and ϕ.

[7] We denote by TM a tangent bundle of M and by T^*M a cotangent bundle. For $p \in M$, T_pM is a tangent space of M at p and T^*_pM a cotangent space of M at p.

[8] $C^\infty(M, N)$ is the set of smooth maps from M into N.

[9] If $\pi : E \to M$ and $\eta : F \to M$ are two vector bundles, we denote by $E \otimes F$ the tensor product of E and F.

[10] Let $\pi : E \to M$ be a smooth vector bundle over M. We denote by $C^\infty(E)$ the vector space of smooth sections of E.

[11] If $u : M \to N$ is a smooth map and $\eta : F \to N$ a smooth vector bundle, we denote by $u^{-1}F$ the pull back bundle.

[12] The Sobolev space $W^{1,2}(\Omega, \mathbb{R}^{n+1})$ is defined to be

$$W^{1,2}(\Omega, \mathbb{R}^{n+1}) = \left\{ u = (u^i)_{1 \le i \le n+1} \;\middle|\; \begin{array}{l} u^i \in L^2(\Omega, \mathbb{R}), \ \dfrac{\partial u^i}{\partial x^\alpha} \in L^2(\Omega, \mathbb{R}) \\ \text{for } 1 \le \alpha \le m, \ \text{ and } 1 \le i \le n+1 \end{array} \right\},$$

where $\partial u^i / \partial x^\alpha$ is the derivative in distribution sense. $W_0^{1,2}(\Omega, \mathbb{R}^{n+1})$ is the closure of $C_0^\infty(\Omega, \mathbb{R}^{n+1})$ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$.

[13] For $\{u_j\}_{j=1}^\infty \subset W^{1,2}(\Omega, \mathbb{R}^{n+1})$ and $u_\infty \in W^{1,2}(\Omega, \mathbb{R}^{n+1})$, by $u_j \to u_\infty$ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$ we mean that u_j converges strongly to u_∞ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$. Also, $u_j \rightharpoonup u_\infty$ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$ means that u_j converges weakly to u_∞ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$.

[14] We define the set $W^{1,2}(\Omega, \mathbb{S}^n)$ of sphere-valued Sobolev maps to be

$$W^{1,2}(\Omega, \mathbb{S}^n) = \{u \in W^{1,2}(\Omega, \mathbb{R}^{n+1}) |\ |u(x)| = 1 \text{ for almost every } x \in \Omega\}.$$

[15] For a map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$, $\xi \in \Omega$ and $0 < \rho < \mathrm{dist}(\xi, \partial\Omega)$, we define the *rescaled map* $u_{\xi,\rho} \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ to be $u_{\xi,\rho}(x) = u(\xi + \rho x)$.

[16] $\mathcal{M}(\Omega)$ is the set of Radon measures on Ω. Given $\mu \in \mathcal{M}(\Omega)$ and any Borel set $A \subset \Omega$, we define another Radon measure $\mu \lfloor A$ to be $(\mu \lfloor A)(B) = \mu(A \cap B)$ for a Borel set B in Ω.

[17] For $\{\mu_j\}_{j=1}^\infty \subset \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, by $\mu_j \rightharpoonup \mu$ in $\mathcal{M}(\Omega)$ we mean that

$$\lim_{j \to \infty} \int_\Omega f \, d\mu_j = \int_\Omega f \, d\mu$$

for any $f \in C_0(\Omega)$.

[18] For $0 \leq s < \infty$, $0 < \delta \leq \infty$, we define the s-dimensional Hausdorff premeasure $\mathcal{H}^s_\delta$ by

$$\mathcal{H}^s_\delta(A) = \mathrm{Inf}\left\{ \sum_{j=1}^{\infty} \omega_s \left(\frac{\mathrm{diam}(C_j)}{2} \right)^2 \,\middle|\, A \subset \bigcup_{j=1}^{\infty} C_j, \quad \mathrm{diam}(C_j) \leq \delta \right\},$$

for each $A \subset \mathbb{R}^m$, where $\omega_s = \pi^{s/2}/\Gamma(s/2+1)$. $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure given by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \mathop{\mathrm{Sup}}_{\delta > 0} \mathcal{H}^s_\delta(A).$$

for each $A \subset \mathbb{R}^m$.

[19] For $1 \leq p < \infty$, we define the p-capacity Cap_p to be

$$\mathrm{Cap}_p(A) = \mathrm{Inf}\left\{ \int_{\mathbb{R}^m} |\nabla \varphi|^p \, dx \,\middle|\, \varphi \in C_0^\infty(\mathbb{R}^m, \mathbb{R}),\ A \subset \mathrm{Int}\{\phi \geq 1\} \right\}$$

for each $A \subset \mathbb{R}^m$, where $\{\phi \geq 1\} = \{x \in \mathbb{R}^m | \phi(x) \geq 1\}$ and $\mathrm{Int}\{\phi \geq 1\}$ is the interior of $\{\phi \geq 1\}$.

Chapter 1

Introduction

§1.1 Smooth harmonic maps

Let M be an m-dimensional compact Riemannian manifold with metric g (with or without boundary), and N be an n-dimensional compact Riemannian manifold with metric h (without boundary). By Nash's embedding theorem, we may assume that N is isometrically embedded in a D-dimensional Euclidean space $\mathbb{R}^D$ for some positive integer D. Let $d\mu_g$ be the canonical measure on M induced by the metric g. We define the Dirichlet energy functional $\mathbf{E} : C^\infty(M,N) \to \mathbb{R}$ to be

$$\mathbf{E}(u) = \frac{1}{2}\int_M |du|^2 \, d\mu_g,$$

where, in terms of a local coordinate system $(x^\alpha)_{1\le\alpha\le m}$ in M and a local coordinate system $(y^i)_{1\le i\le n}$ in N, the *energy density* $|du|^2$ is expressed as

$$|du|^2 = g^{\alpha\beta}(x)h_{ij}(u)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}.$$

Here

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right), \ \ h_{ij} = h\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right),$$

and $(g_{\alpha\beta})^{-1} = (g^{\alpha\beta})$. Hereafter, we use the summation convention of Einstein. Throughout this chapter, repeated Greek indices are understood to be summed from 1 to m, and repeated small Latin indices are to be summed from 1 to n. The manifolds M and N are called the *source* and the *target*, respectively.

We are interested in critical points of $\mathbf{E}$. Let $U \subset \mathrm{Int}(M)$ be a compact set with smooth boundary ∂U, where $\mathrm{Int}(M)$ is the interior of M. For a small $\epsilon > 0$ let $(u_t)_{t\in I}$ be a smooth variation of u satisfying $u_t = u$ in $M\backslash U$ for all $t \in I = (-\epsilon, \epsilon)$. The map $F(x,t) = u_t(x) : M \times I \to N$ is smooth and $u_0 = u$, so we can define a smooth section $V \in C^\infty(u^{-1}TN)$ (here $C^\infty(u^{-1}TN)$ is the vector bundle induced by u) to be

$$V(x) = \frac{d}{dt}u_t(x)\bigg|_{t=0} \qquad \text{for } x \in M.$$

We calculate the first variation of $\mathbf{E}$ at u:

$$\begin{aligned}
\frac{d}{dt}\mathbf{E}(u_t)\bigg|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_M |du_t|^2\, d\mu_g\bigg|_{t=0} \\
&= \frac{1}{2}\int_M \frac{\partial}{\partial t}\langle du_t, du_t\rangle\bigg|_{t=0} d\mu_g = \int_M \langle \nabla_{\frac{\partial}{\partial t}} du_t, du_t\rangle\bigg|_{t=0} d\mu_g,
\end{aligned}$$

where du_t is the differential of u_t along M for a fixed t and $\nabla_{\frac{\partial}{\partial t}}$ is the covariant derivative in $T^*(M \times I) \otimes F^{-1}TN$. Let X be an element of $C^\infty(TM)$. We naturally identify X with an element $(X, 0)$ of $C^\infty(T(M\times I))$. Also we identify $\partial/\partial t \in C^\infty(TM)$ with an element $(0, \partial/\partial t)$ of $C^\infty(T(M \times I))$. Then we have

$$\begin{aligned}
\left(\nabla_{\frac{\partial}{\partial t}} du_t\right)(X) &= \nabla^{F^{-1}TN}_{\frac{\partial}{\partial t}}(du_t(X)) - du_t\left(\nabla^{T(M\times I)}_{\frac{\partial}{\partial t}} X\right) \\
&= \nabla^{F^{-1}TN}_{\frac{\partial}{\partial t}}(dF(X)) - 0 \\
&= \nabla^{F^{-1}TN}_{X}\left(dF\left(\frac{\partial}{\partial t}\right)\right) + dF\left(\left[\frac{\partial}{\partial t}, X\right]\right) \\
&= \nabla^{F^{-1}TN}_{X}\left(\frac{\partial F}{\partial t}\right) + 0.
\end{aligned}$$

Here $\nabla^{F^{-1}TN}$ is the natural connection on $F^{-1}TN$. Thus we have

$$\nabla_{\frac{\partial}{\partial t}} du_t = \nabla^{F^{-1}TN}\frac{\partial F}{\partial t}$$

and

$$\begin{aligned}
\frac{d}{dt}\mathbf{E}(u_t)\bigg|_{t=0} &= \int_M \left\langle \nabla^{F^{-1}TN}\frac{\partial F}{\partial t}, du_t\right\rangle\bigg|_{t=0} d\mu_g \\
&= \int_M \langle \nabla^{u^{-1}TN} V, du\rangle\, d\mu_g.
\end{aligned}$$

For an arbitrary point $p \in M$, we take a normal coordinate system (x^α) with origin at p. For simplicity, we abbreviate

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}, \quad \nabla_\alpha = \nabla_{\frac{\partial}{\partial x^\alpha}}.$$

Then it holds that

$$\nabla_\alpha \partial_\beta = 0, \quad \nabla_\alpha g^{\beta\gamma} = 0 \quad \text{at } p$$

for $1 \leq \alpha, \beta, \gamma \leq m$. Hence we have

$$\begin{aligned}
\langle \nabla^{u^{-1}TN} V, du \rangle &= \sum_{\alpha=1}^{m} \langle \nabla_\alpha^{u^{-1}TN} V, du(\partial_\alpha) \rangle \\
&= \sum_{\alpha=1}^{m} \left\{ \nabla_\alpha (\langle V, du(\partial_\alpha) \rangle) - \langle V, \nabla_\alpha^{u^{-1}TN} (du(\partial_\alpha)) \rangle \right\} \\
&= \sum_{\alpha=1}^{m} \left\{ \nabla_\alpha (\langle V, du(\partial_\alpha) \rangle) - \langle V, \nabla_\alpha du(\partial_\alpha) \rangle \right\}.
\end{aligned}$$

The first term on the right-hand side is the divergence of the vector field $\langle V, \partial_\alpha \rangle \partial_\alpha$, which has a meaning globally. Since V vanishes in $M \backslash U$, the integral of this term vanishes. Consequently, we obtain

$$\left. \frac{d}{dt} \mathbf{E}(u_t) \right|_{t=0} = - \int_M \langle V, \tau(u) \rangle \, d\mu_g, \tag{1.1}$$

where $\tau(u)$ is the smooth section of $u^{-1}TN$ given by

$$\tau(u)(p) = \sum_{\alpha=1}^{m} \nabla_{\boldsymbol{f}_\alpha} du(\boldsymbol{f}_\alpha).$$

Here $(\boldsymbol{f}_\alpha)_{1 \leq \alpha \leq m}$ is an orthonormal basis of T_pM, and $\tau(u)$ is independent of the choice of basis. $\tau(u)$ is said to be the *tension field* of u. We call (1.1) the *first variation formula*. We now define the smooth harmonic map as a critical point of the Dirichlet energy.

Definition 1.1 (Smooth harmonic map) A map $u \in C^\infty(M, N)$ is said to be *smooth and harmonic* if its tension filed $\tau(u)$ vanishes everywhere on M.

Let us give a local expression of the tension field of u. Let (x^α) be a local coordinate system in M and (y^i) be a local coordinate system in N. We denote by ${}^M\Gamma^{\alpha}_{\beta\gamma}$ the Christoffel symbols of the Riemannian connection on M, and by ${}^N\Gamma^i_{jk}$ that on N. First we calculate the components of ∇du:

$$
\begin{aligned}
\nabla_\alpha du &= \nabla_\alpha \left(\frac{\partial u^i}{\partial x^\beta} dx^\beta \otimes \frac{\partial}{\partial y^i} \circ u \right) \\
&= \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} dx^\beta \otimes \frac{\partial}{\partial y^i} \circ u - {}^M\Gamma^\beta_{\alpha\gamma} \frac{\partial u^i}{\partial x^\beta} dx^\gamma \otimes \frac{\partial}{\partial y^i} \circ u \\
&\qquad\qquad + {}^N\Gamma^i_{jk} \frac{\partial u^k}{\partial x^\beta} \frac{\partial u^j}{\partial x^\alpha} dx^\beta \otimes \frac{\partial}{\partial y^i} \circ u \\
&= \left(\frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - {}^M\Gamma^\gamma_{\alpha\beta} \frac{\partial u^i}{\partial x^\gamma} + {}^N\Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} \right) dx^\beta \otimes \frac{\partial}{\partial y^i} \circ u.
\end{aligned}
$$

Therefore we have

$$
(\nabla du)^i_{\alpha\beta} = \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - {}^M\Gamma^\gamma_{\alpha\beta} \frac{\partial u^i}{\partial x^\gamma} + {}^N\Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta}.
$$

This gives a local expression of $\tau(u)$:

$$
\begin{aligned}
\tau(u)^i &= g^{\alpha\beta} (\nabla du)^i_{\alpha\beta} \\
&= g^{\alpha\beta} \left(\frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - {}^M\Gamma^\gamma_{\alpha\beta} \frac{\partial u^i}{\partial x^\gamma} \right) + g^{\alpha\beta}\, {}^N\Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} \\
&= \Delta_M u^i + g^{\alpha\beta}\, {}^N\Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta}.
\end{aligned}
$$

Here Δ_M is the Laplacian on M.

Next, we give another expression of the equation of a smooth harmonic map. We write the equation of a smooth harmonic map in the standard coordinate system of $\mathbb{R}^D$. Let $(u_t)_{t\in I}$ be a smooth variation of u as above, and we define a smooth section $W \in C^\infty(u^{-1}TN)$ to be

$$
W(x) = \left. \frac{d}{dt} u_t(x) \right|_{t=0} \qquad \text{for } x \in M.
$$

We regard u and W as $\mathbb{R}^D$-valued maps and denote

$$
\begin{aligned}
u(x) &= (u^1(x), u^2(x) \cdots, u^D(x)), \\
W(x) &= (W^1(x), W^2(x) \cdots, W^D(x)).
\end{aligned}
$$

Then we have

$$\begin{aligned}
\frac{d}{dt}\mathbf{E}(u_t)\Big|_{t=0} &= \frac{1}{2}\sum_{A=1}^{D}\int_M \frac{\partial}{\partial t}\,\langle du_t^A, du_t^A\rangle\Big|_{t=0}\,d\mu_g \\
&= \sum_{A=1}^{D}\int_M \langle dW^A, du^A\rangle\,d\mu_g = -\sum_{A=1}^{D}\int_M \Delta_M u^A W^A\,d\mu_g \\
&= -\int_M \Delta_M u\cdot W\,d\mu_g,
\end{aligned}$$

where $\Delta_M u = (\Delta_M u^1, \Delta_M u^2, \cdots, \Delta_M u^D)$, and large Latin indices are to be summed from 1 to D. Since we can take an arbitrary $W \in C^\infty(u^{-1}TN)$, u is a smooth harmonic map if and only if the $T_{u(p)}N$-component of $\Delta_M u(p)$ equals 0 for any $p \in M$. For a vector $v \in T_y\mathbb{R}^D$, we denote $v^\top$ the T_yN-component of v and $v^\perp$ the $T_y^\perp N$-component of v, where $T_y^\perp N$ is an orthogonal complement of T_yN in $T_y\mathbb{R}^D$. Let (x^α) be a local coordinate system in M. $\Delta_M u^A$ is expressed as

$$\Delta_M u^A = \frac{1}{\sqrt{\det(g_{\sigma\tau})}}\frac{\partial}{\partial x^\alpha}\left(g^{\alpha\beta}\sqrt{\det(g_{\sigma\tau})}\frac{\partial u^A}{\partial x^\beta}\right) = \frac{1}{\sqrt{\det(g_{\sigma\tau})}}\frac{\partial X_\alpha^A}{\partial x^\alpha}.$$

Here,

$$X_\alpha^A = g^{\alpha\beta}\sqrt{\det(g_{\sigma\tau})}\frac{\partial u^A}{\partial x^\beta},$$

and we define an $\mathbb{R}^D$-valued map of class C^∞ to be $X_\alpha = (X_\alpha^1, X_\alpha^2, \cdots, X_\alpha^D)$. Since

$$X_\alpha = g^{\alpha\beta}\sqrt{\det(g_{\sigma\tau})}\frac{\partial u}{\partial x^\beta} = g^{\alpha\beta}\sqrt{\det(g_{\sigma\tau})}du\left(\frac{\partial}{\partial x^\beta}\right),$$

X_α is a smooth section of $u^{-1}TN$, that is, $X_\alpha(p) \in T_{u(p)}N$ for any $p \in M$. Let $\{Y_1, Y_2, \cdots, Y_n\}$ be a local frame of TN around p. There exist smooth functions $\xi_\alpha^1, \xi_\alpha^2, \cdots, \xi_\alpha^n$ satisfying

$$X_\alpha = \xi_\alpha^i Y_i \circ u$$

around p. Then we have

$$
\begin{aligned}
(\Delta_M u)^\perp &= \left(\frac{1}{\sqrt{\det(g_{\sigma\tau})}}(dX_\alpha)\left(\frac{\partial}{\partial x^\alpha}\right)\right)^\perp \\
&= \frac{1}{\sqrt{\det(g_{\sigma\tau})}}\Big((d\xi^i_\alpha)\Big(\frac{\partial}{\partial x^\alpha}\Big)Y_i \circ u + \xi^i_\alpha (dY_i)\ \Big(du\Big(\frac{\partial}{\partial x^\alpha}\Big)\Big)\Big)^\perp \\
&= \frac{1}{\sqrt{\det(g_{\sigma\tau})}}\xi^i_\alpha \left(dY_i\left(du\left(\frac{\partial}{\partial x^\alpha}\right)\right)\right)^\perp \\
&= \frac{1}{\sqrt{\det(g_{\sigma\tau})}}\xi^i_\alpha \mathbf{A}^N_u \left(Y_i, du\left(\frac{\partial}{\partial x^\alpha}\right)\right) \\
&= \frac{1}{\sqrt{\det(g_{\sigma\tau})}}\mathbf{A}^N_u \left(\sqrt{\det(g_{\sigma\tau})}g^{\alpha\beta} du\left(\frac{\partial}{\partial x^\alpha}\right), du\left(\frac{\partial}{\partial x^\beta}\right)\right) \\
&= g^{\alpha\beta}\mathbf{A}^N_u \left(du\left(\frac{\partial}{\partial x^\alpha}\right), du\left(\frac{\partial}{\partial x^\beta}\right)\right).
\end{aligned}
$$

Here $\mathbf{A}^N$ is the second fundamental form of N. That is, $\mathbf{A}^N_y : T_yN \times T_yN \to T^\perp_y N$ is given by

$$\mathbf{A}^N_y(X, Y) = (\nabla_X Y)^\perp \quad \text{for } X, Y \in T_yN.$$

Consequently, u is a smooth harmonic map if and only if

$$\Delta_M u - \sum_{\alpha=1}^{m} \mathbf{A}_u(du(\boldsymbol{f}_\alpha), du(\boldsymbol{f}_\alpha)) = 0, \tag{1.2}$$

where $(\boldsymbol{f}_\alpha)$ is a local orthonormal frame of TM. In the present thesis, we treat only sphere-valued maps, so we set $N = \mathbb{S}^n \subset \mathbb{R}^{n+1}$. In this case, we obtain the Euler-Lagrange equation as follows. Since $|u| = 1$ in M and $T^\perp_{u(p)}\mathbb{S}^n$ is spanned by $u(p)$, we have

$$
\begin{aligned}
(\Delta_M u)^\top &= \Delta_M u - (\Delta_M u \cdot u)u \\
&= \Delta_M u - \sum_{A=1}^{n+1}(u^A \Delta_M u^A)u \\
&= \Delta_M u - \sum_{A=1}^{n+1}\{\mathrm{div}_M(u^A \mathrm{grad}_M u^A) - |\mathrm{grad}_M u^A|^2\}u \\
&= \Delta_M u + |\mathrm{grad}_M u|^2 u,
\end{aligned}
$$

where div_M is the divergence on M, grad_M is the gradient on M,

$$|\mathrm{grad}_M u^A|^2 = g(\mathrm{grad}_M u^A, \mathrm{grad}_M u^A)$$

for every $1 \leq A \leq n+1$, and

$$|\mathrm{grad}_M u|^2 = \sum_{A=1}^{n+1} |\mathrm{grad}_M u^A|^2.$$

Consequently we have the Euler-Lagrange equation

$$\Delta_M u + |\mathrm{grad}_M u|^2 u = 0 \quad \text{in } M \tag{1.3}$$

of a smooth harmonic map u into spheres.

We give some examples of smooth harmonic maps into spheres. See [7] for details.

Example 1.1 Let M and N be compact Riemann surfaces. If a map $u \in C^\infty(M, N)$ is holomorphic (or anti-holomorphic), then u is a smooth harmonic map. We can construct smooth harmonic maps from $\mathbb{S}^2$ into $\mathbb{S}^2$ by using this fact. Let $\pi : \mathbb{S}^2 \to \mathbb{C}$ be the stereographic projection from the north pole $x^\infty \in \mathbb{S}^2$, where we set $\pi(x^\infty) = \infty$. For any holomorphic (or anti-holomorphic) function $f : \mathbb{C} \to \mathbb{C}$, we define $u_f \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ to be

$$u_f(x) = (\pi^{-1} \circ f \circ \pi)(x). \tag{1.4}$$

Then u_f is a smooth harmonic map.

Example 1.2 We give an example of a non-constant smooth harmonic map from $\mathbb{S}^3$ into $\mathbb{S}^2$. First we regard $\mathbb{S}^2$ and $\mathbb{S}^3$ as the sets

$$\{(\zeta, t) \in \mathbb{C} \times \mathbb{R} | \ |\zeta|^2 + t^2 = 1\}, \tag{1.5}$$

$$\{(z, w) \in \mathbb{C}^2 | \ |z|^2 + |w|^2 = 1\} \tag{1.6}$$

respectively. We define the map $u_H : \mathbb{S}^3 \to \mathbb{S}^2$, called the *Hopf map*, to be

$$u_H(z, w) = (2z\overline{w}, |z|^2 - |w|^2).$$

It is known that the Hopf map is smooth and harmonic.

§1.2 Existence of harmonic maps

We consider the existence problem of critical points of the Dirichlet energy functional. There are two types of problems. One is called the *homotopy problem*.

Problem 1.1 (Homotopy problem) For any given $v \in C^\infty(M,N)$, does there exist a smooth harmonic map $u \in C^\infty(M,N)$ which is homotopic to v?

There are several affirmative answers to this problem. If the manifolds M and N satisfy one of the following conditions, then the homotopy problem is affirmatively solved.

[1] The sectional curvature of N is everywhere non-positive (see Eells-Sampson [8] for the case $\partial M = \phi$ and Hamilton [12] for the case $\partial M \neq \phi$).

[2] $\dim M = 2$ and the second homotopy group $\pi_2(N)$ of N is equal to $\{0\}$ (see Sacks-Uhlenbeck [31]).

[3] $M = N = \mathbb{S}^m$ for $1 \leq m \leq 7$ (see Smith [38]).

On the other hand, there are few results for the case where $m \geq 3$ and the sectional curvature of N is not non-positive.

Another problem is the *Dirichlet problem.*

Problem 1.2 (Dirichlet problem) Suppose that $\partial M \neq \emptyset$. For any smooth map $\zeta \in C^\infty(\partial M, N)$, does there exist a smooth harmonic map $u \in C^\infty(M,N)$ which coincides with ζ on ∂M?

Although this problem seems to be natural, it does not make sense in general. For example, if $M = \mathbb{B}^m$, $N = \mathbb{S}^{m-1}$ and if $\zeta = \mathrm{id}_{\mathbb{S}^{m-1}}$, a topological argument shows that there is no continuous map from $\mathbb{B}^m$ into $\mathbb{S}^{m-1}$ which coincides with ζ on $\partial\mathbb{B}^m$. Therefore we should consider the problem in a weak sense. In other words, we must treat weak solutions to the Euler-Lagrange equation. In the present thesis, we restrict ourselves to the case where M is (a closure of) a bounded domain Ω in the Euclidean space $\mathbb{R}^m$ and N is the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. We shall write down the Euler-Lagrange equation in this situation. At the same time we extend the domain of the functional $\mathbf{E}$ from $C^\infty(\Omega, \mathbb{S}^n)$ to the Sobolev class $W^{1,2}(\Omega, \mathbb{S}^n)$. We define the class $W^{1,2}(\Omega, \mathbb{S}^n)$ of Sobolev maps to be

$$W^{1,2}(\Omega, \mathbb{S}^n) = \{u \in W^{1,2}(\Omega, \mathbb{R}^{n+1}) |\ |u(x)| = 1 \text{ for almost every } x \in \Omega\},$$

and the Dirichlet energy functional $\mathbf{E}$ on $W^{1,2}(\Omega,\mathbb{S}^n)$ to be

$$\mathbf{E}(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,dx.$$

Here, we use the notation

$$\begin{aligned}&\nabla u=\left(\frac{\partial u^i}{\partial x^\alpha}\right)_{1\le\alpha\le m,\ 1\le i\le n+1},\ \langle\nabla u,\nabla v\rangle=\frac{\partial u^i}{\partial x^\alpha}\frac{\partial v^i}{\partial x^\alpha},\\&|\nabla u|^2=\langle\nabla u,\nabla u\rangle.\end{aligned}$$

We shall look for minimum points and critical points of $\mathbf{E}$. First, we define a minimum point of $\mathbf{E}$, that is to say, an energy minimizing map.

Definition 1.2 (Energy minimizing map) A map $u\in W^{1,2}(\Omega,\mathbb{S}^n)$ is said to be an *energy minimizing map*, if

$$\mathbf{E}(u)\le\mathbf{E}(v)\ \text{ for any } v\in W^{1,2}(\Omega,\mathbb{S}^n) \text{ satisfying } u-v\in W_0^{1,2}(\Omega,\mathbb{R}^{n+1}),$$

where $W_0^{1,2}(\Omega,\mathbb{R}^{n+1})$ is the closure of $C_0^\infty(\Omega,\mathbb{R}^{n+1})$ in $W^{1,2}(\Omega,\mathbb{R}^{n+1})$.

This is a natural generalization of the notion of harmonic functions. Then, the existence, uniqueness and regularity of energy minimizing maps are to be discussed. For $\zeta\in W^{1,2}(\Omega,\mathbb{S}^n)$, we define the class $W_\zeta^{1,2}(\Omega,\mathbb{S}^n)$ to be

$$W_\zeta^{1,2}(\Omega,\mathbb{S}^n)=\{u\in W^{1,2}(\Omega,\mathbb{S}^n)|\ u=\zeta \text{ on } \partial\Omega\}.$$

Problem 1.3 For given $\zeta\in W^{1,2}(\Omega,\mathbb{S}^n)$, find a map $u_{\min}$ satisfying

$$\mathbf{E}(u_{\min})=\operatorname*{Inf}_{u\in W_\zeta^{1,2}(\Omega,\mathbb{S}^n)}\mathbf{E}(u),$$

and study its properties.

By the direct method of calculus of variations, we can prove the existence of such $u_{\min}$. The uniqueness, however, is not guaranteed in general.

Example 1.3 Hardt-Kinderlehrer-Lin [13] constructed a map $\zeta\in W^{1,2}(\mathbb{B}^3,\mathbb{S}^2)$ such that there exists a continuum of energy minimizing maps which coincide with ζ on $\partial\mathbb{B}^3$.

Next, we consider the question of regularity. When $m = 2$, Morrey proved the following fact.

Theorem 1.1 ([23]) *Let Ω be a bounded domain in $\mathbb{R}^2$. Then every energy minimizing map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ is smooth on Ω.*

Remark 1.1 Morrey proved a more general fact. Theorem 1.1 is only a part of his result.

When $m \geq 3$, there exists an energy minimizing map with points of discontinuity. Indeed, Lin [21] proved that the map $x/|x| \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^{m-1})$ is energy minimizing if $m \geq 3$. And hence, we can only expect partial regularity results. We define the *regular set* $\mathrm{Reg}(u)$ of u to be

$$\mathrm{Reg}(u) = \{x \in \overline{\Omega} |\ u \text{ is continuous at } x\}.$$

The complement of $\mathrm{Reg}(u)$ in $\overline{\Omega}$ is said to be the *singular set* and denoted by $\mathrm{Sing}(u)$. We call a point of discontinuity of u a *singular point* of u. Concerning the size of the singular set of an energy minimizing map, Schoen-Uhlenbeck proved the following.

Theorem 1.2 ([33]) *Let Ω be a bounded domain in $\mathbb{R}^m$ and $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be an energy minimizing map. Then the Hausdorff dimension of $\mathrm{Sing}(u) \cap \Omega$ is smaller than or equal to $m - 3$. Moreover, $\mathrm{Sing}(u) \cap \Omega$ is a discrete set if $m = 3$.*

Schoen-Uhlenbeck [34] showed the following regularity result near the boundary.

Theorem 1.3 ([34]) *Let Ω be a bounded domain in $\mathbb{R}^m$ having smooth boundary $\partial\Omega$ and $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be an energy minimizing map. Suppose that the boundary value $u|_{\partial\Omega}$ of u is smooth. Then there exists a neighborhood U of $\partial\Omega$ such that $\mathrm{Sing}(u) \cap U = \phi$.*

In the present monograph, we always assume that the maps under consideration are continuous near the boundary, and we discuss only interior singular points.

We give the definition of weakly harmonic maps, *i.e.*, critical points of the Dirichlet energy functional.

Definition 1.3 (Weakly harmonic maps) A map $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ is said to be *weakly harmonic* if it satisfies

$$\Delta u + |\nabla u|^2 u = 0$$

in distribution sense, that is,

$$\int_\Omega \{\langle \nabla u, \nabla \phi \rangle - |\nabla u|^2 u \cdot \phi\}\, dx = 0 \quad \text{for any } \phi \in \mathrm{C}_0^\infty(\Omega, \mathbb{R}^{\mathrm{n}+1}).$$

In the present monograph, we use the terminology "harmonic maps" to mean both in the regular sense and in the weak sense.

Remark 1.2 Energy minimizing maps are always weakly harmonic. The converse, however, is not true in general. Indeed if $1 \leq m \leq 7$, then the equator map $(x/|x|, 0) \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^m)$ is not energy minimizing, but weakly harmonic ([16], [19]).

The following result on smoothness of weakly harmonic maps is due to Schoen.

Theorem 1.4 ([32]) *Let Ω be a bounded domain in $\mathbb{R}^m$ and $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a weakly harmonic map. Then u is smooth on* $\mathrm{Reg}(u)$.

Although the notion of weakly harmonic maps is a natural interpretation of critical points, the weak harmonicity does not imply the regularity at all. Rivière proved that

Theorem 1.5 ([30]) *Let $\zeta \in C^\infty(\partial\mathbb{B}^3, \mathbb{S}^2)$ be an arbitrary non-constant map. Then there exists a weakly harmonic map $u \in W^{1,2}_\zeta(\mathbb{B}^3, \mathbb{S}^2)$ satisfying $u|_{\partial\mathbb{B}^3} = \zeta$ and* $\mathrm{Sing}(u) = \overline{\mathbb{B}^3}$.

To exclude these pathological examples, we treat harmonic maps of a special type. We consider another type of variation. For any $\eta \in C_0^\infty(\Omega, \mathbb{R}^m)$, we define a variation u^t of u to be $u^t(x) = u(x + t\eta(x))$. Here $t \in \mathbb{R}$ is assumed to be of small absolute value, and u^t should not be confused with the t-th component of u. Since

$$\frac{\partial u^t}{\partial x^\alpha}(x) = \frac{\partial u}{\partial x^\alpha}(x + t\eta(x)) + t\frac{\partial u}{\partial x^\beta}(x + t\eta(x))\frac{\partial \eta^\beta}{\partial x^\alpha}(x),$$

we have

$$\begin{aligned}\mathbf{E}(u^t) = \frac{1}{2}\int_\Omega \Bigg\{ & |\nabla u|^2(x + t\eta(x)) \\ & + 2t\frac{\partial u}{\partial x^\alpha}(x + t\eta(x)) \cdot \frac{\partial u}{\partial x^\beta}(x + t\eta(x))\frac{\partial \eta^\beta}{\partial x^\alpha}(x)\Bigg\}\, dx + o(t).\end{aligned}$$

If $|t|$ is small, then the map $\Psi_t(x) = x + t\eta(x)$ is a C^∞-diffeomorphism from Ω into itself satisfying

$$\frac{\partial \Psi_t^\alpha}{\partial x^\beta} = \delta_{\alpha\beta} + t\frac{\partial \eta^\alpha}{\partial x^\beta},$$

$$\det\left(\frac{\partial x^\alpha}{\partial \Psi_t^\beta}\right) = \det\left(\frac{\partial \xi^\alpha}{\partial \Psi_t^\beta}\right)^{-1} = 1 - t\,\mathrm{div}(\eta) + o(t).$$

If we change variables from x to $\xi = x + t\eta(x)$, then we have

$$\begin{aligned}\mathbf{E}(u^t) = \frac{1}{2}\int_\Omega \Bigg\{ & \left(|\nabla u|^2(\xi) + 2t\frac{\partial u}{\partial x^\alpha}(\xi) \cdot \frac{\partial u}{\partial x^\beta}(\xi)\frac{\partial \eta^\alpha}{\partial x^\beta}(\Psi_t^{-1}(\xi))\right) \\ & \times (1 - t\,\mathrm{div}(\eta)(\Psi_t^{-1}(\xi))\Bigg\} d\xi + o(t).\end{aligned}$$

Consequently, we obtain the first variation formula with respect to u^t:

$$\frac{d}{dt}\mathbf{E}(u^t)\bigg|_{t=0} = \frac{1}{2}\int_\Omega \left(- |\nabla u|^2 \mathrm{div}(\eta) + 2\frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial u}{\partial x^\beta}\frac{\partial \eta^\alpha}{\partial x^\beta}\right) dx.$$

And we define a stationary harmonic map as follows.

Definition 1.4 (Stationary harmonic map) Let $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a weakly harmonic map. u is said to be a *stationary harmonic map* if it satisfies

$$\int_\Omega \left(|\nabla u|^2 \mathrm{div}(\eta) - 2\frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial u}{\partial x^\beta}\frac{\partial \eta^\alpha}{\partial x^\beta}\right) dx = 0 \quad \text{for any } \eta \in C_0^\infty(\Omega, \mathbb{R}^m). \tag{1.7}$$

Remark 1.3 Energy minimizing maps are always stationary harmonic, and stationary harmonic maps are weakly harmonic by its own definition. The converse, however, is not true in general (see Remark 1.4 below). We note that smooth weakly harmonic maps are stationary harmonic maps. Indeed, for any $\eta \in C_0^\infty(\Omega, \mathbb{R}^m)$, we take $\phi = (\nabla u)\eta$ as ϕ in Definition 1.3. Then we have

$$\begin{aligned}
0 &= \int_\Omega \{\langle \nabla u, \nabla((\nabla u)\eta) - |\nabla u|^2 u \cdot (\nabla u)\eta\}\, dx \\
&= \int_\Omega \left\{ \frac{\partial u^i}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial u^i}{\partial x^\beta} \eta^\beta \right) - |\nabla u|^2 u^i \frac{\partial u^i}{\partial x^\alpha} \eta^\alpha \right\} dx \\
&= \int_\Omega \left\{ \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \frac{\partial \eta^\alpha}{\partial x^\beta} + \frac{\partial u^i}{\partial x^\alpha} \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} \eta^\beta \right\} dx \\
&= \int_\Omega \left\{ \frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial u}{\partial x^\beta} \frac{\partial \eta^\alpha}{\partial x^\beta} + \frac{1}{2} \frac{\partial}{\partial x^\beta} (|\nabla u|^2 \eta^\beta) - \frac{1}{2} |\nabla u|^2 \mathrm{div}(\eta) \right\} dx \\
&= -\frac{1}{2} \int_\Omega \left(|\nabla u|^2 \mathrm{div}(\eta) - 2 \frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial u}{\partial x^\beta} \frac{\partial \eta^\alpha}{\partial x^\beta} \right) dx.
\end{aligned}$$

In the third equality, we have used the equation $|u| = 1$.

For the case of stationary harmonic maps, we have a result on partial regularity.

Theorem 1.6 ([3], [9]) *Let Ω be a bounded domain in $\mathbb{R}^m$ and $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a stationary harmonic map. Then the $(m-2)$-dimensional Hausdorff measure of $\mathrm{Sing}(u) \cap \Omega$ is zero: $\mathcal{H}^{m-2}(\mathrm{Sing}(u) \cap \Omega) = 0$.*

Remark 1.4 The singular set $\mathrm{Sing}(u)$ of weakly harmonic map u appearing in Theorem 1.5 is $\overline{\mathbb{B}^3}$. Therefore $\mathcal{H}^1(\mathrm{Sing}(u)) = \infty$ and u is not stationary harmonic by Theorem 1.6.

Theorem 1.6 is a consequence of the following ϵ_0-*regularity lemma.*

Lemma 1.1 (ϵ_0-regularity lemma) ([3],[9]) *There exists a constant $\epsilon_0 > 0$ depending only on m and n satisfying the followings. If a stationary harmonic* *map* $u \in W^{1,2}(\Omega, \mathbb{S}^n)$, $\xi \in \Omega$ *and* $0 < \rho < \operatorname{dist}(\xi, \partial\Omega)$ *satisfy*

$$\rho^{2-m} \int_{\mathbb{B}^m_\rho(\xi)} |\nabla u|^2 \, dx < \epsilon_0^2,$$

then $u \in C^\infty(\mathbb{B}^m_{\rho/2}(\xi), \mathbb{S}^n)$. *Moreover, an inequality*

$$\rho^l \operatorname*{Sup}_{\mathbb{B}^m_{\rho/2}(\xi)} |D^l u| \leq C(l, m, n, \epsilon_0)$$

holds for any $l \in \mathbb{N}$, *where* $C(l, m, n, \epsilon_0)$ *is a positive number depending on* l, m, n, ϵ_0 *but neither on* u, ξ *nor on* ρ.

§1.3 Behavior of a harmonic map around isolated singular points

It is very interesting to study what kind of singularity may occur to harmonic maps. In 1987, Brezis-Coron-Lieb [5] analyzed the singular points of an energy minimizing map from a 3-dimensional domain into $\mathbb{S}^2$. They determined the mapping degree of an energy minimizing map around its singular points. To state their result precisely, we give definitions of the mapping degree of a smooth map and that of a Sobolev map around its isolated singular point.

Definition 1.5 (Mapping degree) Let Σ_1 and Σ_2 be compact connected orientable Riemannian manifolds with the same dimensions. For any $u \in C^1(\Sigma_1, \Sigma_2)$, we define the mapping degree $\deg(u)$ of u to be

$$\deg(u) = \frac{1}{\operatorname{vol}(\Sigma_2)} \int_{\Sigma_1} u^* \omega_{\Sigma_2}.$$

Here ω_{Σ_1} is the volume form of Σ_1, and $\operatorname{vol}(\Sigma_2)$ is the volume of Σ_2.

We can define the mapping degree of $u \in C(\Sigma_1, \Sigma_2)$, not necessarily of class C^1, as follows. Take a sequence $\{u_j\}_{j=1}^\infty \subset C^1(\Sigma_1, \Sigma_2)$ such that $u_j \to u$ $(j \to \infty)$ uniformly on Σ_1, and define $\deg(u)$ to be

$$\deg(u) = \lim_{j \to \infty} \deg(u_j).$$

It is known that the right-hand side is independent of the choice of a sequence $\{u_j\}_{j=1}^\infty$. Moreover, this is independent of metrics of Σ_1, Σ_2. Topological arguments show that $\deg(u)$ is an integer and homotopically invariant. That is, if $u, v \in C(\Sigma_1, \Sigma_2)$ are homotopic, then $\deg(u) = \deg(v)$. Moreover, if Σ_2 is a sphere $\mathbb{S}^n$ then $u, v \in C(\Sigma_1, \mathbb{S}^n)$ are homotopic if and only if $\deg(u) = \deg(v)$.

Next we define the mapping degree of a Sobolev map around its isolated singular point.

Definition 1.6 (Mapping degree around an isolated singular point) Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, Σ be an $(m-1)$-dimensional compact orientable Riemannian manifold, and $u \in W^{1,2}(\Omega, \Sigma)$. Suppose that $\xi \in \mathrm{Sing}(u)$ is an isolated singular point of u. Let ρ be a small positive number such that $\mathbb{B}^m_\rho(\xi) \cap \mathrm{Sing}(u) = \{\xi\}$. Then $u|_{\partial \mathbb{B}^m_\rho(\xi)} \in C(\partial \mathbb{B}^m_\rho(\xi), \Sigma)$. We define the degree $\deg(u, \xi)$ of u around ξ to be

$$\deg(u, \xi) = \deg(u|_{\partial \mathbb{B}_\rho(\xi)}).$$

By the homotopic invariance of $\deg(\cdot)$, this value is independent of small ρ.

Now, we state Brezis-Coron-Lieb's result.

Theorem 1.7 ([5]) *Let Ω be a bounded domain in $\mathbb{R}^3$. Suppose that an energy minimizing map $u \in W^{1,2}(\Omega, \mathbb{S}^2)$ has an isolated singular point $\xi \in \mathrm{Sing}(u) \cap \Omega$.*
(1) *Then the mapping degree $\deg(u, \xi)$ is equal to $+1$ or -1.*
(2) *Furthermore, there exists a 3×3 constant orthogonal matrix S such that u behaves like $S(x-\xi)/|x-\xi|$ around ξ in the following sense. For any multi-index (l_1, l_2, l_3), where each l_k is a non-negative integer, the derivative of the rescaled map $u_{\xi,\rho}$*

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1} \left(\frac{\partial}{\partial x^2}\right)^{l_2} \left(\frac{\partial}{\partial x^3}\right)^{l_3} u_{\xi,\rho}$$

converges to

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1} \left(\frac{\partial}{\partial x^2}\right)^{l_2} \left(\frac{\partial}{\partial x^3}\right)^{l_3} S \frac{x}{|x|}$$

uniformly on every compact subset of $\mathbb{B}^3 \backslash \{0\}$ as ρ tends to 0.

Remark 1.5 This is a complete characterization of singular points of energy minimizing maps from a 3-dimensional domain into $\mathbb{S}^2$. We cannot expect the same result for weakly harmonic maps. By Smith's result [38], for any $d \in \mathbb{Z}-\{0\}$, there exists a non-constant smooth harmonic map $v_d \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ satisfying $\deg(v_d) = d$. If we define a map $u_d \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ to be $u_d(x) = v_d(x/|x|)$, then u_d is a weakly harmonic map such that $\mathrm{Sing}(u_d) = \{0\}$ and $\deg(u_d, 0) = d$.

Remark 1.6 Let ζ be an element of $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ satisfying $\deg(\zeta) \neq \pm 1$. Then, from Theorem 1.7, $\zeta(x/|x|) \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ is not an energy minimizing map. On the other hand, by the direct method of calculus of variations, there exists an energy minimizing map $u \in W^{1,2}_\zeta(\mathbb{B}^3, \mathbb{S}^2)$. By a theorem of elementary topology, u has at least two singular points.

It is natural to ask whether the same result holds in higher dimensional cases. In the present monograph we treat the case of maps from $\Omega \subset \mathbb{R}^4$ into $\mathbb{S}^3$. In this case, we can prove a similar result on non-minimizing stationary harmonic maps satisfying a stability condition.

The stability is a condition on the second variation defined below. First, we calculate the second variation of $\mathbf{E}$. Let $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a weakly harmonic map. For any map $\psi \in W^{1,2}_0 \cap L^\infty(\Omega, \mathbb{R}^{n+1})$ satisfying $\psi(x) \cdot u(x) = 0$ for almost every $x \in \Omega$, we denote $u_t = (u + t\psi)/|u + t\psi|$. Since

$$\begin{aligned}
\frac{du_t}{dt} &= \frac{\psi}{|u+t\psi|} - \frac{(u+t\psi, \psi)}{|u+t\psi|^3}(u+t\psi), \\
\frac{d^2u_t}{dt^2} &= -2\frac{(u+t\psi, \psi)}{|u+t\psi|^3} - \frac{|\psi|^2}{|u+t\psi|^3}(u+t\psi), \\
\left.\frac{du_t}{dt}\right|_{t=0} &= \psi, \quad \left.\frac{d^2u_t}{dt^2}\right|_{t=0} = -|\psi|^2 u,
\end{aligned}$$

we obtain

$$\begin{aligned}
\delta^2_u \mathbf{E}(\psi) = \left.\frac{d^2}{dt^2}\mathbf{E}(u_t)\right|_{t=0} &= \int_\Omega \left\{ \left\langle \nabla \left(\left.\frac{d^2u_t}{dt^2}\right|_{t=0} \right), \nabla u \right\rangle + \left| \nabla \left(\left.\frac{du_t}{dt}\right|_{t=0} \right) \right|^2 \right\} dx \\
&= \int_\Omega \{ \langle -\nabla(|\psi|^2 u), \nabla u \rangle + |\nabla \psi|^2 \} \, dx \\
&= \int_\Omega \left\{ |\nabla \psi|^2 - |\nabla u|^2 |\psi|^2 - 2\psi^A \frac{\partial \psi^A}{\partial x^\alpha} u^B \frac{\partial u^B}{\partial x^\alpha} \right\} dx
\end{aligned}$$

$$= \int_\Omega \{|\nabla\psi|^2 - |\nabla u|^2|\psi|^2\}\, dx.$$

Now, we define the weak stability, instability and strict stability as follows.

Definition 1.7 (Weak stability, instability, strict stability [15], [24]) Let $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a weakly harmonic map.
(1) u is said to be *weakly stable* if

$$\delta_u^2 \mathbf{E}(\psi) \geq 0$$

for any $\psi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^{n+1})$ satisfying $u(x) \cdot \psi(x) = 0$ for almost every $x \in \Omega$. Otherwise u is said to be *unstable.*
(2) Suppose that the singular set $\mathrm{Sing}(u)$ of u consists of a finite number of interior points of Ω. u is said to be *strictly stable* if there exists a constant $\lambda > 0$ satifying

$$\delta_u^2 \mathbf{E}(\psi) = \int_\Omega \{|\nabla\psi|^2 - |\nabla u|^2|\psi|^2\}\, dx \geq \lambda \int_\Omega d(x)^{-2}|\psi|^2\, dx \tag{1.8}$$

for any $\psi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^{n+1})$ satisfying $u(x) \cdot \psi(x) = 0$ for almost every $x \in \Omega$. Here $d(x) = \mathrm{dist}(x, \mathrm{Sing}(u))$. And we define $\lambda(u)$ to be

$$\lambda(u) = \operatorname*{Inf}_{\psi} \frac{\delta_u^2 \mathbf{E}(\psi)}{\displaystyle\int_\Omega d(x)^{-2}|\psi|^2\, dx}.$$

Here, ψ runs over the set

$$\{\psi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^{n+1})|\ \psi(x) \cdot u(x) = 0 \text{ for almost every } x \in \Omega\}.$$

In this notation, u is weakly stable if and only if $\lambda(u) \geq 0$ and u is strictly stable if and only if $\lambda(u) > 0$.

Remark 1.7 The weight function $d(x)^{-2}$ appearing in (1.8) is important to study the local behavior of a harmonic map (see Remark 1.8).

To study the local property of harmonic maps around their isolated singular points, we use a weaker condition than the strict stability.

Definition 1.8 (Local strict stability) Let $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a weakly harmonic map and $\xi \in \mathrm{Sing}(u) \cap \Omega$ be an isolated singular point. u is said to be *locally strictly stable* at ξ if there exists $0 < \rho < \mathrm{dist}(\xi, \partial\Omega)$ satisfying $\mathbb{B}^m_\rho(\xi) \cap \mathrm{Sing}(u) = \{\xi\}$ such that the rescaled map $u_{\xi,\rho} \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ is strictly stable. That is, $\lambda(u_{\xi,\rho}) > 0$.

Remark 1.8 If u is strictly stable, then it is locally strictly stable at any isolated singular point $\xi \in \mathrm{Sing}(u)$. Thus we treat only harmonic maps $u \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ such that $\mathrm{Sing}(u) = \{0\}$ and $\deg(u, 0) = d$. We note that $\lambda(u_{\xi,\sigma}) \geq \lambda(u_{\xi,\rho})$ for two small positive numbers $0 < \sigma < \rho$.

Remark 1.9 In the case of maps from $\mathbb{B}^3$ into $\mathbb{S}^2$, there exist many strictly stable harmonic maps u. Indeed, Mou [24] proved that for any $d \in \mathbb{Z} - \{0\}$, there exists a strictly stable, weakly harmonic map $u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ satisfying $\mathrm{Sing}(u) = \{0\}$ and $\deg(u, 0) = d$.

We study the mapping degree of a stable stationary harmonic map from $\mathbb{B}^4$ into $\mathbb{S}^3$. Before stating results, we remark a partial regularity result on weakly stable stationary harmonic maps from $\mathbb{B}^4$ into $\mathbb{S}^3$.

Theorem 1.8 ([17], [18], [35], [28]) *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain and $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ be a weakly stable, stationary harmonic map. Then* $\mathrm{Sing}(u) \cap \Omega$ *is a discrete set.*

Now, we state the main result of the present monograph which will be proved in Chapter 4.

Theorem 1.9 ([26]) *Let Ω be a bounded domain in $\mathbb{R}^4$ with smooth boundary and $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ be a weakly stable, stationary harmonic map. Suppose that u has an isolated singular point $\xi \in \Omega$ and that u is smooth near the boundary of Ω. Then,*

(1) *The mapping degree* $\deg(u, \xi)$ *of u around ξ is equal to $+1$, -1 or 0. In addition if u is locally strictly stable at ξ, then* $\deg(u, \xi) = 0$.

(2) *If* $\deg(u, \xi) = \pm 1$*, then there exist a sequence $\{\rho_j\}_{j=1}^{\infty}$ of positive numbers*

tending to 0 *and a* 4×4 *constant orthogonal matrix* S *such that for any multi-index* (l_1, l_2, l_3, l_4)*, where each* l_k *is a non-negative integer, the derivative of the rescaled map* u_{ξ,ρ_j}

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\left(\frac{\partial}{\partial x^2}\right)^{l_2}\left(\frac{\partial}{\partial x^3}\right)^{l_3}\left(\frac{\partial}{\partial x^4}\right)^{l_4} u_{\xi,\rho_j}$$

converges to

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\left(\frac{\partial}{\partial x^2}\right)^{l_2}\left(\frac{\partial}{\partial x^3}\right)^{l_3}\left(\frac{\partial}{\partial x^4}\right)^{l_4} S\frac{x}{|x|}$$

uniformly on every compact subset of $\mathbb{B}^4\backslash\{0\}$ *as* j *tends to* ∞.

(3) *If* u *is an energy minimizing map, then* S *is independent of a sequence* $\{\rho_j\}_{j=1}^{\infty}$ *and for any multi-index* (l_1, l_2, l_3, l_4)*, where each* l_k *is a non-negative integer, the derivative of the rescaled map* $u_{\xi,\rho}$

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\left(\frac{\partial}{\partial x^2}\right)^{l_2}\left(\frac{\partial}{\partial x^3}\right)^{l_3}\left(\frac{\partial}{\partial x^4}\right)^{l_4} u_{\xi,\rho}$$

converges to

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\left(\frac{\partial}{\partial x^2}\right)^{l_2}\left(\frac{\partial}{\partial x^3}\right)^{l_3}\left(\frac{\partial}{\partial x^4}\right)^{l_4} S\frac{x}{|x|}$$

uniformly on every compact subset of $\mathbb{B}^4\backslash\{0\}$ *as* ρ *tends to 0.*

We state some corollaries derived from Theorem 1.9.

Corollary 1.1 *Suppose that* $\zeta \in C^\infty(\partial\mathbb{B}^4, \mathbb{S}^3)$ *satisfies* $d = \deg(\zeta) \neq 0$*. Then, for any energy minimizing map* $u \in W^{1,2}_\zeta(\mathbb{B}^4, \mathbb{S}^3)$*,* $\mathrm{Sing}(u)$ *is a finite set and the cardinal number of the singular set is greater than or equal to* $|d|$.

From this corollary, we find that the study of singularity of harmonic maps gives not only the local information but also a global one.

Corollary 1.2 *Suppose that* $\zeta \in C^\infty(\partial\mathbb{B}^4, \mathbb{S}^3)$ *satisfies* $\deg(\zeta) \neq 0$*. Then it holds that* $\lambda(u) = 0$ *for any energy minimizing map* $u \in W^{1,2}_\zeta(\mathbb{B}^4, \mathbb{S}^3)$.

Corollary 1.2 shows that we can get the information of stability on an energy minimizing map only from its boundary value.

In Theorem 1.9, the assumption of the weak stability of u is essential for determination of the mapping degree and asymptotic behavior in (2). Indeed we have the following theorem which will be proved in Chapter 4.

Theorem 1.10 *For any $d \in \mathbb{Z}$, there exists a stationary harmonic map $u_d \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ such that $\operatorname{Sing}(u) = \{0\}$ and $\deg(u_d, 0) = d$.*

Therefore, for the case of maps from 4-dimensional domain into 3-sphere, weak stability influences the behavior of stationary harmonic maps around their singular points.

Chapter 2

Monotonicity and Blow-up

In this chapter, we prove an energy identity of stationary harmonic maps and we introduce an important technique to prove the main theorem, *blow-up*. This technique has been developed in the geometric measure theory and used in geometric variational problems ([11], [36]).

§2.1 Monotonicity identity

First, we prove an energy identity that we call the *monotonicity identity*.

Theorem 2.1 ([36], [37]) *Let Ω be a bounded domain in $\mathbb{R}^m$ and $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a stationary harmonic map. Then it holds that*

$$\rho^{2-m} \int_{\mathbb{B}^m_\rho(\xi)} |\nabla u|^2 \, dx - \sigma^{2-m} \int_{\mathbb{B}^m_\sigma(\xi)} |\nabla u|^2 \, dx = 2 \int_{\mathbb{B}^m_\rho(\xi) \setminus \mathbb{B}^m_\sigma(\xi)} r_\xi^{2-m} \left| \frac{\partial u}{\partial r_\xi} \right|^2 dx \tag{2.1}$$

for any $\xi \in \Omega$ and $0 < \sigma < \rho < \operatorname{dist}(\xi, \partial\Omega)$. Here

$$r_\xi = |x - \xi| \quad \textit{and} \quad \frac{\partial}{\partial r_\xi} = \sum_{\alpha=1}^m \frac{x^\alpha - \xi^\alpha}{r_\xi} \frac{\partial}{\partial x^\alpha}.$$

Proof. For small $\epsilon > 0$, we take a cut-off function $\chi_\epsilon \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying $\chi_\epsilon = 1$ for $0 \le t \le 1 - \epsilon$, $\chi_\epsilon = 0$ for $t \ge 1$, $\chi_\epsilon' \le 0$ in $\mathbb{R}$ and $\chi_\epsilon \to \chi_{(-\infty,1]}$ at every point as $\epsilon \to 0$. Here $\chi_{(-\infty,1]}$ is the indicator function of the interval $(-\infty, 1]$. For any $\sigma < \tau < \rho$, we define a function $\eta_\epsilon \in C^\infty(\Omega, \mathbb{R}^m)$ to be

$$\eta_\epsilon(x) = \chi_\epsilon\left(\frac{r_\xi}{\tau}\right)(x - \xi).$$

Substituting η_ϵ into the equation (1.7) of stationary harmonic maps, we have

$$\begin{aligned}
0 &= \int_\Omega \left(\delta_{\alpha\beta}|\nabla u|^2 - 2\frac{\partial u}{\partial x^\alpha}\cdot\frac{\partial u}{\partial x^\beta}\right)\left(\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)\delta^{\alpha\beta}\right)dx \\
&+ \int_\Omega \left(\delta_{\alpha\beta}|\nabla u|^2 - 2\frac{\partial u}{\partial x^\alpha}\cdot\frac{\partial u}{\partial x^\beta}\right)\left(\chi_\epsilon^{'}\left(\frac{r_\xi}{\tau}\right)\frac{1}{\tau}\frac{x^\alpha-\xi^\alpha}{r_\xi}(x^\beta-\xi^\beta)\right)dx \\
&= (m-2)\int_\Omega |\nabla u|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx \\
&+ \int_\Omega \left(\delta_{\alpha\beta}|\nabla u|^2 - 2\frac{\partial u}{\partial x^\alpha}\cdot\frac{\partial u}{\partial x^\beta}\right)\left(\chi_\epsilon^{'}\left(\frac{r_\xi}{\tau}\right)\frac{r_\xi}{\tau}\frac{x^\alpha-\xi^\alpha}{r_\xi}\frac{x^\beta-\xi^\beta}{r_\xi}\right)dx.
\end{aligned}$$

Using the relations

$$\chi_\epsilon^{'}\left(\frac{r_\xi}{\tau}\right) = -\frac{\tau^2}{r_\xi}\frac{\partial}{\partial\tau}\left(\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)\right),$$

and multiplying both sides by τ^{1-m}, we obtain

$$\begin{aligned}
&(m-2)\tau^{1-m}\int_\Omega |\nabla u|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx \\
&= \tau^{2-m}\int_\Omega \left(\delta_{\alpha\beta}|\nabla u|^2 - 2\frac{\partial u}{\partial x^\alpha}\cdot\frac{\partial u}{\partial x^\beta}\right)\frac{\partial}{\partial\tau}\left(\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)\right)\frac{x^\alpha-\xi^\alpha}{r_\xi}\frac{x^\beta-\xi^\beta}{r_\xi}dx \\
&= \tau^{2-m}\int_\Omega \left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r_\xi}\right|^2\right)\frac{\partial}{\partial\tau}\left(\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)\right)dx \\
&= \tau^{2-m}\frac{\partial}{\partial\tau}\left\{\int_\Omega \left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r_\xi}\right|^2 dx\right)\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx\right\}.
\end{aligned} \tag{2.2}$$

We take the limit as $\epsilon \to 0$ in (2.2) in distribution sense. For any $\varphi(\tau) \in C_0^\infty((\sigma,\rho))$, we have

$$\lim_{\epsilon\to 0}\left\langle (m-2)\tau^{1-m}\int_\Omega |\nabla u|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx,\ \varphi\right\rangle = \left\langle (m-2)\tau^{1-m}\int_{\mathbb{B}_\tau^m(\xi)}|\nabla u|^2\,dx,\ \varphi\right\rangle,$$

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left\langle \tau^{2-m}\frac{\partial}{\partial\tau}\left(\int_\Omega |\nabla u|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx\right),\ \varphi\right\rangle \\
&= -\lim_{\epsilon\to 0}\int_\sigma^\rho\left(\int_\Omega |\nabla u|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx\right)\frac{\partial}{\partial\tau}(\tau^{2-m}\varphi)\,d\tau \\
&= -\int_\sigma^\rho\left(\int_{\mathbb{B}_\rho^m(\xi)}|\nabla u|^2\,dx\right)\frac{\partial}{\partial\tau}(\tau^{2-m}\varphi)\,d\tau \\
&= \left\langle \tau^{2-m}\frac{\partial}{\partial\tau}\left(\int_{\mathbb{B}_\rho^m(\xi)}|\nabla u|^2\,dx\right),\ \varphi\right\rangle,
\end{aligned}$$

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\left\langle 2\tau^{2-m}\frac{\partial}{\partial\tau}\left(\int_\Omega\left|\frac{\partial u}{\partial r_\xi}\right|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx\right),\ \varphi\right\rangle\\
&=-2\lim_{\epsilon\to 0}\int_\sigma^\rho\left(\int_\Omega\left|\frac{\partial u}{\partial r_\xi}\right|^2\chi_\epsilon\left(\frac{r_\xi}{\tau}\right)dx\right)\frac{\partial}{\partial\tau}(\tau^{2-m}\varphi)\,d\tau\\
&=-2\int_\sigma^\tau\left(\int_{\mathbb{B}^m_\tau(\xi)}\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\right)\frac{\partial}{\partial\tau}(\tau^{2-m}\varphi)\,d\tau.
\end{aligned}
$$

With the aid of Fubini's theorem we obtain

$$
\begin{aligned}
&-2\int_\sigma^\tau\left(\int_{\mathbb{B}^m_\tau(\xi)}\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\right)\frac{\partial}{\partial\tau}(\tau^{2-m}\varphi)\,d\tau\\
&=-2\int_{\mathbb{B}^m_\rho(\xi)}\left(\int_{r_\xi\vee\sigma}^\rho\frac{\partial}{\partial\tau}(\tau^{2-m}\varphi)\,d\tau\right)\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\\
&=2\int_{\mathbb{B}^m_\rho(\xi)}(r_\xi\vee\sigma)^{2-m}\varphi(r_\xi\vee\sigma)\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\\
&=2\int_{\mathbb{B}^m_\rho(\xi)\setminus\mathbb{B}^m_\sigma(\xi)}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2\varphi(r_\xi)\,dx\\
&=-2\int_{\mathbb{B}^m_\rho(\xi)\setminus\mathbb{B}^m_\sigma(\xi)}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2\left(\int_{r_\xi}^\rho\frac{\partial\varphi}{\partial\tau}\,d\tau\right)dx\\
&=-2\int_{\mathbb{B}^m_\rho(\xi)}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2\left(\int_{r_\xi\vee\sigma}^\rho\frac{\partial\varphi}{\partial\tau}\,d\tau\right)dx.
\end{aligned}
$$

Again we use Fubini's theorem to have

$$
\begin{aligned}
&-2\int_{\mathbb{B}^m_\rho(\xi)}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2\left(\int_{r_\xi\vee\sigma}^\rho\frac{\partial\varphi}{\partial\tau}\,d\tau\right)dx\\
&=-2\int_\sigma^\rho\left(\int_{\mathbb{B}^m_{\tau(\xi)}}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\right)\frac{\partial\varphi}{\partial\tau}\,d\tau\\
&=2\left\langle\frac{\partial}{\partial\tau}\left(\int_{\mathbb{B}^m_\tau(\xi)}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\right),\ \varphi\right\rangle.
\end{aligned}
$$

And hence the relation

$$
\begin{aligned}
&(2-m)\tau^{1-m}\int_{\mathbb{B}^m_\tau(\xi)}|\nabla u|^2\,dx+\tau^{2-m}\frac{\partial}{\partial\tau}\left(\int_{\mathbb{B}^m_\tau(\xi)}|\nabla u|^2\,dx\right)\\
&=2\frac{\partial}{\partial\tau}\left(\int_{\mathbb{B}^m_\tau(\xi)}r_\xi^{2-m}\left|\frac{\partial u}{\partial r_\xi}\right|^2dx\right)
\end{aligned}
\tag{2.3}
$$

holds. Since both

$$\int_{\mathbb{B}^m_\tau(\xi)} |\nabla u|^2 \, dx \quad \text{and} \quad \int_{\mathbb{B}^m_\tau(\xi)} r_\xi^{2-m} \left| \frac{\partial u}{\partial r_\xi} \right|^2 \, dx$$

are absolutely continuous on $[\sigma, \rho]$ with respect to τ, the relation (2.3) holds for $\mathcal{L}^1$-almost every $\tau \in (\sigma, \rho)$. Therefore it holds that

$$\frac{\partial}{\partial \tau} \left(\tau^{2-m} \int_{\mathbb{B}^m_\tau(\xi)} |\nabla u|^2 \, dx \right) = 2 \frac{\partial}{\partial \tau} \left(\int_{\mathbb{B}^m_\tau(\xi)} r_\xi^{2-m} \left| \frac{\partial u}{\partial r_\xi} \right|^2 \, dx \right) \tag{2.4}$$

for $\mathcal{L}^1$-almost every $\tau \in (\sigma, \rho)$. Integrating both sides of (2.4) with respect to τ from σ to ρ, we have the desired result. ∎

§2.2 Blow-up

We shall prove some consequences derived from the monotonicity identity. In this section, Ω is a bounded domain in $\mathbb{R}^m$. Let $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ be a stationary harmonic map, $\xi \in \Omega$ and $0 < \sigma < \rho < \operatorname{dist}(\xi, \partial\Omega)$. From the monotonicity identity, we have

$$\sigma^{2-m} \int_{\mathbb{B}^m_\sigma(\xi)} |\nabla u|^2 \, dx \le \rho^{2-m} \int_{\mathbb{B}^m_\rho(\xi)} |\nabla u|^2 \, dx.$$

If we replace u by rescaled maps $u_{\xi,\rho}, u_{\xi,\sigma} \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ we have

$$\int_{\mathbb{B}^m} |\nabla u_{\xi,\sigma}|^2 \, dx \le \int_{\mathbb{B}^m} |\nabla u_{\xi,\rho}|^2 \, dx,$$

from which we deduce a uniform bound:

$$\operatorname*{Sup}_{0<\rho<\rho_0} \|u_{\xi,\rho}\|_{W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)} \le C(\rho_0)$$

for $0 < \rho_0 < \operatorname{dist}(\xi, \partial\Omega)$. Therefore there exists a sequence $\{\rho_j\}_{j=1}^\infty$ of small positive numbers tending to 0 and a map $u_\infty \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ satisfying

$$u_{\xi,\rho_j} \rightharpoonup u_\infty \quad \text{in } W^{1,2}(\mathbb{B}^m, \mathbb{R}^{n+1}).$$

u_∞ may depend on the sequence $\{\rho_j\}_{j=1}^\infty$. We call this procedure *blow-up* and u_∞ the *blow-up limit.* Since the sequence

$$\left\{ \rho_j^{2-m} \int_{\mathbb{B}^m_{\rho_j}(\xi)} |\nabla u|^2 \, dx \right\}_{j=1}^\infty$$

is monotone decreasing, there exists the limit

$$L = \lim_{j \to 0} \rho_j^{2-m} \int_{\mathbb{B}^m_{\rho_j}(\xi)} |\nabla u|^2 \, dx.$$

From the monotonicity identity, it holds that

$$\rho_j^{2-m} \int_{\mathbb{B}^m_{\rho_j}(\xi)} |\nabla u|^2 \, dx - \rho_k^{2-m} \int_{\mathbb{B}^m_{\rho_k}(\xi)} |\nabla u|^2 \, dx = 2 \int_{\mathbb{B}^m_{\rho_j}(\xi) \setminus \mathbb{B}^m_{\rho_k}(\xi)} r_\xi^{2-m} \left| \frac{\partial u}{\partial r_\xi} \right|^2 dx$$

for any $j, k \in \mathbb{N}$ with $j < k$. Taking the limit as $k \to \infty$, we have

$$\begin{aligned} \rho_j^{2-m} \int_{\mathbb{B}^m_{\rho_j}(\xi)} |\nabla u|^2 \, dx - L &= 2 \int_{\mathbb{B}^m_{\rho_j}(\xi)} r_\xi^{2-m} \left| \frac{\partial u}{\partial r_\xi} \right|^2 dx \\ &= 2 \int_{\mathbb{B}^m} r^{2-m} \left| \frac{\partial u_{\xi,\rho_j}}{\partial r} \right|^2 dx. \end{aligned}$$

Since

$$\frac{\partial u_{\xi,\rho_j}}{\partial r} \rightharpoonup \frac{\partial u_\infty}{\partial r} \quad \text{in } W^{1,2}(\mathbb{B}^m, \mathbb{R}^{n+1}),$$

it follows from the weak lower semi-continuity that

$$\begin{aligned} \int_{\mathbb{B}^m} \left| \frac{\partial u_\infty}{\partial r} \right|^2 dx &\leq \liminf_{j \to \infty} \int_{\mathbb{B}^m} \left| \frac{\partial u_{\xi,\rho_j}}{\partial r} \right|^2 dx \\ &\leq \liminf_{j \to \infty} \int_{\mathbb{B}^m} r^{2-m} \left| \frac{\partial u_{\xi,\rho_j}}{\partial r} \right|^2 dx \\ &= \liminf_{j \to \infty} \int_{\mathbb{B}^m_{\rho_j}(\xi)} r_\xi^{2-m} \left| \frac{\partial u}{\partial r} \right|^2 dx \\ &= \liminf_{j \to \infty} \frac{1}{2} \left(\rho_j^{2-m} \int_{\mathbb{B}^m_{\rho_j}(\xi)} |\nabla u|^2 \, dx - L \right) = 0. \end{aligned}$$

Therefore, we have

$$\frac{\partial u_\infty}{\partial r} = 0 \quad \text{for almost everywhere in } \mathbb{B}^m.$$

A map u_∞ having this property is said to be *homogeneous*. Next, we show that if $n \geq 3$ and if u is weakly stable, then, taking a subsequence if necessary, u_{ξ,ρ_j} converges to u_∞ strongly in $W^{1,2}_{\text{loc}}(\mathbb{B}^m, \mathbb{R}^{n+1})$. More generally, the following fact holds.

Theorem 2.2 ([18]) *Let $n \geq 3$ and $\{u_j\}_{j=1}^{\infty} \subset W^{1,2}(\Omega, \mathbb{S}^n)$ be a sequence of weakly stable stationary harmonic maps. Suppose that u_j's have a uniform energy bound:*

$$E = \operatorname*{Sup}_{j \in \mathbb{N}} \mathbf{E}(u_j) < \infty. \tag{2.5}$$

Then there exist a subsequence, we denote again by $\{u_j\}_{j=1}^{\infty}$, and a weakly stable, stationary harmonic map $u_\infty \in W^{1,2}(\Omega, \mathbb{S}^n)$ such that u_j and ∇u_j converge to u_∞ and ∇u_∞, respectively, in the L^2-norm on every compact subset of Ω as $j \to \infty$.

To prove the theorem above, we need a lemma.

Lemma 2.1 ([18], [28], [35]) *Let Ω be a domain in $\mathbb{R}^m$. Suppose that $n \geq 2$ and that $u \in W^{1,2}(\Omega, \mathbb{S}^n)$ is a weakly stable harmonic map. Then, we have*

$$\frac{n-2}{n} \int_\Omega |\nabla u|^2 |f|^2 \, dx \leq \int_\Omega |\nabla f|^2 \, dx \quad \textit{for } f \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}).$$

Proof of Lemma 2.1. Let $\{e_A\}_{A=1}^{n+1}$ be a constant orthonormal basis of $\mathbb{R}^{n+1}$. We set

$$\psi_{(A)}(x) = f(x)\{e_A - u^A(x)u(x)\}$$

for each A, where $u^A = u \cdot e_A$ and $f \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R})$. The hypothesis of weak stability of u implies $\delta_u^2 \mathbf{E}(\psi_{(A)}) \geq 0$. A computation shows

$$\delta_u^2 \mathbf{E}(\psi_{(A)}) = \int_\Omega P_{(A)}(x) \, dx,$$

where

$$\begin{aligned} P_{(A)}(x) = &|\nabla u|^2 \{2(u^A)^2 - 1\} f^2 + |\nabla u^A|^2 f^2 \\ &+ \{1 - (u^A)^2\} |\nabla f|^2 - \sum_{\alpha=1}^m f \frac{\partial f}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} \{(u^A)^2\}. \end{aligned}$$

Let us sum $P_{(A)}$ with respect to A from 1 to $n+1$. Since

$$\sum_{A=1}^{n+1} \frac{\partial}{\partial x^\alpha} \{(u^A)^2\} = 0 \quad \text{almost everywhere in } \Omega,$$

we have

$$\sum_{A=1}^{n+1} P_{(A)}(x) = n|\nabla f|^2 - (n-2)|\nabla u|^2 f^2 \quad \text{almost everywhere in } \Omega.$$

Therefore we get

$$\sum_{A=1}^{n+1} \delta_u^2 \mathbf{E}(\psi_{(A)}) = \int_\Omega \{n|\nabla f|^2 - (n-2)|\nabla u|^2 f^2\}\, dx \geq 0.$$

This is our desired inequality. ∎

Remark 2.1 We do not know whether the number $(n-2)/n$ appearing in Lemma 2.1 is optimal or not. However, we can prove that this constant is optimal for the case $m = 4$ and $n = 3$. For an energy minimizing map $x/|x| \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^{m-1})$ the inequality in Lemma 2.1 becomes

$$(m-3)\int_{\mathbb{B}^m} |x|^{-2} f^2\, dx \leq \int_{\mathbb{B}^m} |\nabla f|^2\, dx.$$

On the other hand, by Hardy's inequality

$$\frac{(m-2)^2}{4}\int_{\mathbb{B}^m} |x|^{-2} f^2\, dx \leq \int_{\mathbb{B}^m} |\nabla f|^2\, dx$$

holds for any $f \in W_0^{1,2} \cap L^\infty(\mathbb{B}^m, \mathbb{R})$ and the constant $(m-2)^2/4$ is optimal. Therefore if $m = 4$ and $n = 3$, then the constant in Lemma 2.1 is optimal.

The proof of the Theorem 2.2 is organized as follows.
In Step 1, we define the set $\Sigma \subset \Omega$ where the strong convergence may break, and prove that $\Sigma \cap \Lambda$ is a closed set for any compact set $\Lambda \subset \Omega$.
In Step 2, we prove that $\mathcal{H}^{m-2}(\Sigma \cap \Lambda) < \infty$ for any compact set $\Lambda \subset \Omega$.
In Step 3, we prove that $\mathcal{H}^{m-2}(\Sigma \cap \Lambda) = 0$ for any compact set $\Lambda \subset \Omega$.
In Step 4, we prove the strong convergence in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n+1})$.
In Step 5, we prove that the limit map is weakly stable stationary harmonic.

Proof of Theorem 2.2.

Step 1. By (2.5), we may assume that u_j converges to the map $u_\infty \in W^{1,2}(\Omega, \mathbb{S}^n)$ weakly in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$. We define the set $\Sigma \subset \Omega$ to be

$$\Sigma = \bigcap_{R>0} \left\{ \xi \in \Omega \;\middle|\; \liminf_{j\to\infty} R^{2-m} \int_{\mathbb{B}^m_R(\xi)} |\nabla u_j|^2\, dx \geq \epsilon_0^2 \right\}.$$

Here, ϵ_0 is the same constant as in Lemma 1.1, from which we may assume that

$$u_j \to u_\infty \text{ as } j \to \infty \text{ in } C^2_{\mathrm{loc}} \cap W^{1,2}_{\mathrm{loc}}(\Omega \backslash \Sigma, \mathbb{R}^{n+1}). \tag{2.6}$$

In what follows, we denote by Λ an arbitrary compact subset of Ω. We shall prove that $\Sigma \cap \Lambda$ is a closed set. Suppose that a sequence $\{\xi_k\}_{k=1}^{\infty} \subset \Sigma \cap \Lambda$ converges to a point $\xi \in \Lambda$. Since $\xi_k \in \Sigma$, we have

$$\liminf_{j\to\infty} R^{2-m} \int_{\mathbb{B}^m_R(\xi_k)} |\nabla u_j|^2 \, dx \geq \epsilon_0^2 \quad \text{for any } R > 0. \tag{2.7}$$

For any fixed $R > 0$, we set $R_k = R - |\xi - \xi_k|$. Since ξ_k converges to ξ, R_k is positive and $\mathbb{B}^m_{R_k}(\xi_k) \subset \mathbb{B}^m_R(\xi)$ for sufficiently large $k \in \mathbb{N}$. And hence, we have

$$\begin{aligned} R^{2-m} \int_{\mathbb{B}^m_R(\xi)} |\nabla u_j|^2 \, dx &\geq R^{2-m} \int_{\mathbb{B}^m_{R_k}(\xi_k)} |\nabla u_j|^2 \, dx \\ &\geq \left(\frac{R}{R_k}\right)^{2-m} R_k^{2-m} \int_{\mathbb{B}^m_{R_k}(\xi_k)} |\nabla u_j|^2 \, dx \end{aligned}$$

Taking account of (2.7), we have

$$\liminf_{j\to\infty} R^{2-m} \int_{\mathbb{B}^m_R(\xi)} |\nabla u_j|^2 \, dx \geq \left(\frac{R}{R_k}\right)^{2-m} \epsilon_0^2$$

for sufficiently large $k \in \mathbb{N}$. Taking the limit as $k \to \infty$, we obtain

$$\liminf_{j\to\infty} R^{2-m} \int_{\mathbb{B}^m_R(\xi)} |\nabla u_j|^2 \, dx \geq \epsilon_0^2.$$

Therefore $\xi \in \Sigma \cap \Lambda$, and $\Sigma \cap \Lambda$ is a closed set.

Step 2. Let us estimate the $\mathcal{H}^{m-2}$-measure of $\Sigma \cap \Lambda$. By Vitali's covering lemma ([10], [36]), for any $\delta > 0$, there exists a finite disjoint collection $\{\overline{\mathbb{B}^m_{R_k}(\xi_k)}\}_{k=1}^K$ of closed balls with centers in $\Sigma \cap \Lambda$, satisfying

$$\Sigma \cap \Lambda \subset \bigcup_{k=1}^K \overline{\mathbb{B}^m_{5R_k}(\xi_k)},$$

and $0 < R_k < \delta \wedge \mathrm{dist}(\xi_k, \partial\Omega)$. Since $\xi_k \in \Sigma$,

$$R_k^{2-m} \int_{\mathbb{B}^m_{R_k}(\xi_k)} |\nabla u_j|^2 \, dx \geq \frac{\epsilon_0^2}{2}$$

for any $1 \leq k \leq K$ and sufficiently large $j \in \mathbb{N}$. For such j, we have

$$\begin{aligned}
\sum_{k=1}^{K} \omega_{m-2}(5R_k)^{m-2} &= 5^{m-2}\omega_{m-2} \sum_{k=1}^{K} R_k^{m-2} \\
&\leq 5^{m-2}\omega_{m-2} \sum_{k=1}^{K} \frac{2}{\epsilon_0^2} \int_{\mathbb{B}^m_{R_k}(\xi_k)} |\nabla u_j|^2 \, dx \\
&= \frac{2 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \int_{\bigcup_{k=1}^{K} \mathbb{B}^m_{R_k}(\xi_k)} |\nabla u_j|^2 \, dx \\
&\leq \frac{2 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \int_{\Omega} |\nabla u_j|^2 \, dx \\
&\leq \frac{2 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \operatorname*{Sup}_{j \in \mathbb{N}} \mathbf{E}(u_j) < \infty.
\end{aligned}$$

Therefore $\mathcal{H}^{m-2}(\Sigma \cap \Lambda)$ is finite.

Step 3. Since $\mathcal{H}^{m-2}(\Sigma \cap \Lambda) < \infty$, we have $\mathrm{Cap}_2(\Sigma \cap \Lambda) = 0$ (see [10], 4.7.2. Theorem 3). That is, for any $\epsilon > 0$, there exists a function $\varphi_\epsilon \in C_0^\infty(\Omega, \mathbb{R})$ satisfying

$$\Sigma \cap \Lambda \subset \mathrm{Int}\{x \in \Omega |\ \varphi_\epsilon(x) = 1\} \quad \text{and} \quad \int_\Omega |\nabla \varphi_\epsilon|^2 \, dx < \epsilon. \tag{2.8}$$

For any $\xi \in \Sigma \cap \Lambda$, we can take by (2.8) a small constant $\delta(\xi) > 0$ such that

$$\varphi_\epsilon \geq \frac{1}{2} \quad \text{in } \mathbb{B}^m_{\delta(\xi)}(\xi).$$

Since $\Sigma \cap \Lambda$ is compact, there exists a set of a finite number of points $\{\xi_k\}_{k=1}^{K(\epsilon)} \subset \Sigma \cap \Lambda$ satisfying

$$\Sigma \cap \Lambda \subset \bigcup_{k=1}^{K(\epsilon)} \mathbb{B}^m_{\frac{1}{5}\delta(\xi_k)}(\xi_k).$$

By using Vitali's covering lemma, changing indices if necessarily, we may assume that $\{\overline{\mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}(\xi_l)}\}_{l=1}^{L(\epsilon)}$ is disjoint for a subset $\{\xi_l\}_{l=1}^{L(\epsilon)} \subset \{\xi_k\}_{k=1}^{K(\epsilon)}$ and that

$$\Sigma \cap \Lambda \subset \bigcup_{l=1}^{L(\epsilon)} \overline{\mathbb{B}^m_{\delta(\xi_l)}(\xi_l)}. \tag{2.9}$$

By the definition of Σ, for sufficiently large $J(\epsilon) \in \mathbb{N}$,

$$\left(\frac{\delta(\xi_l)}{5}\right)^{2-m} \int_{\mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}} |\nabla u_j|^2 \, dx \geq \frac{\epsilon_0^2}{2} \tag{2.10}$$

for any $1 \leq l \leq L(\epsilon)$ and $j \geq J(\epsilon)$. Combining (2.9) and (2.10), we have for $j \geq J(\epsilon)$,

$$\begin{aligned}
\mathcal{H}^{m-2}_{\epsilon}(\Sigma \cap \Lambda) &\leq \omega_{m-2} \sum_{l=1}^{L(\epsilon)} (\delta(\xi_l))^{m-2} \\
&= 5^{m-2}\omega_{m-2} \sum_{l=1}^{L(\epsilon)} \left(\frac{\delta(\xi_l)}{5}\right)^{m-2} \\
&\leq 5^{m-2}\omega_{m-2} \sum_{l=1}^{L(\epsilon)} \frac{2}{\epsilon_0^2} \int_{\mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}(\xi_l)} |\nabla u_j|^2 \, dx.
\end{aligned}$$

It follows from (2.8) that

$$\int_{\mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}(\xi_l)} |\nabla u_j|^2 \, dx \leq 4 \int_{\mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}(\xi_l)} |\nabla u_j|^2 |\varphi_\epsilon|^2 \, dx.$$

Combining this with Lemma 2.1, we obtain

$$\begin{aligned}
\mathcal{H}^{m-2}_{\epsilon}(\Sigma \cap \Lambda) &\leq 5^{m-2}\omega_{m-2} \sum_{l=1}^{L(\epsilon)} \frac{8}{\epsilon_0^2} \int_{\mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}(\xi_l)} |\nabla u_j|^2 |\varphi|^2 \, dx \\
&= \frac{8 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \int_{\bigcup_{l=1}^{L(\epsilon)} \mathbb{B}^m_{\frac{1}{5}\delta(\xi_l)}(\xi_l)} |\nabla u_j|^2 |\varphi_\epsilon|^2 \, dx \\
&\leq \frac{8 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \int_\Omega |\nabla u_j|^2 |\varphi|^2 \, dx \\
&\leq \frac{8 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \cdot \frac{n}{n-2} \int_\Omega |\nabla \varphi_\epsilon|^2 \, dx \\
&\leq \frac{8 \cdot 5^{m-2}\omega_{m-2}}{\epsilon_0^2} \cdot \frac{n}{n-2} \epsilon.
\end{aligned}$$

Taking the limit as $\epsilon \to 0$, we have the desired result.

Step 4. Since u_j converges to u_∞ weakly in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$, there exists a Radon measure $\nu \in \mathcal{M}(\Omega)$ satisfying

$$|\nabla u_j|^2 dx \to |\nabla u_\infty|^2 dx + \nu \quad \text{in } \mathcal{M}(\Omega).$$

We set $\mu = |\nabla u_\infty|^2 dx + \nu$. From (2.6),

$$\lim_{j \to \infty} \int_U |\nabla u_j|^2 \, dx = \int_U |\nabla u|^2 \, dx$$

for any open set $U \subset\subset \Lambda \backslash \Sigma$. Therefore we have

$$\text{Support}(\nu \lfloor \Lambda) \subset \Sigma \cap \Lambda.$$

We shall prove that $\nu \lfloor \Sigma = 0$ as an element of $\mathcal{M}(\Omega)$.

For any $\xi \in \Omega$ and $0 < \sigma < \rho < \text{dist}(\xi, \partial\Omega)$, we take a small constant $0 < \epsilon < \rho - \sigma$. By the monotonicity identity, we have

$$\begin{aligned}
\sigma^{2-m}\mu(\mathbb{B}^m_{\sigma(\xi)}) &\leq \sigma^{2-m} \liminf_{j\to\infty} \int_{\mathbb{B}^m_\sigma(\xi)} |\nabla u_j|^2 \, dx \\
&\leq (\rho-\epsilon)^{2-m} \liminf_{j\to\infty} \int_{\mathbb{B}^m_{\rho-\epsilon}(\xi)} |\nabla u_j|^2 \, dx \\
&= (\rho-\epsilon)^{2-m} \liminf_{j\to\infty} \int_{\overline{\mathbb{B}^m_{\rho-\epsilon}(\xi)}} |\nabla u_j|^2 \, dx \\
&= (\rho-\epsilon)^{2-m} \mu(\overline{\mathbb{B}^m_{\rho-\epsilon}(\xi)}) \leq (\rho-\epsilon)^{2-m}\mu(\mathbb{B}^m_\rho(\xi)).
\end{aligned}$$

Taking the limit as $\epsilon \searrow 0$, we get

$$\sigma^{2-m}\mu(\mathbb{B}^m_\sigma(\xi)) \leq \rho^{2-m}\mu(\mathbb{B}^m_\rho(\xi)). \tag{2.11}$$

From this, there exists the $(m-2)$-dimensional density function

$$\Theta^{m-2}(\mu, \xi) = \lim_{\sigma\searrow 0} \frac{\mu(\mathbb{B}^m_\sigma(\xi))}{\omega_{m-2}\sigma^{m-2}}$$

with respect to the measure μ for any $\xi \in \Omega$.

We shall give an upper bound of $\Theta(\mu, \cdot)$ on Λ. Define $\rho_0 > 0$ by $\rho_0 = \text{dist}(\Lambda, \partial\Omega)/2$. It follows from (2.11) that

$$\begin{aligned}
\Theta^{m-2}(\mu, \xi) &\leq \frac{\mu(\mathbb{B}^m_{\rho_0}(\xi))}{\omega_{m-2}\rho_0^{m-2}} \leq \frac{\mu(\overline{\mathbb{B}^m_{\rho_0}(\xi)})}{\omega_{m-2}\rho_0^{m-2}} \\
&\leq \frac{1}{\omega_{m-2}\rho_0^{m-2}} \liminf_{j\to\infty} \int_{\overline{\mathbb{B}^m_{\rho_0}(\xi)}} |\nabla u_j|^2 \, dx \leq \frac{1}{\omega_{m-2}\rho_0^{m-2}} E.
\end{aligned}$$

Hence we have a uniform upper bound

$$\operatorname*{Sup}_{a\in\Lambda} \Theta^{m-2}(\mu, \xi) \leq C(m, \Lambda, E),$$

where $C(m, \Lambda, E)$ is a positive number depending only on m, Λ and E. Since $\mathcal{H}^{m-2}(\Sigma \cap \Lambda) = 0$, we have

$$0 \leq \mu(\Sigma \cap \Lambda) \leq C(m, \Lambda, E)\mathcal{H}^{m-2}(\Sigma \cap \Lambda) = 0,$$

and therefore

$$0=\int_{\Sigma\cap\Lambda}|\nabla u_\infty|^2\,dx+\nu(\Sigma\cap\Lambda)=\nu(\Sigma\cap\Lambda).$$

Consequently $\nu\lfloor\Lambda=0$ as an element of $\mathcal{M}(\Omega)$ for any compact set $\Lambda\subset\Omega$. For any open ball $\mathbb{B}^m_\rho(\xi)\subset\subset\Omega$, we have

$$|\nabla u|^2dx(\partial\mathbb{B}^m_\rho(\xi))=0.$$

By the convergence of Radon measures, we obtain

$$\lim_{j\to\infty}\int_{\mathbb{B}^m_\rho(\xi)}|\nabla u_j|^2\,dx=\int_{\mathbb{B}^m_\rho(\xi)}|\nabla u|^2\,dx.$$

This yields

$$\lim_{j\to\infty}\int_{\mathbb{B}^m_\rho(\xi)}|\nabla u_j-\nabla u|^2\,dx=0.$$

By using once more a covering argument, we have the local strong convergence as desired.

Step 5. We can prove that the limit map u_∞ is stationary harmonic by the strong convergence in $W^{1,2}(\Omega,\mathbb{R}^{n+1})$. We are going to prove that u_∞ is weakly stable. For any $\psi\in W^{1,2}_0\cap L^\infty(\Omega,\mathbb{R}^{n+1})$ with $\psi(x)\cdot u_\infty(x)=0$ almost everywhere in Ω, we define $\psi_j\in W^{1,2}_0\cap L^\infty(\Omega,\mathbb{R}^{n+1})$ to be $\psi_j=\psi-(\psi\cdot u_j)u_j$. Then, $\psi_j(x)\cdot u_j(x)=0$ holds for almost every $x\in\Omega$. Since u_j is weakly stable, we have

$$\int_\Omega|\nabla u_j|^2|\psi_j|^2\,dx\le\int_\Omega|\nabla\psi_j|^2\,dx.$$

A simple computation gives

$$\begin{aligned}&\Big|\nabla\psi-\nabla\Big\{(\psi\cdot u_j)u_j\Big\}\Big|^2\\&=|\nabla\psi-(\psi\cdot u_j)\nabla u_j|^2+\sum_{\alpha=1}^m\left(\frac{\partial u_j}{\partial x^\alpha}\cdot\psi\right)^2-\sum_{\alpha=1}^m\left(u_j\cdot\frac{\partial\psi}{\partial x^\alpha}\right)^2.\end{aligned}$$

From this, we have

$$\begin{aligned}&\int_\Omega|\nabla u_j|^2\{|\psi|^2-(u_j\cdot\psi)^2\}\,dx\\&\le\int_\Omega|\nabla\psi-(\psi\cdot u_j)\nabla u_j|^2\,dx+\int_\Omega\sum_{\alpha=1}^m\left(\frac{\partial u_j}{\partial x^\alpha}\cdot\psi\right)^2dx-\int_\Omega\sum_{\alpha=1}^m\left(u_j\cdot\frac{\partial\psi}{\partial x^\alpha}\right)^2dx.\end{aligned}$$

Using the dominated convergence theorem, we have

$$\int_\Omega |\nabla u_\infty|^2 |\psi|^2 \, dx \leq \int_\Omega |\nabla \psi|^2 \, dx.$$

by passing to the limit as $j \to \infty$. Consequently, u_∞ is weakly stable. ∎

Moreover, we can prove (taking a subsequence, if necessarily) that the u_j converges to u_∞ locally uniformly in $\Omega \backslash \mathrm{Sing}(u_\infty)$.

Theorem 2.3 ([32]) *Let $\{u_j\}_{j=1}^\infty \subset W^{1,2}(\Omega, \mathbb{S}^n)$ be a sequence of stationary harmonic maps. Suppose that there exist a stationary harmonic map $u_\infty \in W^{1,2}(\Omega, \mathbb{S}^n)$ such that u_j and ∇u_j converge to u_∞ and ∇u_∞ respectively in the L^2-norm on every compact subset of Ω as j tends to ∞. Then there exists a subsequence $\{u_{j_\nu}\}_{\nu=1}^\infty$ which satisfies the following. For any multi-index $(l_1, \cdots, l_m)$, where each l_k is a non-negative integer,*

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1} \cdots \left(\frac{\partial}{\partial x^m}\right)^{l_m} u_{j_\nu}$$

converges to

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1} \cdots \left(\frac{\partial}{\partial x^m}\right)^{l_m} u_\infty$$

uniformly on every compact subset of $\Omega \backslash \mathrm{Sing}(u_\infty)$ as ν tends to ∞.

Proof. Let $\Lambda \subset \Omega \backslash \mathrm{Sing}(u_\infty)$ be a compact set. Since $u_\infty \in C^\infty(\Omega \backslash \mathrm{Sing}(u_\infty), \mathbb{S}^n)$, for any $\xi \in \Lambda$, there exists a small $R_\xi > 0$ with

$$R_\xi^{2-m} \int_{\mathbb{B}^m_{R_\xi}(\xi)} |\nabla u_\infty|^2 \, dx < \frac{\epsilon_0^2}{2},$$

where ϵ_0 is the constant in Lemma 1.1. The local strong convergence implies

$$\lim_{j \to \infty} R_\xi^{2-m} \int_{\mathbb{B}^m_{R_\xi}(\xi)} |\nabla u_j|^2 \, dx = R_\xi^{2-m} \int_{\mathbb{B}^m_{R_\xi}(\xi)} |\nabla u_\infty|^2 \, dx < \frac{\epsilon_0^2}{2}.$$

Therefore, we can take a large number $J_\xi \in \mathbb{N}$ such that

$$R_\xi^{2-m} \int_{\mathbb{B}^m_{R_\xi}(\xi)} |\nabla u_j|^2 \, dx < \epsilon_0^2 \quad \text{if } j \geq J_\xi.$$

Since Λ is compact, there exists a finite number of points $\xi_1, \xi_2, \cdots, \xi_K \in \Lambda$ such that

$$\Lambda \subset \bigcup_{k=1}^{K} \mathbb{B}^m_{\frac{1}{2}R_k}(\xi_k),$$

where $R_k = R_{\xi_k}$. From Lemma 1.1, we have an inequality

$$\underset{\mathbb{B}^m_{\frac{1}{2}R_k}(\xi_k)}{\mathrm{Sup}} |D^l u_j| \leq \frac{C(l, m, n, \epsilon_0, \Lambda)}{R_k^l}$$

$$\text{if } j \geq J(\Lambda) = \mathrm{Max}\{J_1, J_2 \cdots, J_K\} \quad (J_k = J_{\xi_k}).$$

By Ascoli-Arzelà's theorem, we can extract a subsequence $\{u_j^\Lambda\}_{j=1}^\infty \subset \{u_j\}_{j=1}^\infty$ which converges to u_∞ in $C^k(\Lambda)$ for any non-negative integer k.

Let $\{\Lambda_j\}_{j=1}^\infty$ be an increasing sequence of compact subsets of $\Omega \backslash \mathrm{Sing}(u_\infty)$ satisfying

$$\Omega \backslash \mathrm{Sing}(u_\infty) = \bigcup_{j=1}^{\infty} \Lambda_j.$$

An appropriate subsequence $\{u_j^{\Lambda_j}\}_{j=1}^\infty$ satisfies the assertion. ∎

Now we go back to our problem. Let $\Omega \subset \mathbb{R}^4$ be a bounded domain, $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ be a weakly stable, stationary harmonic map and $\xi \in \mathrm{Sing}(u) \cap \Omega$ be an isolated singular point of u. By Theorem 2.2 and Theorem 2.3, there exist a sequence $\{\rho_j\}_{j=1}^\infty \subset (0, \mathrm{dist}(\xi, \partial\Omega))$ tending to 0 and a weakly stable homogeneous stationary harmonic map $u_\infty \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ which satisfy

$$u_{\xi,\rho_j} \rightharpoonup u_\infty \quad \text{as } j \to \infty \text{ in } W^{1,2}(\mathbb{B}^4, \mathbb{R}^4),$$
$$u_{\xi,\rho_j} \to u_\infty \quad \text{as } j \to \infty \text{ in } W^{1,2}_{\mathrm{loc}}(\mathbb{B}^4, \mathbb{R}^4),$$
$$u_{\xi,\rho_j} \to u_\infty \quad \text{as } j \to \infty \text{ in } C^k_{\mathrm{loc}}(\mathbb{B}^4 \backslash \mathrm{Sing}(u_\infty), \mathbb{R}^4) \quad \text{for any } k \in \mathbb{N} \cup \{0\}.$$

From Theorem 1.8, $\mathrm{Sing}(u_\infty)$ is a discrete set, and the homogeneity of u_∞ implies $\mathrm{Sing}(u_\infty) = \{0\}$. Therefore we have

$$u_{\xi,\rho_j} \to u_\infty \quad \text{as } j \to \infty \text{ in } C_{\mathrm{loc}}(\mathbb{B}^4 \backslash \{0\}, \mathbb{R}^4),$$

and

$$\deg(u,\xi) = \deg(u_{\xi,\rho_j},0) = \deg(u_\infty,0).$$

If u is an energy minimizing map, then u_∞ is independent of a subsequence $\{\rho_j\}_{j=1}^\infty$ due to the following theorem.

Theorem 2.4 ([37]) *Let Ω be a bounded domain of $\mathbb{R}^m$ and N be a real analytic compact Riemannian manifold. Suppose that $u \in W^{1,2}(\Omega, N)$ is an energy minimizing map and $\xi \in \mathrm{Sing}(u)$. If there exists a blow-up limit u_∞ of u at ξ satisfying $\mathrm{Sing}(u_\infty) = \{0\}$, then u_∞ is a unique blow-up limit of u at ξ. That is, for any multi-index $(l_1,\cdots,l_m)$, where each l_k is a non-negative integer,*

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\cdots\left(\frac{\partial}{\partial x^m}\right)^{l_m} u_{\xi,\rho}$$

converges to

$$\left(\frac{\partial}{\partial x^1}\right)^{l_1}\cdots\left(\frac{\partial}{\partial x^m}\right)^{l_m} S\frac{x}{|x|}$$

uniformly on every compact subset of $\mathbb{B}^m\backslash\{0\}$ as ρ tends to 0.

From this, for an energy minimizing map from a 4-dimensional domain Ω into $\mathbb{S}^3$, we need not take a sequence $\{\rho_j\}_{j=1}^\infty$ as in Theorem 2.3.

As a consequence of Theorem 2.3 and Theorem 2.4, to study the degree of a weakly stable stationary harmonic map from a 4-dimensional domain Ω into $\mathbb{S}^3$ around an isolated singular point, we have only to deal with the case $\Omega = \mathbb{B}^4$ and u is a homogeneous map satisfying $\mathrm{Sing}(u) = \{0\}$. Since $u \in C^\infty(\mathbb{B}^4\backslash\{0\}, \mathbb{S}^3)$ in this case, there exists a map $u_0 \in C^\infty(\mathbb{S}^3,\mathbb{S}^3)$ such that $u(x) = u_0(x/|x|)$ for $x \in \mathbb{B}^4\backslash\{0\}$. The Euler-Lagrange equation

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \mathbb{B}^4\backslash\{0\},$$

is then reduced to

$$\Delta_{\mathbb{S}^3} u_0 + |\nabla_{\mathbb{S}^3} u_0|^2 u_0 = 0 \quad \text{in } \mathbb{S}^3.$$

Here $\Delta_{\mathbb{S}^3}$ is the Laplacian on $\mathbb{S}^3$, and $\nabla_{\mathbb{S}^3}$ is the gradient on $\mathbb{S}^3$. Therefore u_0 is a smooth harmonic map from $\mathbb{S}^3$ to itself.

Next, we consider the strict stability. If $\Omega \subset \mathbb{R}^4$ is a bounded domain and $u \in W^{1,2}(\Omega, \mathbb{S}^3)$ is a strictly stable stationary harmonic map, then for any $\xi \in \mathrm{Sing}(u) \cap \Omega$, u is locally strictly stable at ξ. And hence, if $\rho > 0$ is small, the rescaled map $u_{\xi,\rho} \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ is strictly stable and $\lambda(u_{\xi,\rho}) \geq \lambda(u)$. Therefore, it is natural to expect that any blow-up limit $u_\infty \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ of u at ξ is strictly stable and that $\lambda(u_\infty) \geq \lambda(u)$. (For the definition of $\lambda(u)$, see Definition 1.7). Indeed, we have the following result.

Theorem 2.5 *Suppose that $\{u_j\}_{j=1}^\infty \subset W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ is a sequence of strictly stable weakly harmonic maps such that* $\mathrm{Sing}(u_j) = \{0\}$ *and that there exists a constant $\lambda > 0$ satisfying $\lambda(u_j) \geq \lambda$ for any $j \in \mathbb{N}$. If there exists a weakly harmonic map $u_\infty \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^n)$ satisfying* $\mathrm{Sing}(u_\infty) = \{0\}$ *and if u_j converges to u_∞ in $W^{1,2}_{\mathrm{loc}}(\mathbb{B}^m, \mathbb{R}^{n+1})$, then u_∞ is also strictly stable and $\lambda(u_\infty) \geq \lambda$.*

Theorem 2.5 can be proved in the same way as in the Step 5 in the proof of Theorem 2.2, so we omit the proof.

From these considerations, we are lead to the following problem.

Problem 2.1 Let $u_0 \in C^\infty(\mathbb{S}^3, \mathbb{S}^3)$ be a smooth harmonic map. Is a map $u(x) = u_0(x/|x|) \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ a weakly (or strictly) stable stationary harmonic map?

On this problem we need some additional properties of smooth harmonic maps between $\mathbb{S}^3$. No results have been published on complete classification of smooth harmonic maps between $\mathbb{S}^3$. However, we have a partial answer to Problem 2.1 by analyzing the second variation precisely. In the next chapter, we shall prove a very useful result to obtain a lower bound of the Dirichlet energy of smooth harmonic maps between spheres.

Chapter 3

Energies of Smooth Harmonic Maps between Spheres

In this chapter, we shall prove Ramanathan's theorem (Theorem 3.3) on energies of smooth harmonic maps between spheres following his paper [29]. He considered the Dirichlet energies of composition of smooth harmonic maps and conformal diffeomorphisms between spheres. If the source manifold is $\mathbb{S}^2$, then the Dirichlet energy is invariant under composition (see Theorem 3.1 below). However, if the source manifold is $\mathbb{S}^m$ where $m \geq 3$, then the Dirichlet energy decreases in general by composition. Therefore there is a great difference between the case of the domain $\mathbb{S}^2$ and the case of $\mathbb{S}^m$ with $m \geq 3$. Although we do not need here the case of $\mathbb{S}^2$, we discuss both cases for comparison.

We use a notation (ξ^α) as a local coordinate system on $\mathbb{S}^m$ and (x^i) the standard coordinate system of $\mathbb{R}^{m+1}$. Latin indices are understood to be summed from 1 to m and small Latin indices are from 1 to $m+1$ except for the proof of Theorem 3.1.

§3.1 Preliminaries on conformal geometry

First, we define the notion of conformality.

Definition 3.1 Let (M, g) and (N, h) be Riemannian manifolds. An immersion $\varphi \in C^\infty(M, N)$ is said to be *conformal* if there exists a function

$f \in C^\infty(M, \mathbb{R})$ such that

$$\varphi^* h = e^f g \quad \text{in } M.$$

Here, $\varphi^* h$ is the pull-back of the metric h. And we define a set Conf(M) to be

$$\begin{aligned}\mathrm{Conf}(M) = \{\varphi \in C^\infty(M, M)| \\ \varphi \text{ is a conformal diffeomorphism from } M \text{ onto itself}\}.\end{aligned}$$

Conf(M) is a Lie group endowed with the composition of maps as law of multiplication.

We need only Conf($\mathbb{S}^m$). Conf($\mathbb{S}^m$) is isomorphic to a very useful linear group constructed below ([20], [39]). Denote the Lorentz space by $\mathbb{L}^{m+2} = \mathbb{R}^{m+1} \times \mathbb{R}$, which is endowed with the non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{L}^{m+2}} : \mathbb{L}^{m+2} \times \mathbb{L}^{m+2} \to \mathbb{R}$ and where

$$\begin{aligned}\langle v, w \rangle_{\mathbb{L}^{m+2}} = -v^0 w^0 + \sum_{\alpha=1}^{m+1} v^\alpha w^\alpha \\ \text{for } v = {}^t(v^0, v^1, \cdots, v^{m+1}), {}^t(w^0, w^1, \cdots, v^{m+1}) \in \mathbb{L}^{m+2}.\end{aligned}$$

Let $O(1, m+1)$ be

$$\begin{aligned}O(1, m+1) = \{T : \mathbb{L}^{m+2} \to \mathbb{L}^{m+2} \mid T \text{ is a linear transformation satisfying} \\ \langle Tv, Tw \rangle_{\mathbb{L}^{m+2}} = \langle v, w \rangle_{\mathbb{L}^{m+2}} \text{ for } v, w \in \mathbb{L}^{m+2}\}.\end{aligned}$$

It is a Lie group with the composition of linear transformations as law of multiplication. We shall construct a subgroup $\Gamma < O(1, m+1)$ isomorphic to Conf($\mathbb{S}^m$) in the following way. Let C_+ be the subset of $\mathbb{L}^{m+2}$, called the *positive light cone*, defined to be

$$C_+ = \{v \in \mathbb{L}^{m+2} | v^0 > 0,\ \langle v, v \rangle_{\mathbb{L}^{m+2}} = 0\}.$$

Define a subgroup $G < \mathrm{Conf}(\mathbb{S}^m)$ to be

$$G = \{\gamma \in O(1, m+1) |\ \gamma \text{ preserves } C_+ \text{ and satisfies } \det \gamma = 1\}.$$

The next lemma is fundamental for our discussion in this chapter.

Lemma 3.1 $\mathrm{Conf}(\mathbb{S}^m)$ *is isomorphic to* G *as a Lie group.*

See [20] for the proof. We only give here a correspondence. Let $q : \mathbb{S}^m \to C_+$ and $p : C_+ \to \mathbb{S}^m$ be maps defined to be

$$p({}^t(x^1, x^2, \cdots, x^{m+1})) = {}^t(1, x^1, x^2 \cdots, x^{m+1}),$$
$$q({}^t(v^0, v^1, \cdots, v^{m+1})) = {}^t\left(\frac{v^1}{v^0}, \frac{v^2}{v^0}, \cdots, \frac{v^{m+1}}{v^0}\right).$$

Then every element $\gamma \in G$ corresponds to a map $\tilde{\gamma} \in \mathrm{Conf}(\mathbb{S}^{\mathrm{m}})$ defined to be

$$\tilde{\gamma}(x) = p(\gamma \cdot q(x)) \quad \text{for } x \in \mathbb{S}^m.$$

We write the matrix representation of $\gamma \in G$ in terms of the standard basis of $\mathbb{L}^{m+2} \simeq \mathbb{R}^{m+1} \times \mathbb{R}$ as

$$\begin{pmatrix} \theta & c_1 & \cdots & c_{m+1} \\ b^1 & a^1{}_1 & \cdots & a^1{}_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ b^{m+1} & a^{m+1}{}_1 & \cdots & a^{m+1}{}_{m+1} \end{pmatrix} = \begin{pmatrix} \theta & c \\ b & a \end{pmatrix},$$

and we identify γ with this matrix. By the natural embedding

$$SO(m+1) \ni S \to \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \in G,$$

we regard $SO(m+1)$ as a subgroup of G. Then, γ^{-1} has the expression

$$\gamma^{-1} = \begin{pmatrix} \theta & -b^1 & \cdots & -b^{m+1} \\ -c_1 & a^1{}_1 & \cdots & a^{m+1}{}_1 \\ \vdots & \vdots & \ddots & \vdots \\ -c_{m+1} & a^1{}_{m+1} & \cdots & a^{m+1}{}_{m+1} \end{pmatrix} = \begin{pmatrix} \theta & -{}^tb \\ -{}^tc & {}^ta \end{pmatrix}.$$

It holds that

$$\langle \gamma v, w \rangle_{\mathbb{L}^{m+2}} = \langle v, \gamma^{-1} w \rangle_{\mathbb{L}^{m+2}}$$

for any $v, w \in \mathbb{L}^{m+2}$. Equations $\gamma\gamma^{-1} = \gamma^{-1}\gamma = I_{m+2}$ are interpreted as

$$\begin{cases} \theta^2 - |b|^2 = 1, \quad \theta^2 - |c|^2 = 1, \\ \theta c - {}^tba = 0, \quad \theta b - {}^tca = 0, \\ {}^taa - {}^tcc = I_{m+1}, \quad a{}^ta - b{}^tb = I_{m+1}. \end{cases} \tag{3.1}$$

From these relations, we know that $\theta \geq 1$ and that $\gamma \in SO(m+1)$ if and only if $\theta = 1$. On the other hand, the map $\tilde{\gamma}$ is represented as

$$\tilde{\gamma}(x) = \frac{1}{cx+\theta}\begin{pmatrix} {a^1}_A x^A + b^1 \\ \vdots \\ {a^{m+1}}_A x^A + b^{m+1} \end{pmatrix}, \tag{3.2}$$

where $cx = {}^t c \cdot x = \sum_{j=1}^{n+1} c_j x^j$. Since $\tilde{\gamma}$ is a sphere-valued map, we have

$$|ax+b|^2 = (cx+\theta)^2 \quad \text{for } x \in \mathbb{S}^m. \tag{3.3}$$

In this section, we use the notations,

$$a^A = {}^t({a^A}_1, \cdots, {a^A}_{m+1}), \quad a_A = ({a^1}_A, \cdots, {a^{m+1}}_A).$$

By using (3.1), it is easy to check

$$\tilde{\gamma}^* g_{\mathbb{S}^m} = \frac{1}{(cx+\theta)^2} g_{\mathbb{S}^m}. \tag{3.4}$$

Let $\mathbb{H}^{m+1}$ be the $(m+1)$-dimensional hyperbolic space given by

$$\mathbb{H}^{m+1} = \left\{ {}^t(x^0, \cdots, x^{m+1}) \in \mathbb{R}^{m+2} \middle| -(x^0)^2 + \sum_{j=1}^{m+1} (x^j)^2 = -1 \right\}.$$

We define a map $\pi_G : G \to \mathbb{H}^{m+1}$ to be

$$\pi_G\left(\begin{pmatrix} \theta & b \\ c & a \end{pmatrix}\right) = \begin{pmatrix} \theta \\ c \end{pmatrix}.$$

Note that $\pi_G^{-1}({}^t(\theta, c))$ is compact for any ${}^t(\theta, c) \in \mathbb{H}^{m+1}$.

We need a special type of vector fields on spheres.

Definition 3.2 (Conformal vector field) A vector field $V \in C^\infty(T\mathbb{S}^m)$ is said to be *conformal* if there exists a one-parameter family $(\varphi_t)_{t \in I}$, $I = (-\epsilon, \epsilon)$, of conformal diffeomorphisms of $\mathbb{S}^m$ satisfying

$$V(x) = \frac{d}{dt}\varphi_t(x)\bigg|_{t=0} \quad \text{for any } x \in \mathbb{S}^m.$$

The next lemma gives us typical examples of conformal vector fields.

Lemma 3.2 *Let f be the restriction to $\mathbb{S}^m$ of a linear function on $\mathbb{R}^{m+1}$: namely*

$$f(x) = z \cdot x \quad \text{for a fixed point } z \in \mathbb{R}^{m+1},$$

where $\cdot$ is the Euclidean scalar product in $\mathbb{R}^{m+1}$. Then $\nabla_{\mathbb{S}^m} f$ is a conformal vector field.

Proof. First we prove the lemma for the case $z = \boldsymbol{e}_{m+1} = (0, \cdots, 0, 1)$. Let $\pi : \mathbb{S}^m \to \mathbb{R}^m$ be the stereographic projection from the north pole. Then,

$$\pi(x^1, \cdots, x^{m+1}) = (\xi^1, \cdots, \xi^m), \quad \text{where} \quad \xi^\alpha = \frac{x^\alpha}{1 - x^{m+1}} \quad \text{for } 1 \leq \alpha \leq m.$$

The inverse mapping is as follows.

$$\pi^{-1}(\xi^1, \cdots, \xi^m) = (x^1, \cdots, x^{m+1}),$$
$$\text{where} \quad x^\alpha = \frac{2\xi^\alpha}{1 + |\xi|^2} \quad \text{for } 1 \leq \alpha \leq m, \quad x^{m+1} = \frac{|\xi|^2 - 1}{|\xi|^2 + 1}.$$

For $t \in \mathbb{R}$, we define a map $h_t : \mathbb{R}^m \to \mathbb{R}^m$ to be

$$h_t(\xi) = e^t \xi \quad \text{for } \xi \in \mathbb{R}^m,$$

and a map $H_t : \mathbb{S}^m \to \mathbb{S}^m$ to be

$$H_t(x) = (\pi^{-1} \circ h_t \circ \pi)(x).$$

We can show that H_t is a conformal diffeomorphism of $\mathbb{S}^m$ for any $t \in \mathbb{R}$. The components of H_t are given precisely by

$$H_t^\alpha(x) = \frac{2e^t x^\alpha}{(e^{2t} - 1)x^{m+1} + e^{2t} + 1} \quad \text{for } 1 \leq \alpha \leq m,$$
$$H_t^{m+1}(x) = \frac{(e^{2t} + 1)x^{m+1} + e^{2t} - 1}{(e^{2t} - 1)x^{m+1} + e^{2t} + 1}.$$

By a direct calculation, we obtain

$$\begin{aligned} \left.\frac{d}{dt} H_t(x)\right|_{t=0} &= {}^t(-x^1 x^{m+1}, \cdots, -x^m x^{m+1}, 1 - (x^{m+1})^2) \\ &= \boldsymbol{e}_{m+1} - x^{m+1}(x^1, \cdots, x^{m+1}) \\ &= \boldsymbol{e}_{m+1} - (\boldsymbol{e}_{m+1} \cdot x)x. \end{aligned} \tag{3.5}$$

Next we proceed to the general case. If $z = 0$, take $\phi_t = \mathrm{id}_{\mathbb{S}^m}$ for any $t \in \mathbb{R}$. If $z \neq 0$, then there exists an $(m+1) \times (m+1)$ orthogonal matrix Q satisfying $Q\boldsymbol{e}_{m+1} = z/|z|$. We define a map $\varphi_t : \mathbb{S}^m \to \mathbb{S}^m$ to be

$$\varphi_t(x) = QH_{|z|t}(Q^{-1}x) \quad \text{for } t \in \mathbb{R}.$$

Then φ_t is a conformal diffeomorphism of $\mathbb{S}^m$ for any $t \in \mathbb{R}$. From (3.5) we have

$$\begin{aligned}
\frac{d}{dt}\varphi_t(x)\Big|_{t=0} &= |z|Q\Big(\boldsymbol{e}_{m+1} - (\boldsymbol{e}_{m+1} \cdot Q^{-1}x)Q^{-1}x\Big) \\
&= |z|\Big(\frac{z}{|z|} - (Q\boldsymbol{e}_{m+1} \cdot x)x\Big) \\
&= z - (z \cdot x)x = \nabla_{\mathbb{S}^m} f.
\end{aligned}$$

Consequently $\nabla_{\mathbb{S}^m} f$ is a conformal vector field. ∎

§3.2 Energies of smooth harmonic maps between spheres

Let us deal with energies of smooth harmonic maps between spheres. First, we show that, if a source manifold is $\mathbb{S}^2$, then the energy of a smooth harmonic map is invariant under any conformal action (see Theorem 3.1 below). Though we will not use this theorem in this monograph, it is very interesting to compare the case of $\mathbb{S}^2$ with those of $\mathbb{S}^m$, $m \geq 3$ (see Theorem 3.2 below).

Remark 3.1 If γ belongs to $SO(m+1)$, then it holds that

$$\mathbf{E}(u_0 \circ \tilde{\gamma}) = \mathbf{E}(u_0)$$

for any $\gamma \in SO(m+1)$.

A stronger result holds if $m = 2$.

Theorem 3.1 *Let $u_0 \in C^\infty(\mathbb{S}^2, \mathbb{S}^n)$ be a harmonic map. Then the Dirichlet energy is invariant under the composition of u_0 and $\tilde{\gamma}$ for any $\gamma \in G$, that is,*

$$\mathbf{E}(u_0 \circ \tilde{\gamma}) = \mathbf{E}(u_0) \quad \textit{for any } \gamma \in G.$$

Remark 3.2 In this theorem, it is not essential that the target manifold is a sphere. The same conclusion holds if we take any two-dimensional compact Riemannian manifold as a source manifold instead of $\mathbb{S}^2$.

Proof. For any $\gamma \in G$, there exists a function $f \in C^\infty(\mathbb{S}^2)$ satisfying

$$\tilde{\gamma}^* g_{\mathbb{S}^2} = e^f g_{\mathbb{S}^2}.$$

See (3.4) for the precise expression of f. Let (ξ^1, ξ^2) be a local coordinate system on $\mathbb{S}^2$ and $(y^1, \cdots, y^n)$ on $\mathbb{S}^n$. It holds that

$$e^f g_{\alpha\beta} = g_{\sigma\tau} \frac{\partial \tilde{\gamma}^\sigma}{\partial \xi^\alpha} \frac{\partial \tilde{\gamma}^\tau}{\partial \xi^\beta}, \qquad e^{-f} g^{\alpha\beta} = g^{\sigma\tau} \frac{\partial \tilde{\gamma}^\alpha}{\partial \xi^\sigma} \frac{\partial \tilde{\gamma}^\beta}{\partial \xi^\tau}.$$

Therefore the energy density $|d(u_0 \circ \tilde{\gamma})|^2$ is

$$\begin{aligned}
|d(u_0 \circ \tilde{\gamma})|^2 &= g^{\alpha\beta} h_{ij}(u_0 \circ \tilde{\gamma}) \frac{\partial}{\partial x^\alpha}(u_0^i \circ \tilde{\gamma}) \frac{\partial}{\partial x^\beta}(u_0^j \circ \tilde{\gamma}) \\
&= g^{\alpha\beta} h_{ij}(u_0 \circ \tilde{\gamma}) \frac{\partial u_0^i}{\partial x^\sigma} \circ \tilde{\gamma} \frac{\partial \tilde{\gamma}^\sigma}{\partial x^\alpha} \frac{\partial u_0^j}{\partial x^\tau} \circ \tilde{\gamma} \frac{\partial \tilde{\gamma}^\tau}{\partial x^\beta} \\
&= e^{-f} g^{\sigma\tau} h_{ij}(u_0 \circ \tilde{\gamma}) \frac{\partial u_0^i}{\partial x^\sigma} \circ \tilde{\gamma} \frac{\partial u_0^j}{\partial x^\tau} \circ \tilde{\gamma} \\
&= e^{-f} |du_0|^2 \circ \tilde{\gamma}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\tilde{\gamma}^* \mathrm{vol}_{\mathbb{S}^2} &= \sqrt{\det\left(g_{\alpha\beta} \frac{\partial \tilde{\gamma}^\alpha}{\partial x^\sigma} \frac{\partial \tilde{\gamma}^\beta}{\partial x^\tau} \right)} dx^1 \wedge dx^2 \\
&= \sqrt{\det(e^f g_{\sigma\tau})} dx^1 \wedge dx^2 \\
&= e^f \mathrm{vol}_{\mathbb{S}^2}.
\end{aligned}$$

Consequently we obtain the desired result. ∎

On the other hand, if the source manifold is $\mathbb{S}^m$, $m \geq 3$, then we have the following theorem.

Theorem 3.2 *Let $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ be a harmonic map with $m \geq 3$. Then the supremum of Dirichlet energies along the G-orbit of u_0 is attained at u_0, that is,*

$$\mathbf{E}(u_0) = \operatorname*{Sup}_{\gamma \in G} \mathbf{E}(u_0 \circ \tilde{\gamma}). \tag{3.6}$$

Moreover, if u_0 is non-constant and if $\gamma \in G$ satisfies

$$\mathbf{E}(u_0) = \mathbf{E}(u_0 \circ \tilde{\gamma}),$$

then $\gamma \in SO(m+1)$.

Remark 3.3 It is not trivial that the supremum of energies in G-orbit of u_0 is attained, because G is non-compact. Theorem 3.2 shows that, if $m \geq 3$ and if $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ is a non-constant harmonic map, $SO(m+1)$ is the largest subgroup of G whose action preserves the Dirichlet energy of u_0.

The proof of Theorem 3.2 will be given at the end of this chapter. First, we give the precise representation of $\mathbf{E}(u_0 \circ \tilde{\gamma})$.

Lemma 3.3 *Let $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ be an arbitrary map. Then for any $\gamma \in G$, we have*

$$\mathbf{E}(u_0 \circ \tilde{\gamma}) = \frac{1}{2} \int_{\mathbb{S}^m} \frac{1}{(cx+\theta)^2} |du_0|^2 \circ \tilde{\gamma} \, d\mathrm{vol}_{\mathbb{S}^m}(x), \tag{3.7}$$

where $\pi_G(\gamma) = {}^t(\theta, c)$.

Proof. For any $p \in \mathbb{S}^m$, let $\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m\}$ be a local orthonormal frame of $T\mathbb{S}^m$ around p. It follows from (3.4) that

$$(\tilde{\gamma}^* g_{\mathbb{S}^m})(\boldsymbol{f}_\alpha, \boldsymbol{f}_\beta) = \frac{1}{(cx+\theta)^2} g_{\mathbb{S}^m}(\boldsymbol{f}_\alpha, \boldsymbol{f}_\beta) = \frac{1}{(cx+\theta)^2} \delta_{\alpha\beta}.$$

On the other hand we have

$$(\tilde{\gamma}^* g_{\mathbb{S}^m})(\boldsymbol{f}_\alpha, \boldsymbol{f}_\beta) = g_{\mathbb{S}^m}(d\tilde{\gamma}(\boldsymbol{f}_\alpha), d\tilde{\gamma}(\boldsymbol{f}_\beta)).$$

Therefore, $\{(cx+\theta) d\tilde{\gamma}(\boldsymbol{f}_\alpha)\}_{\alpha=1}^m$ is a local orthonormal frame of $T\mathbb{S}^m$ around $\tilde{\gamma}(p)$. Thus it holds that

$$\begin{aligned}
|d(u_0 \circ \tilde{\gamma})|^2 &= \sum_{\alpha=1}^m h(d(u_0 \circ \tilde{\gamma})(\boldsymbol{f}_\alpha), d(u_0 \circ \tilde{\gamma})(\boldsymbol{f}_\alpha)) \\
&= \sum_{\alpha=1}^m h(du_0(d\tilde{\gamma}(\boldsymbol{f}_\alpha)), du_0(d\tilde{\gamma}(\boldsymbol{f}_\alpha)) \\
&= \frac{1}{(cx+\theta)^2} \sum_{\alpha=1}^m h(du_0((cx+\theta) d\tilde{\gamma}(\boldsymbol{f}_\alpha)), du_0((cx+\theta) d\tilde{\gamma}(\boldsymbol{f}_\alpha))) \\
&= \frac{1}{(cx+\theta)^2} |du_0|^2 \circ \tilde{\gamma}.
\end{aligned}$$

Consequently, we have the desired result. ∎

Lemma 3.4 *Let $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ be an arbitrary map with $m \geq 3$. Then for any $\epsilon > 0$, there exists a compact set $K \subset G$ satisfying*

$$\mathbf{E}(u_0 \circ \tilde{\gamma}) < \epsilon \quad \textit{for any } \gamma \in G \backslash K.$$

Remark 3.4 From this lemma, we know that the supremum of $\mathbf{E}$ on the G-orbit of u_0 is attained.

Proof. By Lemma 3.3, we know the estimate

$$\mathbf{E}(u_0 \circ \tilde{\gamma}) \leq \frac{1}{2} \operatorname*{Sup}_{\mathbb{S}^m} |du_0|^2 \int_{\mathbb{S}^m} \frac{1}{(cx+\theta)^2} \, d\mathrm{vol}_{\mathbb{S}^m}(x). \tag{3.8}$$

Since $\pi_G^{-1}({}^t(\theta, c))$ is compact for any ${}^t(\theta, c) \in \mathbb{H}^{m+1}$, it suffices to show that the last integral in (3.8) tends to 0 as $|c| \to \infty$.

Put $\rho = |c|$. Then we have $\theta = \sqrt{\rho^2+1}$ by (3.1). Choose the geodesic coordinate system of $\mathbb{S}^m$ centered at $-c/|c| \in \mathbb{S}^m$, say (r, σ), $r \in [0, \pi)$, $\sigma \in \mathbb{S}^{m-1}$. Then, $g_{\mathbb{S}^{m-1}}$ has the form

$$g_{\mathbb{S}^m} = dr \otimes dr + (\sin^2 r) g_{\mathbb{S}^{m-1}}.$$

In terms of this coordinate system, we have

$$\begin{aligned}\int_{\mathbb{S}^m} \frac{1}{(cx+\theta)^2} \, d\mathrm{vol}_{\mathbb{S}^m} &= \int_0^\pi \int_{\mathbb{S}^{m-1}} \frac{(\sin r)^{m-1}}{(\theta - \rho \cos r)^2} d\mathrm{vol}_{\mathbb{S}^{m-1}} \, dr \\ &= \omega_{m-1} \int_0^\pi \frac{(\sin r)^{m-1}}{(\theta - \rho \cos r)^2} \, dr.\end{aligned}$$

Take a small constant $\delta > 0$, which will be determined later. We estimate the integral by dividing the interval of integration into $[0, \delta]$ and $[\delta, \pi]$. If $\rho \geq 1$, then each integral is dominated as

$$\begin{aligned}\int_0^\delta \frac{(\sin r)^{m-1}}{(\theta - \rho \cos r)^2} \, dr &\leq (\sin \delta)^{m-2} \int_0^\delta \frac{\sin r}{(\theta - \rho \cos r)^2} \, dr \\ &= \frac{(\sin \delta)^{m-2}}{\rho} \left(\frac{1}{\theta - \rho} - \frac{1}{\theta - \rho \cos \delta} \right) \\ &\leq \frac{(\sin \delta)^{m-2}}{\rho} (\sqrt{\rho^2+1} + \rho) \\ &\leq (1+\sqrt{2})(\sin \delta)^{m-2},\end{aligned}$$

$$
\begin{aligned}
\int_\delta^\pi \frac{(\sin r)^{m-1}}{(\theta-\rho\,\cos r)^2}\,dr &\le \int_\delta^\pi \frac{\sin r}{(\theta-\rho\,\cos r)^2}\,dr \\
&\le \frac{1}{\rho}\left(\frac{1}{\theta-\rho\,\cos\delta}-\frac{1}{\theta+\rho}\right) \\
&\le \frac{1}{\rho}\frac{\sqrt{\rho^2+1}+\rho\,\cos\delta}{1+\rho^2\,\sin^2\delta} \\
&\le \frac{1+\sqrt{2}}{1+\rho^2\,\sin^2\delta}.
\end{aligned}
$$

We fix a small positive number $\delta > 0$ satisfying

$$(1+\sqrt{2}\,)(\sin\delta)^{m-2} < \frac{\epsilon}{2},$$

and a positive large number $\rho_0 \ge 1$ satisfying

$$\frac{1+\sqrt{2}}{1+\rho_0^2\,\sin^2\delta} < \frac{\epsilon}{2}.$$

Then it holds that

$$\int_0^\pi \frac{(\sin r)^{m-1}}{(\theta-\rho\,\cos r)^2}\,dr < \epsilon$$

provided that $\rho \ge \rho_0$. This completes the proof. ∎

We need a precise expression of the tension field of a composition map.

Lemma 3.5 *Let $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ and $\gamma \in G$. Then the tension field of $u_0 \circ \tilde{\gamma}$ is given by*

$$\tau(u_0 \circ \tilde{\gamma}) = du_0(\tau(\tilde{\gamma})) + \operatorname{trace}((\nabla du_0)(d\tilde{\gamma}, d\tilde{\gamma})).$$

Proof. Let $\{\boldsymbol{f}_\alpha\}_{\alpha=1}^m$ be a local orthonormal frame of $T\mathbb{S}^m$. Then we have

$$
\begin{aligned}
\tau(u_0 \circ \tilde{\gamma}) &= \Big(\nabla_{\boldsymbol{f}_\alpha} d(u_0\circ\tilde{\gamma})\Big)(\boldsymbol{f}_\alpha) \\
&= \nabla_{\boldsymbol{f}_\alpha}\Big(d(u_0\circ\tilde{\gamma})(\boldsymbol{f}_\alpha)\Big) - d(u_0\circ\tilde{\gamma})\Big(\nabla_{\boldsymbol{f}_\alpha}\boldsymbol{f}_\alpha\Big) \\
&= \nabla_{\boldsymbol{f}_\alpha}\Big(du_0\Big(d\tilde{\gamma}(\boldsymbol{f}_\alpha)\Big)\Big) - du_0\Big(d\tilde{\gamma}\Big(\nabla_{\boldsymbol{f}_\alpha}\boldsymbol{f}_\alpha\Big)\Big) \\
&= \Big(\nabla_{d\tilde{\gamma}(\boldsymbol{f}_\alpha)} du_0\Big)\Big(d\tilde{\gamma}(\boldsymbol{f}_\alpha)\Big) + du_0\Big(\nabla_{\boldsymbol{f}_\alpha}\Big(d\tilde{\gamma}(\boldsymbol{f}_\alpha)\Big)\Big) - du_0\Big(d\tilde{\gamma}\Big(\nabla_{\boldsymbol{f}_\alpha}\boldsymbol{f}_\alpha\Big)\Big) \\
&= \Big(\nabla_{d\tilde{\gamma}(\boldsymbol{f}_\alpha)} du_0\Big)\Big(d\tilde{\gamma}(\boldsymbol{f}_\alpha)\Big) + du_0\Big(\Big(\nabla_{\boldsymbol{f}_\alpha} d\tilde{\gamma}\Big)(\boldsymbol{f}_\alpha)\Big) \\
&= du_0(\tau(\tilde{\gamma})) + \operatorname{trace}\Big((\nabla du_0)(d\tilde{\gamma}, d\tilde{\gamma})\Big).
\end{aligned}
$$
∎

To compute the tension field of $u_0 \circ \tilde{\gamma}$, we calculate that of $\tilde{\gamma}$. First, we compute the energy density of $\tilde{\gamma}$.

Lemma 3.6 *If $\gamma \in G$ is given by the matrix*

$$\begin{pmatrix} \theta & c \\ b & a \end{pmatrix},$$

then the energy density of $\tilde{\gamma}$ is given by

$$|d\tilde{\gamma}|^2(x) = \frac{m}{(cx+\theta)^2}.$$

in the standard coordinate system of $\mathbb{R}^{m+1}$.

Proof. Let $\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m\}$ be an orthonormal basis of $T_x\mathbb{S}^m$. It follows from (3.4) that

$$(\tilde{\gamma}^* g_{\mathbb{S}^m})_x(\boldsymbol{f}_\alpha, \boldsymbol{f}_\beta) = \frac{1}{(cx+\theta)^2}\delta_{\alpha\beta}.$$

On the other hand, it holds that

$$(\tilde{\gamma}^* g_{\mathbb{S}^m})_x(\boldsymbol{f}_\alpha, \boldsymbol{f}_\beta) = (g_{\mathbb{S}^m})_{\tilde{\gamma}(x)}(d\tilde{\gamma}(\boldsymbol{f}_\alpha), d\tilde{\gamma}(\boldsymbol{f}_\beta)).$$

Therefore $\{(cx+\theta)d\tilde{\gamma}(\boldsymbol{f}_\alpha)\}_{\alpha=1}^m$ is an orthonormal basis of $T_{\tilde{\gamma}(x)}\mathbb{S}^m$, and we have

$$\begin{aligned}
|d\tilde{\gamma}|^2(x) &= \sum_{\alpha=1}^m (g_{\mathbb{S}^m})_{\tilde{\gamma}(x)}(d\tilde{\gamma}(\boldsymbol{f}_\alpha), d\tilde{\gamma}(\boldsymbol{f}_\alpha)) \\
&= \frac{1}{(cx+\theta)^2}\sum_{\alpha=1}^m (g_{\mathbb{S}^m})_{\tilde{\gamma}(x)}((cx+\theta)d\tilde{\gamma}(\boldsymbol{f}_\alpha), (cx+\theta)d\tilde{\gamma}(\boldsymbol{f}_\alpha)) \\
&= \frac{m}{(cx+\theta)^2}. \quad \blacksquare
\end{aligned}$$

We calculate the tension field of $\tilde{\gamma}$.

Lemma 3.7 *If $\gamma \in G$ is given by the matrix*

$$\begin{pmatrix} \theta & b \\ c & a \end{pmatrix},$$

then the tension field of $\tilde{\gamma}$ is given by

$$\tau(\tilde{\gamma})^i(x) = \frac{m-2}{cx+\theta}\left\{\frac{(cx)b^i - \theta(a^i x)}{cx+\theta} + \frac{a^i x + b^i}{(cx+\theta)^2}\right\}$$

in the standard coordinate system of $\mathbb{R}^{m+1}$.

Proof. It follows from (1.3) and Lemma 3.6 that

$$\begin{aligned}\tau(\tilde{\gamma}) &= \Delta_{\mathbb{S}^m}\tilde{\gamma} + |d\tilde{\gamma}|^2\tilde{\gamma} \\ &= \Delta_{\mathbb{S}^m}\tilde{\gamma} + \frac{m}{(cx+\theta)^2}\tilde{\gamma}.\end{aligned}$$

Using an elementary relation

$$\mathrm{grad}_{\mathbb{S}^m} x^j = \boldsymbol{e}_j - x^j x,$$

and (3.1) we have

$$\Delta_{\mathbb{S}^m}\tilde{\gamma}^i = -(m-2)\frac{a^i x}{cx+\theta} + (m-2)(cx)\frac{a^i x + b^i}{(cx+\theta)^2} - 2\frac{a^i x + b^i}{(cx+\theta)^3}.$$

Consequently, we can verify

$$\begin{aligned}\tau(\tilde{\gamma})^i &= \frac{m-2}{cx+\theta}\Big\{ - a^i x + (cx)\frac{a^i x + b^i}{cx+\theta} + \frac{a^i x + b^i}{(cx+\theta)^2}\Big\} \\ &= \frac{m-2}{cx+\theta}\Big\{\frac{(cx)b^i - \theta(a^i x)}{cx+\theta} + \frac{a^i x + b^i}{(cx+\theta)^2}\Big\}. \quad \blacksquare\end{aligned}$$

Making use of the following lemma, we obtain a very useful expression of the tension field of $\tilde{\gamma}$ (see (3.9) below).

Lemma 3.8 *If $\gamma \in G$ is given by the matrix*

$$\begin{pmatrix} \theta & c \\ b & a \end{pmatrix},$$

then for the vector field $V_\gamma(x) = \nabla_{\mathbb{S}^m}(cx)$ on $\mathbb{S}^m$, it holds that

$$d\tilde{\gamma}(V_\gamma)^i = \frac{(cx)b^i - \theta(a^i x)}{cx+\theta} + \frac{a^i x + b^i}{(cx+\theta)^2}.$$

Proof. The assertion follows from (3.2) and $dx^j(V_\gamma) = c_j - (cx)x^j$, that is,

$$\begin{aligned}d\tilde{\gamma}(V_\gamma)^i &= \Big\{\frac{a^i{}_j}{cx+\theta}dx^j - \frac{a^i x + b^i}{(cx+\theta)^2}c_j dx^j\Big\}(V_\gamma) \\ &= \frac{a^i c - (cx)(a^i x)}{cx+\theta} - \frac{(|c|^2 - (cx)^2)(a^i x + b^i)}{(cx+\theta)^2} \\ &= \frac{\theta b^i - (cx)(a^i x)}{cx+\theta} - \frac{(-1+\theta^2 - (cx)^2)(a^i x + b^i)}{(cx+\theta)^2}\end{aligned}$$

$$= \frac{(cx)b^i - \theta(a^i x)}{cx+\theta} + \frac{a^i x + b^i}{(cx+\theta)^2}.$$

Here we have used (3.1) in the second and third equalities. ∎

Combining Lemma 3.7 with Lemma 3.8, we have

$$\tau(\tilde{\gamma}) = \frac{m-2}{cx+\theta} d\tilde{\gamma}(V_\gamma). \tag{3.9}$$

Proof of Theorem 3.2. Let $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ be a harmonic map. Since (3.6) holds trivially if u_0 is a constant map, assume that u_0 is non-constant. From Theorem 3.4, there exists a $\gamma \in G$ satisfying

$$\mathbf{E}(u_0 \circ \tilde{\gamma}) = \operatorname*{Max}_{\gamma' \in G} \mathbf{E}(u_0 \circ \tilde{\gamma'}). \tag{3.10}$$

We assume that $\gamma \notin SO(m+1)$. Let the matrix expression of γ be

$$\begin{pmatrix} \theta & b \\ c & a \end{pmatrix}.$$

Lemma 3.5 gives us

$$\begin{aligned} \tau(u_0 \circ \tilde{\gamma}) &= du_0(\tau(\tilde{\gamma})) + \mathrm{trace}((\nabla \mathrm{du_0})(\mathrm{d}\tilde{\gamma}, \mathrm{d}\tilde{\gamma})) \\ &= du_0\Big(\frac{m-2}{cx+\theta} d\tilde{\gamma}(V_\gamma)\Big) + \frac{1}{(cx+\theta)^2}\tau(u_0) \circ \tilde{\gamma} \\ &= \frac{m-2}{cx+\theta} du_0(d\tilde{\gamma}(V_\gamma)), \end{aligned}$$

because u_0 is a harmonic map. Since V_γ is a conformal vector field on $\mathbb{S}^m$, there exists a one-parameter family of conformal diffeomorphisms $(\varphi_t)_{t\in I}$, $I = (-\epsilon, \epsilon) \subset \mathbb{R}$, satisfying

$$\begin{cases} \varphi_0 = \mathrm{id}_{\mathbb{S}^m}, \\ \left.\dfrac{d\varphi_t(x)}{dt}\right|_{t=0} = V_\gamma(x). \end{cases}$$

We note that $\tilde{\gamma} \circ \varphi_t$ belongs to G for any $t \in I$, and

$$\left.\frac{d(u_0 \circ \tilde{\gamma} \circ \varphi_t)}{dt}\right|_{t=0} = d(u_0 \circ \tilde{\gamma})(V_\gamma) = du_0(d\tilde{\gamma}(V_\gamma)).$$

It follows from the first variation formula that

$$\begin{aligned}\frac{d}{dt}\mathbf{E}(u_0 \circ \tilde{\gamma} \circ \varphi_t)\Big|_{t=0} &= \int_{\mathbb{S}^m} \Big\langle \tau(u_0 \circ \tilde{\gamma}), du_0(d\tilde{\gamma}(V_\gamma)) \Big\rangle \, d\mathrm{vol}_{\mathbb{S}^m} \\ &= \int_{\mathbb{S}^m} \frac{m-2}{cx+\theta} |du_0(d\tilde{\gamma}(V_\gamma))|^2 \, d\mathrm{vol}_{\mathbb{S}^m}.\end{aligned}$$

Since $\gamma \notin SO(m+1)$, we have $\theta > 1$ and $c \neq 0$. And hence we have

$$cx + \theta \geq \theta - |c| = \theta - \sqrt{\theta^2 - 1} > 0.$$

Since u_0 is non-constant, $du_0(V_\gamma)$ is not identically equal to 0. Consequently we have

$$\frac{d}{dt}\mathbf{E}(u_0 \circ \tilde{\gamma} \circ \varphi_t)\Big|_{t=0} > 0.$$

This contradicts (3.10) and yields that $\gamma \in SO(m+1)$. And by Remark 3.1, we know that

$$\mathbf{E}(u_0) = \mathbf{E}(u_0 \circ \tilde{\gamma}).$$

And hence we have

$$\mathbf{E}(u_0) = \operatorname*{Max}_{\gamma \in G} \mathbf{E}(u_0 \circ \tilde{\gamma}). \quad \blacksquare$$

Chapter 4

Proof of Main Result

§4.1 Proof of Theorem 1.9

In this chapter, we shall prove Theorem 1.9. By the results in Chapter 2, we have only to consider a weakly stable stationary harmonic map $u \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ satisfying

$$\mathrm{Sing}(u) = \{0\} \text{ and } \frac{\partial u}{\partial r} = 0 \text{ in } \mathbb{B}^4 \backslash \{0\}.$$

Let u_0 be the restriction of u to $\mathbb{S}^3$. Then, $u_0 : \mathbb{S}^3 \to \mathbb{S}^3$ is a harmonic map of class C^∞. The proof of Theorem 1.9 is now reduced to that of a following simplified version.

Proposition 4.1 *Let $u_0 \in C^\infty(\mathbb{S}^3, \mathbb{S}^3)$ be a harmonic map such that the extension*

$$u(x) = u_0\left(\frac{x}{|x|}\right) \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$$

is a weakly stable stationary harmonic map. If $\deg(u_0) \neq 0$, then there exists a 4×4 constant orthogonal matrix S such that

$$u_0(\omega) = S\omega \quad \text{for } \omega \in \mathbb{S}^3.$$

First we prove an upper bound of the Dirichlet energy of u_0.

Lemma 4.1 *If $u_0 \in C^\infty(\mathbb{S}^3, \mathbb{S}^3)$ and $u \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ satisfy the condition in Proposition 4.1 except $\deg(u_0) \neq 0$, we have an inequality*

$$\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2 \, d\,\mathrm{vol}_{\mathbb{S}^3} \leq 3\omega_3(1 - \lambda(u)), \tag{4.1}$$

where $\lambda(u)$ is the number defined in Definition 1.7.

Proof. Since u is weakly stable, we can prove in the same manner as in the proof of Lemma 2.1 that

$$3\lambda(u)\int_{\mathbb{B}^4} r^{-2}|f|^2\,dx + \int_{\mathbb{B}^4} |\nabla u|^2|f|^2\,dx \le 3\int_{\mathbb{B}^4} |\nabla f|^2\,dx$$

for any $f \in W_0^{1,2} \cap L^\infty(\mathbb{B}^4, \mathbb{R})$. Here $r = |x|$. We set $f = \varphi(r)$ where φ is an arbitrary function of class $C_0^\infty(0,1)$. Since u is homogeneous, we have

$$3\lambda(u)\omega_3\int_{\mathbb{S}^3} r|\varphi|^2\,dr + \left(\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2\,d\mathrm{vol}_{\mathbb{S}^3}\right)\left(\int_0^1 r|\varphi|^2\,dr\right) \le 3\omega_3\int_0^1 r^3|\varphi^{'}|^2\,dr.$$

Therefore it holds that

$$\frac{1}{3\omega_3}\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2\,\mathrm{vol}_{\mathbb{S}^3} \le \frac{\displaystyle\int_0^1 r^3|\varphi^{'}|^2\,dr}{\displaystyle\int_0^1 r|\varphi|^2\,dr} - \lambda(u)$$

for any $\varphi \in C_0^\infty(0,1)$ which is not identically zero. Since

$$\operatorname*{Inf}_{\varphi\in C_0^\infty(0,1)\setminus\{0\}} \frac{\displaystyle\int_0^1 r^3|\varphi'|^2\,dr}{\displaystyle\int_0^1 r|\varphi|^2\,dr} = 1$$

by Hardy's inequality, we obtain the desired result. ∎

Next we give a lower bound of the Dirichlet energy of u_0. We introduce some notation. For $p \in \mathbb{S}^m$, let $\overline{p}$ be the antipodal point of p, that is, $\overline{p} = -p$. We write the stereographic projection from $p \in \mathbb{S}^m$ to $T_{\overline{p}}\mathbb{S}^m$ as

$$\pi_p : \mathbb{S}^m \to T_{\overline{p}}\mathbb{S}^m,$$

where we set $\pi_p(p) = \infty$. For $R \ge 1$, let $I_{\overline{p},R} : T_{\overline{p}}\mathbb{S}^m \to T_{\overline{p}}\mathbb{S}^m$ be a dilation, that is,

$$I_{\overline{p},R}(v) = Rv \quad \text{for } v \in T_{\overline{p}}\mathbb{S}^m.$$

And for $p \in \mathbb{S}^m$ and $R \ge 1$, we denote $\eta_{p,R}$ the map from $\mathbb{S}^m$ to $\mathbb{S}^m$ given by

$$\eta_{p,R}(\omega) = (\pi_p^{-1} \circ I_{\overline{p},R} \circ \pi_p)(\omega).$$

For a map $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^m)$ we define the map $H : \mathbb{S}^m \times [1,\infty) \to \mathbb{R}^{m+1}$ to be

$$H(p,R) = \frac{1}{\omega_m}\int_{\mathbb{S}^m} (u_0 \circ \eta_{p,R})(\omega)\,d\mathrm{vol}_{\mathbb{S}^m}(\omega).$$

Here we omit u_0 for simplicity.

Now, we prove a lemma.

Lemma 4.2 *For any $u_0 \in C^\infty(\mathbb{S}^m, \mathbb{S}^m)$ such that $\deg(u_0) \neq 0$, there exist a point $p \in \mathbb{S}^m$ and a number $R \geq 1$ satisfying $H(p, R) = 0$.*

Proof. Suppose that

$$H(p, R) \neq 0 \quad \text{for any } p \in \mathbb{S}^m \text{ and any } R \geq 1.$$

Then we can define a continuous map $\tilde{H} : \mathbb{S}^m \times [1, \infty) \to \mathbb{S}^m$ by $\tilde{H}(p, R) = H(p, R)/|H(p, R)|$. We calculate the values

$$\tilde{H}(p, 1) \text{ and } \tilde{H}(p, \infty) = \lim_{R \to \infty} \tilde{H}(p, R).$$

Since

$$H(p, 1) = \frac{1}{\omega_m} \int_{\mathbb{S}^m} (u_0 \circ \eta_{p,1})(\omega)\, d\mathrm{vol}_{\mathbb{S}^m}(\omega) = \frac{1}{\omega_m} \int_{\mathbb{S}^m} u_0(\omega)\, d\mathrm{vol}_{\mathbb{S}^m}(\omega),$$

$\tilde{H}(\cdot, 1)$ is a constant map, and therefore $\deg(\tilde{H}(\cdot, 1)) = 0$. Since $\tilde{H}$ is continuous on $\mathbb{S}^m \times [1, \infty)$, we have $\deg(H(\cdot, R)) = 0$ for any $R \in [1, \infty)$. On the other hand, by the Lebesgue convergence theorem, we have

$$\begin{aligned}\lim_{R \to \infty} H(p, R) &= \lim_{R \to \infty} \frac{1}{\omega_m} \int_{\mathbb{S}^m} (u_0 \circ \eta_{p,R})(\omega)\, d\mathrm{vol}_{\mathbb{S}^m}(\omega) \\ &= \int_{\mathbb{S}^m} u_0(p)\, d\mathrm{vol}_{\mathbb{S}^m}(\omega) = u_0(p),\end{aligned}$$

which implies

$$\lim_{R \to \infty} \tilde{H}(p, R) = u_0(p) \quad \text{for any } p \in \mathbb{S}^m.$$

We now prove that the convergence is uniform with respect to $p \in \mathbb{S}^m$. For any $\epsilon > 0$, there exists a small constant $\sigma > 0$ satisfying

$$\mathrm{vol}_{\mathbb{S}^m}(\mathbb{D}_\sigma(\overline{p})) < \frac{\omega_m}{4}\epsilon \quad \text{for any } p \in \mathbb{S}^m,$$

where $\mathbb{D}_\sigma(\overline{p})$ is the geodesic ball on $\mathbb{S}^m$ with center $\overline{p}$ and of radius σ. Since u_0 is uniformly continuous on $\mathbb{S}^m$, there exists a small constant $\delta > 0$ satisfying

$$|u_0(p) - u_0(q)| < \frac{1}{2}\epsilon \quad \text{for any } p, q \in \mathbb{S}^m \text{ with } \mathrm{dist}_{\mathbb{S}^m}(p, q) < \delta.$$

Here, $\mathrm{dist}_{\mathbb{S}^m}$ is the geodesic distance on $\mathbb{S}^m$. For a sufficiently large number $R_0 \geq 1$, we have

$$\eta_{p,R}(\mathbb{S}^m \backslash \mathbb{D}_\sigma(\overline{p})) \subset \mathbb{D}_\delta(p) \text{ for any } p \in \mathbb{S}^m \text{ and any } R \geq R_0.$$

Therefore, if $R \geq R_0$, then we have

$$\begin{aligned}
|H(p,R) - u_0(p)| &\leq \frac{1}{\omega_m} \int_{\mathbb{S}^m} |(u_0 \circ \eta_{p,R})(\omega) - u_0(p)|\, d\mathrm{vol}_{\mathbb{S}^m}(\omega) \\
&\leq \frac{1}{\omega_m} \Big(\int_{\mathbb{D}_\sigma(\overline{p})} + \int_{\mathbb{S}^m \backslash \mathbb{D}_\sigma(\overline{p})} \Big) |(u_0 \circ \eta_{p,R})(\omega) - u_0(p)|\, d\mathrm{vol}_{\mathbb{S}^m}(\omega) \\
&\leq \frac{1}{\omega_m} \cdot 2 \cdot \mathrm{vol}_{\mathbb{S}^m}(\mathbb{D}_\sigma(\overline{p})) + \frac{1}{\omega_m} \cdot \frac{1}{2}\epsilon \cdot \mathrm{vol}_{\mathbb{S}^m}(\mathbb{S}^m \backslash \mathbb{D}_\sigma(\overline{p})) \\
&\leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.
\end{aligned}$$

Thus, $H(\cdot, R)$ converges to u_0 uniformly on $\mathbb{S}^m$ as $R \to \infty$, and so does $\tilde{H}(\cdot, R)$. Consequently we obtain

$$\deg(u_0) = \lim_{R \to \infty} \deg(\tilde{H}(\cdot, R)) = 0.$$

This contradicts the assumption $\deg(u_0) \neq 0$. ∎

Proof of Proposition 4.1. By the weak stability, we know $\lambda(u) \geq 0$. We assume that u is strictly stable, that is, $\lambda(u) > 0$. By Lemma 2.1, we have

$$\int_{\mathbb{S}^m} |\nabla_{\mathbb{S}^3} u_0|^2\, d\mathrm{vol}_{\mathbb{S}^3} \leq 3\omega_3(1 - \lambda(u)). \tag{4.2}$$

From Lemma 4.1, there exist $p \in \mathbb{S}^m$ and $R \geq 1$ satisfying

$$\frac{1}{\omega_3} \int_{\mathbb{S}^3} (u_0 \circ \eta_{p,R})(\omega)\, d\mathrm{vol}_{\mathbb{S}^3}(\omega) = 0. \tag{4.3}$$

Since $\eta_{p,R} \in G$ it holds that

$$\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2\, d\mathrm{vol}_{\mathbb{S}^3} \geq \int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3}(u_0 \circ \eta_{p,R})|^2(\omega)\, d\mathrm{vol}_{\mathbb{S}^3}(\omega) \tag{4.4}$$

by Theorem 3.2. And from (4.3) the integral of every component of $u_0 \circ \eta_{p,R}$ over $\mathbb{S}^3$ is equal to 0. Therefore making use of the Poincaré inequality, we have

$$\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3}(u_0 \circ \eta_{p,R})|^2(\omega)\, d\mathrm{vol}_{\mathbb{S}^3}(\omega) \geq 3 \int_{\mathbb{S}^3} |\mathrm{u}_0 \circ \eta_{\mathrm{p,R}}|^2(\omega)\, d\mathrm{vol}_{\mathbb{S}^3} = 3\omega_3. \tag{4.5}$$

Combining (4.4) and (4.5) we have

$$\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2 \, d\,\mathrm{vol}_{\mathbb{S}^3} \geq 3\omega_3. \tag{4.6}$$

Since (4.2) contradicts (4.6), $\lambda(u)$ must be equal to 0. Furthermore, (4.2), (4.4) and (4.5) imply

$$\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3}(u_0 \circ \eta_{p,R})|^2 \, d\mathrm{vol}_{\mathbb{S}^3} = 3 \int_{\mathbb{S}^3} |u_0 \circ \eta_{p,R}|^2 \, d\mathrm{vol}_{\mathbb{S}^3}.$$

Let $\mu_1 < \mu_2 < \cdots$ be the distinct eigenvalues of $-\Delta_{\mathbb{S}^3}$. We note that $\mu_1 = 0$ and $\mu_2 = 3$. Let V_j be the eigenspace associated with μ_j and $P_j : L^2(\mathbb{S}^3) \to V_j$ be the orthogonal projection. Because of (4.3), we have

$$P_1(u_0^i \circ \eta_{p,R}) = 0 \quad \text{for } 1 \leq i \leq 4.$$

If there exist $1 \leq k \leq 4$ and $l \geq 3$ satisfying

$$P_l(u_0^k \circ \eta_{p,R}) \neq 0,$$

then we have

$$\begin{aligned}
\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3}(u_0 \circ \eta_{p,R})|^2 \, d\mathrm{vol}_{\mathbb{S}^3} &= \sum_{i=1}^{4} \sum_{j=2}^{\infty} \mu_j \int_{\mathbb{S}^3} |P_j(u^i \circ \eta_{p,R})|^2 \, d\mathrm{vol}_{\mathbb{S}^3} \\
&> 3 \sum_{i=1}^{4} \sum_{j=2}^{\infty} \int_{\mathbb{S}^3} |P_j(u^i \circ \eta_{p,R})|^2 \, d\mathrm{vol}_{\mathbb{S}^3} \\
&= 3 \int_{\mathbb{S}^3} |u_0 \circ \eta_{p,R}|^2 \, d\mathrm{vol}_{\mathbb{S}^3}.
\end{aligned}$$

This is a contradiction. Therefore it holds that

$$P_j(u^i \circ \eta_{p,R}) = 0 \quad \text{for any } 1 \leq i \leq 4 \text{ and any } j \geq 3.$$

Since eigenfunctions associated with μ_2 are x^i $(1 \leq i \leq 4)$, there exists a 4×4 matrix S' depending on p, R such that

$$u_0 \circ \eta_{p,R}(\omega) = S'\omega \quad \text{for any } \omega \in \mathbb{S}^3.$$

And since $u_0 \circ \eta_{p,R}$ is a sphere-valued map, S' must be an orthogonal matrix. Since $S'\omega$ and u_0 are harmonic maps between $\mathbb{S}^3$ and itself, we obtain $\eta_{p,R} \in SO(4)$ by Lemma 3.2. Set $S = S'\eta_{p,R}^{-1}$, and we have

$$u_0(\omega) = S\omega$$

for any $\omega \in \mathbb{S}^3$. ∎

Remark 4.1 The existence of the sequence $\{\rho_j\}_{j=1}^{\infty}$ and the convergence of the sequence of rescaled maps in Theorem 1.9 follow from Theorem 2.2, Theorem 2.3 and Theorem 2.4.

Remark 4.2 The Dirichlet energy $\mathbf{E}(Sx/|x|)$ is independent of $S \in O(4)$ and $Sx/|x|$ is an energy minimizing map for every $S \in O(4)$. Therefore it seems to be very difficult to analyze the explicit form of S if we do not know the precise behavior of u around an isolated singular point.

We prove a simple corollary to Theorem 1.9.

Proof of Corollary 1.1. From Theorem 1.2, there exists a neighborhood $U \subset \overline{\mathbb{B}^4}$ of $\partial\mathbb{B}^4$ where u is continuous. Also, from Theorem 1.8, $\mathrm{Sing}(u) \cap \mathbb{B}^4$ is a discrete set. Therefore $\mathrm{Sing}(u)$ is itself a discrete set. A topological argument shows that

$$d = \deg(\zeta) = \sum_{\xi \in \mathrm{Sing}(u)} \deg(u, \xi).$$

Since $\deg(u, \xi)$ is equal to $+1$, -1, or 0 for $\xi \in \mathrm{Sing}(u)$ from Theorem 1.9, we obtain the desired result. ∎

Proof of Corollary 1.2. Let $u \in W^{1,2}_{\zeta}(\mathbb{B}^4, \mathbb{S}^3)$ be an energy minimizing map. By the same reason as in the proof of Corollary 1.1, $\mathrm{Sing}(u)$ is non-empty, consists of a finite number of interior points and it holds that

$$0 \neq \deg(\zeta) = \sum_{\xi \in \mathrm{Sing}(u)} \deg(u, \xi).$$

Therefore there exists a singular point $\xi \in \mathrm{Sing}(u)$ around which $\deg(u, \xi) = +1$ or -1. From Theorem 1.9, u is not strictly stable. ∎

Next, we prove Theorem 1.10.

Proof of Theorem 1.10. By Smith's result [38], for any $d \in \mathbb{Z}$ there exists a non-constant harmonic map $v_d \in C^{\infty}(\mathbb{S}^3, \mathbb{S}^3)$ such that $\deg(v_d) = d$. We define the map $u_d \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ to be $u_d(x) = v_d(x/|x|)$. Then, u_d is a weakly harmonic map. We prove that u_d satisfies the equation (1.7) of stationary harmonic maps. For small $\epsilon > 0$, we define the cut-off function $\chi_\epsilon : [0, \infty) \to \mathbb{R}$ to be

$$\begin{cases} \chi_\epsilon(t) = 0 & \text{for } 0 \le t \le 2\epsilon, \\ \chi_\epsilon(t) = 1 & \text{for } t \ge 4\epsilon, \\ \chi_\epsilon(t) = \dfrac{1}{2\epsilon} t - 1 & \text{for } 2\epsilon \le t \le 4\epsilon. \end{cases}$$

For any $\eta \in C_0^\infty(\mathbb{B}^4, \mathbb{R}^m)$, the support of $\chi_\epsilon(r)\eta$ is contained in $\mathbb{B}^4 \backslash \mathbb{B}^4_\epsilon(0)$. Since u is continuous on $\mathbb{B}^4 \backslash \mathbb{B}^4_\epsilon(0)$, $u|_{\mathbb{B}^4 \backslash \mathbb{B}^4_\epsilon(0)}$ is a stationary harmonic map (see Remark 1.3). $\chi_\epsilon \eta$ belongs to $W_0^{1,2} \cap L^\infty(\mathbb{B}^4 \backslash \mathbb{B}^4_\epsilon(0), \mathbb{R}^4)$, and we can take a $\chi_\epsilon \eta$ as a test function in Definition 1.4 by a density argument. Therefore we have

$$\int_{\mathbb{B}^4 \backslash \mathbb{B}^4_\epsilon(0)} \left\{ |\nabla u_d|^2 \operatorname{div}(\chi_\epsilon \eta) - 2 \frac{\partial u_d^i}{\partial x^\alpha} \frac{\partial u_d^i}{\partial x^\beta} \frac{\partial}{\partial x^\beta} (\chi_\epsilon \eta^\alpha) \right\} dx = 0.$$

The homogeneity of u_d implies that

$$\begin{aligned} \frac{\partial u_d^i}{\partial x^\alpha} \frac{\partial u_d^i}{\partial x^\beta} \frac{\partial}{\partial x^\beta} (\chi_\epsilon \eta^\alpha) &= \frac{\partial u_d^i}{\partial x^\alpha} \frac{\partial u_d^i}{\partial x^\beta} \left(\chi_\epsilon^{'}(r) \frac{x^\beta}{r} \eta^\alpha + \chi_\epsilon \frac{\partial \eta^\alpha}{\partial x^\beta} \right) \\ &= \chi_\epsilon \frac{\partial u_d^i}{\partial x^\alpha} \frac{\partial u_d^i}{\partial x^\beta} \frac{\partial \eta^\alpha}{\partial x^\beta}. \end{aligned}$$

Thus it holds that

$$\begin{aligned} &\int_{\mathbb{B}^4 \backslash \mathbb{B}^4_\epsilon(0)} \chi_\epsilon \left\{ |\nabla u_d|^2 \operatorname{div}(\eta) - 2 \frac{\partial u_d^i}{\partial x^\alpha} \frac{\partial u_d^i}{\partial x^\beta} \frac{\partial \eta^\alpha}{\partial x^\beta} \right\} dx \\ =& - \int_{\mathbb{B}^4_{4\epsilon}(0) \backslash \mathbb{B}^4_{2\epsilon}(0)} |\nabla u_d|^2 \frac{1}{2\epsilon} \frac{x}{r} \cdot \eta \, dx \\ =& - \frac{1}{2\epsilon} \int_{2\epsilon}^{4\epsilon} r \, dr \int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2 \omega \cdot \eta \, d\mathrm{vol}_{\mathbb{S}^3}(\omega). \end{aligned} \tag{4.7}$$

Furthermore we have

$$\begin{aligned} &\left| \frac{1}{2\epsilon} \int_{2\epsilon}^{4\epsilon} r \, dr \int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2 \, d\mathrm{vol}_{\mathbb{S}^3} \right| \\ \le& \frac{1}{2\epsilon} \operatorname*{Sup}_{\mathbb{B}^4} |\eta| \int_{2\epsilon}^{4\epsilon} r \, dr \int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2 \, d\mathrm{vol}_{\mathbb{S}^3} \\ =& 3\epsilon \operatorname*{Sup}_{\mathbb{B}^4} |\eta| \int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} u_0|^2 \, d\mathrm{vol}_{\mathbb{S}^3}. \end{aligned}$$

Therefore taking the limit as $\epsilon \searrow 0$ in (4.7), we obtain the desired result. ∎

§4.2 Some remarks

In this section we collect some important remarks.

Theorem 4.1 The map $x/|x| \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ is energy minimizing and satisfies $\lambda(x/|x|) = 0$.

Proof. Due to [21] we know that $x/|x| \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ is an energy minimizing map. In particular, $x/|x|$ is weakly stable and this implies $\lambda(x/|x|) \geq 0$. On the other hand, since $\deg(x/|x|\big|_{\mathbb{S}^3}) = \deg(\mathrm{id}_{\mathbb{S}^3}) = 1$, we have $\lambda(x/|x|) = 0$ by Corollary 1.2. ∎

From this theorem, there exists a sequence $\{\psi_k\}_{k=1}^{\infty} \subset W_0^{1,2} \cap L^\infty(\mathbb{B}^4, \mathbb{S}^3)$ satisfying $\psi_k(x) \cdot x/|x| = 0$ for almost every $x \in \mathbb{B}^4 \backslash \{0\}$ and

$$\frac{\delta^2_{x/|x|}\mathbf{E}(\psi_k)}{\displaystyle\int_{\mathbb{B}^4} r^{-2}|\psi_k|^2\,dx} \to 0, \tag{4.8}$$

where $r = |x| = \mathrm{dist}(x, \mathrm{Sing}(x/|x|))$. However we cannot get such a sequence in the proof above. We shall give an alternative proof by constructing one following [25].

An alternative Proof of Theorem 4.1. We set $u_0 = \mathrm{id}_{\mathbb{S}^3}$, and define $n_j(x)$ $(1 \leq j \leq 3)$ to be

$$n_1(x) = \frac{1}{r}\begin{pmatrix} x^2 \\ -x^1 \\ -x^4 \\ x^3 \end{pmatrix}, \; n_2(x) = \frac{1}{r}\begin{pmatrix} -x^4 \\ x^3 \\ -x^2 \\ x^1 \end{pmatrix}, \; n_3(x) = \frac{1}{r}\begin{pmatrix} -x^3 \\ -x^4 \\ x^1 \\ x^2 \end{pmatrix}, \text{ where } r = |x|.$$

For any $x \in \mathbb{B}^4 \backslash \{0\}$, $\{u(x),\ n_1(x),\ n_2(x),\ n_3(x)\}$ is an orthonormal basis of $\mathbb{R}^4$. We consider the map

$$\psi(x) = \sum_{j=1}^{3} r f_j(x) n_j(x), \quad \text{where} \quad f_j \in C_0^1(\mathbb{B}^4, \mathbb{R}).$$

Then, ψ belongs to $C_0^1(\mathbb{B}^4, \mathbb{R}^4)$ and satisfies $\psi(x) \cdot u(x) = 0$ for any $x \in \mathbb{B}^4 \backslash \{0\}$. We set

$$\frac{\partial}{\partial n_1} = \frac{1}{r}\left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4} \right),$$
$$\frac{\partial}{\partial n_2} = \frac{1}{r}\left(- x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4} \right),$$
$$\frac{\partial}{\partial n_3} = \frac{1}{r}\left(- x^3 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^4} \right).$$

A simple computation shows

$$r^{-2}|\psi|^2 = \sum_{j=1}^{3} |f_j|^2, \tag{4.9}$$

$$\left| \frac{\partial \psi}{\partial r} \right|^2 = \sum_{j=1}^{3} \left(|f_j|^2 + 2r f_j \frac{\partial f_j}{\partial r} + r^2 \left| \frac{\partial f_j}{\partial r} \right|^2 \right), \tag{4.10}$$

$$\left| \frac{\partial \psi}{\partial n_1} \right|^2 = \sum_{j=1}^{3} \left(|f_j|^2 + r^2 \left| \frac{\partial f_j}{\partial n_1} \right|^2 \right) + 2r \left(f_2 \frac{\partial f_3}{\partial n_1} - f_3 \frac{\partial f_2}{\partial n_1} \right), \tag{4.11}$$

$$\left| \frac{\partial \psi}{\partial n_2} \right|^2 = \sum_{j=1}^{3} \left(|f_j|^2 + r^2 \left| \frac{\partial f_j}{\partial n_2} \right|^2 \right) + 2r \left(f_3 \frac{\partial f_1}{\partial n_2} - f_1 \frac{\partial f_3}{\partial n_2} \right), \tag{4.12}$$

$$\left| \frac{\partial \psi}{\partial n_3} \right|^2 = \sum_{j=1}^{3} \left(|f_j|^2 + r^2 \left| \frac{\partial f_j}{\partial n_3} \right|^2 \right) + 2r \left(f_1 \frac{\partial f_2}{\partial n_3} - f_2 \frac{\partial f_1}{\partial n_3} \right). \tag{4.13}$$

From (4.9) – (4.13) we obtain by integration by parts,

$$\begin{aligned} \delta_u^2 \mathbf{E}(\psi) = \sum_{j=1}^{3} \int_{\mathbb{B}^4} (r^2 |\nabla f_j|^2 - 3|f_j|^2)\, dx \\ + 4 \int_{\mathbb{B}^4} r \left(f_1 \frac{\partial f_2}{\partial n_3} + f_2 \frac{\partial f_3}{\partial n_1} + f_3 \frac{\partial f_1}{\partial n_2} \right) dx. \end{aligned} \tag{4.14}$$

We choose f_1, f_2, f_3 in such a way that

$$f_1(x) = a(r)(-x^1 + x^2 + x^3 - x^4),$$
$$f_2(x) = a(r)(x^1 - x^2 + x^3 - x^4),$$
$$f_3(x) = a(r)(x^1 + x^2 - x^3 - x^4),$$

where a is a smooth function on $[0,1]$ satisfying $a(1)=0$. Due to the symmetry of the domain, we have

$$\sum_{j=1}^{3}\int_{\mathbb{B}^4}|f_j|^2dx = 3\int_{\mathbb{B}^4} r^2|a(r)|^2\,dx = 3\omega_3\int_0^1 r^5|a(r)|^2\,dr, \tag{4.15}$$

$$\begin{aligned}\sum_{j=1}^{3}\int_{\mathbb{B}^4} r^2|\nabla f_j|^2\,dx &= 3\int_{\mathbb{B}^4}(r^4|a'(r)|^2+2r^3a(r)a'(r)+4r^2|a(r)|^2)\,dx\\ &= 3\omega_3\int_0^1(r^7|a'(r)|^2-2r^5|a(r)|^2)\,dr,\end{aligned} \tag{4.16}$$

where ω_3 is the volume of $\mathbb{S}^3$. On the other hand, it holds that

$$\frac{\partial f_2}{\partial n_3} = -\frac{1}{r}a(r)(-x^1+x^2+x^3-x^4) = -\frac{1}{r}f_1(x).$$

Therefore by calculating as before, we obtain

$$\int_{\mathbb{B}^4} rf_1\frac{\partial f_2}{\partial n_3}\,dx = -\omega_3\int_0^1 r^5|a(r)|^2\,dr. \tag{4.17}$$

Other terms can be treated in the same way. It follows from (4.14) – (4.17) that

$$\delta_u^2\mathbf{E}(\psi) = 3\omega_3\int_0^1(r^7|a'(r)|^2-9r^5|a(r)|^2)\,dr. \tag{4.18}$$

From (4.9), (4.15) and (4.18) we have

$$\frac{\delta_u^2E(\psi)}{\displaystyle\int_{\mathbb{B}^4} r^{-2}|\psi|^2\,dx} = \frac{\displaystyle\int_0^1 r^7|a'(r)|^2\,dr}{\displaystyle\int_0^1 r^5|a(r)|^2\,dr} - 9.$$

It remains to determine $a(r)$. For a positive integer k, we define $a_k(r)$ to be

$$a_k(r) = (1-r^{\frac{1}{k}})^k.$$

In the following calculation, we use the beta and gamma functions

$$\begin{aligned}&\mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \quad \text{for } p>0,\ q>0,\\ &\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,dt \quad \text{for } p>0,\end{aligned}$$

and the well-known relation

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{for p} > 0,\ \text{q} > 0.$$

By the change of variable $s = r^{\frac{1}{k}}$, we obtain

$$\int_0^1 r^5 |a_k(r)|^2\, dr = k\mathrm{B}\left(6k,\ 2k+1\right),$$
$$\int_0^1 r^7 |a_k'(r)|^2\, dr = k\mathrm{B}\left(6k+2,\ 2k-1\right),$$

and

$$\frac{\displaystyle\int_0^1 r^7 |a_k'(r)|^2\, dr}{\displaystyle\int_0^1 r^5 |a_k(r)|^2\, dr} = \frac{\Gamma\left(6k+2\right)\Gamma(2k-1)}{\Gamma\left(8k+1\right)} \frac{\Gamma(8k+1)}{\Gamma(6k)\Gamma(2k+1)}$$

$$= \frac{6k(6k+1)}{2k(2k-1)} \to 9 \quad \text{as } k \to \infty.$$

Consequently along the sequence

$$\psi_k(x) = \sum_{j=1}^{3} r f_{j,k}(x) n_j(x),$$
$$f_{1,k}(x) = a_k(r)(-x^1 + x^2 + x^3 - x^4),$$
$$f_{2,k}(x) = a_k(r)(x^1 - x^2 + x^3 - x^4),$$
$$f_{3,k}(x) = a_k(r)(x^1 + x^2 - x^3 - x^4),$$

the ratio

$$\frac{\delta_u^2 \mathbf{E}(\psi_k)}{\displaystyle\int_{\mathbb{B}^4} r^{-2} |\psi_k|^2\, dx}$$

tends to 0 as $k \to \infty$. Consequently we have $\lambda(x/|x|) = 0$. ∎

The global frame $\{n_1(\omega), n_2(\omega), n_3(\omega)\}$ on $T\mathbb{S}^3$ makes the computation clear. This frame corresponds to the quaternion algebra. There may be some relation between this structure and stability of harmonic maps. This will be one of our future problems.

In the proof of Theorem 1.9, it is important to use the following fact: $\lambda(u)$ must be equal to 0 for any weakly stable homogeneous harmonic map $u \in W^{1,2}(\mathbb{B}^4, \mathbb{S}^3)$ satisfying $\mathrm{Sing}(u) = \{0\}$ and $\deg(u, 0) \neq 0$. Here we call this *non strict stability phenomenon*. We discuss whether this phenomenon is special to the case of maps from a 4-dimenisonal domain into $\mathbb{S}^3$ or not. In the case of maps from $\mathbb{B}^3$ into $\mathbb{S}^2$, Mou [24] proved the following result. For any homogeneous weakly harmonic map $u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$, it holds that

$$\lambda(u) \geq \frac{1}{4}.$$

Therefore non strict stability phenomenon cannot occur in this case. From this it seems to be difficult to give an alternative proof of Brezis-Coron-Lieb's result by using the method in the present monograph.

Next, we consider the case of maps from $\mathbb{B}^m$ into $\mathbb{S}^{m-1}$ for $m \geq 5$. For a homogenous harmonic map $u \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^{m-1})$, any estimate on $\lambda(u)$ has not yet been established by anyone except the case $u(x) = x/|x|$. Baldes [2] proved an equality

$$\lambda\Big(\frac{x}{|x|}\Big) = \frac{(m-4)^2}{4}. \tag{4.19}$$

(Baldes did not give any minimizing sequence.) From (4.19), it is natural to conjecture that the inequality

$$\lambda(u) \geq \lambda\Big(\frac{x}{|x|}\Big) = \frac{(m-4)^2}{4}$$

holds for any weakly stable homogeneous harmonic map $u \in W^{1,2}(\mathbb{B}^m, \mathbb{S}^{m-1})$ satisfying $\mathrm{Sing}(u) = \{0\}$, and that non strict stability phenomenon does not occur. Because of this, it also seems difficult to analyze the behavior of harmonic maps around singular points in this case by our method.

Non strict stability phenomenon is a special phenomenon for the case of maps from a 4-dimensional domain into $\mathbb{S}^3$. There must be some reason for this phenomenon. And we should try to understand what is special.

For harmonic maps from $\mathbb{B}^m$ into $\mathbb{S}^{m-1}$ for $m \geq 5$, the author does not know what type of singularities may occur. It seems very difficult to analyze these singularities. However, the author believes that these problems will reveal interesting phenomena and will prompt the new development in the future study of singularity of harmonic maps.

Bibliography

[1] F. Almgren Jr. and E. Lieb, *Singularities of energy minimizing maps from balls to the sphere,* Ann. of Math. (2) **128** (1988), 483–530.

[2] A. Baldes, *Stability and uniqueness properties of the equator map from a ball into an ellipsoid,* Math. Z. **185** (1984), 505–516.

[3] F. Bethuel, *On the singular set of stationary harmonic maps,* Manuscripta Math. **78** (1983), 417–443.

[4] F. Bethuel, H. Brezis and J.M. Coron, *Relaxed energies for harmonic maps,* Variational Methods (ed.: H. Berestycki, J. M. Coron and I. Ekeland), 37–52, Progr. Nonlinear Differential Equations Appl. **4**, Birkhäuser, Boston, 1990.

[5] H. Brezis, J. M. Coron and E. Lieb, *Harmonic maps with defects,* Comm. Math. Phys. **107** (1986), 649–705.

[6] J.-M. Coron and R. Gulliver, *Minimizing p-harmonic maps into spheres,* J. Reine Angew. Math. **401** (1989), 82–100.

[7] J. Eells and L. Lemaire, Selected Topics in Harmonic Maps. CBMS Reg. Conf. Ser. Math. **50**, Amer. Math. Soc., Providence, 1983.

[8] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. **86** (1964), 109–160.

[9] L. C. Evans, *Partial regularity for stationary harmonic maps into spheres,* Arch. Ration. Mech. Anal. **116** (1991), 101–113.

[10] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, Stud. Adv. Math., CRC Press, Boca Raton, 1992.

[11] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Monogr. Math. **80**, Birkhäuser-Verlag, Basel, 1984.

[12] R. S. Hamilton, Harmonic maps of manifolds with boundary, Lecture Notes in Math. **471**, Springer-Verlag, Berlin, 1975.

[13] R. Hardt, D. Kinderlehrer and F. H. Lin, *The variety of configurations of static liquid crystals,* Variational Methods (ed.: H. Berestycki, J. M. Coron and I. Ekeland), 115–131, Progr. Nonlinear Differential Equations Appl. **4**, Birkhäuser, Boston, 1990.

[14] R. Hardt and F. H. Lin, *Stability of singularities of minimizing harmonic maps,* J. Differential Geom. **29** (1989), 113–123.

[15] R. Hardt and L. Mou, *Harmonic maps with fixed singular sets,* J. Geom. Anal. **2** (1992), 445–488.

[16] S. Hildebrandt and K. O. Widman, *Some regularity results for quasilinear elliptic systems of second order,* Math. Z. **142** (1975), 67–86.

[17] M. C. Hong, *On the Hausdorff dimension of the singular set of stable stationary harmonic maps,* Comm. Partial Differential Equations **24** (1999), 1967–1985.

[18] M. C. Hong and C. Wang, *On the singular set of stable stationary harmonic maps,* Calc. Var. Partial Differential Equations **9** (1999), 141–156.

[19] W. Jäger and H. Kaul, *Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems,* J. Reine Angew. Math. **343** (1983), 146–161.

[20] R. S. Kulkarni, *Conformal structures and Möbius structures,* Conformal Geometry (ed.: R. S. Kulkarni & U. Pinkall), 1–39, Aspects Math. **E12**, Friedr Vieweg & Sohr, Braunschweig, 1988.

[21] F. H. Lin, *Une remarque sur l'application* $x/|x|$, C. R. Acad. Sci. Paris. Sér. I Math. **305** (1987), 529–531.

[22] S. Luckhaus, *Partial Hölder continuity for minima of certain energies among maps into a Rimannian manifold,* Indiana Univ. Math. J. **37** (1988), 349–367.

[23] C. B. Morrey, Jr., Multiple Integrals in the Calculus of Variations, Grundlehren Math. Wiss. **153**, Springer, Berlin Heidelberg New York, 1966.

[24] L. Mou, *Harmonic maps with prescribed finite singularities,* Comm. Partial Differential Equations **14** (1989), 1509–1540.

[25] T. Nakajima, *Stability of the energy minimizing map* $x/|x|$, Comm. Partial Differential Equations **26** (2001), 1175–1181.

[26] T. Nakajima, *Stability and singularity of harmonic maps into 3-spheres,* to appear in Nonlinear Anal.

[27] T. Nakajima, *Stability and singularities of harmonic maps into spheres,* Doctoral Thesis, Tohoku University, 2002.

[28] T. Okayasu, *Regularity of minimizing harmonic maps into* $\mathbb{S}^4, \mathbb{S}^5$ *and symmetric spaces,* Math. Ann. **298** (1994), 193–205.

[29] J. Ramanathan, *A remark on the energy of harmonic maps between spheres,* Rocky Mountain J. Math. **16** (1986), 783–790.

[30] T. Rivière, *Everywhere discontinuous harmonic maps into spheres,* Acta Math. **175** (1995), 197–226.

[31] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres,* Ann. of Math. (2) **113** (1981), 1–24.

[32] R. Schoen, *Analytic aspects of harmonic map problem.* Seminar on Nonlinear Partial Differential Equations (ed.: S. S. Chern), 321–358, Math. Sci. Res. Inst. Publ. **2**, Springer, New York Berlin, 1984.

[33] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps,* J. Differential Geom. **17** (1982), 307–335.

[34] R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps,* J. Differential Geom. **18** (1983), 253–268

[35] R. Schoen & K. Uhlenbeck, *Regularity of minimizing harmonic maps into the sphere,* Invent. Math. **78** (1984), 89–100.

[36] L. Simon, Lectures on Geometric Measure Theory, Proc. Centre Math. Anal. Austral. Nat. Univ. **3**, Australian National University, Canberra, 1983.

[37] L. Simon, Theorems on Regularity and Singularity of Energy Minimizing Maps, Lectures Math. ETH Zürich, Birkhäuser, Zürich, 1996.

[38] R. T. Smith, *Harmonic mappings of spheres,* Amer. J. Math. **97** (1975), 364–385.

[39] S. Takizawa, Introduction to Geometry (Japanese), Asakura, Tokyo, 1967.

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