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Geometric and analytic properties in the behavior of random walks on nilpotent covering graphs

by

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A thesis presented

by

Satoshi Ishiwata

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Contents

Chapter 1

Introduction

Asymptotic behavior of random walks on various infinite graphs has been studied in many fields such as probability, harmonic analysis, geometry and so on. Especially, many authors have investigated the problem of what kind of structure of underlying graphs affects the behavior of the random walks. For example, it is known that the notion of volume growth plays an important role in the behavior of the symmetric random walks on finitely generated infinite discrete groups ([30], [35], [36]).

In this thesis, we study long time asymptotics of random walks on nilpotent covering graphs and investigate their applications. In our arguments, the polynomial volume growth and the periodicity of the nilpotent covering graphs play an essential role.

On graphs of polynomial volume growth, various estimates for the transition probability are controlled by analytic properties of the graphs. For instance, Coulhon and Grigor'yan [8] proved the equivalence between the Gaussian upper estimate for the transition probability with volume doubling property and an isoperimetric type inequality known as the relative Faber-Krahn inequality (see also Delmotte [10], Hebisch and Saloff-Coste [15], Russ [28]). Moreover, the Gaussian upper estimate is equivalent to the on-diagonal upper bound ([15]), which is applicable to our arguments (see Section 4.1).

On the other hand, Kotani and Sunada obtained several long time asymptotics for random walks on the graphs with abelian periodic structure by certain homogenization ([19], [21], [22]). Generally speaking, the homogenization is a method which relates a long time asymptotic behavior of the heat kernel of a periodic system to the behavior of the heat kernel of the corresponding homogenized system by making use of a scaling relation between the time and the underlying space (see [4], [5], [9]). However, since the notion of the scale change on graphs is not defined, it is not possible to apply them directly to the case of graphs. In order to overcome this difficulty, Kotani and Sunada considered the realization of the graph, preserving the periodicity, in a space on which a scaling is defined. In their method, it is very important to find a suitable space in which the graph is realized.

In view of these on graphs having the geometric structures such as the polynomial volume growth and the periodicity, we may expect to obtain more sophisticated estimates for long time behavior of the random walks than those obtained assuming only one of these structures. This leads us to study the random walks on nilpotent covering graphs. Indeed, we can regard every covering graph of polynomial volume growth as a nilpotent covering graph. To be more precise, let X be a covering graph whose covering transformation group Γ is a finitely generated group of polynomial growth. Then Gromov ([14]) showed that Γ has a finitely generated torsion-free nilpotent subgroup N of finite index so that X is a covering of the finite quotient graph $N\backslash X$ with covering transformation group N (see also [1]). This is the reason why we consider the nilpotent covering graphs.

In this thesis, we first study a central limit theorem on nilpotent covering graphs following the method of Kotani and Sunada ([19], [21], [22]). Realizing the graph in question in a corresponding nilpotent Lie group, we obtain a geometric characterization of the limit operator on the nilpotent Lie group appearing in the limit of the discrete semigroup of the transition operators, as time goes to infinity with a suitable scale change (Theorems 1 and 2). Next, we consider a long time asymptotics of the transition probability which is called a local central limit theorem or a Berry-Esseen type estimate (Theorem 3). This is the main result of this thesis. In the proof of Theorem 3, certain Gaussian bounds for the transition probability is of essential use (Theorem 4). Finally, as an application of Theorems 3 and 4, we prove the L^p boundedness of the Riesz transform on nilpotent covering graphs (Theorem 5) .

1.1 Notation and Results

Let $X = (V, E)$ be a locally finite connected graph, where V represents the set of vertices and E the set of oriented edges. For each oriented edge $e \in E$, the origin and the terminus of e are denoted by $o(e)$ and $t(e)$, respectively, whereas the inverse edge is denoted by $\overline{e}$. Throughout this thesis, we shall assume that X is a nilpotent covering graph, that is, X is a covering graph of a finite graph $X_0 = (V_0, E_0)$, whose covering transformation group Γ is a finitely generated nilpotent group. Without loss of generality, we may assume that Γ is torsion-free (see Alexopoulos [1]).

A symmetric random walk on X with a weight $m : V \to \mathbb{R}_{>0}$ is, by definition, given by a function $p : E \to \mathbb{R}_{>0}$ satisfying

$$\sum_{e \in E_x} p(e) = 1,$$

$$p(e)m(o(e)) = p(\overline{e})m(t(e)),$$

where $E_x = \{e \in E \,|\, o(e) = x\}$. We assume that m and p are Γ-invariant. Then the transition probability for a particle starting at x to reach y at time n is given by

$$p_n(x, y) = \sum_{c=(e_1,e_2,\ldots,e_n)} p(e_1)p(e_2)\cdots p(e_n),$$

where the sum is taken over all paths $c = (e_1, e_2, \ldots, e_n)$ of length n with origin $o(c) = x$ and terminus $t(c) = y$. The transition operator L associated with the random walk is an operator acting on a function f on V defined by

$$Lf(x) = \sum_{e \in E_x} f(t(e))p(e).$$

It is easy to check that the function $k_n(x, y) = p_n(x, y)m(y)^{-1}$ gives rise to the kernel function of L^n, namely, $L^n f(x) = \sum_{y \in V} k_n(x, y)f(y)m(y)$. The assumption of m and p implies that $k_n(x, y) = k_n(y, x)$.

The purpose of Chapter 2 is to analyze the long time behavior of the discrete semigroup $\{L^n\}_{n=0}^{\infty}$ on X by using certain homogenization method, which is developed by Kotani and Sunada ([19], [21], [22]). To be more precise, suppose that X is realized in a suitable space M. Let $C_\infty(X)$ be the set of functions on V vanishing at infinity, and $C_\infty(M)$ the set of continuous functions on M vanishing at infinity, respectively. Then we show that L^n on $C_\infty(X)$ converges to a continuous semigroup on $C_\infty(M)$, as n goes to infinity with a suitable scale change on M.

In [19] and [22], Kotani and Sunada studied the case of a crystal lattice X, which is an abelian covering of a finite graph. In this case, X is realized in an Euclidean space on which the abelian action of X is isomorphic to a lattice.

In the case of a nilpotent covering graph X with covering transformation group Γ, we realize X in a connected and simply connected nilpotent Lie group G_Γ, in which Γ is isomorphic to a lattice. It is known by Malćev [23] that there exists uniquely such a nilpotent Lie group up to isomorphism. Let $\mathfrak{g}$ be the Lie algebra of G_Γ and $\exp : \mathfrak{g} \to G_\Gamma$ the exponential map. Set $n_1 = \mathfrak{g}$ and $n_{i+1} = [\mathfrak{g}, n_i]$ for $i \geq 1$. Since $\mathfrak{g}$ is nilpotent, we then have the filtration :

$$\mathfrak{g} = n_1 \supset n_2 \supset \cdots \supset n_r \neq \{0\} \supset n_{r+1} = \{0\}.$$

We also consider the subspaces $\mathfrak{g}^{(1)}, \ldots, \mathfrak{g}^{(r)} \subset \mathfrak{g}$ defined by

$$n_k = \mathfrak{g}^{(k)} \oplus n_{k+1}. \tag{1.1}$$

A piecewise smooth Γ-equivariant map $\Phi : X \to G_\Gamma$ is said to be a realization of X. In particular, the following notion of the harmonic realization is important to consider the long time behavior of the discrete semigroup $\{L^n\}_{n=0}^\infty$.

Definition (cf. [21]). *A realization $\Phi^h : X \to G_\Gamma$ is said to be harmonic on $\mathfrak{g}^{(1)}$ if for each $x \in V$,*

$$\sum_{e \in E_x} m(e) \left\{ \exp^{-1} \Phi^h(t(e))\big|_{\mathfrak{g}^{(1)}} - \exp^{-1} \Phi^h(o(e))\big|_{\mathfrak{g}^{(1)}} \right\} = 0,$$

where $m(e) = p(e)m(o(e))$.

According to the result in [21] of harmonic maps from a graph to a Riemannian manifold, we have the existence and uniqueness of Φ^h on $\mathfrak{g}^{(1)}$ (see also Section 2.3). By making use of the harmonic realization Φ^h, the limit operator, that is, the limit of the infinitesimal generator of L^n (see Lemma 2.5), is written as

$$\Omega_* = -\frac{1}{2}\sum_{e\in E_0} m(e)\left(\exp^{-1}\Phi^h(o(e))^{-1}\Phi^h(t(e))\big|_{\mathfrak{g}^{(1)}}\right)_*^2, \tag{1.2}$$

where $\left(\exp^{-1}\Phi^h(o(e))^{-1}\Phi^h(t(e))\big|_{\mathfrak{g}^{(1)}}\right)_*$ is the extension of an element of the Lie algebra $\exp^{-1}\Phi^h(o(e))^{-1}\Phi^h(t(e))\big|_{\mathfrak{g}^{(1)}} \in \mathfrak{g}$ to a left invariant vector field on the limit group $(G_\Gamma, *)$ (see Section 2.1). Then, by using Trotter's approximation theory [33], we have the following

Theorem 1 (Central limit theorem). *Let X be the covering graph of a finite graph X_0 whose covering transformation group Γ is a finitely generated torsion-free nilpotent group and $\Phi : X \to G_\Gamma$ a realization of X. Then, for any $f \in C_\infty(G_\Gamma)$, as $n \uparrow \infty, \delta \downarrow 0$ and $n\delta^2 \to m(X_0)t$, we have*

$$\left\| L^n(f\circ(\tau_\delta\Phi)) - (e^{-t\Omega_*}f)\circ(\tau_\delta\Phi)\right\|_\infty \to 0, \tag{1.3}$$

where τ_δ is the dilation on G_Γ (see Section 2.1). In particular, for a sequence $\{x_\delta\}_{\delta>0}$ in X with $\lim_{\delta\downarrow 0}\tau_\delta\Phi(x_\delta) = x$,

$$\lim L^n(f\circ(\tau_\delta\Phi))(x_\delta) = e^{-t\Omega_*}f(x). \tag{1.4}$$

The proof of Theorem 1 is reduced to the case when the realization is harmonic (see the proof).

We remark that Batty, Bratteli, Jørgensen and Robinson considered a homogenization for periodic subelliptic operators on stratified Lie groups in [5]. In their case, a scaling relation between the time and the stratified Lie group (see Section 2.1) is indispensable to obtain the convergence to the homogenized operator. In our proof of Theorem 1, an invariance under the stratifying process (see Lemma 2.2) plays an important role . We also note that, by Pansu [24], the limit group $(G_\Gamma, *)$ is the Gromov-Hausdorff limit of the sequence of metric spaces $(X, \epsilon d_X)$ as ϵ goes to 0, where d_X is the graph distance of X.

By using a relation between $\mathfrak{g}^{(1)}$ and $\mathrm{H}^1(X_0, \mathbb{R})$, the first cohomology group of X_0 (see Section 2.4), we prove the following geometric characterization of the limit operator Ω_*:

Theorem 2. *Ω_* is the sub-Laplacian with respect to the Albanese metric on $\mathfrak{g}^{(1)}$ (see Section 2.4), namely*

$$\Omega_* = -\sum_{i=1}^{d_1} X_{i*}^{(1)} X_{i*}^{(1)},$$

where $\{X_1^{(1)}, \ldots, X_{d_1}^{(1)}\}$ is an orthonormal basis for the Albanese metric on $\mathfrak{g}^{(1)}$ and $X_{i*}^{(1)}$ is the extension of $X_i^{(1)} \in \mathfrak{g}$ to a left invariant vector field on the limit group $(G_\Gamma, *)$.

In Chapter 3, we prove a Berry-Esseen type theorem, which gives an estimate for the speed of convergence of the transition probability to the heat kernel on G_Γ as time goes to infinity. We remark that Alexopoulos proved a Berry-Esseen type theorem on a Cayley graph of a finitely generated discrete group of polynomial growth Γ ([1]). To explain it, let $p_n(x, y)$ be a transition probability associated with the symmetric probability measure on Γ, whose support is finite and generates Γ. Let h_t be the heat kernel of the limit operator associated with the probability measure on the nilpotent Lie group G_Γ (see [1]). Then we have the following

Theorem (Alexopoulos [1, Theorem 10]). *Let Γ be a finitely generated discrete group of polynomial volume growth of order D. Then there exists a constant $C > 0$ such that*

$$\sup_{x,y\in\Gamma} |p_n(x, y) - |G_\Gamma/\Gamma| h_n(x, y)| \leq C n^{-\frac{D+1}{2}}.$$

On the other hand, when X is a crystal lattice, a local central limit theorem is proved by Kotani and Sunada [22].

Theorem (Kotani and Sunada [22]). *Let X be a crystal lattice whose covering transformation group is Γ. For simplicity, we assume that X is non-bipartite. Then we have*

$$\lim_{n\uparrow\infty} \left[(4\pi n)^{D/2} p_n(x, y) m(y)^{-1} - C(X) \exp\left(-\frac{m(X_0)}{4n} d_\Gamma(x, y)^2 \right) \right] = 0,$$

uniformly for all $x, y \in V$, where $C(X)$ is a constant depends on X and d_Γ is the Albanese distance.

We study a generalization of these results to the case of nilpotent covering graphs.

Our strategy for the proof of a Berry-Esseen type theorem on nilpotent covering graphs is much inspired by Alexopoulos [1]. Let p_n be the transition probability on X and h_t the heat kernel of the sub-Laplacian Ω on G_Γ for the Albanese metric on $\mathfrak{g}^{(1)}$ (see Theorem 2, Section 2.4 and [16], [21]). Namely, Ω is defined by

$$\Omega = -\frac{1}{2m(X_0)} \sum_{e \in E_0} m(e) \left(\exp^{-1} \Phi^h(o(e))^{-1} \Phi^h(t(e)) \big|_{\mathfrak{g}^{(1)}} \right)^2,$$

where $\Phi^h : X \to G_\Gamma$ is a harmonic realization of X and $\left(\exp^{-1} \Phi^h(o(e))^{-1} \Phi^h(t(e)) \big|_{\mathfrak{g}^{(1)}} \right)$ is a left invariant vector field on G_Γ identified with $\exp^{-1} \Phi^h(o(e))^{-1} \Phi^h(t(e)) \big|_{\mathfrak{g}^{(1)}} \in \mathfrak{g}$. Then we have

Theorem 3 (Berry-Esseen type theorem). *Let X be a nilpotent covering graph with covering transformation group Γ and $\Phi^h : X \to G_\Gamma$ a harmonic realization of X in the nilpotent Lie group G_Γ. Let D denote the exponent of polynomial growth of X. Then, for any $0 < \epsilon < 1/2$, there exists a constant $C > 0$ such that the following hold:*

1. *If X is a non-bipartite graph, then*

$$\sup_{x,y \in V} \left| p_n(x,y) m(y)^{-1} - \frac{|G_\Gamma/\Gamma|}{m(X_0)} h_n(\Phi^h(x), \Phi^h(y)) \right| \le C n^{-\frac{D+1/2-\epsilon}{2}}.$$

2. *If X is a bipartite graph with a bipartition $V = A \coprod B$, and*

   (a) *if $x, y \in A$ or $x, y \in B$, then $p_n(x,y) = 0$ for odd n and*

   $$\sup_{x,y} \left| p_n(x,y) m(y)^{-1} - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)} h_n(\Phi^h(x), \Phi^h(y)) \right| \le C n^{-\frac{D+1/2-\epsilon}{2}}$$

   for even n,

   (b) *if $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then $p_n(x,y) = 0$ for even n and*

   $$\sup_{x,y} \left| p_n(x,y) m(y)^{-1} - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)} h_n(\Phi^h(x), \Phi^h(y)) \right| \le C n^{-\frac{D+1/2-\epsilon}{2}}$$

   for odd n.

We remark that Alexopoulos proved the following estimate of the difference between h_t and h_{*t}, the heat kernel of Ω_*:

Theorem (Alexopoulos [2, Theorem 1.14.5]). *There is a constant $c > 0$ such that*

$$|h_t(x, y) - h_{*t}(x, y)| \leq ct^{-(D+1)/2}$$

for $x, y \in G_\Gamma$ and $t \geq 1$.

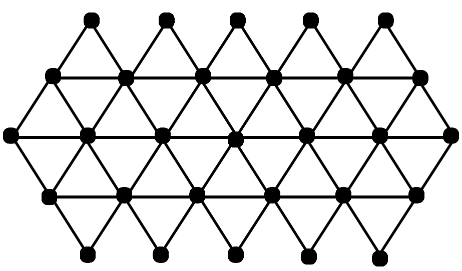
It is not known whether the estimate of Theorem 3 is best possible. In our approach, we have not been able to improve the speed of the convergence better than $Cn^{-\frac{D+1/2-\epsilon}{2}}$, in general. However, if

$$\sum_{e \in E_x} p(e) \exp^{-1} \Phi^h(o(e))^{-1}\Phi^h(t(e))\big|_{\mathfrak{g}^{(2)}} = 0 \tag{1.5}$$

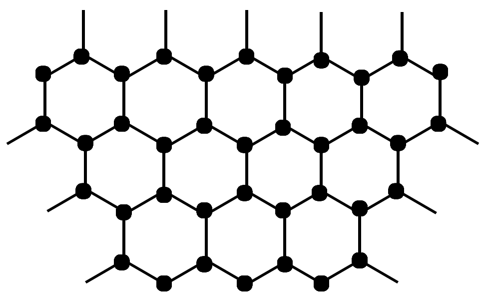
for all $x \in V$, and the second order differential operator on G_Γ

$$\sum_{e \in E_x} p(e) \left(\exp^{-1} \Phi^h(o(e))^{-1}\Phi^h(t(e))\big|_{\mathfrak{g}^{(1)}}\right)^2 \tag{1.6}$$

is independent of the choice of $x \in V$, then the speed of the convergence in Theorem 3 is estimated by $Cn^{-\frac{D+1}{2}}$. Indeed, a simple random walk on a Cayley graph of Γ satisfies (1.5) and (1.6). The triangular lattice and the hexagonal lattice (see Figure 1 and [22]) also satisfy these conditions. However, there exist graphs which do not satisfy them. For example, the Kagome lattice and the $\mathbb{Z}$-lattice with a loop on even vertices (see Figure 2 and [22]) do not satisfy (1.6).

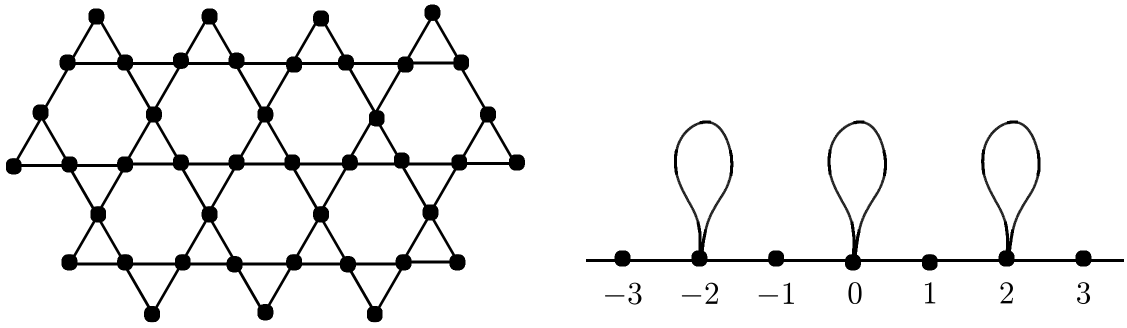


The triangular lattice



The hexagonal lattice

Figure 1. Examples which satisfy (1.5) and (1.6).



The Kagome lattice The $\mathbb{Z}$-lattice with a loop on even vertices

Figure 2. Examples which do not satisfy (1.5) and (1.6).

In the proof of Theorem 3, we employ Gaussian upper estimates for the kernel k_n of L^n and its gradient on nilpotent covering graphs which are obtained in Chapter 4. The definition of the gradient of k_n is given as follows:

1. If X is a non-bipartite graph, then

$$\nabla^y k_n(x,y) = \sup_{d_X(y,z)=1} |k_n(x,z) - k_n(x,y)|.$$

2. If X is a bipartite graph, then

$$\nabla^y k_n(x,y) = \sup_{d_X(y,z)=2} |k_n(x,z) - k_n(x,y)|.$$

We remark that Hebisch and Saloff-Coste [15] gave Gaussian upper estimates for k_n and ∇k_n on a Cayley graph of Γ. Furthermore, Pittet and Saloff-Coste [25] showed that the decay order of the probability of return after $2n$-steps to the starting point does not change under the *quasi-isometry*. Since a nilpotent covering graph X and its covering transformation group Γ are quasi-isometric, the Gaussian upper bound for k_n on X (Theorem 4, (1.7)) is deduced from their results (see also Saloff-Coste [29]). In this thesis, for the sake of completeness, we give a proof of Gaussian estimates for k_n and ∇k_n on X, following the argument by Hebisch and Saloff-Coste [15]. Then we have

Theorem 4 (Gaussian estimates cf. [25], [15]). *There exist constants C and $C' > 0$ such that the following hold:*

1. *If X is a non-bipartite graph,*

$$k_n(x,y) \le Cn^{-\frac{D}{2}} \exp\left(-d_X(x,y)^2/C'n\right) \tag{1.7}$$

$$\nabla^y k_n(x,y) \le Cn^{-\frac{D+1}{2}} \exp\left(-d_X(x,y)^2/C'n\right) \tag{1.8}$$

   for all $x, y \in V$, and all $n = 1, 2, \ldots$.

2. *If X is a bipartite graph with a bipartition $V = A \coprod B$, and*

   (a) *if $x, y \in A$ or $x, y \in B$, then $k_n(x,y) = 0$ for odd n and*

$$k_n(x,y) \le Cn^{-\frac{D}{2}} \exp\left(d_X(x,y)^2/C'n\right),$$
$$\nabla^y k_n(x,y) \le Cn^{-\frac{D+1}{2}} \exp\left(-d_X(x,y)^2/C'n\right)$$

   for even n,

   (b) *if $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then $k_n(x,y) = 0$ for even n and*

$$k_n(x,y) \le Cn^{-\frac{D}{2}} \exp\left(-d_X(x,y)^2/C'n\right),$$
$$\nabla^y k_n(x,y) \le Cn^{-\frac{D+1}{2}} \exp\left(-d_X(x,y)^2/C'n\right)$$

   for odd n.

As a corollary of Theorem 4, by using the same argument as in [15], we prove a Gaussian lower bound for k_n.

Corollary (cf. [15]). *There exist constants C, C' and $C'' > 0$ such that the following hold:*

1. *If X is non-bipartite graph, then*

$$k_n(x,y) \ge Cn^{-\frac{D}{2}} \exp\left(-d_X(x,y)^2/C'n\right)$$

   for all $n \ge \max_{x \in V} \min\{$length of the odd cycle from $x\}$ and $d_X(x,y) \le n/C''$.

2. *If X is a bipartite graph with a bipartition $V = A \coprod B$, and*

(a) *if* $x, y \in A$ *or* $x, y \in B$, *then* $k_n(x,y) = 0$ *for odd* n *and*

$$k_n(x,y) \geq Cn^{-\frac{D}{2}} \exp\left(-d_X(x,y)^2/C'n\right)$$

for even $n \geq 2$ *and* $d_X(x,y) \leq n/C''$,

(b) *if* $x \in A$, $y \in B$ *or* $x \in B$, $y \in A$, *then* $k_n(x,y) = 0$ *for even* n *and*

$$k_n(x,y) \geq Cn^{-\frac{D}{2}} \exp\left(-d_X(x,y)^2/C'n\right)$$

for odd $n \geq 3$ *and* $d_X(x,y) \leq n/C''$.

We note that various applications of this type of estimates have been discussed (for instance, see [8], [10], [34] and [36]).

In Chapter 5, we study the L^p boundedness of the Riesz transform on nilpotent covering graphs which is defined by $\nabla\Delta^{-1/2}$. This is a discrete analogue of $\partial/\partial x_j \Delta^{-1/2}$, the Riesz transform on $\mathbb{R}^d$. It is known that the Riesz transform on $\mathbb{R}^d$ is bounded on L^p for $1 < p < \infty$, which gives an equivalence of the Sobolev space defined by $\partial/\partial x_j$ and $\Delta^{1/2}$ in L^p (see Duoandikoetxea [13], Stein [32]). When X is a Cayley graph of Γ, Alexopoulos [1] proved the L^p boundedness for $1 < p < \infty$ and weak-$(1,1)$. When X is a graph with volume doubling property and the Gaussian upper estimate (1.7) holds, Russ [28] proved that the Riesz transform is bounded on L^p for $1 < p \leq 2$ and weak-$(1,1)$. We remark that nilpotent covering graphs satisfy the assumptions of Russ's theorem.

Consequently, we prove the following result:

Theorem 5 (L^p boundedness of the Riesz transform). *Let* X *be a nilpotent covering graph and assume that* X *is non-bipartite. Then the Riesz transform is bounded on* L^p *for* $1 < p < \infty$ *and weak-*$(1,1)$, *which means that there exists a constant* $C_p > 0$ *such that for all finitely supported functions* f *on* V,

$$\|\nabla f\|_p \leq C_p \|\Delta^{1/2} f\|_p, \quad 1 < p < \infty,$$

and

$$\sup_{\lambda > 0} \lambda m\left(\{x \in V \,:\, |\nabla f(x)| > \lambda\}\right) \leq C_1 \|\Delta^{1/2} f\|_1.$$

Remark 1. There are some developments for the Riesz transform on complete Riemannian manifolds. In [11], Dungey proved that the Riesz transform is L^p bounded for $1 < p < \infty$ on *nilpotent covering manifolds*. His argument can be adapted to the case of nilpotent covering graphs. Moreover, Auscher, Coulhon, Duong, Hofmann [3] obtained a sufficient and necessary condition to be the L^p boundedness for the Riesz transform on some complete Riemannian manifolds (see also [7]).

Throughout this thesis, unless necessary, different constants may be denoted by the same letter C. When their dependence or independence is significant, it will be clearly stated.

Chapter 2

Central limit theorem

In this chapter, we prove a central limit theorem on nilpotent covering graphs by the method of Kotani and Sunada. To prove the convergence of the semigroup, we use the approximation theory due to Trotter [33], that is, we show the convergence of its infinitesimal generator (Lemma 2.5).

First, we will introduce the notion of the limit group, which is obtained by stratifying the original product on a nilpotent Lie group (see also [1], [12]). We remark that an invariance under the stratifying process (see Lemma 2.2) plays an important role in our proof of the central limit theorem.

2.1 Limit group

Let $(G, \cdot)$ be a connected, simply connected nilpotent Lie group and $\mathfrak{g}$ its Lie algebra. We set $n_1 = \mathfrak{g}$ and $n_{i+1} = [\mathfrak{g}, n_i]$ for $i \geq 1$. Since $\mathfrak{g}$ is nilpotent, we have the filtration : $\mathfrak{g} = n_1 \supset n_2 \supset \ldots \supset n_r \neq \{0\} \supset n_{r+1} = \{0\}$. We consider subspaces $\mathfrak{g}^{(1)}, \ldots, \mathfrak{g}^{(r)} \subset \mathfrak{g}$ defined by

$$n_k = \mathfrak{g}^{(k)} \oplus n_{k+1}.$$

By this decomposition, each element $X \in \mathfrak{g}$ can be written uniquely as $X = X^{(1)} + X^{(2)} + \cdots + X^{(k)} + \cdots + X^{(r)}$ with $X^{(k)} \in \mathfrak{g}^{(k)}$. For $\epsilon > 0$, we define a linear operator $T_\epsilon : \mathfrak{g} \to \mathfrak{g}$

by

$$T_\epsilon(X^{(1)} + X^{(2)} + \cdots + X^{(k)} + \cdots + X^{(r)}) = \epsilon X^{(1)} + \epsilon^2 X^{(2)} + \cdots + \epsilon^k X^{(k)} + \cdots + \epsilon^r X^{(r)}.$$

We also define a Lie bracket $[\ ,\ ]^*$ on $\mathfrak{g}$ by setting

$$[X,Y]^* = \lim_{\epsilon \to 0} T_\epsilon [T_{\epsilon^{-1}} X, T_{\epsilon^{-1}} Y].$$

Then, for any $X^{(k)} \in \mathfrak{g}^{(k)}$ and $X^{(\ell)} \in \mathfrak{g}^{(\ell)}$, we have

$$[X^{(k)}, X^{(\ell)}]^* = [X^{(k)}, X^{(\ell)}]\big|_{\mathfrak{g}^{(k+\ell)}} . \tag{2.1}$$

Define the dilation $\tau_\epsilon : G \to G$ by

$$\tau_\epsilon(x) = \exp\left(T_\epsilon\left(\exp^{-1} x\right)\right), \tag{2.2}$$

where $\exp : \mathfrak{g} \to G$ is the exponential map. We define a product $*$ on G by setting

$$x * y = \lim_{\epsilon \to 0} \tau_\epsilon(\tau_{\epsilon^{-1}} x \cdot \tau_{\epsilon^{-1}} y).$$

Then it is known that $(G, *)$ is a nilpotent Lie group, whose Lie algebra is isomorphic to $(\mathfrak{g}, [\ ,\ ]^*)$. We call $(G, *)$ the *limit group* of $(G, \cdot)$. We note that the limit group $(G, *)$ has the following properties (see Alexopoulos [1]):

(a) For $X, Y \in \mathfrak{g}$, $\exp X * \exp Y = \exp\left(X + Y + \frac{1}{2}[X,Y]^* + \cdots [\quad,\quad]^* \cdots\right)$.

(b) The exponential map from $(\mathfrak{g}, [\ ,\ ]^*)$ to $(G, *)$ coincides with the original exponential map.

(c) $(G, *)$ is a stratified Lie group. Namely, the Lie algebra $(\mathfrak{g}, [\ ,\ ]^*)$ of $(G, *)$ has a direct sum decomposition $\oplus_{k=1}^r \mathfrak{g}^{(k)}$ satisfying

   (i) if $k + \ell \le r$, then $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}]^* \subset \mathfrak{g}^{(k+\ell)}$,
   if $k + \ell > r$, then $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}]^* = \{0\}$,

   (ii) $\mathfrak{g}^{(1)}$ generates $\mathfrak{g}$.

(d) $\tau_\delta(x * y) = \tau_\delta x * \tau_\delta y$.

For the sake of completeness, we prove (d). For a fixed $\delta > 0$, we have

$$\begin{aligned}\tau_\delta(x * y) &= \tau_\delta \lim_{\epsilon\to 0} \tau_\epsilon \left(\tau_{\epsilon^{-1}} x \cdot \tau_{\epsilon^{-1}} y\right)\\ &= \lim_{\epsilon\to 0} \tau_{\delta\epsilon} \left(\tau_{(\delta\epsilon)^{-1}} \tau_\delta x \cdot \tau_{(\delta\epsilon)^{-1}} \tau_\delta y\right)\\ &= \tau_\delta x * \tau_\delta y.\end{aligned}$$

By the definition of $*$, we easily obtain

$$\begin{aligned}\exp^{-1}(x * y)\big|_{\mathfrak{g}^{(1)}} &= \exp^{-1}(x \cdot y)\big|_{\mathfrak{g}^{(1)}},\\ \exp^{-1}(x * y)\big|_{\mathfrak{g}^{(2)}} &= \exp^{-1}(x \cdot y)\big|_{\mathfrak{g}^{(2)}}\end{aligned}$$

for any $x, y \in G$. Note that for $k \geq 3$, $\exp^{-1}(x * y)|_{\mathfrak{g}^{(k)}}$ does not coincide with $\exp^{-1}(x \cdot y)|_{\mathfrak{g}^{(k)}}$ in general. The above invariance for $k = 1, 2$ is important to show the central limit theorem.

For each $k \leq r$, let $\{X_1^{(k)}, X_2^{(k)}, \ldots, X_{d_k}^{(k)}\}$ be a basis of $\mathfrak{g}^{(k)}$. We have the following two identifications of G with $\mathbb{R}^n$ as a differentiable manifold, given respectively by

$$(x_{d_r}^{(r)}, x_{d_r-1}^{(r)}, \ldots, x_1^{(1)}) \mapsto \exp x_{d_r}^{(r)} X_{d_r}^{(r)} \cdot \exp x_{d_r-1}^{(r)} X_{d_r-1}^{(r)} \cdot \cdots \cdot \exp x_1^{(1)} X_1^{(1)},$$

and

$$(x_{d_r*}^{(r)}, x_{d_r-1*}^{(r)}, \ldots, x_{1*}^{(1)}) \mapsto \exp x_{d_r*}^{(r)} X_{d_r}^{(r)} * \exp x_{d_r-1*}^{(r)} X_{d_r-1}^{(r)} * \cdots * \exp x_{1*}^{(1)} X_1^{(1)}.$$

We call them $(\cdot)$-coordinates and $(*)$-coordinates of second kind, respectively. For $x \in G$, we denote $P_i^{(k)}(x) = x_i^{(k)}$ and $P_{i*}^{(k)}(x) = x_{i*}^{(k)}$. The following lemma illustrates the relation among these coordinates.

Lemma 2.1. *For $x \in G$, we have*

$$P_{i*}^{(1)}(x) = P_i^{(1)}(x), \tag{2.3}$$

$$P_{i*}^{(2)}(x) = P_i^{(2)}(x), \tag{2.4}$$

$$P_{i*}^{(k)}(x) = P_i^{(k)}(x) + \sum_{0<|K|\leq k-1} C_K P^K(x) \tag{2.5}$$

for some constants C_K, where K denotes a multi-index $((i_1, k_1), \ldots, (i_n, k_n))$ and $P^K(x) = P_{i_1}^{(k_1)}(x) P_{i_2}^{(k_2)}(x) \cdots P_{i_n}^{(k_n)}(x)$. We call $|K| = \sum_{i=1}^n k_n$ the order *of $P^K(x)$.*

Proof. (2.3) and (2.4) are immediate by comparing $(\cdot)$-coordinates and $(*)$-coordinates of $x \in G$. We will show (2.5) by induction in k of $P^{(k)}_{i*}(x)$. Note that the cases $k=1$ and $k=2$ are obvious. We assume that it is true in the case $P^{(\ell)}_{i*}(x)$ for $\ell \leq k-1$. Then the (i,k)-component of x is given by

$$\begin{aligned}\exp^{-1} x\big|_{X^{(k)}_i} &= P^{(k)}_{i*}(x) + \sum_{|K|=k} C_K Pr^{(k)}_i [X^K]^* P^K_*(x)\\ &= P^{(k)}_i(x) + \sum_{0<|K|\leq k} C_K Pr^{(k)}_i [X^K] P^K(x)\end{aligned}$$

for some constants C_K, where $[X^K] = [X^{(k_1)}_{i_1}, [X^{(k_2)}_{i_2}, [X^{(k_3)}_{i_3}, \cdots, X^{(k_n)}_{i_n}]\cdots]$, $[X^K]^* = [X^{(k_1)}_{i_1}, [X^{(k_2)}_{i_2}, [X^{(k_3)}_{i_3}, \cdots, X^{(k_n)}_{i_n}]^* \cdots]^*$ and $Pr^{(k)}_i X = X|_{X^{(k)}_i}$. By the induction hypothesis, the lower order terms do not affect for this claim. Since $C_K Pr^{(k)}_i [X^K]^* = C_K Pr^{(k)}_i [X^K]$ for $|K| = k$ by (2.1), the terms of order k are cancelled. Consequently, we have

$$P^{(k)}_{i*}(x) = P^{(k)}_i(x) + \sum_{0<|K|\leq k-1} C_K P^K(x).$$

□

By using Lemma 2.1, we have the following relation between the $(\cdot)$-coordinates and the $(*)$-coordinates:

Lemma 2.2.

$$P^{(1)}_{i*}(x * y) = P^{(1)}_i(x \cdot y), \tag{2.6}$$

$$P^{(2)}_{i*}(x * y) = P^{(2)}_i(x \cdot y), \tag{2.7}$$

$$P^{(k)}_{i*}(x * y) = P^{(k)}_i(x \cdot y) + \sum_{\substack{|K_1|+|K_2|\leq k-1,\\ |K_2|>0}} C_{K_1K_2} P^{K_1}_*(x) P^{K_2}(x \cdot y). \tag{2.8}$$

Proof. From (2.1), Lemma 2.1 together with the Campbell-Hausdorff formula, (2.6) and (2.7) are obtained easily. We will show (2.8) inductively. By the definition of $*$, Lemma 2.1 and the induction hypothesis, the difference of $P^{(k)}_{i*}(x*y)$ and $P^{(k)}_i(x \cdot y)$ is the terms with order less than k. Namely,

$$P^{(k)}_{i*}(x * y) = P^{(k)}_i(x \cdot y) + \sum_{0<|K_1|+|K_2|\leq k-1} C_{K_1K_2} P^{K_1}(x) P^{K_2}(y). \tag{2.9}$$

By using

$$P_i^{(k)}(y) = P_i^{(k)}(x \cdot y) - P_i^{(k)}(x) - \sum_{0<|K_1|+|K_2|\le k} C_{K_1K_2} P^{K_1}(x) P^{K_2}(y),$$

we can replace $P^{K_2}(y)$ with

$$P^{K_2}(x \cdot y) - \sum_{0<|K_3|+|K_4|\le |K_2|} C_{K_3K_4} P^{K_3}(x) P^{K_4}(x \cdot y) + \sum_{0<|K|\le |K_2|} C_K P^K(x).$$

Hence we refine (2.9) to

$$P_{i*}^{(k)}(x * y) = P_i^{(k)}(x \cdot y) + \sum_{\substack{|K_1|+|K_2|\le k-1, \\ |K_2|>0}} C_{K_1K_2} P^{K_1}(x) P^{K_2}(x \cdot y) + \sum_{0<|K|\le k-1} C_K P^K(x).$$

But $\sum_{0<|K|\le k-1} C_K P^K(x)$ vanishes, since if $y = x^{-1}$, then $x * y = x \cdot y = e$. Moreover, $P^{K_1}(x)$ can be replaced with $P_*^{K_1}(x)$ because of Lemma 2.1. Consequently,

$$P_{i*}^{(k)}(x * y) = P_i^{(k)}(x \cdot y) + \sum_{\substack{|K_1|+|K_2|\le k-1, \\ |K_2|>0}} C_{K_1K_2} P_*^{K_1}(x) P^{K_2}(x \cdot y).$$

□

Example 2.3. *For $k = 3$, we have*

$$\begin{aligned} P_{i*}^{(3)}(x) =& P_i^{(3)}(x) - \frac{1}{2} \sum_{i_1>i_2} Pr_i^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}] P_{i_1}^{(1)}(x) P_{i_2}^{(1)}(x), \\ P_{i*}^{(3)}(x * y) =& P_i^{(3)}(x \cdot y) - \frac{1}{2} \sum_{i_1>i_2} Pr_i^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}] \Big\{ P_{i_1*}^{(1)}(x) P_{i_2}^{(1)}(x \cdot y) \\ & - P_{i_1}^{(1)}(x \cdot y) P_{i_2*}^{(1)}(x) + P_{i_1}^{(1)}(x \cdot y) P_{i_2}^{(1)}(x \cdot y) \Big\}. \end{aligned} \tag{2.10}$$

To show (2.10), we use the following:

$$
\begin{aligned}
P_i^{(3)}(x\cdot y) &= P_j^{(3)}(x) + P_j^{(3)}(y) \\
&+ \frac{1}{2}\sum_{i_1>i_2} Pr_j^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}]\left(P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(x) + P_{i_1}^{(1)}(y)P_{i_2}^{(1)}(y)\right) \\
&+ \frac{1}{2}\sum_{i_1,i_2} Pr_i^{(3)}[X_{i_1}^{(2)}, X_{i_2}^{(1)}]\left(P_{i_1}^{(2)}(x)P_{i_2}^{(1)}(x) + P_{i_1}^{(2)}(y)P_{i_2}^{(1)}(y)\right) \\
&+ \frac{1}{2}\sum_{i_1,i_2} Pr_j^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}]P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(y) + \frac{1}{2}\sum_{i_1,i_2} Pr_j^{(3)}[X_{i_1}^{(2)}, X_{i_2}^{(1)}]P_{i_1}^{(2)}(x)P_{i_2}^{(1)}(y) \\
&+ \frac{1}{2}\sum_{i_1,i_2} Pr_j^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(2)}]P_{i_1}^{(1)}(x)P_{i_2}^{(2)}(y) - \frac{1}{2}\sum_{i_1>i_2} Pr_j^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}]P_{i_1}^{(1)}(x\cdot y)P_{i_2}^{(1)}(x\cdot y) \\
&- \frac{1}{2}\sum_{i_1,i_2} Pr_j^{(3)}[X_{i_1}^{(2)}, X_{i_2}^{(1)}] \\
&\quad\times\left(P_{i_1}^{(2)}(x) + P_{i_1}^{(2)}(y) - \sum_{\nu<\lambda} Pr_{i_1}^{(2)}[X_\lambda^{(1)}, X_\nu^{(1)}]P_\nu^{(1)}(x)P_\lambda^{(1)}(y)\right)P_{i_2}^{(1)}(x\cdot y) \\
&+ \frac{1}{4}\sum_{i_1>i_2>i_3} Pr_j^{(3)}[X_{i_1}^{(1)}, [X_{i_2}^{(1)}, X_{i_3}^{(1)}]]\left(P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(x)P_{i_3}^{(1)}(x) + P_{i_1}^{(1)}(y)P_{i_2}^{(1)}(y)P_{i_3}^{(1)}(y)\right) \\
&+ \frac{1}{12}\sum_{i>i_1,i_2} Pr_j^{(3)}[[X_i^{(1)}, X_{i_1}^{(1)}], X_{i_2}^{(1)}]\left(P_i^{(1)}(x)P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(x) + P_i^{(1)}(y)P_{i_1}^{(1)}(y)P_{i_2}^{(1)}(y)\right) \\
&- \frac{1}{12}\sum_{i>i_1} Pr_j^{(3)}[[X_i^{(1)}, X_{i_1}^{(1)}], X_i^{(1)}]\left(P_i^{(1)}(x)P_{i_1}^{(1)}(x)P_i^{(1)}(x) + P_i^{(1)}(y)P_{i_1}^{(1)}(y)P_i^{(1)}(y)\right) \\
&+ \frac{1}{4}\sum_{i,i_1>i_2} Pr_j^{(3)}[X_i^{(1)}, [X_{i_1}^{(1)}, X_{i_2}^{(1)}]]P_i^{(1)}(x)P_{i_1}^{(1)}(y)P_{i_2}^{(1)}(y) \\
&+ \frac{1}{4}\sum_{i,i_1>i_2} Pr_j^{(3)}[[X_{i_1}^{(1)}, X_{i_2}^{(1)}], X_i^{(1)}]P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(x)P_i^{(1)}(y) \\
&+ \frac{1}{12}\sum_{i_1,i_2,i_3} Pr_j^{(3)}[[X_{i_1}^{(1)}, X_{i_2}^{(1)}], X_{i_3}^{(1)}]\left(P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(y)P_{i_3}^{(1)}(y) - P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(y)P_{i_3}^{(1)}(x)\right) \\
&- \frac{1}{4}\sum_{i_1>i_2>i_3} Pr_j^{(3)}[X_{i_1}^{(1)}, [X_{i_2}^{(1)}, X_{i_3}^{(1)}]]P_{i_1}^{(1)}(x\cdot y)P_{i_2}^{(1)}(x\cdot y)P_{i_3}^{(1)}(x\cdot y) \\
&- \frac{1}{12}\sum_{i>i_1,i_2} Pr_j^{(3)}[[X_i^{(1)}, X_{i_1}^{(1)}], X_{i_2}^{(1)}]P_i^{(1)}(x\cdot y)P_{i_1}^{(1)}(x\cdot y)P_{i_2}^{(1)}(x\cdot y) \\
&+ \frac{1}{12}\sum_{i>i_1} Pr_j^{(3)}[[X_i^{(1)}, X_{i_1}^{(1)}], X_i^{(1)}]P_i^{(1)}(x\cdot y)P_{i_1}^{(1)}(x\cdot y)P_i^{(1)}(x\cdot y).
\end{aligned}
$$

2.2 Proof of CLT

Recall that X is a nilpotent covering graph whose covering transformation group is Γ. Let G_Γ be the nilpotent Lie group such that Γ is isomorphic to a lattice of G_Γ. It is known by Malćev [23] that there exists uniquely such a connected and simply connected nilpotent Lie group up to isomorphism, and Γ is a cocompact lattice (cf. Raghunathan [26]).

Let $\mathfrak{g}$ be the Lie algebra of G_Γ and denote by $\mathfrak{g}^{(1)}, \dots, \mathfrak{g}^{(r)}$ the subspaces of $\mathfrak{g}$ as in Section 2.1. We define a map $P_\delta : C_\infty(G_\Gamma) \to C_\infty(X)$ by $P_\delta f(x) = f(\tau_\delta \Phi(x))$, where $C_\infty(G_\Gamma)$ is the set of continuous functions on G_Γ vanishing at infinity, $C_\infty(X)$ is the set of functions on V vanishing at infinity and $\tau_\delta : G_\Gamma \to G_\Gamma$ is the dilation defined by (2.2). We remark that $(C_\infty(G_\Gamma), \|\cdot\|_\infty)$ and $(C_\infty(X), \|\cdot\|_\infty)$ are Banach spaces, where $\|\cdot\|_\infty$ is the supremum norm. Take a basis $\{X_1^{(k)}, \dots, X_{d_k}^{(k)}\}$ of $\mathfrak{g}^{(k)}$ for each $k \le r$ and we identify $X_i^{(k)}$ with the left invariant vector field on G_Γ. We denote by d_{cc} the Carnot-Carathéodory distance on G_Γ associated with the basis $\{X_1^{(1)}, \dots, X_{d_1}^{(1)}\}$. More precisely, let C be the set of all absolutely continuous paths $c : [0,1] \to G_\Gamma$ satisfying $\dot{c}(t) = \sum_{i \le d_1} a_i(t) X_i^{(1)}(c(t))$ for almost every $t \in [0,1]$. We set

$$|c| = \int_0^1 \left(\sum_{i \le d_1} a_i^2(t) \right)^{1/2} dt,$$

and define for $x, y \in G_\Gamma$,

$$d_{cc}(x,y) = \inf\{\, |c| \mid c \in C,\ c(0) = x,\ c(1) = y \}.$$

Then d_{cc} gives rise to a left invariant distance, which induces the topology of G_Γ (see [35]).

Lemma 2.4. $\{(C_\infty(X), P_\delta)\}_{\delta > 0}$ *is a sequence of Banach spaces approximating* $C_\infty(G_\Gamma)$. *Namely, for any* $f \in C_\infty(G_\Gamma)$, *we have*

$$\|P_\delta f\|_\infty \le \|f\|_\infty, \tag{2.11}$$

$$\|P_\delta f\|_\infty \to \|f\|_\infty \quad as \quad \delta \to 0. \tag{2.12}$$

Proof. Since (2.11) is trivial, we consider (2.12). Fix $a \in G_\Gamma$ such that $|f(a)| = \|f\|_\infty$. Then we have

$$\begin{aligned}\|P_\delta f\| &= \sup_{x\in X} |f(\tau_\delta \Phi(x)) - f(a) + f(a)| \\ &\geq |f(a)| - \inf_{x \in X} |f(a) - f(\tau_\delta \Phi(x))|.\end{aligned}$$

On the other hand, since $\Gamma \subset G_\Gamma$ is a cocompact lattice and Φ is Γ-equivariant, we have

$$\inf_{x\in X} d_{cc}(a, \tau_\delta \Phi(x)) = \delta \inf_{x \in X} d_{cc}(\tau_{\delta^{-1}} a, \Phi(x)) < \delta M$$

for $M = \sup_{g \in \mathcal{D}, x \in F} d_{cc}(g, \Phi(x)) < \infty$, where $\mathcal{D} \subset G_\Gamma$ and $F \subset X$ are fundamental domains of G_Γ and X for the action Γ, respectively. Since f is continuous at a, for any $\epsilon > 0$, there exists $\delta' > 0$ such that if $d_{cc}(a, y) < \delta'$, then $|f(a) - f(y)| < \epsilon$. For $\delta = \delta'/M$, there exists $x' \in X$ such that $d_{cc}(a, \tau_\delta \Phi(x')) < \delta'$. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{x\in X} |f(a) - f(\tau_\delta \Phi(x))| \leq |f(a) - f(\tau_\delta \Phi(x'))| < \epsilon.$$

Consequently, we have $\|P_\delta f\|_\infty \to \|f\|_\infty$ as $\delta \to 0$. □

According to a theorem of Trotter ([33], Theorem 5.3), to deduce the assertion of Theorem 1, it suffices to show the following lemma, which yields the convergence of the sequence of the infinitesimal generators.

Lemma 2.5 (cf. Lemma 3.1, Kotani [19]). *Let $\Phi^h : X \to G_\Gamma$ be a harmonic realization of X. Then, for any $f \in C_0^\infty(G_\Gamma)$ and $N \uparrow \infty$, $\delta \downarrow 0$ with $N^2\delta \to 0$, there exists a limit operator Ω_* such that*

$$\left\| \frac{m(X_0)}{N\delta^2}(I - L^N) P_\delta^h f - P_\delta^h \Omega_* f \right\|_\infty \to 0,$$

where $P_\delta^h f(x) = f(\tau_\delta \Phi^h(x))$. In addition, Ω_* is given by (1.2).

Proof. By the definition of the transition operator, we have

$$\frac{m(X_0)}{N\delta^2}(I - L^N) P_\delta^h f(x) = \frac{m(X_0)}{N\delta^2} \sum_{c \in C_{x,N}} p(c) \left\{ f(\Phi_\delta^h(x)) - f(\Phi_\delta^h(t(c))) \right\},$$

where $C_{x,N}$ is a set of paths $(e_1, \dots, e_N)$ with $o(e_1) = x$, $p(c) = p(e_1)p(e_2)\cdots p(e_N)$ and $\Phi^h_\delta = \tau_\delta \Phi^h$. By the same argument as in Alexopoulos [1] and Kotani [19], we apply the Taylor formula for the $(*)$-coordinates of second kind to $f'(g) = f(\Phi^h_\delta(x) * g)$ with $g = \Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))$. Then we have

$$
\begin{aligned}
(2.13)\quad & \frac{m(X_0)}{N\delta^2}(I - L^N)P^h_\delta f(x) = \\
& \frac{m(X_0)}{N\delta^2} \sum_{c \in C_{x,N}} p(c) \Bigg\{ -\sum_{(i,k)} X^{(k)}_{i*} f(\Phi^h_\delta(x)) P^{(k)}_{i*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) \\
& - \frac{1}{2}\Bigg(\sum_{(i_1,k_1)\geq(i_2,k_2)} X^{(k_1)}_{i_1*} X^{(k_2)}_{i_2*} + \sum_{(i_2,k_2)>(i_1,k_1)} X^{(k_2)}_{i_2*} X^{(k_1)}_{i_1*} \Bigg) f(\Phi^h_\delta(x)) \\
& \times P^{(k_1)}_{i_1*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) P^{(k_2)}_{i_2*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) \\
& - \frac{1}{6} \sum_{(i_1,k_1),(i_2,k_2),(i_3,k_3)} \frac{\partial^3 f'}{\partial x^{(k_1)}_{i_1*} \partial x^{(k_2)}_{i_2*} \partial x^{(k_3)}_{i_3*}}(\theta) P^{(k_1)}_{i_1*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) \\
& \times P^{(k_2)}_{i_2*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) P^{(k_3)}_{i_3*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) \Bigg\}
\end{aligned}
$$

for some $\theta \in G_\Gamma$ satisfying $|P^{(k)}_{i*}(\theta)| \leq |P^{(k)}_{i*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c)))|$, where $(i_1, k_1) > (i_2, k_2)$ means either $k_1 > k_2$ or $k_1 = k_2$, $i_1 > i_2$. Since $(G_\Gamma, *)$ is a stratified Lie group,

$$P^{(k)}_{i*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) = \delta^k P^{(k)}_{i*}(\Phi^h(x)^{-1} * \Phi^h(t(c))).$$

We denote by $\mathrm{Ord}_\delta(k)$ the terms of (2.13) whose order of δ is k. Then (2.13) is rewritten as

$$(2.14)\qquad \frac{m(X_0)}{N\delta^2}(I - L^N)P^h_\delta f(x) = \mathrm{Ord}_\delta(-1) + \mathrm{Ord}_\delta(0) + \sum_{k\geq 1} \mathrm{Ord}_\delta(k).$$

We will consider three terms in (2.14) separately.

Estimate of $\mathbf{Ord_\delta(-1)}$. From Lemmas 2.1 and 2.2 together with the harmonicity of

Φ^h, we have inductively

$$\sum_{c\in C_{x,N}} p(c)P^{(1)}_{i*}(\Phi^h(x)^{-1} * \Phi^h(t(c)))$$

$$= \sum_{c'\in C_{x,N-1}} p(c') \sum_{e\in E_{t(c')}} p(e) \Big\{ \exp^{-1} \Phi^h(x)^{-1} \cdot \Phi^h(t(c'))\big|_{X^{(1)}_i}$$

$$+ \exp^{-1} \Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))\big|_{X^{(1)}_i} \Big\}$$

$$= \sum_{c'\in C_{x,N-1}} p(c')P^{(1)}_i(\Phi^h(x)^{-1} * \Phi^h(t(c'))) = 0.$$

This shows that $\mathrm{Ord}_\delta(-1)$ vanishes.

Estimate of $\mathbf{Ord}_{\boldsymbol\delta}\mathbf{(0)}$. Let us first observe the coefficient of $X^{(2)}_{i*} f(\Phi^h_\delta(x))$. Then we have

$$(2.15)\quad \frac{m(X_0)}{N} \sum_{c\in C_{x,N}} p(c)\Big\{P^{(2)}_{i*}(\Phi^h(x)^{-1} * \Phi^h(t(c))) - \frac{1}{2}\sum_{i_2>i_1} Pr^{(2)}_i[X^{(1)}_{i_1}, X^{(1)}_{i_2}]^*$$

$$\times P^{(1)}_{i_1*}(\Phi^h(x)^{-1} * \Phi^h(t(c)))P^{(1)}_{i_2*}(\Phi^h(x)^{-1} * \Phi^h(t(c)))\Big\}$$

$$= \frac{m(X_0)}{N} \sum_{c\in C_{x,N}} p(c) \exp^{-1} \Phi^h(x)^{-1} * \Phi^h(t(c))\big|_{X^{(2)}_i}$$

$$= \frac{m(X_0)}{N} \sum_{k=0}^{N-1} \sum_{c\in C_{x,k}} p(c) \sum_{e\in E_{t(c)}} p(e) \exp^{-1} \Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))\big|_{X^{(2)}_i}$$

$$= \frac{m(X_0)}{N} \sum_{k=0}^{N-1} \sum_{c\in C_{x,k}} p(c)F(t(c)),$$

where $F(x) = \sum_{e\in E_x} p(e) \exp^{-1} \Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))\big|_{X^{(2)}_i}$. Since $F(\gamma x) = F(x)$, there exists a function $f_0 : X_0 \to \mathbb{R}$ such that $f_0(\pi(x)) = F(x)$, where $\pi : X \to X_0$ is the covering map. Let L_0 be the transition operator on $C(X_0)$. By the ergodicity (cf. [19]), we have

$$\frac{m(X_0)}{N} \sum_{k=0}^{N-1} \sum_{c\in C_{x,k}} p(c)F(t(c)) = \frac{m(X_0)}{N} \sum_{k=0}^{N-1} L^k_0 f_0(\pi(x))$$

$$= \sum_{x_0\in X_0} f_0(x_0)m(x_0) + O\left(\frac{1}{N}\right)$$

$$= \sum_{e\in E_0} m(e) \exp^{-1} \Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))\big|_{X^{(2)}_i} + O\left(\frac{1}{N}\right).$$

Since

$$\sum_{\overline{e}\in E_0} m(\overline{e}) \exp^{-1} \Phi^h(o(\overline{e}))^{-1} \cdot \Phi^h(t(\overline{e}))\big|_{X_i^{(2)}} = -\sum_{e\in E_0} m(e) \exp^{-1} \Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))\big|_{X_i^{(2)}},$$

$\sum_{e\in E_0} m(e) \exp^{-1} \Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))\big|_{X_i^{(2)}} = 0$ so that (2.15) goes to 0.

By the harmonicity and ergodicity, the coefficient of $X_{i_1*}^{(1)} X_{i_2*}^{(1)} f(\Phi_\delta^h(x))$ is given by

$$\begin{aligned}
&-\frac{m(X_0)}{N} \sum_{i_1,i_2\le d_1} \frac{1}{2} X_{i_1*}^{(1)} X_{i_2*}^{(1)} f(\Phi_\delta^h(x)) \\
&\quad\times \sum_{c\in C_{x,N}} p(c) P_{i_1*}^{(1)}(\Phi^h(x)^{-1} * \Phi^h(t(c))) P_{i_2*}^{(1)}(\Phi^h(x)^{-1} * \Phi^h(t(c))) \\
&\quad= -\sum_{i_1,i_2\le d_1} \frac{1}{2} \sum_{e\in E_0} m(e) P_{i_1}^{(1)}(\Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))) P_{i_2}^{(1)}(\Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))) \\
&\quad\times X_{i_1*}^{(1)} X_{i_2*}^{(1)} f(\Phi_\delta^h(x)) + O\left(\frac{1}{N}\right).
\end{aligned}$$

By the definition of Ω_* (1.2), $\mathrm{Ord}_\delta(0)$ converges to $P_\delta^h \Omega_* f(x)$.

Estimate of $\sum_{k\ge 1} \mathrm{Ord}_\delta(k)$. We observe the coefficient of $X_{i*}^{(k)} f(\Phi_\delta^h(x))$. By Lemma 2.2 and

$$|P_i^{(k)}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c)))| \le CN^k,$$

we have

$$\begin{aligned}
(2.16)\quad &\frac{m(X_0)\delta^{k-2}}{N} \sum_{c\in C_{x,N}} p(c) P_{i*}^{(k)}(\Phi^h(x)^{-1} * \Phi^h(t(c))) \\
&= \frac{m(X_0)\delta^{k-2}}{N} \sum_{c\in C_{x,N}} p(c) \Bigg\{ P_i^{(k)}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \\
&\quad + \sum_{\substack{|K_1|+|K_2|\le k-1,\\ |K_2|>0}} C_{K_1K_2} P_*^{K_1}(\Phi^h(x)^{-1}) P^{K_2}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \Bigg\} \\
&\le M_i^{(k)}(\Phi_\delta^h(x)) \left(\delta^{k-2} N^{k-1} + \sum_{\substack{|K_1|+|K_2|\le k-1,\\ |K_2|\ge 2}} \delta^{k-2-|K_1|} N^{|K_2|-1} \right)
\end{aligned}$$

for a continuous function $M_i^{(k)}$ on G_Γ, since

$$\sum_{c\in C_{x,N}} p(c) P^{K_2}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) = 0,$$

when $|K_2| = 1$. By the assumptions of N and δ, (2.16) converges to 0.

By the same argument as above, each coefficient of $X^{(k_1)}_{i_1*}X^{(k_2)}_{i_2*}f(\Phi^h_\delta(x))$ for $k_1+k_2 \geq 3$ converges to 0.

Finally, we consider the coefficient of $\frac{\partial^3 f'}{\partial x^{(k_1)}_{i_1*}\partial x^{(k_2)}_{i_2*}\partial x^{(k_3)}_{i_3*}}(\theta)$. Since $f \in C_0^\infty(G_\Gamma)$ and

$$\mathrm{supp}\frac{\partial^3 f'}{\partial x^{(k_1)}_{i_1*}\partial x^{(k_2)}_{i_2*}\partial x^{(k_3)}_{i_3*}} \subset \mathrm{supp} f' = \Phi^h_\delta(x)^{-1} * \mathrm{supp} f,$$

it suffices to show that, for a continuous function $M^{(k)}_i$ on G_Γ,

$$\left|P^{(k)}_{i*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c)))\right| \leq M^{(k)}_i(\Phi^h_\delta(x) * \theta)\delta N,$$

if $\delta N < 1$. For $k = 1$ and 2, this is true. Assume that it holds up to $k-1$. Then

$$\begin{aligned}
P^{(k)}_{i*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c))) &= \delta^k P^{(k)}_{i*}(\Phi^h(x)^{-1} * \Phi^h(t(c))) \\
=&\delta^k \Bigg(P^{(k)}_i(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \\
&+ \sum_{\substack{|K_1|+|K_2|\leq k-1,\\ |K_2|>0}} C_{K_1K_2} P^{K_1}_*(\Phi^h(x)^{-1}) P^{K_2}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c)))\Bigg).
\end{aligned}$$

Since

$$\begin{aligned}
P^{(k_1)}_{i_1*}(\Phi^h_\delta(x)^{-1}) =& P^{(k_1)}_{i_1*}(\theta * (\Phi^h_\delta(x) * \theta)^{-1}) \\
=& P^{(k_1)}_{i_1*}(\theta) + P^{(k_1)}_{i_1*}((\Phi^h_\delta(x) * \theta)^{-1}) \\
&+ \sum_{\substack{|L_1|+|L_2|=k_1,\\ |L_1|,|L_2|>0}} C_{L_1L_2} P^{L_1}_*(\theta) P^{L_2}_*((\Phi^h_\delta(x) * \theta)^{-1}),
\end{aligned}$$

we have inductively $|P^{(k_1)}_{i_1*}(\Phi^h_\delta(x)^{-1})| \leq M(\Phi^h_\delta(x) * \theta)$ for $k_1 \leq k-1$. So we conclude

$$\begin{aligned}
&|P^{(k)}_{i*}(\Phi^h_\delta(x)^{-1} * \Phi^h_\delta(t(c)))| \\
&\qquad \leq C\Bigg(\delta^k N^k + \sum_{\substack{|K_1|+|K_2|\leq k-1,\\ |K_2|>0}} M(\Phi^h_\delta(x) * \theta)\delta^{k-|K_1|} N^{|K_2|}\Bigg) \\
&\qquad \leq M^{(k)}_i(\Phi^h_\delta(x) * \theta)\delta N.
\end{aligned}$$

From these estimates, it follows that $\sum_{k\geq 1}\mathrm{Ord}_\delta(k)$ converges to 0. Hence the proof of the lemma is completed. $\square$

We remark that by a theorem of Robinson ([27], p.304), for some $\lambda > 0$, the range of $\Omega_* + \lambda$ in $C_\infty(G_\Gamma)$ is dense. Then we apply the argument of Kotani [19] to prove Theorem 1. Let Φ^h be a harmonic realization of X. Then we have

$$\begin{aligned}(2.17)\qquad \left\|L^n P_\delta f - P_\delta e^{-t\Omega_*} f\right\|_\infty \leq & \left\|L^n (P_\delta f - P_\delta^h f)\right\|_\infty \\ & + \left\|L^n P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\right\|_\infty \\ & + \left\|P_\delta^h e^{-t\Omega_*} f - P_\delta e^{-t\Omega_*} f\right\|_\infty .\end{aligned}$$

Since f and $e^{-t\Omega_*}f$ are uniformly continuous and

$$d(\tau_\delta \Phi(x), \tau_\delta \Phi^h(x)) = \delta d(\Phi(x), \Phi^h(x)) \leq \delta M$$

for $M = \sup_{x\in X} d(\Phi(x), \Phi^h(x)) < \infty$, the first and third terms of the right hand side of (2.17) converges to 0 as $\delta \to 0$.

Take $N \uparrow \infty$ and $\delta \downarrow 0$ such that $N^2\delta \to 0$. Then it follows from Lemma 2.5, Robinson ([27]) and Trotter ([33], Theorem 5.3) that for any $f \in C_\infty(G_\Gamma)$,

$$(2.18)\qquad \left\|(L^N)^{k_N} P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\right\|_\infty \to 0,$$

as $k_N N\delta^2 \to m(X_0)t$. Now we prove that the second term of the right hand side of (2.17) converges to 0. Let $N(n)$ be the integer with $n^{1/5} \leq N(n) \leq n^{1/5} + 1$ and k_N and r_N are the quotient and remainder of n/N, respectively. Then $n \uparrow \infty$ and $\delta \downarrow 0$ imply $N \to \infty$, $N^2\delta \leq (n^{1/5}+1)^2\delta \to 0$ and $k_N N\delta^2 = n\delta^2 - r_N\delta^2$. We also see $k_N N\delta^2 \to m(X_0)t$, since $r_N < N$ and $r_N\delta^2 \leq N\delta^2 \leq (n^{1/5}+1)\delta^2 \to 0$. Hence we have

$$\begin{aligned}\left\|L^n P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\right\|_\infty &= \left\|L^{k_N N + r_N} P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\right\|_\infty \\ &\leq \left\|L^{k_N N}(L^{r_N} - \mathrm{I})\, P_\delta^h f\right\|_\infty + \left\|L^{Nk_N} P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\right\|_\infty .\end{aligned}$$

From the property of N, δ and k_N, (2.18) holds. Since $r_N^2\delta \leq (n^{1/5}+1)^2\delta \to 0$ and by Lemma 2.5,

$$\left\|\frac{m(X_0)}{r_N\delta^2}\left(\mathrm{I} - L^{r_N}\right) P_\delta^h \varphi - P_\delta^h \Omega_* \varphi\right\|_\infty \to 0$$

for any $\varphi \in C_0^\infty(G_\Gamma)$. This implies that $\left\| L^{k_N N}\left(L^{r_N} - \mathrm{I}\right) P_\delta^h f \right\|_\infty \to 0$. Then we conclude (1.3).

Finally, (1.4) is obtained by

$$\begin{aligned} &\left| L^n P_\delta f(x_\delta) - e^{-t\Omega_*} f(x) \right| \\ &\qquad \le \left\| L^n P_\delta f - P_\delta e^{-t\Omega_*} f \right\|_\infty + \left| e^{-t\Omega_*} f(\Phi_\delta(x_\delta)) - e^{-t\Omega_*} f(x) \right| \to 0. \end{aligned}$$

Hence Theorem 1 follows.

2.3 Existence and uniqueness of the harmonic realization

In the previous section, we have proved the existence of the limit operator Ω_* by assuming the existence of a harmonic realization. In this section, we consider the existence and uniqueness of such harmonic realizations.

Let I be a homomorphism from G_Γ to an additive group $\mathfrak{g}^{(1)}$ given by

$$\mathrm{I}(g) = \exp^{-1} g \big|_{\mathfrak{g}^{(1)}}.$$

Then we have the following.

Lemma 2.6. $\mathrm{I}(\Gamma)$ *is a lattice in* $\mathfrak{g}^{(1)}$. *Namely, the following hold.*

(i) $\mathrm{I}(\Gamma) \subset \mathfrak{g}^{(1)}$ *is a* $\mathbb{Z}$*-module, and* $\mathrm{I}(\Gamma) \otimes \mathbb{R} = \mathfrak{g}^{(1)}$.

(ii) $\mathrm{I}(\Gamma)$ *is a discrete subgroup of* $\mathfrak{g}^{(1)}$.

Proof. It is easy to show that $\mathrm{I}(\Gamma) \subset \mathfrak{g}^{(1)}$ is a $\mathbb{Z}$-module. Since Γ is a cocompact lattice of G_Γ, there exists a compact subset $U \subset G_\Gamma$ such that $\Gamma U = G_\Gamma$. By restricting ΓU to $\mathfrak{g}^{(1)}$, we have

$$\mathfrak{g}^{(1)} = \exp^{-1}(\Gamma U)\big|_{\mathfrak{g}^{(1)}} = \mathrm{I}(\Gamma) + \exp^{-1} U\big|_{\mathfrak{g}^{(1)}}.$$

Since $\exp^{-1} U|_{\mathfrak{g}^{(1)}}$ is also compact, $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ is compact. So **(i)** is obtained.

For **(ii)**, we first show that $0 \in \mathrm{I}(\Gamma) \subset \mathfrak{g}^{(1)}$ is an isolated point. It is known that $[G_\Gamma, G_\Gamma] \cap \Gamma \subset [G_\Gamma, G_\Gamma]$ is also a lattice (Raghunathan [26], p.31, Corollary 1). Hence there exists a fundamental domain $F' \subset [G_\Gamma, G_\Gamma]$ such that $[G_\Gamma, G_\Gamma] = F'[G_\Gamma, G_\Gamma] \cap \Gamma$. Since $F' \cap (\Gamma \backslash ([G_\Gamma, G_\Gamma] \cap \Gamma)) = \emptyset$ and Γ is discrete, there exists a neighborhood V of $e \in G_\Gamma$ such that $(VF') \cap (\Gamma \backslash ([G_\Gamma, G_\Gamma] \cap \Gamma)) = \emptyset$, which means that

$$(VF') \cap \Gamma \subset [G_\Gamma, G_\Gamma] \cap \Gamma.$$

As a neighborhood of $0 \in \mathrm{I}(\Gamma)$, we take $\mathrm{I}(V)$. If there exists $\gamma \in \Gamma$ such that $\mathrm{I}(\gamma) \neq 0$ and $\mathrm{I}(\gamma) \in \mathrm{I}(V)$, then $\gamma \in \mathrm{I}^{-1}\mathrm{I}(V) = V[G_\Gamma, G_\Gamma]$ because $\mathrm{Ker}\,\mathrm{I} = [G_\Gamma, G_\Gamma]$. This implies that $\gamma \in VF'([G_\Gamma, G_\Gamma] \cap \Gamma)$. So there exists $\gamma' \in [G_\Gamma, G_\Gamma] \cap \Gamma$ such that $\gamma\gamma' \in VF'$, which shows that $\gamma\gamma' \in (VF') \cap \Gamma \subset [G_\Gamma, G_\Gamma] \cap \Gamma$. However, since $\mathrm{I}(\gamma) \neq 0$, we have $\gamma \notin [G_\Gamma, G_\Gamma] \cap \Gamma$. This means that $\gamma\gamma' \notin [G_\Gamma G_\Gamma] \cap \Gamma$, which is a contradiction. Hence $0 \in \mathrm{I}(\Gamma)$ is an isolated point.

Next, we show that $\mathrm{I}(\gamma)$ is an isolated point for any $\gamma \in \Gamma$. Take $\mathrm{I}(\gamma V)$ as a neighborhood of $\mathrm{I}(\gamma)$. If there exists $\eta \in \Gamma$ such that $\mathrm{I}(\eta) \neq \mathrm{I}(\gamma)$ and $\mathrm{I}(\eta) \in \mathrm{I}(\gamma V)$, then we have $\mathrm{I}(\gamma^{-1}\eta) \neq 0$ and $\mathrm{I}(\gamma^{-1}\eta) \in \mathrm{I}(V)$, which is a contradiction. In consequence, we conclude that $\mathrm{I}(\Gamma) \in \mathfrak{g}^{(1)}$ is discrete. Hence $\mathrm{I}(\Gamma)$ is a lattice of $\mathfrak{g}^{(1)}$. □

Let $\Phi : X \to G_\Gamma$ be a realization of X. Since Φ is Γ-equivariant, we define the projection $\exp^{-1}\Phi|_{\mathfrak{g}^{(1)}} : X_0 \to \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$, where $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ is compact by the previous lemma. Hence we may apply results in [21] to the map from X_0 to $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$. Fix a flat metric on the torus $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$. Given a piecewise smooth map $F : X_0 \to \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$, we define the energy $E(F)$ of F by

$$E(F) = \frac{1}{2} \sum_{e \in E_0} m(e) \int_0^1 \left\| \frac{dF_e}{dt}(t) \right\|^2 dt, \tag{2.19}$$

where $F_e : [0,1] \to \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ is the restriction of F to $e \in E_0$ such that $F_e(0) = o(e), F_e(1) = t(e)$. Then the following facts are proved by Kotani and Sunada ([21]):

Lemma 2.7 (First variation formula). *The following* **(a)** *and* **(b)** *are equivalent.*

(a) *F is a critical point.*

(b) *For any $x_0 \in X_0$, it holds that*

$$\begin{cases} \displaystyle\sum_{e \in E_{x_0}} m(e) \frac{dF_e}{dt}(0) = 0, \\ \displaystyle\frac{D}{dt}\frac{dF_e}{dt}(t) = 0. \end{cases}$$

We remark that critical points of the energy functional E do not depend on the choice of a flat metric on $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$. Moreover, a realization $\Phi : X \to G_\Gamma$ is harmonic on $\mathfrak{g}^{(1)}$ if and only if the projection $\exp^{-1} \Phi|_{\mathfrak{g}^{(1)}} : X_0 \to \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ is a critical point of E. From these results, we have

(i) (Kotani and Sunada [22]) Each homotopy class of piecewise smooth maps of X_0 into $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ contains at least one harmonic map.

(ii) (Kotani and Sunada [22]) If two harmonic maps $F_i : X_0 \to \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$, $i = 1, 2$, are homotopic, then there exists $a \in \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ such that $F_1 - F_2 = a$.

(iii) There exists a harmonic realization $\Phi : X \to G_\Gamma$ of X. Moreover, if Φ_1 and Φ_2 are harmonic realizations of X, then

$$\exp^{-1} \Phi_1\big|_{\mathfrak{g}^{(1)}} - \exp^{-1} \Phi_2\big|_{\mathfrak{g}^{(1)}} = \text{constant}.$$

We show **(iii)** by using **(i)**,**(ii)**. Let C be a homotopy class of X_0 into $\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ such that for any $F \in C$, $F_* : \pi_1(X_0) \to \pi_1(\mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)) = \mathrm{I}(\Gamma)$ satisfies

$$F_*([c]) = \mathrm{I}(\sigma_c).$$

Here $\sigma_c \in \Gamma$ satisfies $\sigma_c o(\tilde{c}) = t(\tilde{c})$ for a lift $\tilde{c}$ of c to X. From **(i)**, there exists a harmonic map F^h in C. Then the lift $\widetilde{F^h} : X \to \mathfrak{g}^{(1)}$ of F^h is I-equivariant. Namely, $\widetilde{F^h}(\gamma x) = \widetilde{F^h}(x) + \mathrm{I}(\gamma)$ for any $x \in X$ and $\gamma \in \Gamma$. We define Φ such that $\exp^{-1} \Phi(x)|_{\mathfrak{g}^{(1)}} = \widetilde{F^h}(x)$ for a vertex x in a fundamental domain $F_X \subset X$. Next we define $\Phi(\gamma x) = \gamma\Phi(x)$ for all $\gamma \in \Gamma$. Iterating these processes for all vertices in F_X, we can realize all vertices of X to G_Γ. Finally, for any $e \in E$, we define a smooth map $\Phi(e) : [0, 1] \to G_\Gamma$ such that

$\Phi(e)(0) = \Phi(o(e))$ and $\Phi(e)(1) = \Phi(t(e))$. This Φ is a harmonic realization. Also, by **(ii)**, if Φ_1, Φ_2 are both harmonic, then

$$\exp^{-1} \Phi_1\big|_{\mathfrak{g}^{(1)}} - \exp^{-1} \Phi_2\big|_{\mathfrak{g}^{(1)}} = \text{constant}.$$

2.4 The characterization of Ω_*

First, we consider the following diagram.

$$\begin{array}{ccccccc}
\mathfrak{g}^{(1)} & \simeq & \Gamma/\left([G_\Gamma, G_\Gamma] \cap \Gamma\right) \otimes \mathbb{R} & \twoheadleftarrow & \Gamma/[\Gamma, \Gamma] \otimes \mathbb{R} & \twoheadleftarrow & \mathrm{H}_1(X_0, \mathbb{R}) \\
\updownarrow \text{dual} & & \updownarrow \text{dual} & & \updownarrow \text{dual} & & \updownarrow \text{dual} \\
\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) & \simeq & \mathrm{Hom}(\Gamma/\left([G_\Gamma, G_\Gamma] \cap \Gamma\right), \mathbb{R}) & \hookrightarrow & \mathrm{Hom}(\Gamma/[\Gamma, \Gamma], \mathbb{R}) & \hookrightarrow & \mathrm{H}^1(X_0, \mathbb{R}),
\end{array}$$

where $\mathrm{H}^1(X_0, \mathbb{R})$ is the first cohomology group of X_0 with real coefficients. We identify $\mathrm{H}^1(X_0, \mathbb{R})$ with the set of harmonic 1-forms on X_0 by the discrete analogue of Hodge-Kodaira's theorem. Namely,

$$\mathrm{H}^1(X_0, \mathbb{R}) \simeq \Big\{\omega : E_0 \to \mathbb{R} \;\Big|\; \omega(\overline{e}) = -\omega(e),\ \sum_{e \in E_x} \omega(e) = 0\Big\}.$$

We have an inner product on the set of harmonic 1-forms given by

$$\langle\!\langle \omega, \eta \rangle\!\rangle = \frac{1}{2} \sum_{e \in E_0} m(e)\omega(e)\eta(e)$$

for any harmonic 1-forms ω, η. By the above identification, we define an inner product on $\mathrm{H}^1(X_0, \mathbb{R})$. The surjective homomorphisms $\rho_1 : \Gamma/[\Gamma, \Gamma] \to \Gamma/\left([G_\Gamma, G_\Gamma] \cap \Gamma\right)$ and $\rho_2 : \mathrm{H}_1(X_0, \mathbb{Z}) \to \Gamma/[\Gamma, \Gamma]$ are given respectively by $\rho_1(\gamma[\Gamma, \Gamma]) = \gamma[G_\Gamma, G_\Gamma] \cap \Gamma$ and $\rho_2([c]) = [\sigma_c]$, where $\sigma_c \in \Gamma$ satisfies $\sigma_c o(\tilde{c}) = t(\tilde{c})$ for $\tilde{c}$ a lift of c to X. Since $\mathrm{I}(\Gamma) \simeq \Gamma/([G_\Gamma, G_\Gamma] \cap \Gamma)$ and $\mathrm{I}(\Gamma)$ is a lattice in $\mathfrak{g}^{(1)}$ (Lemma 2.6), we have $\mathfrak{g}^{(1)} \simeq \Gamma/([G_\Gamma, G_\Gamma] \cap \Gamma) \otimes \mathbb{R}$. Hence the surjective homomorphism $\rho_1 \circ \rho_2 : \mathrm{H}_1(X_0, \mathbb{R}) \to \mathfrak{g}^{(1)}$ is defined. By the induced injective homomorphism ${}^t(\rho_1 \circ \rho_2) : \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \to \mathrm{H}^1(X_0, \mathbb{R})$, we induce the metric $\langle\!\langle\, , \rangle\!\rangle$ to $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$. We call the dual metric on $\mathfrak{g}^{(1)}$ the **Albanese metric**. We define $\mathrm{Alb} : X \to \mathfrak{g}^{(1)}$ by

$$\mathrm{Alb}(x)\omega = \int_{x_0}^{x} \widetilde{\omega} \qquad (\omega \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}))$$

for a base point $x_0 \in V$, where $\widetilde{\omega}$ is the lift of ω to X. We call Alb the **Albanese map**. For an orthonormal basis $\{\omega_1, \ldots, \omega_{d_1}\}$ of $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$ and the dual basis $\{X_1^{(1)}, \ldots, X_{d_1}^{(1)}\}$ on $\mathfrak{g}^{(1)}$, we have

$$\mathrm{Alb}(x) = \left(\int_{x_0}^{x} \widetilde{\omega}_1, \ldots, \int_{x_0}^{x} \widetilde{\omega}_{d_1} \right) = \sum_{i \le d_1} \int_{x_0}^{x} \widetilde{\omega}_i X_i^{(1)}.$$

Since $\int_c \widetilde{\omega} = 0$ for any closed path c on X and $\omega \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$, Alb is well-defined. For any $x \in X$, $\gamma \in \Gamma$ and $\omega \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$, Alb satisfies

$$\mathrm{Alb}(\gamma x)\omega = \int_{x_0}^{x} \widetilde{\omega} + \int_{x}^{\gamma x} \widetilde{\omega} = \mathrm{Alb}(x)\omega + \int_{[c_\gamma]} \omega.$$

Since $\mathfrak{g}^{(1)} \simeq \Gamma / \left([G_\Gamma, G_\Gamma] \cap \Gamma \right) \otimes \mathbb{R}$, we have $\int_{[c_\gamma]} \omega = \mathrm{I}(\gamma)\omega$. Thus Alb is an I-equivariant map and the projection $\mathrm{Alb} : X_0 \to \mathfrak{g}^{(1)}/\mathrm{I}(\Gamma)$ is a critical point for the energy functional E given by (2.19). Hence we can define a harmonic realization $\Phi^h : X \to G_\Gamma$ of X such that $\exp^{-1} \Phi^h(x)\big|_{\mathfrak{g}^{(1)}} = \mathrm{Alb}(x)$. From a theorem of Kotani and Sunada [21], for any harmonic realization $\Phi^h : X \to G_\Gamma$, there exists $X^{(1)} \in \mathfrak{g}^{(1)}$ such that $\exp^{-1} \Phi^h\big|_{\mathfrak{g}^{(1)}} = \mathrm{Alb} + X^{(1)}$. Then we conclude

$$\begin{aligned}
\Omega_* &= -\frac{1}{2} \sum_{e \in E_0} m(e) \left(\exp^{-1} \Phi^h(o(e))^{-1} \Phi^h(t(e)) \big|_{\mathfrak{g}^{(1)}} \right)_*^2 \\
&= -\frac{1}{2} \sum_{e \in E_0} m(e) \left(\mathrm{Alb}(t(e)) - \mathrm{Alb}(o(e)) \right)_*^2 \\
&= - \sum_{i,j \le d_1} \frac{1}{2} \sum_{e \in E_0} m(e) \omega_i(e) \omega_j(e) X_{i*}^{(1)} X_{j*}^{(1)}.
\end{aligned}$$

Hence Theorem 2 follows.

Chapter 3

Berry-Esseen type theorem

As we already mentioned in the introduction, Alexopoulos [1] proved a Berry-Esseen type theorem for convolution powers on a Cayley graph of a finitely generated discrete group of polynomial growth. In this chapter, we aim to generalize his results to our case.

In his proof, the following three results play crucial roles:

R1: An estimate established in [1, Corollary 7] (see Lemma 3.1).

R2: Gaussian estimates for the heat kernel on a nilpotent Lie group (Varopoulos [35, Theorem IV. 4.2]) .

R3: Gaussian estimates for convolution powers on a discrete group of polynomial growth (Hebisch, Saloff-Coste [15, Theorem 5.1]).

Hence we consider an analogue of these results for a nilpotent covering graph.

Let h_t be the heat kernel of the sub-Laplacian on a nilpotent Lie group G_Γ with respect to an inner product on $\mathfrak{g}^{(1)}$. Then we use **R2**:

Theorem (Varopoulos [35, Theorem IV. 4.2]). *Let* $|K| = k_1 + k_2 + \cdots + k_\ell$. *Then*

$$\left|\partial_t^s X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} \cdots X_{i_\ell}^{(k_\ell)} h_t(g_1, g_2)\right| \leq Ct^{\frac{D+2s+|K|}{2}} \exp(-d_{cc}(g_1, g_2)^2/c't), \tag{3.1}$$

where $X_i^{(k)}$ *is the left invariant vector field identified with* $X_i^{(k)} \in \mathfrak{g}^{(k)}$ *(see section* (2.1)*) and* $d_{cc}(g_1, g_2)$ *is the Carnot-Caratheodory distance associated with the sub-Laplacian on* G_Γ *(see* [35]*).*

We will show a similar result to **R3** for a nilpotent covering graph in the next chapter. Now we try to establish **R1** in our case.

Let $\Phi : X \to G_\Gamma$ be a harmonic realization of X. For $u \in C^\infty(\mathbb{R}_{\geq 0} \times G_\Gamma)$ and $x \in V$, let $\partial_N u(t, \Phi(x)) = u(t+N, \Phi(x)) - u(t, \Phi(x))$ and $\Phi^* u(t, x) = u(t, \Phi(x))$. We denote

$$C_{x,n} = \{c = (e_1, e_2, \dots, e_n) \,|\, e_i \in E, o(e_1) = x, t(e_i) = o(e_{i+1}),\, i = 1, \dots, n-1\},$$

and $t(c) = t(e_n)$ for $c = (e_1, e_2, \dots, e_n) \in C_{x,n}$. As an analogue of **R1**, we have

Lemma 3.1 (cf. Lemma 2.5, [1, Corollary 7] and [19, Theorem 3]). *There exists a constant $C > 0$ such that, for any $u \in C^\infty(\mathbb{R}_{\geq 0} \times G_\Gamma)$ and $J \geq 4$, the following hold:*

$$\begin{aligned}(3.2)\quad &\left|\left(\partial_N + (I - L^N)\right) \Phi^* u(t,x) - N\left(\partial_t + \Omega\right) u(t, \Phi(x))\right| \\ &\leq C \sup_{\theta \in [0,1],\, g \in U_N} \Big(N^2 \left| \frac{\partial^2}{\partial t^2} u(t + \theta N, \Phi(x)) \right| + X^2 u(t, \Phi(x)) \\ &\quad + \sum_{j=3}^{J-1} N^{j-1} X^j u(t, \Phi(x)) + \sum_{k=J}^{Jr} N^k X^k u(t, \Phi(x) g) \Big).\end{aligned}$$

Here,

$$X^k u(t, \Phi(x)) = \sum_{\ell=1}^{k} \sum_{k_1 + k_2 + \cdots + k_\ell = k} \left| X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} \cdots X_{i_\ell}^{(k_\ell)} u(t, \Phi(x)) \right|,$$

and U_N is a set of all $g \in G_\Gamma$ satisfying that there exists $c \in C_{x,N}$ such that

$$\left|P_i^{(k)}(g)\right| \leq \left|P_i^{(k)}(\Phi(x)^{-1}\Phi(t(c)))\right|$$

for all (i,k).

Proof. Let $u'(t, g) = u(t, \Phi(x) g)$. By Taylor's formula with respect to the $(\cdot)$-coordinates of second kind (see Section 2.1), there exist $\theta \in [0,1]$ and $g_c \in U_N$ such that

$$
\begin{aligned}
\left(\partial_N + (I - L^N)\right) \Phi^* u(t,x) &= N\frac{\partial u}{\partial t}(t,\Phi(x)) + \frac{N^2}{2}\frac{\partial^2 u}{\partial t^2}(t+\theta N,\Phi(x)) \\
&+ \sum_{c\in C_{x,N}} p(c)\Big\{ -\frac{\partial u'}{\partial x_i^{(k)}}(t,e)P_i^{(k)}(\Phi(x)^{-1}\Phi(t(c))) \\
&-\frac{1}{2}\frac{\partial^2 u'}{\partial x_{i_1}^{(k_1)}\partial x_{i_2}^{(k_2)}}(t,e)P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c)))P_{i_2}^{(k_2)}(\Phi(x)^{-1}\Phi(t(c))) \\
&-\sum_{j=3}^{J-1}\frac{1}{j!}\frac{\partial^j u'}{\partial x_{i_1}^{(k_1)}\partial x_{i_2}^{(k_2)}\cdots\partial x_{i_j}^{(k_j)}}(t,e)P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c))) \\
&\times P_{i_2}^{(k_2)}(\Phi(x)^{-1}\Phi(t(c)))\cdots P_{i_j}^{(k_j)}(\Phi(x)^{-1}\Phi(t(c))) \\
&-\frac{1}{J!}\frac{\partial^J u'}{\partial x_{i_1}^{(k_1)}\partial x_{i_2}^{(k_2)}\cdots\partial x_{i_J}^{(k_J)}}(t,g_c)P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c))) \\
&\times P_{i_2}^{(k_2)}(\Phi(x)^{-1}\Phi(t(c)))\cdots P_{i_J}^{(k_J)}(\Phi(x)^{-1}\Phi(t(c)))\Big\}.
\end{aligned}
$$

We observe now that

$$
\begin{aligned}
\frac{\partial u'}{\partial x_i^{(k)}}(t,e) &= X_i^{(k)}u(t,\Phi(x)), \\
\frac{\partial^2 u'}{\partial x_{i_1}^{(k_1)}\partial x_{i_2}^{(k_2)}}(t,e) &= X_{i_1}^{(k_1)}X_{i_2}^{(k_2)}u(t,\Phi(x)), \quad (i_1,k_1)\geq(i_2,k_2).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\left(\partial_N + (I - L^N)\right) \Phi^* u(t,x) &= N\frac{\partial u}{\partial t}(t,\Phi(x)) + \frac{N^2}{2}\frac{\partial^2 u}{\partial t^2}(t+\theta N,\Phi(x)) \\
&-\sum_{(i,k)} X_i^{(k)}u(t,\Phi(x)) \sum_{c\in C_{x,N}} p(c)P_i^{(k)}(\Phi(x)^{-1}\Phi(t(c))) \\
&-\frac{1}{2}\Big(\sum_{(i_1,k_1)\geq(i_2,k_2)} X_{i_1}^{(k_1)}X_{i_2}^{(k_2)} + \sum_{(i_2,k_2)>(i_1,k_1)} X_{i_2}^{(k_2)}X_{i_1}^{(k_1)}\Big)u(t,\Phi(x)) \\
&\times\sum_{c\in C_{x,N}} p(c)P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c)))P_{i_2}^{(k_2)}(\Phi(x)^{-1}\Phi(t(c))) \\
&-\sum_{j=3}^{J-1}\frac{1}{j!}\frac{\partial^j u'}{\partial x_{i_1}^{(k_1)}\partial x_{i_2}^{(k_2)}\cdots\partial x_{i_j}^{(k_j)}}(t,e)\sum_{c\in C_{x,N}} p(c)P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c))) \\
&\times P_{i_2}^{(k_2)}(\Phi(x)^{-1}\Phi(t(c)))\cdots P_{i_j}^{(k_j)}(\Phi(x)^{-1}\Phi(t(c))) \\
&-\frac{1}{J!}\sum_{c\in C_{x,N}} p(c)\frac{\partial^J u'}{\partial x_{i_1}^{(k_1)}\partial x_{i_2}^{(k_2)}\cdots\partial x_{i_J}^{(k_J)}}(t,g_c)P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c))) \\
&\times P_{i_2}^{(k_2)}(\Phi(x)^{-1}\Phi(t(c)))\cdots P_{i_J}^{(k_J)}(\Phi(x)^{-1}\Phi(t(c))).
\end{aligned}
$$

From the harmonicity of Φ,

$$\sum_{c \in C_{x,N}} p(c) P_i^{(1)}(\Phi(x)^{-1}\Phi(t(c))) = 0.$$

By using the ergodicity (see Lemma 2.5, [16] and [19]) and the harmonicity of Φ, there exists $C > 0$ independent of N such that

$$\text{(3.3)} \quad \Big| X_i^{(2)} u(t, \Phi(x)) \sum_{k=0}^{N-1} \sum_{c \in C_{x,k}} p(c) \sum_{e \in E_{t(c)}} p(e) \exp^{-1} \Phi(o(e))^{-1}\Phi(t(e))\big|_{X_i^{(2)}} \Big| \\ \leq C X^2 u(t, \Phi(x)),$$

and

$$\text{(3.4)} \quad \Big| -\frac{1}{2} \sum_{i_1, i_2 \leq d_1} X_{i_1}^{(1)} X_{i_2}^{(1)} u(t, \Phi(x)) \sum_{k=0}^{N-1} \sum_{c \in C_{x,k}} p(c) \\ \times \sum_{e \in E_{t(c)}} p(e) \exp^{-1} \Phi(o(e))^{-1}\Phi(t(e))\big|_{X_{i_1}^{(1)}} \exp^{-1} \Phi(o(e))^{-1}\Phi(t(e))\big|_{X_{i_2}^{(1)}} \\ - N\Omega f(\Phi(x)) \Big| \leq C X^2 u(t, \Phi(x)).$$

By the harmonicity of Φ and the definition of $P_i^{(k)}$ (see also [16]), we have

$$\sum_{c \in C_{x,N}} p(c) P_{i_1}^{(k_1)}(\Phi(x)^{-1}\Phi(t(c))) \cdots P_{i_j}^{(k_j)}(\Phi(x)^{-1}\Phi(t(c))) \leq C N^{|K|-1},$$

where $|K| = k_1 + k_2 + \cdots + k_j$. Since $g_c \in U_N$, there exists a constant $C'_J > 0$ such that

$$\left| \frac{\partial^J u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)} \cdots \partial x_{i_J}^{(k_J)}}(t, g_c) \right| \leq C'_J \sum_{k \geq k_1 + k_2 + \cdots + k_J}^{Jr} N^{k - k_1 - k_2 - \cdots - k_J} X^k u(t, \Phi(x) g_c).$$

Hence the lemma follows. □

Remark 2. If both of (1.5) and (1.6) are satisfied, then (3.3) and (3.4) are zero, so that $X^2 u(t, \Phi(x))$ is vanished in (3.2).

For the proof of Theorem 3, we introduce some notations. We define

$$S_t(x, y) = \frac{|G_\Gamma / \Gamma|}{m(X_0)} h_t(\Phi(x), \Phi(y)) \quad (x, y \in V),$$

$$S'_t(x, y) = \frac{1}{m(X_0)} \int_{\mathcal{D}} h_t(\Phi(x)\eta, \Phi(y)) d\eta \quad (x, y \in V),$$

where $\mathcal{D}$ is a fundamental domain in G_Γ for the action of Γ. We shall denote

$$k \cdot S(x,y) = \sum_{z \in V} k(x,z)S(z,y)m(z).$$

Let us also denote, for $T \geq 0$,

$$\delta(n) = \sup_{x,y \in V} |k_n(x,y) - S_n(x,y)|,$$

$$\delta_T(n) = \sup_{x,y \in V} |(k_n - S_n) \cdot S'_T(x,y)|.$$

By using Gaussian bounds for k_n, ∇k_n (Theorem 4) and h_t ([35]), we have

Lemma 3.2 (cf. [1, Lemma 11] and [31, Lemma 1]). *We assume that X is a non-bipartite graph. Then there exist constants $\alpha, \beta \geq 0$ independent of n and T such that*

$$\delta(n) \leq \alpha \delta_T(n) + \beta \sqrt{T} n^{-\frac{D+1}{2}}.$$

Proof. Let us assume that

$$\delta(n) = -\min_{x,y \in V} (k_n - S_n)(x,y).$$

The case $\delta(n) = \max_{x,y \in V}(k_n - S_n)(x,y)$ is treated in the same way. Then there exist $x', y' \in V$ such that $(k_n - S_n)(x', y') = -\delta(n)$. Hence we have

$$\begin{aligned}
-\delta_T(n) \leq & \sum_{z \in V} (k_n - S_n)(x', z) \cdot S'_T(z, y')m(z) \\
= & (k_n - S_n)(x', y') \sum_{d(y',z) \leq c\sqrt{t}} S'_T(z, y')m(z) \\
& + \sum_{d(y',z) \leq c\sqrt{t}} \{(k_n - S_n)(x', z) - (k_n - S_n)(x', y')\} \cdot S'_T(z, y')m(z) \\
& + \sum_{d(y',z) > c\sqrt{t}} (k_n - S_n)(x', z) \cdot S'_T(z, y')m(z) \\
\leq & -\delta(n) \sum_{d(y',z) \leq c\sqrt{t}} S'_T(z, y')m(z) \\
& + c\sqrt{t} \|\nabla^y (k_n - S_n)(x', \cdot)\|_\infty \sum_{d(y',z) \leq c\sqrt{t}} S'_T(z, y')m(z) \\
& + \|(k_n - S_n)(x', \cdot)\|_\infty \sum_{d(y',z) > c\sqrt{t}} S'_T(z, y')m(z).
\end{aligned}$$

Since $\sum_{z\in V} S'_T(z,y')m(z) = 1$ and by Theorem 4 (1.7), if

$$\lambda = \sum_{d(y',z)\le c\sqrt{t}} S'_T(z,y')m(z),$$

then

$$-\delta_T(n) \le -\delta(n)\lambda + c\sqrt{t}\lambda n^{-\frac{D+1}{2}} + \delta(n)(1-\lambda).$$

By choosing c large enough so that $\lambda > 1/2$, we get

$$\delta(n) \le \frac{1}{2\lambda-1}\delta_T(n) + \frac{c\lambda}{2\lambda-1}\sqrt{T}n^{-\frac{D+1}{2}},$$

which proves the lemma. □

As an analogue of [1, Proposition 12], we have

Lemma 3.3. *We assume that X is a non-bipartite graph. Let $q > 0$ and $J \ge 4$. If there exists a constant $A > 0$ such that*

$$\delta(i) \le Ai^{-\frac{D+q}{2}} \tag{3.5}$$

for all $i = 1, 2, \dots, n-1$, then there exists a constant $C > 0$ independent of q, A such that

$$\begin{aligned}\delta(n) \le\, & C\Big(n^{-\frac{D+1}{2}} + N^{-1}n^{-\frac{D}{2}} + \sum_{j=3}^{J-1} N^{j-2}n^{-\frac{D+j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1}n^{-\frac{D+k-2}{2}} \\ & + \sum_{j=3}^{J-1} N^{j-1}n^{-\frac{D+j}{2}} + \sum_{k=J}^{Jr} N^{k}n^{-\frac{D+k}{2}} + T^{\frac{1}{2}}n^{-\frac{D+1}{2}} \\ & + An^{-\frac{D+q}{2}}\Big[N^{-1}\log(n+T) + \sum_{j=3}^{J-1} N^{j-2}T^{-\frac{j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1}T^{-\frac{k-2}{2}}\exp\Big(\frac{N^2}{c'T}\Big) \\ & + \sum_{j=3}^{J-1} N^{j-1}T^{-\frac{j}{2}} + \sum_{k=J}^{Jr} N^{k}T^{-\frac{k}{2}}\exp\Big(\frac{N^2}{c'T}\Big)\Big]\Big)\end{aligned}$$

for sufficiently smaller $N \in \mathbb{N}$ than n and for all $T \in \mathbb{N}$.

Proof. By the previous lemma, we study $\delta_T(n)$. We first prove that

$$\|S_{n+T} - S_n \cdot S'_T\|_\infty \le Cn^{-\frac{D+1}{2}}. \tag{3.6}$$

Let F be a fundamental domain in X for the action of Γ such that $\Phi(F) \subset \mathcal{D}$. Since Φ is Γ-equivariant, we get

$$\begin{aligned}
&S_{n+T}(x,y) - S_n \cdot S'_T(x,y) \\
&\quad = \frac{|G_\Gamma/\Gamma|}{m(X_0)} \sum_{\gamma\in\Gamma, z_0\in F} \Big[\frac{1}{m(X_0)}\int_{\mathcal{D}} \Big(h_n(\Phi(x), \gamma\Phi(z_0)\eta) h_T(\gamma\Phi(z_0)\eta, \Phi(y)) \\
&\qquad - h_n(\Phi(x), \gamma\Phi(z_0)) h_T(\gamma\Phi(z_0)\eta, \Phi(y))\Big) d\eta\Big] m(z_0) \\
&\quad \le \frac{|G_\Gamma/\Gamma|}{m(X_0)^2} \sum_{\gamma\in\Gamma, z_0\in F} \Big[\sup_{\eta\in\mathcal{D}} |h_n(\Phi(x), \gamma\Phi(z_0)\eta) - h_n(\Phi(x), \gamma\Phi(z_0))| \\
&\qquad \times \int_{\mathcal{D}} h_T(\gamma\Phi(z_0)\eta, \Phi(y)) d\eta\Big] m(z_0) \\
&\quad \le C n^{-\frac{D+1}{2}}.
\end{aligned}$$

Hence it is enough to estimate $\|S_{n+T} - k_n \cdot S'\|_\infty$. Let $I \in \mathbb{N}$ be a quotient of n by N. Then we have

$$\begin{aligned}
&S_{n+T}(x,y) - k_n \cdot S'_T(x,y) \\
&\quad = \sum_{0\le i\le I-2} \Big\{k_{iN} \cdot S_{n-iN+T} - k_{(i+1)N} \cdot S_{n-(i+1)N+T}\Big\}(x,y) \\
&\qquad + k_{(I-1)N} \cdot S_{n-(I-1)N+T}(x,y) - k_n \cdot S'_T(x,y) \\
&\quad = \sum_{0\le i\le \frac{I-2}{2}} k_{iN} \cdot \Big(S_{n-iN+T} - k_N \cdot S_{n-(i+1)N+T}\Big)(x,y) \\
&\qquad + \sum_{\frac{I-2}{2}<i\le I-2} \Big(k_{iN} - S_{iN}\Big) \cdot \Big(S_{n-iN+T} - k_N \cdot S_{n-(i+1)N+T}\Big)(x,y) \\
&\qquad + \sum_{\frac{I-2}{2}<i\le I-2} S_{iN} \cdot \Big(S_{n-iN+T} - k_N \cdot S_{n-(i+1)N+T}\Big)(x,y) \\
&\qquad + \Big(k_{(I-1)N} - S_{(I-1)N}\Big) \cdot \Big(S_{n-(I-1)N+T} - k_{n-(I-1)N} \cdot S'_T\Big)(x,y) \\
&\qquad + S_{(I-1)N} \cdot \Big(S_{n-(I-1)N+T} - k_{n-(I-1)N} \cdot S'_T\Big)(x,y) \\
&\quad = E_1 + E_2 + E_3 + E_4 + E_5.
\end{aligned}$$

Using Hölder's inequality,

$$E_1 \le \sum_{0\le i\le\frac{I-2}{2}} \|k_{iN}(x,\cdot)\|_{L^1} \cdot \Big\|\Big(S_{n-iN+T} - k_N \cdot S_{n-(i+1)N+T}\Big)(\cdot,y)\Big\|_\infty .$$

By making use of (3.1) and (3.2), we have

$$E_1 \le \sum_{0\le i\le \frac{I-2}{2}} C\Big\{N^2(n-(i+1)N+T)^{-\frac{D+4}{2}} + (n-(i+1)N+T)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-1}(n-(i+1)N+T)^{-\frac{D+j}{2}} + \sum_{k=J}^{Jr} N^k(n-(i+1)N+T)^{-\frac{D+k}{2}}\Big\}.$$

Since $IN/2 < n/2$, we get

$$E_1 \le C'\Big(Nn^{-\frac{D+2}{2}} + N^{-1}n^{-\frac{D}{2}} + \sum_{j=3}^{J-1} N^{j-2}n^{-\frac{D+j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1}n^{-\frac{D+k-2}{2}}\Big).$$

To estimate E_2, using Hölder's inequality and (3.5),

$$\begin{aligned} E_2 &\le \sum_{\frac{I-2}{2}<i\le I-2} \|(k_{iN}-S_{iN})(x,\cdot)\|_\infty \|(S_{n-iN+T} - k_N\cdot S_{N-(i+1)N+T})(\cdot,y)\|_{L^1} \\ &\le \sum_{\frac{I-2}{2}<i\le I-2} A(iN)^{-\frac{D+q}{2}} \|\{\partial_N + (I-L^N)\}S_{n-(i+1)N+T}(\cdot,y)\|_{L^1}. \end{aligned}$$

By using (3.1) and (3.2), we have

$$\begin{aligned} &\|\{\partial_N + (I-L^N)\}S_{n-(i+1)N+T}(\cdot,y)\|_{L^1} \\ \le& C'\Big(\sup_{\theta\in[0,1]} N^2\left|\frac{\partial^2}{\partial t^2}h_{n-(i+1)N+T+\theta N}(\Phi(z),\Phi(y))\right| \\ &+ X^2 h_{n-(i+1)N+T}(\Phi(z),\Phi(y)) + \sum_{j=3}^{J-1} N^{j-1}X^j h_{n-(i+1)N+T}(\Phi(z),\Phi(y)) \\ &+ \sup_{g\in U_N}\sum_{k=J}^{Jr} N^k X^k h_{n-(i+1)N+T}(\Phi(z)g,\Phi(y))\Big)m(z) \\ \le& C'\sum_{z\in V}\Big[N^2(n-(i+1)N+T)^{-\frac{D+4}{2}}\exp\Big(-\frac{d(\Phi(z),\Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \\ &+ (n-(i+1)N+T)^{-\frac{D+2}{2}}\exp\Big(-\frac{d(\Phi(z),\Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \\ &+ \sum_{j=3}^{J-1} N^{j-1}(n-(i+1)N+T)^{-\frac{D+j}{2}}\exp\Big(-\frac{d(\Phi(z),\Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \\ &+ \sup_{g\in U_N}\sum_{k=J}^{Jr} N^k(n-(i+1)N+T)^{-\frac{D+k}{2}}\exp\Big(-\frac{d(\Phi(z)g,\Phi(y))^2}{c'(n-(i+1)N+T)}\Big)\Big]m(z). \end{aligned}$$

Since the order of polynomial growth of X is D, there exists a constant $C > 0$ independent of n, i, N, T and $\Phi(y)$ such that

$$(n-(i+1)N+T)^{-\frac{D}{2}} \sum_{z \in V} \exp\Big(-\frac{d_{cc}(\Phi(z),\Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \le C,$$
$$\sup_{g \in U_N} (n-(i+1)N+T)^{-\frac{D}{2}} \sum_{z \in V} \exp\Big(-\frac{d_{cc}(\Phi(z)g,\Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \le C \exp\Big(\frac{N^2}{c'T}\Big).$$

These imply that

$$\begin{aligned}
&\|\{\partial_N + (I-L^N)\}S_{n-(i+1)N+T}(\cdot,y)\|_{L^1} \\
&\le C'\Big(N^2(n-(i+1)N+T)^{-\frac{4}{2}} + (n-(i+1)N+T)^{-\frac{2}{2}} \\
&\quad + \sum_{j=3}^{J-1} N^{j-1}(n-(i+1)N+T)^{-\frac{j}{2}} + \sum_{k=J}^{Jr} N^k (n-(i+1)N+T)^{-\frac{k}{2}} \exp\Big(\frac{N^2}{c'T}\Big)\Big).
\end{aligned}$$

Hence we conclude

$$\begin{aligned}
E_2 \le& C'A(n-2N)^{-\frac{D+q}{2}} \int_{\frac{I}{2}-1}^{I-1} \Big\{N^2(n-(x+1)N+T)^{-2} \\
&+ (n-(x+1)N+T)^{-1} + \sum_{j=3}^{J-1} N^{j-1}(n-(x+1)N+T)^{-j/2} \\
&+ \sum_{k=J}^{Jr} N^k(n-(x+1)N+T)^{-\frac{k}{2}} \exp\Big(\frac{N^2}{c'T}\Big)\Big\} dx \\
\le& C'A(n-2N)^{-\frac{D+q}{2}} \Big(NT^{-1} + N^{-1}\log(n+T) \\
&+ \sum_{j=3}^{J-1} N^{j-2}T^{-\frac{j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1}T^{-\frac{k-2}{2}} \exp\Big(\frac{N^2}{c'T}\Big)\Big).
\end{aligned}$$

E_4 is estimated by

$$\begin{aligned}
E_4 &\le \|(k_{(I-1)N} - S_{(I-1)N})(x,\cdot)\|_\infty \|(S_{n-(I-1)N+T} - k_{n-(I-1)N}\cdot S'_T)(\cdot,y)\|_{L^1} \\
&\le A((I-1)N)^{-\frac{D+q}{2}} \|(S_{n-(I-1)N+T} - k_{n-(I-1)N}\cdot S'_T)(\cdot,y)\|_{L^1}.
\end{aligned}$$

By the Gaussian estimates for h_t [35, Theorem IV. 4.2], we have

$$
\begin{aligned}
&\|(S_{n-(I-1)N+T} - k_{n-(I-1)N}\cdot S'_T)(\cdot,y)\|_{L^1}\\
=&\sum_{x\in V}\frac{1}{m(X_0)}\int_{\mathcal{D}}\Big(h_{n-(I-1)N+T}(\Phi(x),\Phi(y)) - h_{n-(I-1)N+T}(\Phi(x)\eta,\Phi(y))\\
&+\{\partial_{n-(I-1)N} + (I-L^{n-(I-1)N})\}h_T(\Phi(\cdot)\eta,\Phi(y))|_x\Big)d\eta\\
\le& C'\sup_{\substack{\eta\in\mathcal{D}'\\ g\in U_N}}\sum_{\gamma\in\Gamma,x_0\in F}\Big[(n-(I-1)N+T)^{-\frac{D+1}{2}}\exp\Big(-\frac{d_{cc}(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'(n-(I-1)N+T)}\Big)\\
&+(n-(I-1)N)^2T^{-\frac{D+4}{2}}\exp\Big(-\frac{d_{cc}(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'T}\Big)\\
&+T^{-\frac{D+2}{2}}\exp\Big(-\frac{d_{cc}(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'T}\Big)\\
&+\sum_{j=3}^{J-1}(n-(I-1)N)^{j-1}T^{-\frac{D+j}{2}}\exp\Big(-\frac{d_{cc}(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'T}\Big)\\
&+\sum_{k=J}^{Jr}(n-(I-1)N)^{k}T^{-\frac{D+k}{2}}\exp\Big(-\frac{d_{cc}(\gamma\Phi(x_0)g\eta,\Phi(y))^2}{c'T}\Big)\Big]\\
\le& C'\Big(T^{-\frac{1}{2}} + N^2T^{-2} + T^{-1} + \sum_{j=3}^{J-1}N^{j-1}T^{-\frac{j}{2}} + \sum_{k=J}^{Jr}N^kT^{-\frac{k}{2}}\exp\Big(\frac{N^2}{c'T}\Big)\Big),
\end{aligned}
$$

where $\mathcal{D}'$ is a compact subset in G_Γ.

Next, we study $E_3 + E_5$. Let $[a]$ be the greatest integer not greater than a. Then we have

$$
\begin{aligned}
E_3 + E_5 =&(S_{[\frac{I}{2}]N}\cdot S_{n-[\frac{I}{2}]N+T} - S_{(I-1)N}\cdot k_{n-(I-1)N}\cdot S'_T)(x,y)\\
&+\sum_{\frac{I-2}{2}<i\le I-2}(S_{(i+1)N} - S_{iN}\cdot k_N)\cdot S_{n-(i+1)N+T}(x,y)\\
=&E'_3 + E'_5.
\end{aligned}
$$

By using Hölder's inequality,

$$
\begin{aligned}
E_5' \leq &\sum_{\frac{I-2}{2}<i\leq I-2} \|(S_{(i+1)N} - S_{iN}\cdot k_N)(x,\cdot)\|_\infty \|S_{n-(i+1)N+T}(\cdot,y)\|_{L^1} \\
\leq & C' \sum_{\frac{I-2}{2}<i\leq I-2} \Big(N^2 (iN)^{-\frac{D+4}{2}} + (iN)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-1}(iN)^{-\frac{D+j}{2}} + \sum_{k=J}^{Jr} N^k (iN)^{-\frac{D+k}{2}} \Big) \\
\leq & C' n \Big(N(n-2N)^{-\frac{D+4}{2}} + N^{-1}(n-2N)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-2}(n-2N)^{-\frac{D+j}{2}} \\
& + \sum_{k=J}^{Jr} N^{k-1}(n-2N)^{-\frac{D+k}{2}} \Big).
\end{aligned}
$$

E_3' is estimated by

$$
\begin{aligned}
E_3' \leq & \|S_{[\frac{I}{2}]N} \cdot S_{n-[\frac{I}{2}]N+T} - S_{n+T}\|_\infty + \|S_{n+T} - S_n \cdot S_T'\|_\infty \\
& + \|(S_n - S_{(I-1)N}\cdot k_{n-(I-1)N})\cdot S_T'\|_\infty.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&(S_{[\frac{I}{2}]N} \cdot S_{n-[\frac{I}{2}]N+T} - S_{n+T})(x,y) \\
&= \frac{|G_\Gamma/\Gamma|}{m(X_0)^2} \sum_{\gamma\in\Gamma, z_0\in F} \int_{\mathcal{D}} \Big[h_{[\frac{I}{2}]N}(\Phi(x), \gamma\Phi(z_0)) h_{n-[\frac{I}{2}]N+T}(\gamma\Phi(z_0), \Phi(y)) \\
&\quad - h_{[\frac{I}{2}]N}(\Phi(x), \gamma\eta) h_{n-[\frac{I}{2}]N+T}(\gamma\eta, \Phi(y)) \Big] d\eta\, m(z_0) \\
&\leq \frac{|G_\Gamma/\Gamma|}{m(X_0)^2} \sum_{\gamma\in\Gamma, z_0\in F} \Big[\sup_{\eta\in\mathcal{D}} |h_{n-[\frac{I}{2}]N+T}(\gamma\Phi(z_0), \Phi(y)) - h_{n-[\frac{I}{2}]N+T}(\gamma\eta, \Phi(y))| \\
&\quad \times \int_{\mathcal{D}} h_{[\frac{I}{2}]N}(\Phi(x), \gamma\Phi(z_0)) d\eta + \sup_{\eta\in\mathcal{D}} |h_{[\frac{I}{2}]N}(\Phi(x), \gamma\Phi(z_0)) - h_{[\frac{I}{2}]N}(\Phi(x), \gamma\eta)| \\
&\quad \times \int_{\mathcal{D}} h_{n-[\frac{I}{2}]N+T}(\gamma\eta, \Phi(y)) d\eta \Big] m(z_0) \\
&\leq C' \Big(\Big(\frac{n}{2}\Big)^{-\frac{D+1}{2}} + \Big(\frac{n}{2} - \frac{3}{2}N\Big)^{-\frac{D+1}{2}} \Big).
\end{aligned}
$$

By (3.6),

$$
\|S_{n+T} - S_n \cdot S_T'\|_\infty \leq C n^{-\frac{D+1}{2}}.
$$

Hence, $\|(S_n - S_{(I-1)N} \cdot k_{n-(I-1)N}) \cdot S'_T\|_\infty$ is estimated by

$$\begin{aligned}
&\big(S_n - S_{(I-1)N} \cdot k_{n-(I-1)N}\big) \cdot S'_T(x,y) \\
&\qquad \le \|\big(S_n - S_{(I-1)N} \cdot k_{n-(I-1)N}\big)(x,\cdot)\|_\infty \|S'_T(\cdot,y)\|_{L^1} \\
&\qquad \le C'\Big[N^2(n-2N)^{-\frac{D+4}{2}} + (n-2N)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-1}(n-2N)^{-\frac{D+j}{2}} \\
&\qquad\quad + \sum_{k=J}^{Jr} N^k (n-2N)^{-\frac{D+k}{2}}\Big].
\end{aligned}$$

By the hypothesis on N, the lemma follows. □

3.1 Proof of the Berry-Esseen type theorem

First, we investigate the case when X is a non-bipartite graph. We note that if both of (1.5) and (1.6) hold, then the terms with $N^{-1}n^{-\frac{D}{2}}$ and $N^{-1}\log(n+T)$ in Lemma 3.3 are vanished. Hence we can use the same arguments as in Alexopoulos [1] by putting $N = 1$ and $q = 1$. However, if both of (1.5) and (1.6) do not hold, then we put $N = [n^{(J-2)/(4J-6)}]$, $T = T_0 \cdot [n^{(J-1)/(2J-3)}]$ for $T_0 \in \mathbb{N}$ and $q = (J-2)/(2J-3)$. In this case, by Lemma 3.3, if $\delta(i) \le Ai^{-\frac{D+(J-2)/(2J-3)}{2}}$ for $i = 1, 2, \dots n-1$, then there exists a constant $\alpha_J > 1$ and a sequence $\{\beta_{T_0}(n)\}_{n\in\mathbb{N}}$ which converges to zero as $n \uparrow \infty$ such that

$$\delta(n) \le \alpha_J \left(1 + T_0^{1/2} + A\left(\beta_{T_0}(n) + T_0^{-(J-2)/2}\exp(1/c'T_0)\right)\right) n^{-\frac{D+(J-2)/(2J-3)}{2}}.$$

Hence we make use of the induction in n to prove Theorem 3. Fix $s_J \in \mathbb{R}$ such that $1 - 1/\alpha_J < s_J < 1$. Let K_J and T_J be positive integers such that

$$\left(\beta_{T_J}(n) + T_J^{-(J-2)/2}\exp(1/c'T_J)\right) < 1 - s_J$$

for all $n \ge K_J$. Since $\delta(n)$ is uniformly bounded, there exists a constant $A_J > 0$ such that

$$\delta(n) \le A_J n^{-\frac{D+(J-2)/(2J-3)}{2}}$$

for all $n < K_J$. By Lemma 3.3 and the assumption of K_J, we have

$$\delta(K_J) \le \alpha_J \left(1 + T_J^{1/2} + A_J(1-s_J)\right) K_J^{-\frac{D+(J-2)/(2J-3)}{2}}.$$

Put $C_J = \max\{A_J, (1+T_J^{1/2})(1/\alpha_J + s_J - 1)^{-1}\}$. Then clearly we have

$$\delta(n) \leq C_J n^{-\frac{D+(J-2)/(2J-3)}{2}}$$

for all $n \leq K_J$.

When $n > K_s$, we assume that

$$\delta(i) \leq C_J i^{-\frac{D+(J-2)/(2J-3)}{2}}$$

for $i = 1, 2, \ldots n-1$. By Lemma 3.3 and the assumption of C_J, we conclude

$$\begin{aligned}\delta(n) \leq & \alpha_J (1 + T_J^{1/2} + C_J(1-s_J)) n^{-\frac{D+(J-2)/(2J-3)}{2}} \\ \leq & C_J n^{-\frac{D+(J-2)/(2J-3)}{2}}.\end{aligned}$$

3.2 Bipartite case

Next, we investigate the case when X is a bipartite graph. Suppose that m and p are a weight and a transition probability on X which gives a symmetric random walk. The bipartition of V is denoted by $V = A \coprod B$. Let $X_A = (A, E_A)$ be an oriented graph, where $E_A = \{(e_1, e_2) \in C_{x,2} \,|\, x \in A\}$. For $e = (e_1, e_2) \in E_A$, let $o(e) = o(e_1)$, $t(e) = t(e_2)$ and $\overline{e} = (\overline{e_2}, \overline{e_1})$. Then a weight m_A and a transition probability p^A on X_A is denoted by

$$\begin{aligned} m_A(x) =& m(x) \quad x \in A, \\ p^A(e) =& p(e_1)p(e_2) \quad e = (e_1, e_2) \in E_A, \end{aligned}$$

respectively. It is easy to show that m_A and p^A give a symmetric random walk on X_A. The transition probability starting at x reaches y at time n on X_A is denoted by $p_n^A(x, y)$. Then the kernel function k_n^A of the transition operator on X_A is written by $k_n^A(x, y) = p_n^A(x, y) m_A(y)^{-1}$. By using the argument of [20], X_A is also a nilpotent covering graph of a finite graph X_{A1} whose covering transformation group Γ_1 is Γ or a subgroup of Γ of index two. We note that X_A have a loop for each vertex. Hence we conclude

$$\sup_{x,y \in A} \left| p_n^A(x,y) m(y)^{-1} - \frac{|G_\Gamma/\Gamma_1|}{m(X_{A1})} h_n^A(\Phi(x), \Phi(y)) \right| \leq C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}},$$

where h_n^A is the heat kernel with respect to m_A and p^A. Since $p_n^A = p_{2n}$, $h_n^A = h_{2n}$, and $|G_\Gamma/\Gamma_1|/m(X_{A1}) = 2|G_\Gamma/\Gamma|/m(X_0)$, the theorem is proved when $x, y \in A$ for even n. If $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then we have

$$\begin{aligned}
&p_{2n+1}(x,y)m(y)^{-1} - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n+1}(\Phi(x),\Phi(y)) \\
&\quad= \sum_{z\in A} k_{2n}(x,z)k(z,y)m(z) - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n+1}(\Phi(x),\Phi(y)) \\
&\quad= \sum_{z\in A} \left(k_{2n}(x,z) - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n}(\Phi(x),\Phi(z))\right) k(z,y)m(z) \\
&\qquad+ \sum_{z\in A} 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n}(\Phi(x),\Phi(y))k(z,y)m(z) - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n+1}(\Phi(x),\Phi(y)) \\
&\quad\le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}} + |\left(\partial_1 + (I - L_y)\right) S_{2n}(x,y)| \\
&\quad\le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}} + Cn^{-\frac{D+2}{2}} \le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}}.
\end{aligned}$$

Hence we complete the proof of Theorem 3.

Chapter 4

Gaussian estimates

Recall that k_n is the kernel of transition operator L^n on a nilpotent covering graph X. In this chapter, we prove Gaussian estimates for k_n.

4.1 Gaussian upper estimate for k_n

First, we consider a Gaussian upper estimate for k_n. Since k_n is symmetric, we can use the following result:

Theorem (Hebisch, Saloff-Coste [15, Theorem 2.1]). *Let X be a measurable space endowed with a positive σ-finite measure with a measurable distance. Denote by $B(x,r)$, $x \in X$, $r > 0$, the ball of center x and radius r. Let $k(x,y)$, $(x,y) \in X \times X$, be a bounded symmetric Markov kernel such that*

$$\{y \in X \mid k(x,y) \neq 0\} \subset B(x,r_0),\ x \in X$$

for some fixed $r_0 > 0$ and assume that

$$\sup_{x,y}\{k_n(x,y)\} \leq C_0 n^{-D/2}, \quad n = 1,2,\ldots. \tag{4.1}$$

Then there exist constants $C, C' > 0$ such that

$$k_n(x,y) \leq CC_0 n^{-D/2} \exp(-d_X(x,y)^2/C'n)$$

for all $x, y \in X$, and $n = 1,2,\ldots$.

Hence it is enough to show (4.1) in our case.

The next simple lemma plays an important role for our proof of Gaussian upper estimates for k_n and ∇k_n.

Lemma 4.1 (cf. [15, Lemma 3.2]). *Let $\ell, n \in \mathbb{N}$ and $f \in L^2(X)$. There exists a constant $C_\ell > 0$ such that*

$$\|(I - L^{2\ell})^{1/2} L^n f\|_2 \leq C_\ell n^{-1/2} \|f\|_2.$$

The following result is also crucial for our proof of (4.1).

Lemma 4.2 (cf. [15, Theorem 4.2]). *Assume that X is a non-bipartite graph. Let F be a fundamental domain in X. Then there exists a constant $C_0 > 0$ such that*

$$|k_{2n+m}(x, y) - k_{2n+m}(x, x)| \leq C_0 d_X(x, y) m^{-1/2} \sup_{z \in F} k_n(z, z).$$

Proof. We define

$$\nabla_2^y k_n(x, y) = \Big(\sum_{d_X(y,z) \leq 2} |k_n(x, y) - k_n(x, z)|^2 m(z) \Big)^{1/2}.$$

By the same argument as in [15], it is easy to show that

$$\nabla^y k_n(x, y) \leq C \sup_{d_X(y,z) \leq 1} \nabla_2^y k_n(x, z). \tag{4.2}$$

There exist $y_0 = y, y_1, \ldots, y_\ell = x \in V$ such that $d_X(y_i, y_{i+1}) = 1$ for $0 \leq i \leq \ell - 1$ and $\ell = d_X(x, y)$. Hence we have

$$\begin{aligned} |k_{2n+m}(x, y) - k_{2n+m}(x, x)| \leq & |k_{2n+m}(x, y) - k_{2n+m}(x, y_1)| \\ & + |k_{2n+m}(x, y_1) - k_{2n+m}(x, y_2)| \\ & \cdots + |k_{2n+m}(x, y_{\ell-1}) - k_{2n+m}(x, x)| \\ \leq & d_X(x, y) \sup_{z \in V} \nabla^z k_{2n+m}(x, z). \end{aligned}$$

From (4.2), it is enough to show that

$$\sup_{y \in V} \nabla_2^y k_{2n+m}(x, y) \leq C m^{-1/2} \sup_{x \in F} k_{2n}(x, x).$$

By using the Cauchy-Schwarz inequality,

$$\begin{aligned}\nabla_2^y k_{2n+m}(x,y) \leq & \|k_n(x,\cdot)\|_2 \|\nabla_2^y k_{n+m}(\cdot,y)\|_2 \\ = & k_{2n}(x,x)^{1/2} \|\nabla_2^y k_{n+m}(\cdot,y)\|_2.\end{aligned}$$

Since X is a non-bipartite graph, there exists $n_0 \in \mathbb{N}$ such that

$$\inf\{k_{2n_0}(z',z_3) \,|\, d_X(z',z_3) \leq 2, z_3 \in F_y\} > 0,$$

where $F_y = \gamma(y)F$ for $\gamma(y) \in \Gamma$ so that $y \in \gamma(y)F$. Then we have

$$\begin{aligned}\|\nabla_2^y k_{n+m}(\cdot,y)\|_2 \leq & C\Big(\sum_{z_3 \in F_y}\sum_{z \in V} \big|\nabla_2^y k_{n+m}(z,z_3)\big|^2 m(z)m(z_3)\Big)^{1/2} \\ \leq & C\Big(\sum_{\substack{z_3 \in F_y, z \in X, \\ d_X(z_3,z') \leq 2}} \big|k_{n+m}(z,z_3) - k_{n+m}(z,z')\big|^2 k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\Big)^{1/2} \\ \leq & C\Big(\sum_{\substack{z_1,z_2,z_3 \in F_y, \\ \gamma_1,\gamma_2 \in \Gamma}} k_{n+m}(\gamma_1 z_1, z_3)\big(k_{n+m}(\gamma_1 z_1, z_3) - k_{n+m}(\gamma_1 z_1, \gamma_2 z_2)\big) \\ & \times k_{2n_0}(\gamma_2 z_2, z_3)m(z_2)m(z_1)m(z_3) \\ & + \sum_{\substack{z_1,z_2,z_3 \in F_y, \\ \gamma_1,\gamma_2 \in \Gamma}} k_{n+m}(\gamma_1 z_1, z_2)\big(k_{n+m}(\gamma_1 z_1, z_2) - k_{n+m}(\gamma_2\gamma_1 z_1, z_3)\big) \\ & \times k_{2n_0}(\gamma_2 z_2, z_3)m(z_2)m(z_1)m(z_3)\Big)^{1/2}.\end{aligned}$$

The definition of L and the symmetry of k_n imply that

$$\begin{aligned}\|\nabla_2^y k_{n+m}(\cdot,y)\|_2 \leq & C\Big(\sum_{z_3 \in F_y}\sum_{z \in V} k_{n+m}(z,z_3)\big(I - L^{2n_0}\big)k_{n+m}(z,z_3)m(z_3) \\ & + \sum_{\substack{z_1,z_2,z_3 \in F_y, \\ \gamma_1,\gamma_2 \in \Gamma}} k_{n+m}(\gamma_1 z_1, z_2)\big(k_{n+m}(\gamma_1 z_1, z_2) - k_{n+m}(\gamma_1 z_1, \gamma_2 z_3)\big) \\ & \times k_{2n_0}(\gamma_2 z_3, z_2)m(z_3)m(z_1)m(z_2)\Big)^{1/2} \\ = & \sqrt{2}C\Big(\sum_{z_3 \in F_y} \|\big(I - L^{2n_0}\big)k_{n+m}(\cdot,z_3)\|_2^2 m(z_3)\Big)^{1/2}.\end{aligned}$$

By Lemma 4.1, we conclude

$$\begin{aligned}\|\nabla_2^y k_{n+m}(\cdot,y)\|_2 \le& C\Big(\sum_{z_3\in F_y} m^{-1}k_{2n}(z_3,z_3)m(z_3)\Big)^{1/2}\\ \le& Cm^{-1/2}\sup_{z_3\in F_y} k_{2n}(z_3,z_3)^{1/2}.\end{aligned}$$

Since $k_n(\gamma x,\gamma x)=k_n(x,x)$ for all $\gamma\in\Gamma$, the lemma follows. □

To prove Theorem 4, we note

$$\begin{aligned}\sup_{x,y\in V} k_{2n}(x,y) =& \sup_{x\in F} k_{2n}(x,x),\\ \sup_{x,y\in V} k_{2n+1}(x,y) \le& \sup_{x,y\in F} k_{2n}(x,x)^{1/2}k_{2n+2}(y,y)^{1/2},\\ \sup_{x\in F} k_{2n+1}(x,x) \le& \sup_{x\in F} k_{2n}(x,x),\\ \sup_{x\in F} k_{2n+2}(x,x) \le& \sup_{x\in F} k_{2n}(x,x).\end{aligned}\tag{4.3}$$

Hence it is enough to show $A(n):=\sup_{x\in F}k_n(x,x)\le Cn^{-D/2}$. Let $n,m\in\mathbb{N}$ and

$$r(n,m):=\frac{m^{1/2}A(2n+m)}{2C_0A(2n)},$$

where C_0 is a constant defined by Lemma 4.2. For $y\in X$ satisfying

$$\sup_{x\in F} d_X(x,y)\le r(n,m),$$

we have

$$\begin{aligned}\sup_{x\in F}|k_{2n+m}(x,y)-k_{2n+m}(x,x)| \le& C_0 r(n,m)m^{-1/2}A(2n)\\ \le& \frac{1}{2}A(2n+m).\end{aligned}$$

This implies

$$\frac{1}{2}A(2n+m)\le \inf_{x\in F} k_{2n+m}(x,y).$$

By integrating in $\{y\in X\mid d_X(x,y)\le r(n,m)\}$, we have

$$1\ge \sum_{d_X(x,y)\le r(n,m)} k_{2n+m}(x,y)m(y)\ge \frac{1}{2}A(2n+m)V_x(r(n,m)),$$

where $V_x(r) = \sum_{d_X(x,y)\le r} m(y)$. Since $V_x(n) \sim n^D$,

$$A(2n+m) \le Cr(n,m)^{-D} \le C\left(\frac{m^{1/2}A(2n+m)}{A(2n)}\right)^{-D}.$$

Put $m = 2n$ and $\theta = D/(D+1)$. Then we have

$$A(4n) \le \left(Cn^{-1/2}A(2n)\right)^{\theta}.$$

For $n \ge 3$, define $\sigma(n)$ to be the smallest integer such that $2^{-\sigma(n)-1}n \le 1$. Since $n > 2^{\sigma(n)}$, we conclude

$$\begin{aligned} A(n) \le& A(2^{\sigma(n)}) \le \Pi_{i=1}^{\sigma(n)-1}\{C^{\theta^i}2^{\theta^i(i-\sigma(n))/2}\}A(2^{\theta^{\sigma(n)-1}}) \\ \le& C2^{-D\sigma(n)/2} \le C(n/2)^{-D/2}. \end{aligned}$$

By using a theorem of Hebisch and Saloff-Coste [15, Theorem 2.1], we complete the proof of Theorem 4 (1.7).

In the case when X is a bipartite graph, by making use of the argument in Section 3.2, we have

$$k_n^A(x,y) \le Cn^{-\frac{D}{2}}\exp(-d_{X_A}(x,y)^2/C'n)$$

for $x, y \in A$. Since $k_n^A(x,y) = k_{2n}(x,y)$ and $d_{X_A}(x,y) = d_X(x,y)/2$, we obtain a Gaussian upper estimate for k_n if $x, y \in A$ or $x, y \in B$. If $x \in A$, $y \in B$ or $x \in B$, $y \in A$, we conclude

$$\begin{aligned} k_{2n+1}(x,y) =& \sum_{z\in V} k(x,z)k_{2n}(z,y)m(z) \\ \le& \sup_{d_X(x,z)\le 1} Cn^{-\frac{D}{2}}\exp(-d_X(z,y)^2/C'n) \\ \le& Cn^{-\frac{D}{2}}\exp(-d_X(x,y)^2/C'n). \end{aligned}$$

4.2 Gaussian upper estimate for ∇k_n

Next, we prove a Gaussian estimate for ∇k_n. First, we assume that X is a non-bipartite graph. We employ the same argument as in [15, Theorem 5.1]. As an easy consequence of (1.7), we have

Lemma 4.3 (cf. [15, Lemma 5.2]). *Set* $\omega_s(x,y) = \exp(sd_X(x,y))$ $(x, y \in V)$. *Then we have*

$$\|k_n(x,\cdot)\omega_s(x,\cdot)\|_q \le Cn^{-\frac{D}{2}(1-1/q)}\exp(C's^2n). \tag{4.4}$$

From (4.2), we consider $\nabla_2^y k_n(x,y)$. Fix $s > 0$, $\nu = n+m$, and note that $\omega_s(x,y) \le \omega_s(x,z)\omega_s(z,y)$. This implies that

$$\omega_s(x,y)\nabla_2^y k_\nu(x,y) \le \|k_m(x,\cdot)\omega_s(x,\cdot)\|_2 \|\nabla_2^y k_n(\cdot,y)\omega_s(\cdot,y)\|_2.$$

Lemma 4.3 yields a good bound for $\|k_m(x,\cdot)\omega_s(x,\cdot)\|_2$. The second factor can be estimated by

$$\begin{aligned}
\|\omega_s(\cdot,y)\nabla_2^y k_n(\cdot,y)\|_2^2 &\le C\sum_{z_3\in F_y}\|\omega_s(\cdot,z_3)\nabla_2^{z_3}k_n(\cdot,z_3)\|_2^2 m(z_3)\\
&= C\sum_{z_3\in F_y}\sum_{z\in V}\omega_{2s}(z,z_3)\sum_{d(z_3,z')\le 2}|k_n(z,z_3)-k_n(z,z')|^2 m(z')m(z)m(z_3).
\end{aligned}$$

Since X is a non-bipartite graph, there exists $n_0 \in \mathbb{N}$ such that

$$\inf\{k_{2n_0}(z',z_3) \mid d_X(z_3,z') \le 2,\ z_3 \in F\} > 0.$$

Hence we have

$$\begin{aligned}
&\|\omega_s(\cdot,y)\nabla_2^y k_n(\cdot,y)\|_2^2\\
&\le C'\sum_{z_3\in F_y}\sum_{z\in V}\omega_{2s}(z,z_3)\sum_{d(z_3,z')\le 2}|k_n(z,z_3)-k_n(z,z')|^2\\
&\quad \times k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\
&\le C'\sum_{z_3\in F_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)\left(k_n(z,z_3)^2 - 2k_n(z,z_3)k_n(z,z') + k_n(z,z')^2\right)\\
&\quad \times k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\
&= 2C'\sum_{z_3\in F_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)k_n(z,z_3)\left(k_n(z,z_3)-k_n(z,z')\right)\\
&\quad \times k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\
&\quad + C'\Big(\sum_{z_3\in F_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)k_n(z,z')^2 k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\
&\qquad - \sum_{z_3\in F_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)k_n(z,z_3)^2 k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\Big)\\
&= B_1 + B_2.
\end{aligned}$$

By using Lemmas 4.1 and 4.3, B_1 is estimated by

$$\begin{aligned}B_1 =&2C' \sum_{z_3\in F_y} \omega_{2s}(z,z_3)k_n(z,z_3)\left(I-L^{2n_0}\right)k_n(z,z_3)m(z)m(z_3)\\ \le&2C'\|\omega_{2s}(\cdot,z_3)k_n(\cdot,z_3)\|_2\cdot\|\left(I-L^{2n_0}\right)k_n(\cdot,z_3)\|_2 m(z_3)\\ \le&Cn^{-\frac{D}{4}}\exp(C's^2n)\cdot n^{-1}\cdot n^{-\frac{D}{4}}=Cn^{-1-\frac{D}{2}}\exp(C's^2n).\end{aligned}$$

Since every $z\in V$ can be written as $z=\gamma z_0$ ($\gamma\in\Gamma, z_0\in F_y$), and the weight m is Γ-invariant, we have

$$\begin{aligned}B_2 =&C'\Big(\sum_{z_3\in F_y}\sum_{\substack{z_1,z_2\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}}\omega_{2s}(\gamma_1 z_1,z_3)k_n(\gamma_1 z_1,\gamma_2 z_2)^2 k_{2n_0}(\gamma_2 z_2,z_3)m(z_2)m(z_1)m(z_3)\\ &-\sum_{z_3\in F_y}\sum_{\substack{z_1,z_2\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}}\omega_{2s}(\gamma_1 z_1,z_2)k_n(\gamma_1 z_1,z_2)^2 k_{2n_0}(z_2,\gamma_2^{-1}z_3)m(z_3)m(z_1)m(z_2)\Big).\end{aligned}$$

By replacing γ_1 with $\gamma_2^{-1}\gamma_1$ in the second term,

$$\begin{aligned}B_2 =&C'\Big(\sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}}\omega_{2s}(\gamma_1 z_1,z_3)k_n(\gamma_1 z_1,\gamma_2 z_2)^2 k_{2n_0}(\gamma_2 z_2,z_3)m(z_3)m(z_2)m(z_1)\\ &-\sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}}\omega_{2s}(\gamma_2^{-1}\gamma_1 z_1,z_2)k_n(\gamma_2^{-1}\gamma_1 z_1,z_2)^2 k_{2n_0}(\gamma_2 z_2,z_3)m(z_3)m(z_2)m(z_1)\Big)\\ =&C'\sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}}\big(\omega_{2s}(\gamma_1 z_1,z_3)-\omega_{2s}(\gamma_1 z_1,\gamma_2 z_2)\big)k_n(\gamma_1 z_1,\gamma_2 z_2)^2\\ &\times k_{2n_0}(\gamma_2 z_2,z_3)m(z_3)m(z_2)m(z_1).\end{aligned}$$

By inverting z_2 and z_3, replacing $\gamma_2^{-1}\gamma_1$ with γ_1 and γ_2 with γ_2^{-1}, B_2 is written as

$$\begin{aligned}B_2 =&C'\sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}}\big(\omega_{2s}(\gamma_1 z_1,\gamma_2 z_2)-\omega_{2s}(\gamma_1 z_1,z_3)\big)k_n(\gamma_1 z_1,z_3)^2\\ &\times k_{2n_0}(\gamma_2 z_2,z_3)m(z_3)m(z_2)m(z_1).\end{aligned}$$

Since $|\omega_s(x,y)-\omega_s(x,z)|\le r_0|s|(\omega_s(x,y)+\omega(x,z))$ for $d_X(y,z)\le r_0$ (see [15, Lemma

2.3]), we have

$$
\begin{aligned}
B_2 =&\frac{C'}{2} \sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}} (\omega_{2s}(\gamma_1 z_1, z_3) - \omega_{2s}(\gamma_1 z_1, \gamma_2 z_2)) \\
&\times \left(k_n(\gamma_1 z_1, \gamma_2 z_2)^2 - k_n(\gamma_1 z_1, z_3)^2\right) k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1) \\
\le& C|s| \sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}} (\omega_{2s}(\gamma_1 z_1, z_3) + \omega_{2s}(\gamma_1 z_1, \gamma_2 z_2)) \\
&\times \left|k_n(\gamma_1 z_1, \gamma_2 z_2)^2 - k_n(\gamma_1 z_1, z_3)^2\right| k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1).
\end{aligned}
$$

By using the Cauchy-Schwarz inequality and Lemma 4.3,

$$
\begin{aligned}
B_2 \le& C|s| \Bigg(\sum_{\substack{z_1,z_2,z_3\in F_y,\\ \gamma_1,\gamma_2\in\Gamma}} \Big\{ k_n(\gamma_1 z_1, z_2)\big(k_n(\gamma_1 z_1, z_2) - k_n(\gamma_2\gamma_1 z_1, z_3)\big) k_{2n_0}(\gamma_2 z_2, z_3) \\
&+ k_n(\gamma_1 z_1, z_3)\big(k_n(\gamma_1 z_1, z_3) - k_n(\gamma_1 z_1, \gamma_2 z_2)\big) k_{2n_0}(\gamma_2 z_2, z_3)\Big\} \\
&\times m(z_3) m(z_2) m(z_1)\Bigg)^{1/2} \\
&\times \Bigg[\Big(\sum_{z_2\in F_y z'\in V} \|\omega_{2s}(\cdot, z_2) k_n(\cdot, z_2)\|_2^2 \omega_{4s}(z_2, z') k_{2n_0}(z_2, z') m(z') m(z_2)\Big)^{1/2} \\
&+ n^{-\frac{D}{4}} \exp(C' s^2 n) + n^{-\frac{D}{4}} \exp(C' s^2 n) \\
&+ \Big(\sum_{z_3\in F_y z'\in V} \|\omega_{2s}(\cdot, z_3) k_n(\cdot, z_3)\|_2^2 \omega_{4s}(z_3, z') k_{2n_0}(z_3, z') m(z') m(z_3)\Big)^{1/2} \Bigg].
\end{aligned}
$$

Then it follows from Lemma 4.1 that

$$
\begin{aligned}
B_2 \le& C|s| \big(\sum_{z_3\in F_y} \| \left(I - L^{2n_0}\right)^{1/2} k_n(\cdot, z_3)\|_2^2 m(z_3)\big)^{1/2} \times n^{-\frac{D}{4}} \exp(C' s^2 n) \\
\le& C|s| n^{-\frac{1}{2}-\frac{D}{2}} \exp(C' s^2 n).
\end{aligned}
$$

By choosing $n = m$ or $n = m + 1$ depending on whether ν is even or odd, we obtain

$$
\omega_s(x, y) \nabla_2^y k_\nu(x, y) \le C(1 + s\sqrt{\nu})^{1/2} \nu^{-D/2-1/2} \exp(C' s^2 \nu).
$$

Choosing $s = d_X(x, y)/2C'\nu$ in this last inequality yields the estimate

$$
\nabla_2^y k_\nu(x, y) \le C\nu^{-1/2-D/2} \exp(-d_X(x, y)^2/C'\nu).
$$

Hence we conclude Theorem 4 (1.8).

Next, we study a Gaussian bound for ∇k_n in the case when X is a bipartite graph. By the same argument as in the last of Section 2, we have

$$\begin{aligned}\nabla^y k_{2n}(x,y) =& \sup_{d_X(y,z)=2} |k_{2n}(x,y) - k_{2n}(x,z)| \\ =& \sup_{d_{X_A}(y,z)=1} |k_n^A(x,y) - k_n^A(x,z)| \\ \leq& C n^{-\frac{D+1}{2}} \exp(-d_X(x,y)^2/C'n)\end{aligned}$$

for $x, y \in A$. If $x \in A$, $y \in B$ or $x \in B$, $y \in A$, we conclude

$$\begin{aligned}\nabla^y k_{2n+1}(x,y) =& \sup_{d_X(y,z)=2} |\sum_{\omega \in V} k(x,\omega)(k_{2n}(\omega,y) - k_{2n}(\omega,z))m(z)| \\ \leq& \sup_{d_X(x,\omega)\leq 1} C n^{-\frac{D+1}{2}} \exp(-d_X(\omega,y)^2/C'n) \\ \leq& C n^{-\frac{D+1}{2}} \exp(-d_X(x,y)^2/C'n).\end{aligned}$$

Hence we complete the proof of Theorem 4.

Finally, we consider Corollary, which gives a Gaussian lower bound for k_n for the sake of completeness. We assume that X is a non-bipartite graph. We can treat the bipartite case in the same way (see Section 3.2).

First, we prove that there exists a constant $C > 0$ such that for any $n \geq n_0$ and $x \in V$, we have

$$k_n(x,x) \geq C n^{-\frac{D}{2}}. \tag{4.5}$$

By Theorem 4 (1.7), for fixed $x \in V$, we have

$$\begin{aligned}\sum_{d_X(x,y)^2 \geq An} k_{2n}(x,y)m(y) =& \sum_{i=0}^{\infty} \sum_{A2^i n \leq d_X(x,y)^2 < A2^{i+1}n} k_{2n}(x,y)m(y) \\ \leq& C A^{\frac{D}{2}} \sum_{i=0}^{\infty} 2^{(i+1)\frac{D}{2}} e^{-\frac{2^i A}{c}} \\ \leq& 1/2\end{aligned}$$

for large $A > 0$. Then by (4.3),

$$1/2 \leq \sum_{d_X(x,y)^2 < An} k_{2n}(x,y)m(y) \leq C(An)^{\frac{D}{2}} \sup_{x \in F} k_{2n}(x,x).$$

Hence we obtain

$$\sup_{x \in F} k_{2n}(x,x) \geq (Cn)^{-\frac{D}{2}}, \quad n \in \mathbb{N}.$$

For any $x \in V$ and a large n, let x_0 be the vertex in F_x which satisfies $\sup_{x \in F} k_{2n}(x,x) = k_{2n}(x_0,x_0)$ and $d_X(x,x_0) = a \leq \mathrm{diam}F$. Then there exists $C > 0$ such that

$$\begin{aligned} k_{2n}(x,x) =& \sum_{y \in V} k_{2n-2a}(x,y)k_{2a}(y,x)m(y) \\ \geq& k_{2n-2a}(x,x_0)k_{2a}(x_0,x)m(x_0) \\ \geq& (Cn)^{-\frac{D}{2}}. \end{aligned}$$

For odd $n > n_0$, we have

$$\begin{aligned} k_n(x,x) =& \sum_{y \in V} k_{n-n_0}(x,y)k_{n_0}(y,x)m(y) \\ \geq& k_{n-n_0}(x,x)k_{n_0}(x,x)m(x) \\ \geq& Cn^{-\frac{D}{2}}. \end{aligned}$$

Hence (4.5) is proved.

Since X is non-bipartite, by Theorem 4 (1.8), we have

$$k_n(x,x) - Cn^{-\frac{D+1}{2}} d_X(x,y) \leq k_n(x,y).$$

By (4.5), there exist positive constants C_0, C_1 such that

$$k_n(x,y) \geq (C_0 n)^{-\frac{D}{2}} \tag{4.6}$$

for $n \geq n_0$ and $d_X(x,y) \leq \sqrt{n}/C_1 + 1$. Then we prove the Gaussian lower bound by the chain argument (see Hebisch and Saloff-Coste [15]). Fix $n \geq n_0$. Since it is trivial when $d_X(x,y) \leq \sqrt{n}/C_1 + 1$ by (4.6), we assume that $\sqrt{n}/C_1 + 1 < d_X(x,y) \leq n/10C_1$. Let $j \leq n$ be the smallest integer such that $\sqrt{j} \geq 10C_1 d_X(x,y)/\sqrt{n}$. Then we fix integers

$n_i \in [n/j-1, n/j+1]$ for $1 \le i \le j$ such that $n = n_1 + \cdots + n_j$ and $y_0, \ldots, y_j \in V$ such that $y_0 = x$, $y_j = y$ and $d_X(y_i, y_{i+1}) \le d_X(x,y)/j + 1$. Let $B_i = B_V(y_i, \sqrt{n_i}/10C_1)$. Since

$$\begin{aligned} d_X(z_i, z_{i+1}) \le & d_X(z_i, y_i) + d_X(y_i, y_{i+1}) + d_X(y_{i+1}, z_{i+1}) \\ \le & \frac{\sqrt{n_i}}{10C_1} + \frac{d_X(x,y)}{j} + 1 + \frac{\sqrt{n_{i+1}}}{10C_1} \\ \le & \frac{\sqrt{n_i}}{C_1} + 1 \end{aligned}$$

for $z_i \in B_i$, $z_{i+1} \in B_{i+1}$ and (4.6), we have

$$\inf\{k_{n_i}(z_i, z_{i+1}) \,|\, z_i \in B_i, z_{i+1} \in B_{i+1}\} \ge (C_0 n_i)^{-\frac{D}{2}}.$$

Hence there exist constants $C_2 > 1$ and $C_3 > 0$ such that

$$\begin{aligned} k_n(x,y) = & \sum_{z_1 \ldots z_{j-1}} k_{n_1}(x, z_1) \cdots k_{n_{j-1}}(z_{j-1}, y) m(z_1) \cdots m(z_{j-1}) \\ \ge & \sum_{z_i \in B_i, 1 \le i \le j-1} k_{n_1}(x, z_1) \cdots k_{n_{j-1}}(z_{j-1}, y) m(z_1) \cdots m(z_{j-1}) \\ \ge & \inf\{k_{n_1}(x, z_1) \,|\, z_1 \in B_1\} (1/C_2)^{j-1} \\ \ge & C_3 (n/j)^{-\frac{D}{2}} (1/C_2)^{j-1}. \end{aligned}$$

Since $j - 1 \le C_4 d_X(x,y)^2/n$, we conclude

$$k_n(x,y) \ge C_3 n^{-\frac{D}{2}} \left(\frac{d_X(x,y)^2}{n}\right)^{\frac{D}{2}} \exp\left((j-1)\log 1/C_2\right) \ge C n^{-\frac{D}{2}} \exp\left(-\frac{d_X(x,y)^2}{C'n}\right)$$

for $n \ge n_0$ and $d_X(x,y) \le n/10C_1$.

Chapter 5

Riesz transform

For a nilpotent covering graph $X = (V, E)$ and $1 \le p < \infty$, let

$$L^p(V) := \{f : V \to \mathbb{R} \mid \|f\|_p < \infty\},$$

and $L^p(E)$ the space of L^p 1-forms on E defined by

$$L^p(E) = \{\omega : E \to \mathbb{R} \mid \omega(\bar{e}) = -\omega(e),\ \|\omega\|_p < \infty\},$$

where

$$\|f\|_p = \left(\sum_{x\in V} |f(x)|^p m(x)\right)^{1/p},$$
$$\|\omega\|_p = \left(\frac{1}{2}\sum_{e\in E} |\omega(e)|^p m(e)\right)^{1/p}.$$

In particular, inner products on $L^2(V)$ and $L^2(E)$ are defined by

$$\langle f, g\rangle = \sum_{x\in V} f(x)g(x)m(x), \quad f, g \in L^2(V),$$
$$\langle\langle \omega, \eta\rangle\rangle = \frac{1}{2}\sum_{e\in E} \omega(e)\eta(e)m(e), \quad \omega, \eta \in L^2(E),$$

respectively. Let d be the *coboundary operator* from functions on V to 1-forms on E defined by $df(e) = f(t(e)) - f(o(e))$. Since $\|\nabla f\|_p \sim \|df\|_p$, $1 < p < \infty$ and $|\nabla f(x)| \le \sup_{e\in E_x} |df(e)|$ for all finitely supported functions f on V, we study the L^p

boundedness of $R = d\Delta^{-1/2}$. By the spectral theory, the kernel of R is written as $r(e,x) = \sum_{n\geq 0} a_n dk_n(e,x)$ in L^2, where a_n is given by

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} a_n x^n.$$

Let $R^* : L^2(E) \to L^2(V)$ be the adjoint operator of R and $r^*(x,e)$ its kernel. We note that $r^*(x,e) = r(e,x)$. According to the Calderon-Zygmund theory (cf. [13]), the L^p boundedness of R follows from an estimate of r.

5.1 Berry-Esseen type estimate for dk_n

First, we prove an estimate for the derivative of the kernel $k_n(x,y)$ of transition operator L^n by making use of Theorems 3, 4 and Lemma 3.2.

Recall that h_n is the heat kernel of the sub-Laplacian Ω on G_Γ for the Albanese metric on $\mathfrak{g}^{(1)}$ (see Section 1.1) and

$$S_n(x,y) = \frac{|G_\Gamma/\Gamma|}{m(X_0)} h_n(\Phi(x), \Phi(y))$$

for $x, y \in V$, where $\Phi : X \to G_\Gamma$ is a harmonic realization of X. By the same argument as in Alexopoulos [1, Theorem 13], we have

Lemma 5.1. *For any $0 < \epsilon < 1/2$, there exists a constant $C > 0$ such that*

$$\sup_{x\in V, e\in E} |dk_n(x,e) - dS_n(x,e)| \leq Cn^{-\frac{D+3/2-\epsilon}{2}}.$$

Proof. Let $V_n(x,y) = k_n(x,y) - S_n(x,y)$ and denote

$$V_n \cdot k_T(x,y) = \sum_{z\in V} V_n(x,z) k_T(z,y) m(z).$$

For $N < n$, let I be the quotient of n by N. Then we have

$$\begin{aligned} V_n(x,y) &= \sum_{0\leq i<[\frac{I}{2}]} \Big(V_{n-iN} \cdot k_{iN}(x,y) - V_{n-(i+1)N} \cdot k_{(i+1)N}(x,y) \Big) \\ &\quad + V_{n-[\frac{I}{2}]N} \cdot k_{[\frac{I}{2}]N}(x,y) \\ &= \sum_{0\leq i<[\frac{I}{2}]} \Big(V_{n-iN} - V_{n-(i+1)N} \cdot k_N \Big) \cdot k_{iN}(x,y) \\ &\quad + V_{n-[\frac{I}{2}]N} \cdot k_{[\frac{I}{2}]}(x,y), \end{aligned}$$

where $[I/2]$ is the greatest integer not greater than $I/2$. Since

$$\begin{aligned} dV_n(x,e) =& V_n(x,t(e)) - V_n(x,o(e)) \\ =& \sum_{0\le i<[\frac{I}{2}]} \Big(V_{n-iN} - V_{n-(i+1)N}\cdot k_N\Big)\cdot dk_{iN}(x,e) \qquad + V_{n-[\frac{I}{2}]N}\cdot dk_{[\frac{I}{2}]N}(x,e), \end{aligned}$$

we have

$$\begin{aligned} |dV_n(x,e)| \le& \sum_{0\le i<[\frac{I}{2}]} \Big\|\Big(V_{n-iN} - V_{n-(i+1)N}\cdot k_N\Big)(x,\cdot)\Big\|_\infty \|dk_{iN}(\cdot,e)\|_1 \\ &+ \|V_{n-[\frac{I}{2}]N}(x,\cdot)\|_\infty \|dk_{[\frac{I}{2}]N}(\cdot,e)\|_1 \\ =& C\Big\|\Big\{\partial_N + (I-L^N)\Big\}S_{n-N}(x,\cdot)\Big\|_\infty \\ &+ \sum_{0<i<[\frac{I}{2}]} \Big\|\Big\{\partial_N + (I-L^N)\Big\}S_{n-(i+1)N}(x,\cdot)\Big\|_\infty \|dk_{iN}(\cdot,e)\|_1 \\ &+ \|V_{n-[\frac{I}{2}]N}(x,\cdot)\|_\infty \|dk_{[\frac{I}{2}]N}(\cdot,e)\|_1. \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} |dV_n(x,e)| \le& C\Big(N^2(n-N)^{-\frac{D+3}{2}} + \sum_{j=4}^{4r} N^j(n-N)^{-\frac{D+j}{2}} \exp\Big(\frac{N^2}{c(n-N)}\Big)\Big) \\ &+ C\sum_{0<i<[\frac{I}{2}]} \Big(N^2(n-(i+1)N)^{-\frac{D+4}{2}} + (n-(i+1)N)^{-\frac{D+2}{2}} \\ &+ N^2(n-(i+1)N)^{-\frac{D+3}{2}} \\ &+ \sum_{j=4}^{4r} N^j(n-(i+1)N)^{-\frac{D+j}{2}} \exp\Big(\frac{N^2}{c(n-(i+1)N)}\Big)\Big)(iN)^{-\frac{1}{2}} \\ &+ C\left(n-[I/2]\,N\right)^{-\frac{D+1/2-\epsilon}{2}} \left([I/2]\,N\right)^{-\frac{1}{2}}. \end{aligned}$$

By choosing $N \sim n^{1/4}$, we conclude

$$\begin{aligned} |dV_n(x,e)| \le& Cn^{-\frac{D+3/2-\epsilon}{2}} + C\sum_{1\le i\le[\frac{I}{2}]} \Big(n^{-\frac{D+2}{2}} + \sum_{j=4}^{4r} n^{-\frac{D+j/2}{2}}\Big)N^{-1/2}i^{-1/2} \\ \le& Cn^{-\frac{D+3/2-\epsilon}{2}} + Cn^{-\frac{D+2}{2}}N^{-1/2}\big(n/N\big)^{1/2} \le C'n^{-\frac{D+3/2-\epsilon}{2}}. \end{aligned}$$

□

By using this lemma, we show that the kernel $r^*(x,e)$ is a *standard kernel* (see [13]).

Lemma 5.2. *Let us assume that* $d_X(x_1,x_2) \le d_X(x_1,o(e))/2$ *and* $d_X(x_1,o(e)) > 2$. *For any* $0 < \epsilon < 1/2$ *and* $0 < \delta < 1$, *there exists* $C > 0$ *such that*

$$|r^*(x_1,e) - r^*(x_2,e)| \le C\bigg(\frac{1}{d_X(x_1,o(e))^{D+\delta(1/2-\epsilon)}} + \sum_{l=1}^{r} \frac{d_X(x_1,x_2)^{l\delta}}{d_X(x_1,o(e))^{D+l\delta}}\bigg).$$

Proof. First, we prove that

$$\begin{aligned} &|dk_n(x_1,e) - dk_n(x_2,e)| \\ &\le C\left(n^{\frac{1/2+\epsilon}{2}} + \sum_{l=1}^{r} d_X(x_1,x_2)^l n^{-\frac{l-1}{2}}\right)^{\delta} n^{-\frac{D+1+\delta}{2}} \exp\left(-\frac{d_X(x_1,o(e))^2}{cn}\right). \end{aligned} \tag{5.1}$$

By using the Gaussian estimate of the gradient of k_n (Theorem 4) and by the assumptions on x_1, x_2 and e, it is easy to show that

$$|dk_n(x_1,e) - dk_n(x_2,e)| \le Cn^{-\frac{D+1}{2}} \exp\left(-\frac{d_X(x_1,o(e))^2}{cn}\right). \tag{5.2}$$

On the other hand, by the previous lemma, we have

$$\begin{aligned} |dk_n(x_1,e) - dk_n(x_2,e)| \le & |dk_n(x_1,e) - dS_n(x_1,e)| + |dS_n(x_1,e) - dS_n(x_2,e)| \\ & + |dS_n(x_2,e) - dk_n(x_2,e)| \\ \le & Cn^{-\frac{D+3/2-\epsilon}{2}} + |dS_n(x_1,e) - dS_n(x_2,e)|. \end{aligned}$$

By Taylor's formula, there exist $g_1, g_2, g_3 \in G_\Gamma$ such that

$$\begin{aligned} dS_n(x_1,e) - dS_n(x_2,e) \quad = \quad & \frac{|G_\Gamma/\Gamma|}{m(X_0)}\Big(\sum_{\substack{k,l \\ i,j}} X_j^{(l)} X_i^{(k)} h_n(\Phi(x_1)g_3, \Phi(o(e))) \\ & \times P_j^{(l)}(\Phi(x_1)^{-1}\Phi(x_2)) P_i^{(k)}(\Phi(o(e))^{-1}\Phi(t(e))) \\ & + \frac{1}{2}\sum_{\substack{k_1,k_2 \\ i_1,i_2}} X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} h_n(\Phi(x_1), \Phi(o(e))g_1) \\ & \times P_{i_1}^{(k_1)}(\Phi(o(e))^{-1}\Phi(t(e))) P_{i_2}^{(k_2)}(\Phi(o(e))^{-1}\Phi(t(e))) \\ & - \frac{1}{2}\sum_{\substack{k_1,k_2 \\ i_1,i_2}} X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} h_n(\Phi(x_2), \Phi(o(e))g_2) \\ & \times P_{i_1}^{(k_1)}(\Phi(o(e))^{-1}\Phi(t(e))) P_{i_2}^{(k_2)}(\Phi(o(e))^{-1}\Phi(t(e)))\Big). \end{aligned}$$

The Gaussian estimates for h_n by Varopoulos [35] imply that

$$(5.3)\qquad |dk_n(x_1,e) - dk_n(x_2,e)| \le C\Big(n^{\frac{1/2+\epsilon}{2}} + \sum_{l=1}^{r} d_X(x_1,x_2)^l n^{-\frac{l-1}{2}}\Big) n^{-\frac{D+2}{2}}.$$

By interpolating (5.2) and (5.3), we obtain (5.1). Finally, we have

$$\begin{aligned}
|r^*(x_1,e) - r^*(x_2,e)| \le C\Bigg(& \frac{1}{d_X(x_1,o(e))^{D+\delta(1/2-\epsilon)}} \\
& \times \sum_{n=1}^{\infty} a_n n^{-1/2} \left(\frac{d_X(x_1,o(e))^2}{n}\right)^{\frac{D+\delta(1/2-\epsilon)}{2}} \exp\left(-\frac{d_X(x_1,o(e))^2}{cn}\right) \\
& + \sum_{l=1}^{r} \frac{d_X(x_1,x_2)^{l\delta}}{d_X(x_1,o(e))^{D+l\delta}} \\
& \times \sum_{n=1}^{\infty} a_n n^{-1/2} \left(\frac{d_X(x_1,o(e))^2}{n}\right)^{\frac{D+l\delta}{2}} \exp\Big(-\frac{d_X(x_1,o(e))^2}{cn}\Big)\Bigg) \\
\le & C\left(\frac{1}{d_X(x_1,o(e))^{D+\delta(1/2-\epsilon)}} + \sum_{l=1}^{r} \frac{d_X(x_1,x_2)^{l\delta}}{d_X(x_1,o(e))^{D+l\delta}}\right).
\end{aligned}$$

□

5.2 Proof of the L^p boundedness of the Riesz transform

We show the L^p boundedness of the adjoint operator R^* for $1 < p \le 2$. We can treat the L^p boundedness of R for $1 < p \le 2$ in the same way. It is easy to see that R^* is bounded on L^2 and $L^p \subset L^2$ for $1 \le p \le 2$. By Marcinkiewicz interpolation theorem (cf. [13]), it suffices to show that the adjoint operator R^* is weak-$(1,1)$:

$$m\big(\{x \in V \,:\, |R^*\omega(x)| > \lambda\}\big) \le \frac{C}{\lambda}\|\omega\|_1.$$

Let $E = E_1 \coprod E_2$ be a decomposition such that $\overline{E_1} = E_2$. Then the fundamental domain $F \subset E$ for the action of Γ is decomposed to $F = F_1 \coprod F_2$. Then, each ω in $L^1(E)$ can be written as $\omega = \sum_{e_1 \in F_1} \omega^{e_1}$, where

$$\omega^{e_1}(e) = \begin{cases} \omega(e) & \text{if } e \in \Gamma e_1 \cap \Gamma\overline{e_1}, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that ω^{e_1} can be identified with an element in $L^1(\Gamma)$ by $\omega^{e_1}(\gamma) = \omega^{e_1}(\gamma e_1)$. Let $S = \{s_1, \ldots, s_n\}$ be a symmetric finite generator of Γ. A distance d_Γ on Γ is defined by

$$d_\Gamma(\gamma_1, \gamma_2) := \min\{k \in \mathbb{N} \,:\, \gamma_1 = s_{i_1} s_{i_2} \cdots s_{i_k} \gamma_2,\ s_{i_j} \in S,\, 1 \le i_j \le n\}.$$

We denote

$$B_\Gamma(\gamma, d) := \{\eta \in \Gamma \,:\, d_\Gamma(\gamma, \eta) \le d\}.$$

Here we apply the following theorem to $\omega^{e_1} \in L^1(\Gamma)$:

Theorem (Coifman and Weiss [6]). *There exists a constant $C > 0$ such that, for any $\omega^{e_1} \in L^1(\Gamma)$ and $\lambda > 0$, ω^{e_1} is decomposed by $g^{e_1} + b^{e_1}$ with $b^{e_1} = \sum_{i \in I} b_i^{e_1}$ so that*

(a) $|g^{e_1}(\gamma)| \le C\lambda$, $\gamma \in \Gamma$.

(b) *For any $i \in I$, there exists $B_\Gamma(\gamma_i, d_i)$ so that the support of b_i is contained in $B_\Gamma(\gamma_i, d_i)$, $\sum_{\gamma \in \Gamma} |b_i^{e_1}(\gamma)| \le C\lambda |B_\Gamma(\gamma_i, d_i)|$ and $\sum_{\gamma \in \Gamma} b_i^{e_1}(\gamma) = 0$.*

(c) $\sum_{i \in I} |B_\Gamma(\gamma_i, d_i)| \le C \|\omega^{e_1}\|_1 / \lambda$.

We denote

$$M = \sup_{x \in V, \gamma \in \Gamma} \frac{d_X(x, \gamma x)}{d_\Gamma(id, \gamma)},$$

and $A^{e_1} = \cup_{i \in I} B_V(\gamma_i o(e_1), 2M d_i) = \cup_{i \in I} B_i$. Hence we have

$$\begin{aligned} m(\{x \in V \,:\, |R^* \omega(x)| > \lambda\}) &\le \sum_{e_1 \in F_1} m\left(\left\{x \in V \,:\, |R^* \omega^{e_1}(x)| > \frac{\lambda}{\# F_1}\right\}\right) \\ &\le \sum_{e_1 \in F_1} \left[m\left(\left\{x \in V \,:\, |R^* g^{e_1}(x)| > \frac{\lambda}{2\# F_1}\right\}\right) \right. \\ &\qquad \left. + m\left(\left\{x \in V \,:\, |R^* b^{e_1}(x)| > \frac{\lambda}{2\# F_1}\right\}\right) \right]. \end{aligned}$$

Then we have

$$m\left(\left\{x \in V \,:\, |R^* g^{e_1}(x)| > \frac{\lambda}{2\# F_1}\right\}\right) \le \left(\frac{2\# F_1}{\lambda}\right)^2 \sum_{x \in V} |R^* g^{e_1}(x)|^2 m(x).$$

Since R^* is bounded on L^2, we obtain

$$m\left(\left\{x \in V \,:\, |R^* g^{e_1}(x)| > \frac{\lambda}{2\#F_1}\right\}\right) \leq \frac{C}{\lambda}\|\omega^{e_1}\|_1.$$

Next, we consider $m\left(\{x \in V \,:\, |R^* b^{e_1}(x)| > \lambda/(2\#F_1)\}\right)$. By the assumption of b^{e_1}, we have

$$\begin{aligned}
m&\left(\left\{x \in V \,:\, |R^* b^{e_1}(x)| > \frac{\lambda}{2\#F_1}\right\}\right) \\
&\leq |A^{e_1}| + m\left(\left\{x \in V \backslash A^{e_1} \,:\, |R^* b^{e_1}(x)| > \frac{\lambda}{2\#F_1}\right\}\right) \\
&\leq |A^{e_1}| + \frac{2\#F_1}{\lambda}\sum_{x \in V\backslash A^{e_1}} |R^* b^{e_1}(x)| m(x) \\
&\leq \frac{C}{\lambda}\|\omega^{e_1}\|_1 + \frac{2\#F_1}{\lambda}\sum_{i \in I}\sum_{x \in V\backslash B_i}\left|\sum_{e \in E_1} r^*(x,e) b_i^{e_1}(e) m(e)\right| m(x) \\
&= \frac{C}{\lambda}\|\omega^{e_1}\|_1 + \frac{2\#F_1}{\lambda}\sum_{i \in I}\sum_{x \in V\backslash B_i}\left|\sum_{\gamma \in \Gamma} r^*(x,\gamma e_1) b_i^{e_1}(\gamma e_1) m(e_1)\right| m(x).
\end{aligned}$$

Since $b_i^{e_1}$ has a zero integral, we have

$$\begin{aligned}
m&\left(\left\{x \in V \,:\, |R^* b^{e_1}(x)| > \frac{\lambda}{2\#F_1}\right\}\right) \\
&\leq \frac{C}{\lambda}\|\omega^{e_1}\|_1 + \frac{2\#F_1}{\lambda}\sum_{i \in I}\sum_{\gamma \in B_\Gamma(\gamma_i, d_i)} |b_i^{e_1}(\gamma e_1)| m(e_1) \sum_{x \in V\backslash B_i} |r^*(x,\gamma e_1) - r^*(x,\gamma_i e_1)| m(x).
\end{aligned}$$

By Lemma 5.2,

$$|r^*(x,\gamma e_1) - r^*(x,\gamma_i e_1)| \leq C\left(\frac{1}{d_X(x,\gamma_i o(e_1))^{D+\delta(1/2-\epsilon)}} + \sum_{l=1}^{r}\frac{M^{l\delta} d_i^{l\delta}}{d_X(x,\gamma_i o(e_1))^{D+l\delta}}\right).$$

Consequently, we have

$$\begin{aligned}
\frac{2\#F_1}{\lambda}&\sum_{i \in I}\sum_{\gamma \in B_\Gamma(\gamma_i, d_i)} |b_i^{e_1}(\gamma e_1)| m(e_1) \sum_{x \in V\backslash B_i} |r^*(x,\gamma e_1) - r^*(x,\gamma_i e_1)| m(x) \\
&\leq \frac{C'}{\lambda}\sum_{i \in I}\sum_{\gamma \in B_\Gamma(\gamma_i, d_i)} |b_i^{e_1}(\gamma e_1)| m(e_1) \\
&\quad \times \sum_{x \in V\backslash B_i}\left(\frac{1}{d_X(x,\gamma_i o(e_1))^{D+\delta(1/2-\epsilon)}} + \sum_{l=1}^{r}\frac{M^{l\delta} d_i^{l\delta}}{d_X(x,\gamma_i o(e_1))^{D+l\delta}}\right) \\
&\leq \frac{C}{\lambda}\|\omega^{e_1}\|_1.
\end{aligned}$$

Hence the proof of Theorem 5 is completed.

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