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Homogeneous Kähler Einstein manifolds of nonpositive curvature operator

by

Wakako Obata

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A thesis presented

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Wakako OBATA

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Contents

Introduction

Given a homogeneous Riemannian manifold M of nonpositive curvature, it has been a central problem to find geometric conditions for M to be a Riemannian symmetric space of noncompact type. Indeed, it was in the 1970's that the structure of homogeneous Riemannian manifolds of nonpositive curvature was determined. More precisely, in 1974, Heintze [4] proved that a connected, simply connected homogeneous Riemannian manifold of nonpositive curvature can be identified with a simply connected solvable Lie group with a left invariant metric. In consequence, to classify the structure of these manifolds it suffices to determine the structure of solvable Lie algebras $\mathfrak{g}$ with inner product $\langle\ ,\ \rangle$ of nonpositive curvature.

In this direction, Heintze [4] studied a necessary and sufficient condition for a metric solvable Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ to have strictly negative sectional curvature, and obtained the condition that $(\mathfrak{g}, \langle\ ,\ \rangle)$ be isomorphic to the metric Lie algebra associated with a Riemannian symmetric space of negative curvature. Subsequently, in 1976, Azencott and Wilson [1] succeeded in determining the structure of metric solvable Lie algebras $(\mathfrak{g}, \langle\ ,\ \rangle)$ of nonpositive curvature. Moreover, it is well-known that the Killing form associated with a Riemannian symmetric space M of noncompact type induces an Einstein metric of nonpositive curvature on M.

With these foregoing results understood, let (M, g) be a homogeneous *Einstein* manifold of nonpositive curvature. By virtue of the result of Heintze mentioned above, it suffices to investigate the structure of the metric solvable Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ associated with M. Also, it should be remarked that, since the metric $\langle\ ,\ \rangle$ is Einstein, the scalar curvature of $(\mathfrak{g}, \langle\ ,\ \rangle)$ is either strictly negative or zero. In the case when the scalar curvature of $(\mathfrak{g}, \langle\ ,\ \rangle)$ vanishes, we know that the Ricci curvature also vanishes. Then it was proved by Heber [3] that this eventually implies $(\mathfrak{g}, \langle\ ,\ \rangle)$ being flat. On the other hand, if the Ricci curvature of $(\mathfrak{g}, \langle\ ,\ \rangle)$ is strictly negative, then Heber [3] also proved that $\mathfrak{g}$ is a non-unimodular Lie algebra, that is, there exists a non-zero vector

$H \in \mathfrak{g}$ such that $\langle H, X \rangle = \operatorname{tr} X$ for all $X \in \mathfrak{g}$. It is immediate that H is perpendicular to the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. Moreover, we obtain the following

Lemma 5.3 (1998, Heber [3]). *Let $\mathfrak{g}$ be a non-unimodular solvable Lie algebra with an Einstein metric $\langle\ ,\ \rangle$, $\mathfrak{a}$ the orthogonal complement of the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$, and $H \in \mathfrak{g}$ a vector defined by $\langle H, X \rangle = \operatorname{tr} \operatorname{ad} X$ for any $X \in \mathfrak{g}$. Assume that $\mathfrak{a}$ is abelian. Then the following holds*:

(1) *For any $A \in \mathfrak{a}$, the symmetric part D_A and the skew-symmetric part S_A of the adjoint representation* $\operatorname{ad} A$ *are derivations of $\mathfrak{g}$. Moreover, $\{D_A, S_A \mid A \in \mathfrak{a}\}$ is abelian.*

(2) *$D_A \neq 0$ for any $A \in \mathfrak{a}$, and the restriction $D_H|_{\mathfrak{n}}$ of D_H to $\mathfrak{n}$ is positive definite.*

Now, let (M, g) be a Riemannian manifold, and $\bigwedge^2 T_pM$ denote the space of skew-symmetric $(2,0)$-tensors on the tangent space T_pM of M at a point $p \in M$. The curvature tensor R of M then gives rise to the *curvature operator* $\hat{R} : \bigwedge^2 T_pM \to \bigwedge^2 T_pM$ defined by

$$\langle\!\langle \hat{R}(X \wedge Y), Z \wedge W \rangle\!\rangle = g(R(Z, W)Y, X), \quad X, Y, Z, W \in T_pM.$$

The symmetry properties of R imply that $\hat{R}$ is self-adjoint with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$, so that the eigenvalues of $\hat{R}$ are all real. We say that M has *nonpositive* (resp. *negative*) *curvature operator* if all eigenvalues of $\hat{R}$ are nonpositive (resp. negative) everywhere. For instance, the Einstein metric induced by the Killing form on a Riemannian symmetric space of noncompact type has nonpositive curvature operator.

In 1998, Wolter [11] conjectured that *a simply connected homogeneous Einstein manifold with nonpositive curvature operator must be a Riemannian symmetric space.* The primary object of this thesis is to study the structure of homogeneous Einstein manifolds of nonpositive curvature operator.

The nonpositivity of the curvature operator immediately implies that the sectional curvature is nonpositive everywhere. In 1990, it was proved by D'Atri and Dotti Miatello [2] that a homogeneous manifold has an invariant Riemannian metric of negative curvature if and only if it admits an invariant Riemannian metric of negative curvature operator. However, in the case of nonpositive sectional curvature, a homogeneous Riemannian manifold of nonpositive curvature does not always admit an invariant Riemannian metric of nonpositive curvature operator. Concerning this, in 1998, Wolter

[11] obtained a necessary and sufficient condition for a homogeneous Riemannian manifold of nonpositive curvature to have nonpositive curvature operator.

Noticing that there exist many examples of Kähler symmetric spaces with nonpositive curvature operator, we study in this thesis Wolter's conjecture in the case of Kähler manifolds, and prove it affirmatively. Namely, we prove

Main Theorem. *A homogeneous Kähler Einstein manifold of nonpositive curvature operator is a Riemannian symmetric space.*

To be more precise, let (M, J, g) be a homogeneous Kähler Einstein manifold of nonpositive curvature. Recall that M is identified with a simply connected solvable Lie group G with a left invariant almost complex structure J and a left invariant Kähler metric $\langle\ ,\ \rangle$, so that it suffices to study the structure of its Lie algebra $(\mathfrak{g}, J, \langle\ ,\ \rangle)$. Note that $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ satisfies the following conditions:

(K1) $J^2 = -\operatorname{id}$,

(K2) $\langle JX, Y\rangle = -\langle X, JY\rangle$,

(K3) $\langle [X, Y], JZ\rangle + \langle [Y, Z], JX\rangle + \langle [Z, X], JY\rangle = 0$,

(K4) $[JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0$

for any $X, Y, Z \in \mathfrak{g}$. Also, by a result of Azencott and Wilson [1], we know that the orthogonal complement $\mathfrak{a}$ of the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is abelian.

As remarked above, in the Ricci-flat case, it is obvious that the conjecture is true. Hence it suffices to prove the conjecture in the case where $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ is not Ricci flat. Applying recent results of Heber [3], we first prove

Proposition 6.1. *Let $\mathfrak{g}$ be a solvable Lie algebra with an endomorphism J and an Einstein metric $\langle\ ,\ \rangle$ satisfying Conditions* (K1)–(K4). *Suppose that $(\mathfrak{g}, \langle\ ,\ \rangle)$ has nonpositive sectional curvature and is not Ricci flat. Then the following hold*:

(a) *There exists an orthogonal basis $\{H_a\}_{a\in\Lambda}$ of $\mathfrak{a}$ with respect to $\langle\ ,\ \rangle$ such that*

$$\begin{aligned} &[H_a, JH_a] = \lambda_a JH_a && \text{for some } \lambda_a > 0, \\ &[H_b, JH_a] = 0 && \text{if } a \neq b. \end{aligned}$$

Moreover, setting $H = \sum_{a\in\Lambda} H_a$, we have $\langle H, X\rangle = \operatorname{tr}\operatorname{ad} X$ for any $X \in \mathfrak{g}$.

(b) *Define a linear function* $\lambda_a\colon \mathfrak{a} \to \mathbb{R}$ *by* $\lambda_a(H_b) = \delta_{ab}\lambda_a$ *for any* $b \in \Lambda$. *Let* $\mathfrak{n}_a^{\pm b}$ *and* $\mathfrak{n}_a^0$ *be subspaces of* $\mathfrak{n}$ *defined by*

$$\mathfrak{n}_a^{\pm b} = \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\left(\lambda_a(A) \pm \lambda_b(A)\right) X \quad \text{for any } A \in \mathfrak{a} \right\},$$
$$\mathfrak{n}_a^0 = \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\lambda_a(A) X \quad \text{for any } A \in \mathfrak{a} \right\},$$

where $\lambda_b(H) < \lambda_a(H)$, *and set*

$$\mathfrak{n}_a = \bigoplus_{\lambda_b(H)<\lambda_a(H)} \left(\mathfrak{n}_a^{+b} \oplus \mathfrak{n}_a^{-b}\right) \oplus \mathfrak{n}_a^0.$$

Then $\mathfrak{g}$ *is decomposed into a direct sum* $\mathfrak{g} = \bigoplus_a \mathbb{R}\{H_a\} \oplus \mathfrak{n}_a \oplus \mathbb{R}\{JH_a\}$ *which satisfies the following*:

(i) $J\mathfrak{n}_a^{\pm b} = \mathfrak{n}_a^{\mp b}$.

(ii) $[X, Y] = \dfrac{\lambda_a(H_a)}{|H_a|^2}\langle JX, Y\rangle JH_a$ *for* $X, Y \in \mathfrak{n}_a$.

(iii) $[JH_b, X] = -\lambda_b(H_b) JX$ *for* $X \in \mathfrak{n}_a^{-b}$.

(iv) $[Y, X] = -J[JY, X]$ $\quad |[Y, X]|^2 = \dfrac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2|X|^2$ *for* $X \in \mathfrak{n}_a^{-b}, Y \in \mathfrak{n}_b$.

(v) $[Y, X] = [JY, JX]$, $\quad |[Y, X]|^2 = \dfrac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2|X|^2$ *for* $X \in \mathfrak{n}_a^{\mp c}, Y \in \mathfrak{n}_b^{\pm c}$.

(vi) $[Y, X] = [JY, JX]$, $\quad |[Y, X]| = |[Y, JX]|$ *for* $X \in \mathfrak{n}_a^0, Y \in \mathfrak{n}_b^0$.

(vii) *Set* $\Lambda_c = \{a \in \Lambda \mid \mathfrak{n}_a^{\pm c} \neq \{0\}\} \cup \{c\}$ *for* $c \in \Lambda$, *and let* $a, b \in \Lambda_c$. *If* $a \neq b$, *then* $\lambda_a(H) \neq \lambda_b(H)$. *Moreover, if* $\lambda_a(H) > \lambda_b(H)$, *then* $\mathfrak{n}_a^{\pm b} \neq \{0\}$.

Then, concerning the necessary and sufficient condition for $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ to be symmetric, we obtain the following

Proposition 7.1. *Let* $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ *be as in Proposition* 6.1. *Then the following conditions are equivalent*:

(a) $\nabla R \equiv 0$.

(b) *For each* $c \in \Lambda$, *let* Λ_c *denote the subset* $\{a \in \Lambda \mid \mathfrak{n}_a^{\pm c} \neq \{0\}\} \cup \{c\}$ *of* Λ. *Then there exists a subset* $\{a_1, \ldots, a_m\}$ *of* Λ *satisfying that* $\Lambda_{a_1} \cup \ldots \cup \Lambda_{a_m} = \Lambda$ *and that* $\Lambda_{a_i} \cap \Lambda_{a_j} = \{0\}$ *if* $i \neq j$. *Moreover, the following hold*:

(i) *If there exists* a_i *such that* $\mathfrak{n}_{a_i}^0 = \{0\}$, *then* $\mathfrak{n}_b^0 = \{0\}$ *for any* $b \in \Lambda_{a_i}$.

(ii) $\dfrac{\lambda_b(H_b)}{|H_b|} = \dfrac{\lambda_c(H_c)}{|H_c|}$ *for any* $b, c \in \mathfrak{n}_{a_i}$.

Finally, by making full use of these conditions, we obtain the following proposition which suffices to prove our Main Theorem.

Proposition 8.1 *Let* $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ *be as in Proposition* 6.1. *If* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *has nonpositive curvature operator, then* $\nabla R = 0$.

The present thesis is organized as follows.

In Chapter 1, after giving relevant definitions, we recall the conjecture proposed by Wolter [11] in Section 1.

Section 2 is devoted to the statement of our Main Theorem.

In Section 3, we review the structure of homogeneous Kähler manifolds of nonpositive curvature.

Section 4 is devoted to the computation of several curvature functions on metric solvable Lie algebras.

In Section 5, we review fundamental results obtained by Heber [3].

In Section 6, we prove Proposition 6.1 using the results of Heber in Section 5.

In Section 7, by making use of Proposition 6.1, we obtain a necessary and sufficient condition for a metric solvable Lie algebra under consideration to be symmetric.

Finally, in Section 8, we prove our main Theorem.

In Chapter 2, we determine the curvature operator of classical type irreducible symmetric Kähler manifolds of noncompact type.

Chapter 1

Homogeneous Kähler Einstein manifolds of nonpositive curvature operator

In this chapter, we study the structure of homogeneous Kähler Einstein manifolds of nonpositive curvature operator.

1 Wolter's Conjecture

Let (M, g) be a Riemannian manifold, and $\bigwedge^2 T_pM$ denote the space of skew-symmetric $(2, 0)$-tensors on the tangent space T_pM of M at a point $p \in M$. For any $X, Y \in T_pM$, we define an element $X \wedge Y \in \bigwedge^2 T_pM$ by

$$X \wedge Y(Z, W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z), \quad Z, W \in T_pM,$$

and an inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\bigwedge^2 T_pM$ by

$$\langle\!\langle X \wedge Y, Z \wedge W \rangle\!\rangle = g(X, Z)g(Y, W) - g(X, W)g(Y, Z), \quad X, Y, Z, W \in T_pM.$$

The curvature tensor R of M then gives rise to the *curvature operator* $\hat{R} : \bigwedge^2 T_pM \to \bigwedge^2 T_pM$ defined by

$$\langle\!\langle \hat{R}(X \wedge Y), Z \wedge W \rangle\!\rangle = g(R(Z, W)Y, X)$$

for any $X, Y, Z, W \in T_pM$. It is easy to see that $\hat{R}$ is self-adjoint with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$, so that the eigenvalues of $\hat{R}$ are all real. We say that M has *nonpositive curvature operator* if all eigenvalues of $\hat{R}$ are nonpositive everywhere.

Recall that for each 2-plane π in T_pM, the sectional curvature $K(\pi)$ for π is defined by

$$K(\pi) = \langle R(X,Y)Y, X\rangle = \langle\!\langle \hat{R}(X \wedge Y), X \wedge Y\rangle\!\rangle,$$

where $\{X, Y\}$ is an orthonormal basis for π. From this definition it is immediate to see the following.

Remark 1.1. If the curvature operator $\hat{R}$ of (M, g) is nonpositive, then (M, g) has nonpositive sectional curvature everywhere.

However, the converse of Remark 1.1 is not true in general, even in the case of homogenous manifolds. Indeed, as the following example shows, we have many solvable Lie groups with left invariant metric, which have nonpositive sectional curvature everywhere but do not have nonpositive curvature operator.

Example 1.1 (1991, Wolter [10]). Let $\mathfrak{n}$ be a two step nilpotent Lie algebra. We call $\mathfrak{n}$ a *uniform Lie algebra of type* (m, n, r) if it has a basis $\{V_1, \ldots, V_n, Z_1, \ldots, Z_m\}$ satisfying the following conditions, where $1 \leq i, j, k \leq n$ and $1 \leq l \leq, m$:

(K1) $[V_i, V_j] \in \{0, \pm Z_1, \ldots, \pm Z_m\}$ and $[V_i, Z_l] = [Z_k, Z_l] = 0$.

(K2) If $[V_i, V_j] = \pm[V_i, V_k] \neq 0$, then $V_j = V_k$.

(K3) For any Z_l, the cardinality of $\{(V_i, V_j) \mid [V_i, V_j] = Z_l\}$ is r.

(K4) For any V_i, the cardinality of $\{V_j \mid [V_i, V_j] \neq 0\}$ is s.

Note that, from Condition (3), the cardinality of $\{(V_i, V_j) \mid [V_i, V_j] \neq 0\}$ is $2rm$. On the other hand, Condition (4) implies that the cardinality of $\{(V_i, V_j) \mid [V_i, V_j] \neq 0\}$ is sn. So we have $s = 2rm/n$.

Let $\mathfrak{n} = \text{span}\{V_1, \ldots, V_n, Z_1, \ldots, Z_m\}$ be a uniform Lie algebra of type (m, n, r) with an inner product for which $V_1, \ldots, V_n, Z_1, \ldots, Z_m$ are orthonormal. Let $\text{Alt}(\mathfrak{v})$ denote the space of alternating linear transformations on $\mathfrak{v}$ with respect to $\langle\ ,\ \rangle$. Setting $\mathfrak{v} = \text{span}\{V_1, \ldots, V_n\}$ and $\mathfrak{z} = \text{span}\{Z_1, \ldots, Z_m\}$, we define a linear operator $j\colon \mathfrak{z} \to \text{Alt}(\mathfrak{v})$ by

$$\langle j(Z)V, W\rangle = \langle [V, W], Z\rangle,$$

where $V, W \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. Also, we assume that $n = 2r$ and that $j(Z_k)j(Z_l) = -j(Z_l)j(Z_k)$ for $k \neq l$. Then we have $j(Z_k)V_i \in \{V_1, \ldots, V_n\}$ and $\langle j(Z_k)V_i, j(Z_l)V_i \rangle = \delta_{kl}$. Moreover, it holds that $\langle j(Z)V, j(Z)V' \rangle = |Z|^2 \langle V, V' \rangle$ for any $Z \in \mathfrak{z}$ and $V, V' \in \mathfrak{v}$.

With these understood, let $\mathfrak{s} = \mathbb{R}\{A\} \oplus \mathfrak{n}$ be the direct sum of $\mathbb{R}\{A\}$ and $\mathfrak{n}$. We define on $\mathfrak{s}$ an inner product $\langle\ ,\ \rangle$ and a Lie bracket $[\ ,\ ]$ by

$$\begin{aligned} &\langle aA + V + Z, bA + V' + Z' \rangle = ab + \langle V, V' \rangle + \langle Z, Z' \rangle, \\ &\operatorname{ad} A|_{\mathfrak{v}} = \frac{1}{2}\,\mathrm{id}, \quad \operatorname{ad} A|_{\mathfrak{z}} = \mathrm{id}, \end{aligned}$$

where id denotes the identity map on $\mathfrak{s}$. Then $\mathfrak{s}$ becomes a solvable Lie algebra with inner product $\langle\ ,\ \rangle$.

Now, let S be a solvable Lie group with Lie algebra $\mathfrak{s}$, and extend the inner product $\langle\ ,\ \rangle$ on $\mathfrak{s}$ to a left invariant metric $\langle\ ,\ \rangle$ on S. Then the Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K on S define respectively the corresponding Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of $\mathfrak{s}$.

Given a 2-plane π in $\mathfrak{s}$ spanned by an orthonormal basis $\{aA + V + Z, V' + Z'\}$ with $a \geq 0$, $V, V' \in \mathfrak{v}$ and $Z, Z' \in \mathfrak{z}$, it is immediate to see that the sectional curvature $K(\pi)$ for π is given by

$$K(\pi) = -\frac{3}{4}|[V, V'] + aZ'|^2 - \frac{1}{4} - \frac{3}{4}|Z|^2|Z'|^2 - \frac{3}{4}\langle Z, Z' \rangle^2 - \frac{3}{2}\langle j(Z)V, j(Z')V' \rangle.$$

Note that a function $f\colon \left[0, \sqrt{1-a^2}\,\right] \times [0,1] \to \mathbb{R}$ defined by $f(s,t) = -(1/4) - (3/4)s^2t^2 + (3/2)st\sqrt{1-a^2-s^2}\sqrt{1-t^2}$ is nonpositive everywhere, that is, $f(s,t) \leq 0$. It follows from this that

$$\begin{aligned} K(\pi) &\leq -\frac{1}{4} - \frac{3}{4}|Z|^2|Z'|^2 - \frac{3}{2}\langle j(Z)V, j(Z')V' \rangle \\ &\leq -\frac{1}{4} - \frac{3}{4}|Z|^2|Z'|^2 + \frac{3}{2}|j(Z)V||j(Z')V'| \\ &= -\frac{1}{4} - \frac{3}{4}|Z|^2|Z'|^2 + \frac{3}{2}|Z||V||Z'||V'| \\ &= -\frac{1}{4} - \frac{3}{4}|Z|^2|Z'|^2 + \frac{3}{2}|Z||Z'|\sqrt{1-a^2-|Z|^2}\sqrt{1-|Z'|^2} \\ &\leq 0. \end{aligned}$$

Hence $(\mathfrak{s}, \langle\ ,\ \rangle)$ has nonpositive sectional curvature.

On the other hand, it was proved by Heintze [4] that $\nabla R = 0$ if and only if $\mathfrak{s}$ satisfies $j(Z_k)j(Z_l)V \in \mathrm{span}\{j(Z_1)V, \ldots, j(Z_m)V\}$ for all $V \in \mathfrak{v}$ and $k \neq l$. This implies that $\nabla R = 0$ if and only if $\mathfrak{s}$ satisfies $j(Z_k)j(Z_l)V_i \in \{\pm j(Z_1)V_i, \ldots, \pm j(Z_m)V_i\}$ for all $i = 1, \ldots, n$ and $k \neq l$. Then the following was proved by Wolter [10].

Claim 1.1. $\hat{R} \leq 0$ *if and only if* $\nabla R = 0$.

Proof. The 'only if' part is obvious, since it is well-known that a Riemannian symmetric space of noncompact type has nonpositive curvature operator.

To see the converse, assume that $\hat{R} \leq 0$. If $\nabla R \neq 0$, then there exist V_i and $k \neq l$ such that $j(Z_k)j(Z_l)V_i \notin \{\pm j(Z_1)V_i, \ldots, \pm j(Z_m)V_i\}$. Set $V = V_1$ and $V' = j(Z_k)j(Z_l)V_i$. Then V and V' are orthogornal, and $j(Z_l)V = -j(Z_k)V'$. Since $\{j(Z_1)V_i, \ldots, j(Z_m)V_i\} \subset \{V_1, \ldots, V_n\}$ and $j(Z_k)j(Z_l)V_i \in \{V_1, \ldots, V_n\}$, we have $[V, V'] = 0$. Now, let $\omega \in \bigwedge^2 \mathfrak{s}$ be an element defined by $\omega = V \wedge V' + 1/2\, Z_l \wedge Z_k$. Then, by an easy computation, we see that $\langle \hat{R}(\omega), \omega \rangle = 0$. Since $\hat{R} \leq 0$, this implies that ω lies in the eigenspace of $\hat{R}$ with eigenvalue 0, that is, $\hat{R}(\omega) = 0$. However, we have $\langle \hat{R}(\omega), j(Z_l)V \wedge j(Z_k)V \rangle = 3/4$, which contradicts $\hat{R} \leq 0$. □

A Riemannian manifold (M, g) is called an *Einstein manifold*, or g is said to be an *Einstein metric*, if the Ricci tensor Ric of M is proportional to g, that is, $\mathrm{Ric} = cg$ for some constant c.

It should be noted that the metric given in Example 1.1 is an Einstein metric. On the other hand, it is known that each Riemannian symmetric space of noncompact type admits an Einstein metric, induced by the Killing form, with nonpositive curvature operator. These observations motivated T. Wolter to propose the following

Conjecture (1991, Wolter [11]). *A (simply connected) homogeneous Einstein manifold with nonpositive curvature operator is a Riemannian symmetric space.*

2 Main Theorem

An *almost complex structure* on a real differentiable manifold M is a tensor field J which is, at every point $p \in M$, an endomorphism of the tangent space T_pM such that $J^2 = -\mathrm{id}$, where id denotes the identity transformation of T_pM. A manifold with a fixed almost complex structure is called an *almost complex manifold*. The Nijenhuis tensor N of an almost complex manifold (M, J) is a tensor field of type $(1, 2)$ defined by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{1.1}$$

where X and Y are vector fields on M.

Let M be an n-dimensional complex manifold and $(z^1, \ldots, z^n)$ a complex local coordinate system in M. We set $z^i = x^i + \sqrt{-1}y^i$ for $i = 1, \ldots, n$. A complex structure J of M is an almost complex structure J on M defined by

$$J\left(\left(\frac{\partial}{\partial x^i}\right)_p\right) = \left(\frac{\partial}{\partial y^i}\right)_p, \quad J\left(\left(\frac{\partial}{\partial y^i}\right)_p\right) = -\left(\frac{\partial}{\partial x^i}\right)_p$$

for each $p \in M$ and $i = 1, \ldots, n$. It is known that an almost complex structure is a complex structure if and only if N vanishes identically.

A *Hermitian metric* on an almost complex manifold (M, J) is a Riemannian metric g invariant by the almost complex structure J, that is, $g(JX, JY) = g(X, Y)$ for any vector fields X, Y on M. An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an *almost Hermitian manifold* (resp. a *Hermitian manifold*). The *fundamental* 2*-form* Φ of an almost Hermitian manifold $M = (M, J, g)$ is defined by $\Phi(X, Y) = g(X, JY)$ for any vector fields X, Y of M.

An almost Hermitian manifold M is called a *Kähler manifold* if the fundamental 2-form Φ of M is closed and the Nijenhuis tensor N of M vanishes identically. In this case, a Hermitian metric g on M is called a *Kähler metric*. A Kähler manifold (M, J, g) is called *homogeneous* if the group of holomorphic isometries of M acts transitively on M.

In this thesis, we study Wolter's Conjecture in the case of Kähler manifolds and prove the following

Main Theorem. *A homogeneous Kähler Einstein manifold with nonpositive curvature operator is a Riemannian symmetric space.*

3 Structure of homogeneous Kähler manifolds with $K \leq 0$

Let (M, J, g) be a connected, simply connected homogeneous Kähler manifold with non-positive curvature, that is, the sectional curvature K of M is nonpositive everywhere.

It is known, by a result of Heintze [4], that in the group of holomorphic isometries of M there exists a solvable Lie subgroup G which acts simply transitively on M. More precisely, we have the following

Theorem 3.1. *A connected, simply connected homogeneous Kähler manifold* (M, J, g) *with nonpositive curvature is identified with a connected solvable Lie group equipped with a left invariant complex structure* J *and a left invariant Kähler metric* $\langle\ ,\ \rangle$.

Proof. First, by a result of Wolf [9], we know that in the group of holomorphic isometries of M there exists a connected, closed, solvable Lie subgroup G acting transitively on M. Thus M is represented as $M = G/H$, where H is the isotropy subgroup at a given point $p \in M$ and hence is a compact subgroup of G.

By the structure theory of solvable Lie groups ([7]), we know that there exist a closed normal k-solvable subgroup L of G and a compact subgroup K of G such that G is the semidirect product $G = L \cdot K$. Note that a k-solvable subgroup L is a solvable Lie group for which the coset manifold L/T by the compact normal subgroup T in L is simply connected, where T is the unique maximal compact subgroup in the center of L. On the other hand, K is also a compact subgroup of the group of isometries of M. By a theorem of Cartan ([6]), K has a fixed point $p_0 \in M$, since M is simply connected and has nonpositive sectional curvature. This implies that L acts on M transitively, and hence M is represented as $M = L/H'$ with an isotropy subgroup H' of L. Since H' is compact, it is contained in the maximal compact subgroup T. Hence H' is a normal subgroup of L. Note that L acts effectively on M, so that $H' = \{e\}$, where e is a identity element of L. Hence $M = L$, and M is identified with a solvable Lie group L.

Moreover, since L is a subgroup of holomorphic isometries of M, the complex structure J of M induces a left invariant complex structure J of L. Also, the Kähler metric g on M induces a left invariant Kähler metric $\langle\ ,\ \rangle$ on L. □

Our first goal is to determine the structure of a connected, simply connected homogeneous Kähler manifold (M, J, g) with nonpositive curvature. By Theorem 3.1, we see that such M is represented as a simply connected solvable Lie group G with a left invariant complex structure J and a left invariant Kähler metric $\langle\ ,\ \rangle$. Note that, since G is simply connected, the structure of G is determined by its Lie algebra $\mathfrak{g}$ up to isomorphism. Then we obtain the following

Lemma 3.1. *Let $(G, J, \langle\ ,\ \rangle)$ be a connected, simply connected homogeneous Kähler manifold with nonpositive curvature, and $\mathfrak{g}$ the solvable Lie algebra consisting of left invariant vector fields on G. Then the left invariant complex structure J and the left invariant Kähler metric $\langle\ ,\ \rangle$ on G induce, respectively, an endomorphism J and an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfying the following conditions*:

(K1) $J^2 = -\operatorname{id}$,

(K2) $\langle JX, Y\rangle = -\langle X, JY\rangle$,

(K3) $\langle [X,Y], JZ\rangle + \langle [Y,Z], JX\rangle + \langle [Z,X], JY\rangle = 0$,

(K4) $[JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0$

for any $X, Y, Z \in \mathfrak{g}$.

Proof. (K1) is obvious, and (K2) is immediate, since $\langle\ ,\ \rangle$ is Hermitian. Also, (K4) follows from the fact that the Nijenhuis tensor N equals 0.

For (K3), it suffices to recall that the fundamental 2-form $\Phi(X,Y) = \langle X, JY\rangle$ of G is closed, so that for any $X, Y, Z \in \mathfrak{g}$

$$\begin{aligned}
0 &= 3d\Phi(X,Y,Z) \\
&= X\Phi(Y,Z) - Y\Phi(X,Z) + Z\Phi(X,Y) - \Phi([Y,Z],X) + \Phi([X,Z],Y) - \Phi([X,Y],Z) \\
&= X\langle Y, JZ\rangle - Y\langle X, JZ\rangle + Z\langle X, JY\rangle - \langle [Y,Z], JX\rangle + \langle [X,Z], JY\rangle - \langle [X,Y], JZ\rangle \\
&= -\langle [Y,Z], JX\rangle + \langle [X,Z], JY\rangle - \langle [X,Y], JZ\rangle.
\end{aligned}$$

□

4 Curvature functions on solvable Lie algebras

Let $\mathfrak{g}$ be a solvable Lie algebra with inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and extend the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ to a left invariant metric $\langle\ ,\ \rangle$ on G. Regarding $\mathfrak{g}$ as the Lie algebra consisting of left invariant vector fields on G, the Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of G defines respectively the corresponding Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of $\mathfrak{g}$. We first note the following

Claim 4.1. *For any $X, Y \in \mathfrak{g}$, the Levi-Civita connection ∇ is given by*

$$\nabla_X Y = \frac{1}{2}[X, Y] + U(X, Y),$$
$$U(X, Y) = -\frac{1}{2}((\operatorname{ad} X)^* Y + (\operatorname{ad} Y)^* X),$$

where ad *denotes the adjoint representation of $\mathfrak{g}$ and * the transpose with respect to $\langle\ ,\ \rangle$.*

Proof. It follows from the definition of the Levi-Civita connection ∇ that

$$\begin{aligned}\langle \nabla_X Y, Z\rangle =& \frac{1}{2}(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & - \langle [Y, Z], X\rangle - \langle [X, Z], Y\rangle + \langle [X, Y], Z\rangle) \\ =& \frac{1}{2}(\langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle + \langle [X, Y], Z\rangle)\end{aligned}$$

for any $X, Y, Z \in \mathfrak{g}$, since $\langle X, Y\rangle, \langle Y, Z\rangle, \langle Z, X\rangle$ are constant functions on G. Hence we have

$$\nabla_X Y = U(X, Y) + \frac{1}{2}[X, Y].$$

It should be remarked that $U(X, Y)$ (resp. $(1/2)[X, Y]$) gives the symmetric (resp. the skew-symmetric) part of $\nabla_X Y$. □

As a consequence of Claim 4.1, we see that the curvature tensor $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ of $\mathfrak{g}$ is determined by the bracket product of $\mathfrak{g}$. Namely, the following holds.

Claim 4.2. *For any $X, Y \in \mathfrak{g}$, we have*

$$\begin{aligned}\langle R(X, Y)Y, X\rangle =& |U(X, Y)|^2 - \langle U(X, X), U(Y, Y)\rangle - \frac{3}{4}\,|[X, Y]|^2 \\ & - \frac{1}{2}\langle [X, [X, Y]], Y\rangle - \frac{1}{2}\langle [Y, [Y, X]], X\rangle,\end{aligned}$$

where $|\cdot|$ denotes the norm defined by $\langle\ ,\ \rangle$.

Proof. It follows from Claim 4.1 that for any $X, Y \in \mathfrak{g}$

$$\begin{aligned}\langle R(X, Y)Y, X\rangle &= \langle \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y, X\rangle \\ &= \frac{1}{2}\left(\langle X, [X, \nabla_Y Y]\rangle + \langle \nabla_Y Y, [X, X]\rangle + \langle [X, \nabla_Y Y], X\rangle\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\left(\langle Y, [X, \nabla_X Y]\rangle + \langle \nabla_X Y, [X, Y]\rangle + \langle [Y, \nabla_X Y], X\rangle\right) \\
&\quad - \frac{1}{2}\left(\langle [X, Y], [X, Y]\rangle + \langle Y, [X, [X, Y]]\rangle + \langle [[X, Y], Y], X\rangle\right) \\
&= -\langle U(X, X), \nabla_Y Y\rangle + \langle U(X, Y), \nabla_X Y\rangle - \frac{1}{2}\langle U(X, Y), [X, Y]\rangle \\
&\quad - \frac{1}{4}|[X, Y]|^2 - \frac{1}{2}|[X, Y]|^2 - \frac{1}{2}\langle [[Y, X], X], Y\rangle - \frac{1}{2}\langle [[X, Y], Y], X\rangle \\
&= -\langle U(X, X), U(Y, Y)\rangle + \langle U(X, Y), U(X, Y)\rangle \\
&\quad + \frac{1}{2}\langle U(X, Y), [X, Y]\rangle - \frac{1}{2}\langle U(X, Y), [X, Y]\rangle \\
&\quad - \frac{3}{4}|[X, Y]|^2 - \frac{1}{2}\langle [[X, Y], Y], X\rangle - \frac{1}{2}\langle [[Y, X], X], Y\rangle \\
&= -\langle U(X, X), U(Y, Y)\rangle + |U(X, Y)|^2 - \frac{3}{4}|[X, Y]|^2 \\
&\quad - \frac{1}{2}\langle [[X, Y], Y], X\rangle - \frac{1}{2}\langle [[Y, X], X], Y\rangle.
\end{aligned}
$$

□

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$, and B the Killing form of $\mathfrak{g}$. We now define $H \in \mathfrak{g}$ by

$$\langle H, X\rangle = \operatorname{tr}\operatorname{ad} X, \quad X \in \mathfrak{g}.$$

Then we see that H is orthogonal to the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Indeed, for any $X, Y \in \mathfrak{g}$, we have

$$\langle H, [X, Y]\rangle = \operatorname{tr}\operatorname{ad}[X, Y] = \operatorname{tr}[\operatorname{ad} X, \operatorname{ad} Y] = 0.$$

Claim 4.3. *Let B be the Killing form of $(\mathfrak{g}, \langle\ ,\ \rangle)$. Then the Ricci tensor* Ric *and the scalar curvature* sc *of $\mathfrak{g}$ can be expressed as follows*:

(1) $\operatorname{Ric}(X, X) = -\langle \operatorname{ad}(H)X, X\rangle - \frac{1}{2}B(X, X) - \frac{1}{2}\operatorname{tr}\operatorname{ad} X \circ \operatorname{ad} X^* + \frac{1}{4}\sum_{i,j=1}^{n}\langle [e_i, e_j], X\rangle^2$

for all $X \in \mathfrak{g}$.

(2) $\operatorname{sc} = -\langle H, H\rangle - \frac{1}{2}\sum_{i=1}^{n} B(e_i, e_i) - \frac{1}{4}\sum_{i=1}^{n}\operatorname{tr}(\operatorname{ad} e_i)^* \circ \operatorname{ad} e_i.$

Proof. (1) Let $X \in \mathfrak{g}$. It follows from the definition of $H \in \mathfrak{g}$ and Claim 4.1 that

$$\operatorname{tr}\operatorname{ad}\nabla_X X = \langle H, \nabla_X X\rangle = \frac{1}{2}\left(\langle H, [X, X]\rangle - \langle H, (\operatorname{ad} X)^* X\rangle - \langle H, (\operatorname{ad} X)^* X\rangle\right)$$

$$= -\langle [X, H], X\rangle = \langle \operatorname{ad} H(X), X\rangle.$$

This together with Claim 4.2 then yields

$$\begin{aligned}
\operatorname{Ric}(X,X) &= \sum_{i=1}^{n} \langle R(e_i, X)X, e_i\rangle \\
&= \sum_{i=1}^{n} \Big(|U(e_i, X)|^2 - \langle U(e_i, e_i), U(X,X)\rangle \\
&\qquad - \frac{3}{4}\,|[e_i, X]|^2 - \frac{1}{2}\langle [e_i, [e_i, X]], X\rangle - \frac{1}{2}\langle [X, [X, e_i]], e_i\rangle \Big) \\
&= \sum_{i=1}^{n} \left(\frac{1}{4}\sum_{j=1}^{n} \left(\langle X, [e_i, e_j]\rangle + \langle e_i, [X, e_j]\rangle\right)^2 + \langle (\operatorname{ad} e_i)^* e_i, \nabla_X X\rangle \right. \\
&\qquad \left. - \frac{3}{4}\langle (\operatorname{ad} X)^* \circ \operatorname{ad} X(e_i), e_i\rangle - \frac{1}{2}\langle [e_i, [e_i, X]], X\rangle - \frac{1}{2}\langle [X, [X, e_i]], e_i\rangle \right) \\
&= \frac{1}{4}\sum_{i,j=1}^{n} \langle X, [e_i, e_j]\rangle^2 + \frac{1}{2}\sum_{i=1}^{n} \langle X, [[X, e_i], e_j]\rangle + \frac{1}{4}\sum_{i=1}^{n} \langle [X, e_i], [X, e_i]\rangle \\
&\qquad - \operatorname{tr}\operatorname{ad} \nabla_X X - \frac{3}{4}\operatorname{tr}(\operatorname{ad} X)^* \circ \operatorname{ad} X - \frac{1}{2}\sum_{i=1}^{n} \langle [e_i, [e_i, X]], X\rangle - \frac{1}{2}B(X,X) \\
&= \frac{1}{4}\sum_{i,j=1}^{n} \langle X, [e_i, e_j]\rangle^2 - \frac{1}{2}\operatorname{tr}\operatorname{ad} X \circ (\operatorname{ad} X)^* - \langle \operatorname{ad} H(X), X\rangle - \frac{1}{2}B(X,X).
\end{aligned}$$

(2) Using (1), we see that the scalar curvature sc of $\mathfrak{g}$ is given by

$$\begin{aligned}
\mathrm{sc} &= \sum_{k=1}^{n} \operatorname{Ric}(e_k, e_k) \\
&= \sum_{k=1}^{n} \left(-\langle \operatorname{ad} H e_k, e_k\rangle - \frac{1}{2}B(e_k, e_k) - \frac{1}{2}\operatorname{tr}\operatorname{ad} e_k \circ \operatorname{ad} {e_k}^* + \frac{1}{4}\sum_{i,j=1}^{n} \langle [e_i, e_j], e_k\rangle^2 \right) \\
&= -\operatorname{tr}\operatorname{ad} H - \frac{1}{2}\sum_{k=1}^{n} B(e_k, e_k) - \frac{1}{2}\sum_{k=1}^{n} \operatorname{tr}\operatorname{ad} e_k \circ \operatorname{ad} {e_k}^* + \frac{1}{4}\sum_{i,j=1}^{n} \langle [e_i, e_j], [e_i, e_j]\rangle \\
&= -\langle H, H\rangle - \frac{1}{2}\sum_{k=1}^{n} B(e_k, e_k) - \frac{1}{4}\sum_{k=1}^{n} \operatorname{tr}\operatorname{ad} e_k \circ \operatorname{ad} {e_k}^*.
\end{aligned}$$

□

5 Results of Heber

This section is devoted to recalling several relevant results proved by Heber [3] which will be used throughout this thesis.

Let $\mathfrak{g}$ be a Lie algebra with inner product Q, and let $\mathfrak{k}$ and $\mathfrak{p}$ denote the spaces of skew-symmetric and symmetric derivations of $\mathfrak{g}$ with respect to Q, respectively. Note that the direct sum $\mathfrak{k} \oplus \mathfrak{p}$ yields a subalgebra of the Lie algebra $\operatorname{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$.

We now define an involutive Lie algebra automorphism θ of $\mathfrak{k} \oplus \mathfrak{p}$ by $\theta(X+Y) = X - Y$ for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, and an inner product $\langle \ , \ \rangle$ on $\mathfrak{k} \oplus \mathfrak{p}$ by

$$\langle A, B \rangle = -\operatorname{tr}_Q \theta(A) \circ B, \quad A, B \in \mathfrak{k} \oplus \mathfrak{p}.$$

Then, for $A_1, A_2, A_3 \in \mathfrak{k} \oplus \mathfrak{p}$, we have

$$\begin{aligned} \langle [A_1, A_2], A_3 \rangle &= -\operatorname{tr}_Q \theta([A_1, A_2]) \circ A_3 = -\operatorname{tr}_Q [\theta(A_1), \theta(A_2)] \circ A_3 \\ &= -\operatorname{tr}_Q (\theta(A_1) \circ \theta(A_2) - \theta(A_2) \circ \theta(A_1)) \circ A_3 \\ &= -\operatorname{tr}_Q (\theta(A_2) \circ A_3 \circ \theta(A_1) - \theta(A_2) \circ \theta(A_1) \circ A_3 \\ &= -\operatorname{tr}_Q \theta(A_2) \circ [A_3, \theta(A_1)] \\ &= -\langle A_2, [\theta(A_1), A_3] \rangle, \end{aligned} \tag{1.2}$$

which shows that if $A \in \mathfrak{k}$, then $\operatorname{ad} A$ is skew-symmetric with respect to $\langle \ , \ \rangle$. Similarly, it also holds that if $A \in \mathfrak{p}$, then $\operatorname{ad} A$ is symmetric with respect to $\langle \ , \ \rangle$.

Claim 5.1. *$\mathfrak{k} \oplus \mathfrak{p}$ is a reductive subalgebra of $\operatorname{Der}(\mathfrak{g})$, that is, $\mathfrak{k} \oplus \mathfrak{p}$ is decomposed into a direct sum $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p}) \oplus [\mathfrak{k} \oplus \mathfrak{p}, \mathfrak{k} \oplus \mathfrak{p}]$ of the center $\mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$ of $\mathfrak{k} \oplus \mathfrak{p}$ and a semisimple ideal $[\mathfrak{k} \oplus \mathfrak{p}, \mathfrak{k} \oplus \mathfrak{p}]$.*

Proof. For any $Z \in \mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$ and $X, Y \in \mathfrak{k} \oplus \mathfrak{p}$, it follows from (1.2) that

$$\langle Z, [X, Y] \rangle = -\langle [\theta(X), Z], Y \rangle = 0,$$

which implies that $\mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$ is orthogonal to $[\mathfrak{k} \oplus \mathfrak{p}, \mathfrak{k} \oplus \mathfrak{p}]$. Conversely, choose an element $Z \in \mathfrak{k} \oplus \mathfrak{p}$ which is orthogonal to $[\mathfrak{k} \oplus \mathfrak{p}, \mathfrak{k} \oplus \mathfrak{p}]$. For $X, Y \in \mathfrak{k} \oplus \mathfrak{p}$ and Z, (1.2) then yields that

$$\langle [X, Z], Y \rangle = -\langle Z, [\theta(X), Y] \rangle = 0,$$

and hence $Z \in \mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$. In consequence, $\mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$ is an orthogonal complement of $[\mathfrak{k} \oplus \mathfrak{p}, \mathfrak{k} \oplus \mathfrak{p}]$, that is, $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p}) \oplus [\mathfrak{k} \oplus \mathfrak{p}, \mathfrak{k} \oplus \mathfrak{p}]$.

Since θ is a Lie algebra automorphism of $\mathfrak{k}\oplus\mathfrak{p}$, $[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$ is invariant by θ. Let B denote the Killing form of $[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$, and $\{E_k\}$ an orthonormal basis of $[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$ with respect to $\langle\ ,\ \rangle$. For $X\in[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$, we have

$$\begin{aligned} B(\theta(X),X) &= \operatorname{tr}\operatorname{ad}\theta(X)\circ\operatorname{ad}X = \sum_k\langle[\theta(X),[X,E_k]],E_k\rangle \\ &= -\sum_k\langle[X,E_k],[\theta^2(X),E_k]\rangle = -\sum_k\langle[X,E_k],[X,E_k]\rangle. \end{aligned}$$

Assume that B is degenerate. Then there exists some $X_0\in[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$ such that $B(X_0,\cdot)=0$. In particular, we have $B(\theta(X_0),X_0)=0$, which implies that $[X_0,E_k]=0$ for any k. Hence $X\in\mathfrak{z}(\mathfrak{k}\oplus\mathfrak{p})$, contradicting that $X\in[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$. Therefore, B is non-degenerate, that is, $[\mathfrak{k}\oplus\mathfrak{p},\mathfrak{k}\oplus\mathfrak{p}]$ is semisimple. ☐

Claim 5.2. *Let $\mathfrak{g}$ be a solvable Lie algebra with inner product Q. If there exists $X\in\mathfrak{g}$ such that* $\operatorname{ad}X\in\mathfrak{k}\oplus\mathfrak{p}$*, then* $(\operatorname{ad}X)_{\mathfrak{k}},(\operatorname{ad}X)_{\mathfrak{p}}\in\mathfrak{z}(\mathfrak{k}\oplus\mathfrak{p})$*, where* $(\operatorname{ad}X)_{\mathfrak{k}}$ *(resp.* $(\operatorname{ad}X)_{\mathfrak{k}}$*) denotes the* $\mathfrak{k}$ *(resp.* $\mathfrak{p}$*) component of* $\operatorname{ad}X$*.*

Proof. Let $\mathfrak{h}=\operatorname{ad}(\mathfrak{g})\cap\mathfrak{k}\oplus\mathfrak{p}$, a subspace of $\mathfrak{k}\oplus\mathfrak{p}$. Since $\mathfrak{g}$ is solvable, $\mathfrak{h}$ is a solvable ideal of $\mathfrak{k}\oplus\mathfrak{p}$. Moreover, in Claim 5.1 we see that $\mathfrak{k}\oplus\mathfrak{p}$ is reductive. Hence $\mathfrak{z}(\mathfrak{k}\oplus\mathfrak{p})$ is a radical, and hence $\mathfrak{h}\subset\mathfrak{z}(\mathfrak{k}\oplus\mathfrak{p})$.

Let $X\in\mathfrak{g}$ for which $\operatorname{ad}X\in\mathfrak{h}$, and let $(\operatorname{ad}X)_{\mathfrak{k}}$ and $(\operatorname{ad}X)_{\mathfrak{p}}$ be as above. Then it is easy to see that $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{p},\mathfrak{k}]\subset\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$. Hence we have $[(\operatorname{ad}X)_{\mathfrak{k}},\mathfrak{k}]\subset\mathfrak{k}$ and $[(\operatorname{ad}X)_{\mathfrak{p}},\mathfrak{k}]\subset\mathfrak{p}$. It then follows from $[\operatorname{ad}X,\mathfrak{k}]=0$ that $[(\operatorname{ad}X)_{\mathfrak{k}},\mathfrak{k}]=[(\operatorname{ad}X)_{\mathfrak{p}},\mathfrak{k}]=0$. Similarly, we have $[(\operatorname{ad}X)_{\mathfrak{k}},\mathfrak{p}]=[(\operatorname{ad}X)_{\mathfrak{p}},\mathfrak{p}]=0$. Consequently, $(\operatorname{ad}X)_{\mathfrak{k}},(\operatorname{ad}X)_{\mathfrak{p}}\in\mathfrak{z}(\mathfrak{k}\oplus\mathfrak{p})$. ☐

Let $\operatorname{Sym}(\mathfrak{g})$ denote the space of symmetric bilinear forms on $\mathfrak{g}$, and let $\mathrm{GL}^+(\mathfrak{g})$ be the group of linear endomorphisms of $\mathfrak{g}$ with positive determinant. Also, we denote by $\mathbf{P}\subset\operatorname{Sym}(\mathfrak{g})$ the open convex cone of inner products on $\mathfrak{g}$.

For any $Q\in\mathbf{P}$ and $h\in\operatorname{Sym}(\mathfrak{g})$, there exists a symmetric endomorphism $C\in\operatorname{End}(\mathfrak{g})$ relative to Q such that $h=Q(C\cdot,\cdot)$. We now define a curve $Q(t)\in\mathbf{P}$ by

$$Q(t)=Q(e^{tC}\cdot,\cdot)\in\mathbf{P},\quad e^{tC}=\sum_{k=0}^{\infty}\frac{t^k}{k!}C^k.$$

Then the differential of $Q(t)$ at $t=0$ is given by

$$Q'(0)(\cdot,\cdot)=\frac{d}{dt}Q(e^{tC}\cdot,\cdot)\bigg|_{t=0}=Q(C\cdot,\cdot)=h,$$

which implies that $\mathrm{Sym}(\mathfrak{g})$ is a subspace of the tangent space $T_Q\mathbf{P}$ of $\mathbf{P}$ at $Q \in \mathbf{P}$. On the other hand, obviously $T_Q\mathbf{P}$ is a subspace of $\mathrm{Sym}(\mathfrak{g})$. Hence we have $T_Q\mathbf{P} = \mathrm{Sym}(\mathfrak{g})$.

We now define an inner product g_Q on $T_Q\mathbf{P} = \mathrm{Sym}(\mathfrak{g})$ by

$$g_Q(h,k) = \sum_{i,j} h(e_i,e_j)k(e_i,e_j), \quad h,k \in T_Q\mathbf{P},$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to Q. Note that g_Q is well-defined, that is, independent of the choice of an orthonormal basis of $\mathfrak{g}$. Setting $g = \{g_Q\}_Q$, we obtain a Riemannian metric g on $\mathbf{P}$.

Given $Q \in \mathbf{P}$ and $a \in \mathrm{GL}^+(\mathfrak{g})$, we define an action $a \cdot Q$ of $\mathrm{GL}^+(\mathfrak{g})$ on $\mathbf{P}$ by

$$(a \cdot Q)(X,Y) = Q(a^{-1}X, a^{-1}Y).$$

Note that this action of $\mathrm{GL}^+(\mathfrak{g})$ is transitive and isometric on $\mathbf{P}$ with respect to g. Moreover, the isotropy subgroup $\mathrm{GL}^+(\mathfrak{g})_Q$ of $\mathrm{GL}^+(\mathfrak{g})$ at Q coincides with the special orthogonal group $SO(\mathfrak{g},Q)$ of $\mathfrak{g}$ with respect to Q, and hence is compact.

We now fix $Q \in \mathbf{P}$, and define an involutive automorphism $\sigma\colon \mathrm{GL}^+(\mathfrak{g}) \to \mathrm{GL}^+(\mathfrak{g})$ by $\sigma(g) = (g^*)^{-1}$, where * denotes the transpose with respect to Q. Then the set of fixed points of σ coincides with $SO(\mathfrak{g},Q)$. Consequently, $(\mathrm{GL}^+(\mathfrak{g}), SO(\mathfrak{g},Q))$ is a Riemannian symmetric pair, and hence $(\mathbf{P},g) = (\mathrm{GL}^+(\mathfrak{g})_Q/SO(\mathfrak{g},Q), g)$ is a Riemannian symmetric space. Note that, for any geodesic $\tilde{Q}(t) \in \mathbf{P}$ with $\tilde{Q}(0) = Q$, there exists a symmetric endomorphism C of $\mathfrak{g}$ with respect to Q such that $\tilde{Q}(t) = e^{-t/2C} \cdot Q \in \mathbf{P}$.

Let B_Q denote the Killing form of $(\mathfrak{g},Q)$. We define $H_Q \in \mathfrak{g}$ by

$$Q(H_Q, X) = \mathrm{tr}\,\mathrm{ad}\,X, \quad X \in \mathfrak{g}.$$

We also define the following functions on $\mathbf{P}$:

$$\begin{aligned}
\mathrm{Ric}_Q &= \text{the Ricci tensor of } (\mathfrak{g},Q),\\
\mathrm{sc}(Q) &= \text{the scalar curvature of } (\mathfrak{g},Q),\\
h(Q) &= Q(H_Q,H_Q),\\
b(Q) &= \mathrm{tr}_Q\, B_Q,\\
n(Q) &= \mathrm{sc}(Q) + h(Q) + \frac{1}{2}b(Q).
\end{aligned}$$

Recall that the scalar curvature $\mathrm{sc}(Q)$ is given by

$$\mathrm{sc}(Q) = -Q(H_Q,H_Q) - \frac{1}{2}\sum_{i=1}^n B_Q(e_i,e_i) - \frac{1}{4}\sum_{i=1}^n \mathrm{tr}_Q(\mathrm{ad}\,e_i)^* \circ \mathrm{ad}\,e_i,$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to Q, and * denotes the transpose with respect to Q. Then the function n is given by

$$n(Q) = -\frac{1}{4}\sum_{i=1}^{n} \operatorname{tr}_Q (\operatorname{ad} e_i)^*_Q \circ \operatorname{ad} e_i.$$

Claim 5.3. (1) $(\operatorname{grad} n)_Q = -\operatorname{Ric}_Q - Q(D_{H_Q}\cdot, \cdot) - (1/2)\, B_Q$ *holds at any point* $Q \in \mathbf{P}$*, where* D_{H_Q} *denotes the symmetric part of* $\operatorname{ad} H_Q$ *with respect to* Q*.*

(2) *The function* n *is concave on* $\mathbf{P}$*, that is,* $(n \circ Q)'' \leq 0$ *holds along any geodesic* $Q(t)$ *in* $\mathbf{P}$*.*

(3) *Define a curve* $Q(t) = e^{-t\,C/2} \cdot Q$ *in* $\mathbf{P}$*, where* C *is a symmetric endomorphism of* $\mathfrak{g}$ *with respect to* Q*. If* $(n \circ Q)''(0) = 0$*, then* C *is a derivation of* $\mathfrak{g}$*.*

Proof. (2) Fix $Q \in \mathbf{P}$. Let C be a symmetric endomorphism of $\mathfrak{g}$ with respect to Q, and $\{e_1, \dots, e_n\}$ an orthonormal basis of $\mathfrak{g}$ with respect to Q. Without loss of generality, we may suppose that each e_i is an eigenvector of C, that is, $Ce_i = \mu_i e_i$ for $\mu_i \in \mathbb{R}$.

Now, consider a geodesic $Q(t) = e^{-t/2\,C} \cdot Q \in \mathbf{P}$. Note that $\{e^{-t/2\,\mu_1} e_1, \dots, e^{-t/2\,\mu_n} e_n\}$ yields an orthonormal basis of $\mathfrak{g}$ with respect to $Q(t)$. Since $A^*_{Q(t)}$ denotes the transpose of an endomorphism A of $\mathfrak{g}$ with respect to $Q(t)$, we obtain

$$\begin{aligned}
n \circ Q(t) &= -\frac{1}{4}\sum_{i=1}^{n} \operatorname{tr}_{Q(t)} \left(\operatorname{ad}\left(e^{-t\mu_i/2} e_i\right)\right)^*_{Q(t)} \circ \operatorname{ad}\left(e^{-t\mu_i/2} e_i\right) \\
&= -\frac{1}{4}\sum_{i,j=1}^{n} Q(t) \left(\left(\operatorname{ad}\left(e^{-t\mu_i/2} e_i\right)\right)^*_{Q(t)} \circ \operatorname{ad}\left(e^{-t\mu_i/2} e_i\right) e^{-t\mu_j/2} e_j, e^{-t\mu_j/2} e_j\right) \\
&= -\frac{1}{4}\sum_{i,j=1}^{n} Q(t) \left(\left[e^{-t\mu_i/2} e_i, e^{-t\mu_j/2} e_j\right], \left[e^{-t\mu_i/2} e_i, e^{-t\mu_j/2} e_j\right]\right) \\
&= -\frac{1}{4}\sum_{i,j=1}^{n} e^{-t(\mu_i+\mu_j)} Q\left(e^{tC/2}[e_i, e_j], e^{tC/2}[e_i, e_j]\right) \\
&= -\frac{1}{4}\sum_{i,j,k=1}^{n} e^{-t(\mu_i+\mu_j)} Q\left(e^{tC/2}[e_i, e_j], e_k\right)^2 \\
&= -\frac{1}{4}\sum_{i,j,k=1}^{n} e^{-t(\mu_i+\mu_j-\mu_k)} Q\left([e_i, e_j], e_k\right)^2.
\end{aligned}$$

Hence the second order derivative of $n \circ Q(t)$ is given by

$$(n \circ Q)''(t) = -\frac{1}{4} \sum_{i,j,k=1}^{n} (\mu_i + \mu_j - \mu_k)^2 e^{-t(\mu_i+\mu_j-\mu_k)} Q\left([e_i, e_j], e_k\right)^2,$$

which shows that $(n \circ Q)''(t) \leq 0$. Hence n is concave on $\mathbf{P}$.

(3) Let $\mathfrak{g}_{\mu_i}$ denote the eigenspace relative to an eigenvalue μ_i. It follows from (2) that

$$(n \circ Q)''(0) = -\frac{1}{4} \sum_{i,j,k=1}^{n} (\mu_i + \mu_j - \mu_k)^2 Q\left([e_i, e_j], e_k\right)^2 = 0.$$

If $\mu_i + \mu_j - \mu_k \neq 0$, then $Q\left([e_i, e_j], e_k\right) = 0$, that is, $[\mathfrak{g}_{\mu_i}, \mathfrak{g}_{\mu_j}]$ is orthogonal to $\mathfrak{g}_{\mu_k}$. On the other hand, if $\mu_i + \mu_j - \mu_l = 0$, then for $\mu_k \neq \mu_l$, $\mu_i + \mu_j - \mu_k \neq 0$ holds. Hence we have $[\mathfrak{g}_{\mu_i}, \mathfrak{g}_{\mu_j}] \subset \mathfrak{g}_{\mu_l}$ with $\mu_i + \mu_j = \mu_l$. Consequently, C is a derivation of $\mathfrak{g}$.

(1) The first order derivative of $n \circ Q(t)$ at $t = 0$ is given by

$$\begin{aligned}
\langle \operatorname{grad} n, Q'(0) \rangle_Q = (n \circ Q)'(0) &= \frac{1}{4} \sum_{i,j,k=1}^{n} (\mu_i + \mu_j - \mu_k) Q\left([e_i, e_j], e_k\right)^2 \\
&= \frac{1}{2} \sum_{i,j,k=1}^{n} \mu_i Q\left([e_i, e_j], e_k\right)^2 - \frac{1}{4} \sum_{i,j,k=1}^{n} \mu_k Q\left([e_i, e_j], e_k\right)^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{n} \mu_i Q\left([e_i, e_j], [e_i, e_j]\right) - \frac{1}{4} \sum_{i,j,k=1}^{n} \mu_i Q\left([e_j, e_k], e_i\right)^2 \\
&= \sum_{i=1}^{n} \mu_i \left\{ \frac{1}{2} \operatorname{tr}_Q (\operatorname{ad} e_i)^*_Q \circ \operatorname{ad} e_i - \frac{1}{4} \sum_{j,k=1}^{n} Q\left([e_j, e_k], e_i\right)^2 \right\} \\
&= \sum_{i=1}^{n} Q(Ce_i, e_i) \left(-\operatorname{Ric}_Q(e_i, e_i) - Q(D_{H_Q} e_i, e_i) - \frac{1}{2} B(e_i, e_i) \right) \\
&= \left\langle Q'(0), -\operatorname{Ric}_Q - Q(D_{H_Q} \cdot, \cdot) - \frac{1}{2} B \right\rangle_Q .
\end{aligned}$$

Hence we have $(\operatorname{grad} n)_Q = -\operatorname{Ric}_Q - Q(D_{H_Q} \cdot, \cdot) - (1/2)B$. □

Claim 5.4. *Let $a \in GL^+(\mathfrak{g})$ be a Lie algebra automorphism of $\mathfrak{g}$ with positive determinant. Then the function n satisfies $n(a \cdot Q) = n(Q)$ for any $Q \in \mathbf{P}$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to Q, and note that $\{ae_1, \ldots, ae_n\}$ yields an orthonormal basis with respect to $a \cdot Q$. Since a is an automorphism of $\mathfrak{g}$, $n(a \cdot Q)$ is given by

$$
\begin{aligned}
n(a \cdot Q) &= -\frac{1}{4} \sum_{i=1}^{n} \operatorname{tr}_{a \cdot Q} \left(\operatorname{ad}(ae_i)\right)^*_{a \cdot Q} \circ \operatorname{ad}(ae_i) \\
&= -\frac{1}{4} \sum_{i,j=1}^{n} (a \cdot Q) \Big(\left(\operatorname{ad}(ae_i)\right)^*_{a \cdot Q} \circ \operatorname{ad}(ae_i)(ae_j), (ae_j) \Big) \\
&= -\frac{1}{4} \sum_{i,j=1}^{n} (a \cdot Q) \left([ae_i, ae_j], [ae_i, ae_j] \right) \\
&= -\frac{1}{4} \sum_{i,j=1}^{n} (a \cdot Q) \left(a[e_i, e_j], a[e_i, e_j] \right) \\
&= -\frac{1}{4} \sum_{i,j=1}^{n} Q \left([e_i, e_j], [e_i, e_j] \right) \\
&= -\frac{1}{4} \sum_{i=1}^{n} \operatorname{tr}_Q \left(\operatorname{ad} e_i\right)^*_Q \circ \operatorname{ad} e_i = n(Q).
\end{aligned}
$$

Since Q is arbitrary, we obtain Claim 5.4. □

Note that each derivation A of $\mathfrak{g}$ induces a Lie algebra automorphism e^{tA} of $\mathfrak{g}$ with positive determinant. We define a 1-parameter group $\phi\colon \mathbb{R} \times \mathbf{P} \to \mathbf{P}$ of transformations of $\mathbf{P}$ by $\phi_t(Q) = e^{tA} \cdot Q$ for any $Q \in \mathbf{P}$ and $t \in \mathbb{R}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to Q. For any $h \in T_Q\mathbf{P} = \operatorname{Sym}(\mathfrak{g})$, we then have

$$
(d\phi_t)_Q h = \frac{d}{ds} \phi_t(\exp sh) \Big|_{s=0} = \frac{d}{ds} (e^{tA} \cdot \exp sh) \Big|_{s=0} = e^{tA} \cdot h.
$$

Since $\{e^{tA}e_1, \ldots, e^{tA}e_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $\phi_t(Q)$, we have for $h, k \in T_Q\mathbf{P}$

$$
\begin{aligned}
\langle (d\phi_t)_Q h, (d\phi_t)_Q k \rangle &= \langle e^{tA} \cdot h, e^{tA} \cdot k \rangle \\
&= \sum_{i,j=1}^{n} \left(e^{tA} \cdot h\right) \left(e^{tA}e_i, e^{tA}e_j\right) \left(e^{tA} \cdot k\right) \left(e^{tA}e_i, e^{tA}e_j\right) \\
&= \sum_{i,j=1}^{n} h(e_i, e_j) k(e_i, e_j) = \langle h, k \rangle_Q.
\end{aligned}
$$

This shows that ϕ_t is an isometry on $\mathbf{P}$ for any $t \in \mathbb{R}$. Hence the infinitesimal transformation $\tilde{A}$ of ϕ is a Killing vector filed on $\mathbf{P}$.

Claim 5.5. *The Hessian of n in the direction $\tilde{A}_Q$ at $Q \in \mathbf{P}$ is given by*

$$\begin{aligned}&\operatorname{Hess} n(\tilde{A}_Q, \tilde{A}_Q)\\ &\quad = -\sum_i \operatorname{Ric}_Q([A, A_Q^*]e_i, e_i) - \operatorname{tr}_Q D_{H_Q} \circ [A, A_Q^*] - \frac{1}{2}\sum_i B([A, A^*]e_i, e_i),\end{aligned}$$

where D_{H_Q} denotes the symmetric part of $\operatorname{ad} H_Q$ with respect to Q.

Proof. We define a function f on $\mathbf{P}$ by $f(Q) = \langle \tilde{A}_Q, \tilde{A}_Q\rangle_Q$. Since $\phi_t(Q) = Q(e^{-tA}\cdot, e^{-tA}\cdot)$ $= Q(e^{-tA_Q^*} \cdot e^{-tA}\cdot, \cdot)$, $\tilde{A}$ is given by

$$\tilde{A}_Q = \frac{d}{dt}\phi_t(Q)\Big|_{t=0} = \frac{d}{dt}Q(e^{-tA_Q^*}e^{-tA}\cdot, \cdot)\Big|_{t=0} = Q(-(A_Q^* + A)\cdot, \cdot).$$

Fix $Q \in \mathbf{P}$, and let $\{e_q, \dots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to Q. Then f is given by

$$\begin{aligned}f(Q) &= \big\langle Q(-(A_Q^* + A)\cdot, \cdot), Q(-(A_Q^* + A)\cdot, \cdot)\big\rangle_Q\\ &= \sum_{i,j=1}^n Q(-(A_Q^* + A)e_i, e_j)^2\\ &= \sum_{i=1}^n Q((A_Q^* + A)^2 e_i, e_i) = \operatorname{tr}_Q(A_Q^* + A)^2\\ &= 2\operatorname{tr}_Q(A^2 + A_Q^* A).\end{aligned}$$

Given a symmetric endomorphism C of $\mathfrak{g}$ with respect to Q, we look at a curve $Q(t) = e^{-(t/2)C}Q \in \mathbf{P}$ to obtain

$$\begin{aligned}&\langle \operatorname{grad} f,\ Q'(0)\rangle_Q\\ &\quad = \frac{d}{dt} f \circ Q(t)\Big|_{t=0} = 2\frac{d}{dt}\operatorname{tr}_{Q(t)}(A^2 + A_{Q(t)}^* A)\Big|_{t=0}\\ &\quad = 2\frac{d}{dt}\sum_{i=1}^n Q(t)\left((A^2 + A_{Q(t)}^* A)e^{-tC/2}e_i, e^{-tC/2}e_i\right)\Big|_{t=0}\\ &\quad = 2\frac{d}{dt}\sum_{i=1}^n \left(Q(t)(A^2 e^{-tC/2}e_i, e^{-tC/2}e_i) + Q(t)(Ae^{-tC/2}e_i, Ae^{-tC/2}e_i)\right)\Big|_{t=0}\\ &\quad = 2\frac{d}{dt}\sum_{i=1}^n \left(Q(e^{tC/2}A^2 e^{-tC/2}e_i, e_i) + Q(e^{tC/2}Ae^{-tC/2}e_i, e^{tC/2}Ae^{-tC/2}e_i)\right)\Big|_{t=0}\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{d}{dt}\operatorname{tr}_Q\left(e^{tC/2}A^2e^{-tC/2} + e^{-tC/2}A_Q^*e^{tC}Ae^{-tC/2}\right)\Big|_{t=0} \\
&= 2\frac{d}{dt}\operatorname{tr}_Q\left(A^2 + A_Q^*e^{tC}Ae^{-tC}\right)\Big|_{t=0} \\
&= 2\operatorname{tr}_Q(A_Q^*CA - A_Q^*AC) = 2\operatorname{tr}_Q[A, A_Q^*]C \\
&= 2\sum_{i=1}^n Q([A, A_Q^*]Ce_i, e_i) = 2\sum_{i,j=1}^n Q([A, A_Q^*]e_j, e_i)Q(Ce_i, e_j) \\
&= 2\langle Q([A, A_Q^*]\cdot, \cdot), Q'(0)\rangle_Q,
\end{aligned}
$$

which implies that $(\operatorname{grad} f)_Q = 2Q([A, A_Q^*]\cdot, \cdot)$.

On the other hand, since $\tilde{A}$ is a Killing vector field, we have for $X \in T_Q\mathbf{P}$

$$
\begin{aligned}
\langle \nabla_{\tilde{A}}\tilde{A}, X\rangle_Q &= \tilde{A}\langle \tilde{A}, X\rangle_Q - \langle \tilde{A}, \nabla_{\tilde{A}}X\rangle_Q \\
&= \langle [\tilde{A}, \tilde{A}], X\rangle_Q + \langle \tilde{A}, [\tilde{A}, X]\rangle_Q - \langle \tilde{A}, [\tilde{A}, X] + \nabla_X\tilde{A}\rangle_Q \\
&= -\langle \tilde{A}, \nabla_X\tilde{A}\rangle_Q = -\frac{1}{2}X\langle \tilde{A}, \tilde{A}\rangle_Q \\
&= -\frac{1}{2}Xf = -\frac{1}{2}\langle \operatorname{grad} f, X\rangle_Q \\
&= -\langle Q([A, A_Q^*]\cdot, \cdot), X\rangle_Q,
\end{aligned}
$$

which implies that $(\nabla_{\tilde{A}}\tilde{A})_Q = -Q([A, A_Q^*]\cdot, \cdot)$. Moreover, it follows from Claim 5.4 that

$$
\tilde{A}_Q \cdot n = \frac{d}{dt} n \circ \phi_t(Q)\Big|_{t=0} = \frac{d}{dt} n(e^{tA}\cdot Q)\Big|_{t=0} = \frac{d}{dt} n(Q)\Big|_{t=0} = 0.
$$

Consequently, it follows from Claim 5.3 (1) that

$$
\begin{aligned}
&\operatorname{Hess} n(\tilde{A}_Q, \tilde{A}_Q) \\
&\quad = \tilde{A}_Q(\tilde{A}\cdot n) - \nabla_{\tilde{A}_Q}\tilde{A}\cdot n \\
&\quad = -\langle \operatorname{grad} n, \nabla_{\tilde{A}_Q}\tilde{A}\rangle_Q \\
&\quad = -\left\langle -\operatorname{Ric}_Q - Q(D_{H_Q}\cdot, \cdot) - \frac{1}{2}B_Q, -Q([A, A_Q^*]\cdot, \cdot)\right\rangle \\
&\quad = \sum_{i,j=1}^n \left(-\operatorname{Ric}_Q(e_i, e_j) - Q(D_{H_Q}e_i, e_j) - \frac{1}{2}B_Q(e_i, e_j)\right) Q([A, A_Q^*]e_i, e_j) \\
&\quad = \sum_{i=1}^n \left(-\operatorname{Ric}_Q(e_i, [A, A_Q^*]e_i) - Q(D_{H_Q}e_i, [A, A_Q^*]e_i) - \frac{1}{2}B_Q(e_i, [A, A_Q^*]e_i)\right)
\end{aligned}
$$

$$= -\sum_{i=1}^{n} \mathrm{Ric}_Q(e_i, [A, A_Q^*]e_i) - \mathrm{tr}_Q\, D_{H_Q} \circ [A, A_Q^*] - \frac{1}{2}\sum_i B([A, A_Q^*]e_i, e_i),$$

which proves Claim 5.5. □

Lemma 5.1. *Let $\mathfrak{g}$ be a solvable Lie algebra with an Einstein metric Q_0, and $H_0 = H_{Q_0} \in \mathfrak{g}$ a vector defined by $Q_0(H_0, X) = \mathrm{tr}_{Q_0}\, \mathrm{ad}\, X$ for any $X \in \mathfrak{g}$. Then, for any derivation A, the following inequality holds*:

$$\mathrm{tr}\, \mathrm{ad}\, A(H_0) \circ A^* + \frac{1}{2}\sum_i B(A^*Ae_i, e_i) \leq 0, \tag{1.3}$$

where $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to Q_0, B denotes the Killing form of $\mathfrak{g}$, and * denotes the transpose with respect to Q_0. In particular, the equality holds if and only if A^* is a derivation of $\mathfrak{g}$.

Proof. Since Q_0 is an Einstein metric, the Ricci tensor Ric_{Q_0} relative to Q_0 satisfies $\mathrm{Ric}_{Q_0} = \lambda Q_0$ for some constant λ. Hence we have

$$\sum_{i=1}^{n} \mathrm{Ric}_{Q_0}(e_i, [A, A_{Q_0}^*]e_i) = \sum_{i=1}^{n} \lambda Q_0(e_i, [A, A_{Q_0}^*]e_i) = \lambda\, \mathrm{tr}_{Q_0}[A, A_{Q_0}^*] = 0.$$

For a derivation A of $\mathfrak{g}$, let $\tilde{A}$ denote the infinitesimal transformation of a 1-parameter group $\phi_t(Q) = e^{tA} \cdot Q$ for $t \in \mathbb{R}$ and $Q \in \mathbf{P}$. Let S_{H_0} denote the skew-symmetric part of $\mathrm{ad}\, H_0$ with respect to Q_0. Since $AA_{Q_0}^*$ and $A_{Q_0}^*A$ are symmetric with respect to Q_0, we have $\mathrm{tr}_{Q_0}\, S_{H_0}AA_{Q_0}^* = \mathrm{tr}_{Q_0}\, S_{H_0}A_{Q_0}^*A = 0$. Hence we obtain

$$\begin{aligned}\mathrm{tr}_{Q_0}\, D_{H_0} \circ [A, A_{Q_0}^*] &= \mathrm{tr}_{Q_0}\, \mathrm{ad}\, H_0 \circ [A, A_{Q_0}^*] \\ &= -\,\mathrm{tr}_{Q_0}[A, \mathrm{ad}\, H_0] \circ A_{Q_0}^* \\ &= -\,\mathrm{tr}_{Q_0}\, \mathrm{ad}\, A(H_0) \circ A_{Q_0}^*.\end{aligned}$$

Since $\mathfrak{g}$ is solvable, $A(\mathfrak{g})$ is a subalgebra of a maximal nilpotent ideal of $\mathfrak{g}$. Thus $B(AA_{Q_0}^*e_i, e_i) = 0$ holds for all i, and hence we have

$$\sum_i B([A, A_{Q_0}^*]e_i, e_i) = -\sum_i B(A_{Q_0}^*Ae_i, e_i).$$

Consequently, it follows from Claim 5.5 that

$$\mathrm{Hess}\, n(\tilde{A}_{Q_0}, \tilde{A}_{Q_0}) = \mathrm{tr}_{Q_0}\, \mathrm{ad}\, A(H_0) \circ A_{Q_0}^* + \frac{1}{2}\sum_i B(A_{Q_0}^*Ae_i, e_i).$$

Recall that the function n is concave on $\mathbf{P}$ by Claim 5.3. Hence we have $\operatorname{Hess} n(\tilde{A}_{Q_0}, \tilde{A}_{Q_0}) \leq 0$, that is,

$$\operatorname{tr}_{Q_0} \operatorname{ad} A(H_0) \circ A^*_{Q_0} + \frac{1}{2} \sum_i B(A^*_{Q_0} A e_i, e_i) \leq 0.$$

Now, recalling that $\tilde{A}_Q = Q(-(A^*_Q + A)\cdot, \cdot)$, let C be a symmetric endomorphism defined by $C = -(A^*_{Q_0} + A)$ with respect to Q_0. Setting $Q_0(t) = e^{-tC/2} Q_0$, we have $Q_0'(0) = Q_0(-(A^*_{Q_0} + A)\cdot, \cdot) = \tilde{A}_{Q_0}$. If $\operatorname{Hess} n(\tilde{A}_{Q_0}, \tilde{A}_{Q_0}) = 0$, then $(n \circ Q_0)''(0) = 0$. Hence Claim 5.3 (3) shows that $C = -(A^*_{Q_0} + A)$ is a derivation of $\mathfrak{g}$. Since A is a derivation, $A^*_{Q_0}$ is also a derivation. □

A solvable Lie algebra $\mathfrak{g}$ with an inner product Q is called *unimodular* if $\operatorname{tr}_Q \operatorname{ad} X = 0$ for all $X \in \mathfrak{g}$. Note that $H_Q = 0$ if and only if $\mathfrak{g}$ is unimodular.

Lemma 5.2. *Let $\mathfrak{g}$ be a solvable Lie algebra with Einstein metric Q_0 on $\mathfrak{g}$. Then the following are equivalent*:

(1) *$\mathfrak{g}$ is unimodular.*

(2) *$(\mathfrak{g}, Q_0)$ is flat.*

(3) *$(\mathfrak{g}, Q_0)$ is Ricci flat.*

(4) *The orthogonal complement $\mathfrak{a}$ of the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ is abelian, and $\operatorname{ad} A$ is skew-symmetric with respect to Q_0 for any $A \in \mathfrak{a}$.*

Proof. (1) $\Rightarrow$ (3) Since $\mathfrak{g}$ is solvable, the orthogonal complement $\mathfrak{a}$ of the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is not zero. Let $A \in \mathfrak{a}$, and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to Q_0 such that $D_A e_i = \lambda_i e_i$ for $i = 1, \ldots, n$. Applying Claim 4.3 to A, we have

$$\begin{aligned}
\operatorname{Ric}_{Q_0}(A, A) &= -\frac{1}{2} B_{Q_0}(A, A) - \frac{1}{2} \operatorname{tr}_{Q_0} \operatorname{ad} A \circ \operatorname{ad} A^* + \frac{1}{4} \sum_{i,j=1}^{n} Q_0([e_i, e_j], A)^2 \\
&= -\frac{1}{2} \operatorname{tr}_{Q_0} \operatorname{ad} A \circ \operatorname{ad} A - \frac{1}{2} \operatorname{tr}_{Q_0} \operatorname{ad} A \circ \operatorname{ad} A^* \\
&= -\operatorname{tr}_{Q_0} \operatorname{ad} A \circ D_A = -\sum_{i=1}^{n} Q_0(\operatorname{ad} A \circ D_A e_i, e_i) \\
&= -\sum_{i=1}^{n} \lambda_i Q_0(\operatorname{ad} A e_i, e_i) = -\sum_{i=1}^{n} \lambda_i Q_0(D_A e_i, e_i)
\end{aligned}$$

$$= -\sum_{i=1}^{n} \lambda_i^2 \ \le \ 0.$$

On the other hand, it is proved by Miatello [8] that no unimodular solvable Lie algebra admits inner product of strictly negative Ricci curvature. Hence, since Q_0 is Einstein, we have $\mathrm{Ric}_{Q_0} = 0$, that is, $(\mathfrak{g}, Q_0)$ is Ricci flat.

(3) $\Rightarrow$ (4) It is proved by Jensen [5] that for a solvable Lie algebra with inner product Q, its scalar curvature $\mathrm{sc}(Q)$ satisfies $\mathrm{sc}(Q) \le \mathrm{sc}(Q) + Q(H_Q, H_Q) \le 0$. Moreover, if $\mathrm{sc}(Q) + Q(H_Q, H_Q) = 0$, then $\mathfrak{g}$ satisfies Condition (4).

Since Q_0 is a Ricci flat Einstein metric, the scalar curvature of $(\mathfrak{g}, Q_0)$ vanishes, that is, $\mathrm{sc}(Q_0) = 0$. Hence we also have $\mathrm{sc}(Q_0) + Q(H_{Q_0}, H_{Q_0}) = 0$, which implies the result.

(4) $\Rightarrow$ (1) It follows from (3) that $\mathrm{tr}\,\mathrm{ad}\, A = 0$ for $A \in \mathfrak{a}$. Moreover, it is easy to see that $\mathrm{tr}\,\mathrm{ad}\, Y = 0$ for $Y \in \mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Hence we have $\mathrm{tr}\,\mathrm{ad}\, X = 0$ for all $X \in \mathfrak{g}$.

(4) $\Rightarrow$ (2) It is not hard to see that $\nabla_A = \mathrm{ad}\, A$ for $A \in \mathfrak{a}$, and $\nabla_X = 0$ for $X \in \mathfrak{n}$. A straightforward computation then yields that $R(A, A') = 0, R(A, X) = 0$ and $R(X, X') = 0$ for any $A, A' \in \mathfrak{a}$ and $X, X' \in \mathfrak{n}$.

(2) $\Rightarrow$ (3) is trivial. □

Claim 5.6. *If $\mathfrak{g}$ be a non-unimodular solvable Lie algebra with Einstein metric Q_0, then the Ricci curvature of $(\mathfrak{g}, Q_0)$ is strictly negative.*

Proof. As stated above, it is proved by Jensen [5] that any solvable Lie algebra with inner product has nonpositive scalar curvature. Since Q_0 is an Einstein metric, its scalar curvature $\mathrm{sc}(Q_0)$ is zero or strictly negative. If $\mathrm{sc}(Q_0) = 0$, then $(\mathfrak{g}, Q_0)$ is Ricci flat, and hence $\mathfrak{g}$ is unimodular by Lemma 5.2, which is a contradiction. Hence, $\mathrm{sc}(Q_0) < 0$, and the Ricci curvature of $(\mathfrak{g}, Q_0)$ is strictly negative. □

Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n}$ with respect to Q. For any $A \in \mathfrak{a}$, we denote by D_A and S_A the symmetric and skew-symmetric parts of $\mathrm{ad}\, A$ with respect to Q, respectively.

Lemma 5.3. *Let $\mathfrak{g}$ be a non-unimodular solvable Lie algebra with Einstein metric Q_0, $\mathfrak{a}$ the orthogonal complement of the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$, and $H_0 = H_{Q_0} \in \mathfrak{g}$ the vector defined by $Q_0(H_0, X) = \mathrm{tr}\,\mathrm{ad}\, X$ for any $X \in \mathfrak{g}$. Assume that $\mathfrak{a}$ is abelian. Then the following holds*:

(1) *For any* $A \in \mathfrak{a}$*,* D_A *and* S_A *are derivations of* $\mathfrak{g}$*. Moreover,* $\{D_A, S_A \mid A \in \mathfrak{a}\}$ *is abelian.*

(2) $D_A \neq 0$ *for any* $A \in \mathfrak{a}$*, and the restriction* $D_{H_0}|_{\mathfrak{n}}$ *of* D_{H_0} *to* $\mathfrak{n}$ *is positive definite.*

Proof. (1) Let $\{\mathfrak{n}^{(i)}\}$ be the lower central series of $\mathfrak{n}$ defined by

$$\mathfrak{n}^{(1)} = \mathfrak{n} \supseteq \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supseteq \cdots \supseteq \mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}] \supseteq \cdots,$$

and $\mathfrak{a}_i$ be the orthogonal complement of $\mathfrak{n}^{(i+1)}$ in $\mathfrak{n}^{(i)}$ with respect to Q_0. Since $\mathfrak{g}$ is solvable, $\mathfrak{n}$ is nilpotent. Hence there exists $r > 0$ such that $\mathfrak{n}^{(r)} \neq \{0\}$ and $\mathfrak{n}^{(r+1)} = \{0\}$. Then $\mathfrak{g}$ is decomposed into a direct sum $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$. We set $\mathfrak{a}_0 = \mathfrak{a}$, and let $\{e_p^i\}$ be an orthonormal basis of $\mathfrak{a}_p$ with respect to Q_0 for $p = 0, 1, \ldots, r$.

Let $A \in \mathfrak{a}$, and consider the derivation $\operatorname{ad} A$ of $\mathfrak{g}$. Recall that H_0 is orthogonal to $\mathfrak{n}$, that is, $H_0 \in \mathfrak{a}$. Substituting $\operatorname{ad} A$ for A in the left hand side of (1.3), we have

$$\begin{aligned} &\operatorname{tr} \operatorname{ad}(\operatorname{ad} A(H_0)) \circ (\operatorname{ad} A)^* + \frac{1}{2} \sum_{i,p} B((\operatorname{ad} A)^* \operatorname{ad} A e_p^i, e_p^i) \\ &= \frac{1}{2} \sum_{i,j,p,q} Q_0 \left(\left[(\operatorname{ad} A)^*([A, e_p^i]), [e_p^i, e_q^j] \right], e_q^j \right) = 0. \end{aligned}$$

Hence it follows from Lemma 5.1 that D_A and S_A are derivations of $\mathfrak{g}$.

Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) denote the space of skew-symmetric (resp. symmetric) derivations of $\mathfrak{g}$ with respect to Q_0, and $\mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$ be the center of a Lie algebra $\mathfrak{k} \oplus \mathfrak{p}$. Then it follows from Claim 5.2 that $D_A, S_A \in \mathfrak{z}(\mathfrak{k} \oplus \mathfrak{p})$, since $\operatorname{ad} A \in \mathfrak{k} \oplus \mathfrak{p}$. This implies that $\{D_A, S_A \mid A \in \mathfrak{a}\}$ is abelian, thereby proving (1).

(2) Since D_A is a derivation of $\mathfrak{g}$ for each $A \in \mathfrak{a}$, we define a new Lie bracket $[\ ,\ ]^+$ on $\mathfrak{g}$ by

$$[A, X]^+ = D_A X, \quad [X, Y]^+ = [X, Y] \quad A \in \mathfrak{a}, X, Y \in \mathfrak{n}.$$

Then we have a new solvable Lie algebra $\mathfrak{g}^+ = (\mathfrak{g}, [\ ,\ ]^+)$. Moreover, it is easy to see that Q_0 yields an Einstein metric on $\mathfrak{g}^+$.

Let ad^+ denote the adjoint representation of $\mathfrak{g}^+$, and B^+ the Killing form of $\mathfrak{g}^+$. Let $H_0^+ \in \mathfrak{g}^+$ be defined by $Q_0(H_0^+, X) = \operatorname{tr} \operatorname{ad}^+ X$ for any $X \in \mathfrak{g}^+$. Then, for $X \in \mathfrak{g}^+$, we have

$$\operatorname{tr} \operatorname{ad}^+ X = \sum_i Q_0[X, e_i]^+, e_i) = \sum_i Q_0([X, e_i], e_i) = \operatorname{tr} \operatorname{ad} X,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathfrak{g}^+$ with respect to Q_0. Hence we have $H_0^+ = H_0$.

Let $\{e_p^i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to Q_0 given in (1). Then, for $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$, we obtain

$$
\begin{aligned}
&B^+(A+X, A+X)\\
&\quad = \operatorname{tr} \operatorname{ad}^+(A+X) \circ \operatorname{ad}^+(A+X)\\
&\quad = \operatorname{tr} \big(\operatorname{ad}^+ A \circ \operatorname{ad}^+ A + 2 \operatorname{ad}^+ X \circ \operatorname{ad}^+ A + \operatorname{ad}^+ X \circ \operatorname{ad}^+ X\big)\\
&\quad = \sum_{i,p} \big(Q_0(D_A D_A e_p^i, e_p^i) + 2Q_0([X, D_A e_p^i]^+, e_p^i) + Q_0([X,[X,e_p^i]^+]^+, e_p^i)\big)\\
&\quad = \sum_{i,p} Q_0(D_A e_p^i, D_A e_p^i) \geq 0,
\end{aligned}
$$

which shows that B^+ is positive definite.

Choose a unit vector $X \in \mathfrak{n}$ such that $\operatorname{ad}^+ H_0(X) = \alpha X$. Since $(\operatorname{ad}^+ X)^* \circ \operatorname{ad}^+ X$ is symmetric with respect to Q_0, there exists an orthonormal basis $\{e_i\}$ of $\mathfrak{g}$ with respect to Q_0, satisfying $(\operatorname{ad}^+ X)^* \circ \operatorname{ad}^+ X(e_i) = \mu_i e_i$. We first note that $\mu_i \geq 0$ for all i. Indeed, we have

$$
\mu_i Q_0(e_i, e_i) = Q_0\big((\operatorname{ad}^+ X)^* \circ \operatorname{ad}^+ X(e_i), e_i\big) = Q_0(\operatorname{ad}^+ X(e_i), \operatorname{ad}^+ X(e_i)) \geq 0.
$$

Then, substituting $\operatorname{ad}^+ X$ for A in (1.3), the left side of (1.3) is given by

$$
\begin{aligned}
&\operatorname{tr} \operatorname{ad}^+(\operatorname{ad}^+ X(H_0)) \circ (\operatorname{ad}^+ X)^* + \frac{1}{2}\sum_i B^+((\operatorname{ad}^+ X)^* \circ \operatorname{ad}^+ X e_i, e_i)\\
&\qquad = -\alpha \operatorname{tr} \operatorname{ad}^+ X \circ (\operatorname{ad}^+ X)^* + \frac{1}{2}\sum_i \mu_i B^+(e_i, e_i)\\
&\qquad = -\alpha \sum_i Q_0(((\operatorname{ad}^+ X)^* \circ \operatorname{ad}^+ X) e_i, e_i) + \frac{1}{2}\sum_i \mu_i B^+(e_i, e_i)\\
&\qquad = -\alpha \sum_i \mu_i + \frac{1}{2}\sum_i \mu_i B^+(e_i, e_i).
\end{aligned}
$$

Hence we have

$$
-\alpha \sum_i \mu_i \leq -\frac{1}{2}\sum_i \mu_i B^+(e_i, e_i) \leq 0,
$$

which implies that $\alpha \geq 0$.

Assume that $\alpha = 0$. Then from the above inequality we have $\sum \mu_i B^+(e_i, e_i) = 0$, which implies that

$$
\operatorname{tr} \operatorname{ad}^+(\operatorname{ad}^+ X(H_0)) \circ (\operatorname{ad}^+ X)^* + \frac{1}{2}\sum_i B^+((\operatorname{ad}^+ X)^* \circ \operatorname{ad}^+ X e_i, e_i) = 0.
$$

Hence, by Lemma 5.1, the symmetric and the skew-symmetric parts of $\mathrm{ad}^+ X$ are commutative derivations of $\mathfrak{g}$. Since $\mathfrak{n}$ is nilpotent, $\mathrm{ad}^+ X|_{\mathfrak{n}}$, the restriction of $\mathrm{ad}^+ X$ to $\mathfrak{n}$, is also nilpotent. Hence $\mathrm{ad}^+ X$ is nilpotent on $\mathfrak{g}$, since $\mathrm{ad}^+ X(\mathfrak{g}) \subset \mathfrak{n}$. Note that $(\mathrm{ad}^+ X)^*$ is also nilpotent on $\mathfrak{g}$. Since $(\mathrm{ad}^+ X)^* \circ \mathrm{ad}^+ X = \mathrm{ad}^+ X \circ (\mathrm{ad}^+ X)^*$ holds, the symmetric and the skew-symmetric parts of $\mathrm{ad}^+ X$ are both nilpotent on $\mathfrak{g}$. Hence the symmetric part of $\mathrm{ad}^+ X$ vanishes, so that $\mathrm{ad}^+ X$ is a skew-symmetric derivation of $\mathfrak{g}$ with respect to Q_0. Then the Ricci curvature $\mathrm{Ric}(X, X)$ in the direction X is given by

$$\begin{aligned}
\mathrm{Ric}(X, X) =& - Q_0(\mathrm{ad}^+(H_0)X, X) - \frac{1}{2}B^+(X, X) \\
& - \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}^+ X \circ (\mathrm{ad}^+ X)^* + \frac{1}{4}\sum_{i,j} Q_0([e_i, e_j]^+, X)^2 \\
=& - \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}^+ X \circ \mathrm{ad}^+ X + \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}^+ X \circ \mathrm{ad}^+ X + \frac{1}{4}\sum_{i,j} Q_0([e_i, e_j]^+, X)^2 \\
=& \frac{1}{4}\sum_{i,j} Q_0([e_i, e_j]^+, X)^2 \geq 0.
\end{aligned}$$

This contradicts Claim 5.6. Therefore, we have $\alpha > 0$, that is, the restriction $D_{H_0}|_{\mathfrak{n}}$ of D_{H_0} to $\mathfrak{n}$ is positive definite.

Assume that there exists $A \in \mathfrak{a}$ such that $D_A = 0$, that is, $\mathrm{ad}^+ A = 0$. The Ricci curvature $\mathrm{Ric}(A, A)$ in the direction A is then given by

$$\begin{aligned}
\mathrm{Ric}(A, A) =& - Q_0(\mathrm{ad}^+(H_0)A, A) - \frac{1}{2}B^+(A, A) \\
& - \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}^+ A \circ (\mathrm{ad}^+ A)^* + \frac{1}{4}\sum_{i,j} Q_0([e_i, e_j]^+, A)^2 \\
=& - \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}^+ A \circ \mathrm{ad}^+ A = 0,
\end{aligned}$$

which contradicts Claim 5.6. Hence $D_A \neq 0$ for any $A \in \mathfrak{a}$. □

6 Structure of homogeneous Kähler Einstein manifolds with $K \leq 0$

Let $M = (M, J, g)$ be a connected, simply connected homogeneous Kähler manifold with nonpositive sectional curvature $K \leq 0$. Recall that M is identified with a simply connected solvable Lie group G with a left invariant complex structure J and a left invariant Kähler metric $\langle\ ,\ \rangle$ on G (cf. Theorem 3.1 of §3). By Lemma 3.1, the Lie algebra $\mathfrak{g}$ of G admits an endomorphism J and an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfying Conditions (K1)–(K4). Also, the Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of $\mathfrak{g}$ are defined in the natural way.

Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle$. For any $A \in \mathfrak{a}$, we denote by D_A and S_A the symmetric and the skew-symmetric parts of $\operatorname{ad} A\colon \mathfrak{g} \to \mathfrak{n}$, respectively.

From now on we assume that $\langle\ ,\ \rangle$ is an Einstein metric. Since the sectional curvature of $(\mathfrak{g}, \langle\ ,\ \rangle)$ is nonpositive, the Ricci curvature of $(\mathfrak{g}, \langle\ ,\ \rangle)$ is either strictly negative or zero. We have already seen in Lemma 5.2 that if $(\mathfrak{g}, \langle\ ,\rangle)$ is Ricci flat, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is flat. Suppose that $(\mathfrak{g}, \langle\ ,\ \rangle)$ is not Ricci flat. In order to describe basic properties of $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ in the language of Lie algebra, we first prove

Proposition 6.1. *Let $\mathfrak{g}$ be a solvable Lie algebra with an endomorphism J and an Einstein metric $\langle\ ,\ \rangle$ satisfying Conditions* (K1)–(K4). *Suppose that $(\mathfrak{g}, \langle\ ,\ \rangle)$ has nonpositive sectional curvature and is not Ricci flat. Then the following hold*:

(a) *There exists an orthogonal basis $\{H_a\}_{a\in\Lambda}$ of $\mathfrak{a}$ with respect to $\langle\ ,\ \rangle$ such that*

$$\begin{aligned} &[H_a, JH_a] = \lambda_a JH_a && \text{for some } \lambda_a > 0, \\ &[H_b, JH_a] = 0 && \text{if } a \neq b. \end{aligned}$$

Moreover, setting $H = \sum_{a\in\Lambda} H_a$, we have $\langle H, X\rangle = \operatorname{tr}\operatorname{ad} X$ for any $X \in \mathfrak{g}$.

(b) *Define a linear function $\lambda_a\colon \mathfrak{a} \to \mathbb{R}$ by $\lambda_a(H_b) = \delta_{ab}\lambda_a$ for any $b \in \Lambda$. Let $\mathfrak{n}_a^{\pm b}$ and $\mathfrak{n}_a^0$ be the subspaces of $\mathfrak{n}$ defined by*

$$\mathfrak{n}_a^{\pm b} = \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\left(\lambda_a(A) \pm \lambda_b(A)\right) X \quad \text{for any } A \in \mathfrak{a} \right\},$$

$$\mathfrak{n}_a^{0} = \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\lambda_a(A) X \quad \text{for any } A \in \mathfrak{a} \right\},$$

where $\lambda_b(H) < \lambda_a(H)$, and set

$$\mathfrak{n}_a = \bigoplus_{\lambda_b(H)<\lambda_a(H)} \left(\mathfrak{n}_a^{+b} \oplus \mathfrak{n}_a^{-b}\right) \oplus \mathfrak{n}_a^0.$$

Then $\mathfrak{g}$ is decomposed into a direct sum $\mathfrak{g} = \bigoplus_a \mathbb{R}\{H_a\} \oplus \mathfrak{n}_a \oplus \mathbb{R}\{JH_a\}$ which satisfies the following:

(i) $J\mathfrak{n}_a^{\pm b} = \mathfrak{n}_a^{\mp b}$.

(ii) $[X,Y] = \dfrac{\lambda_a(H_a)}{|H_a|^2}\langle JX, Y\rangle JH_a$ *for* $X, Y \in \mathfrak{n}_a$.

(iii) $[JH_b, X] = -\lambda_b(H_b)JX$ *for* $X \in \mathfrak{n}_a^{-b}$.

(iv) $[Y,X] = -J[JY,X], \quad |[Y,X]|^2 = \dfrac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2|X|^2$ *for* $X \in \mathfrak{n}_a^{-b}, Y \in \mathfrak{n}_b$.

(v) $[Y,X] = [JY,JX], \quad |[Y,X]|^2 = \dfrac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2|X|^2$ *for* $X \in \mathfrak{n}_a^{\mp c}, Y \in \mathfrak{n}_b^{\pm c}$.

(vi) $[Y,X] = [JY,JX], \quad |[Y,X]| = |[Y,JX]|$ *for* $X \in \mathfrak{n}_a^0, Y \in \mathfrak{n}_b^0$.

(vii) *Set* $\Lambda_c = \{a \in \Lambda \mid \mathfrak{n}_a^{\pm c} \neq \{0\}\} \cup \{c\}$ *for* $c \in \Lambda$, *and let* $a, b \in \Lambda_c$. *If* $a \neq b$, *then* $\lambda_a(H) \neq \lambda_b(H)$. *Moreover, if* $\lambda_a(H) > \lambda_b(H)$, *then* $\mathfrak{n}_a^{\pm b} \neq \{0\}$.

Proof. (a) Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle$. Then it is known by Azencott and Wilson [1] that $\mathfrak{a}$ is abelian, since $\mathfrak{g}$ has nonpositive sectional curvature $K \leq 0$.

Since $(\mathfrak{g}, \langle\ ,\ \rangle)$ is not Ricci flat, $\mathfrak{g}$ is non-unimodular by Lemma 5.2. Hence there exists a non-zero vector $H \in \mathfrak{g}$ such that $\langle H, X\rangle = \operatorname{tr}\operatorname{ad} X$ for all $X \in \mathfrak{g}$. We have already seen in §4 that H is orthogonal to $\mathfrak{n}$, and hence $H \in \mathfrak{a}$.

Moreover, by Lemma 5.3, $\{D_A, S_A \mid A \in \mathfrak{a}\}$ is a commuting family of derivations of $\mathfrak{g}$ that annihilate $\mathfrak{a}$. Also, $D_A \neq 0$ for any $A \in \mathfrak{a}$, and D_H is positive definite on $\mathfrak{n}$.

Claim 6.1. (1) $S_AJ - JS_A = 0$ *for any* $A \in \mathfrak{a}$.

(2) $S_AJ\mathfrak{a} = \{0\}$ *for any* $A \in \mathfrak{a}$.

(3) $D_BJA - D_AJB = 0$ *for any* $A, B \in \mathfrak{a}$.

(4) $[JA, JB] = 0$ *for any* $A, B \in \mathfrak{a}$.

Proof. (1) It follows from Conditions (K3) and (K4) with $X, Y \in \mathfrak{g}$ and $A \in \mathfrak{a}$ that

$$\begin{aligned}
0 &= \langle [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y], JA\rangle \\
&= -\langle [JY,A], J^2X\rangle - \langle [A,JX], J^2Y\rangle + \langle [Y,A], JX\rangle + \langle [A,X], JY\rangle \\
&= -\langle \operatorname{ad} AJY, X\rangle + \langle \operatorname{ad} AJX, Y\rangle - \langle \operatorname{ad} AY, JX\rangle + \langle \operatorname{ad} AX, JY\rangle \\
&= \langle Y, J(\operatorname{ad} A)^*X\rangle + \langle \operatorname{ad} AJX, Y\rangle - \langle Y, (\operatorname{ad} A)^*JX\rangle - \langle J\operatorname{ad} AX, Y\rangle \\
&= 2\langle (S_AJ - JS_A)X, Y\rangle.
\end{aligned}$$

Since X and Y are arbitrary in $\mathfrak{g}$, we have $S_A J - J S_A = 0$.

(2) Let $A, B \in \mathfrak{a}$. Then as stated above, we have $S_A B = 0$. Hence, setting $X = B$ in (1), we obtain $S_A J B = J S_A B = 0$. This proves (2).

(3) Let $A, B \in \mathfrak{a}$, and $X \in \mathfrak{g}$. By Condition (K3) together with (2), we have

$$\begin{aligned} 0 &= \langle [A, B], JX \rangle + \langle [B, X], JA \rangle + \langle [X, A], JB \rangle \\ &= \langle (D_B + S_B)X, JA \rangle - \langle (D_A + S_A)X, JB \rangle \\ &= \langle X, (D_B - S_B)JA \rangle - \langle X, (D_A - S_A)JB \rangle \\ &= \langle X, D_B JA - D_A JB \rangle, \end{aligned}$$

implying that $D_B JA - D_A JB = 0$.

(4) For any $A, B \in \mathfrak{a}$, it follows from Condition (K4) together with (2) and (3) that

$$\begin{aligned} 0 &= [JA, JB] - J[JA, B] - J[A, JB] - [A, B] \\ &= [JA, JB] + J(D_B JA + S_B JA) - J(D_A JB + S_A JB) \\ &= [JA, JB] + J(D_B JA - D_A JB) \\ &= [JA, JB]. \end{aligned}$$

□

Let $\{\mathfrak{n}^{(i)}\}$ be the lower central series of $\mathfrak{n}$ defined by

$$\mathfrak{n}^{(1)} = \mathfrak{n} \supseteq \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supseteq \cdots \supseteq \mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}] \supseteq \cdots .$$

Note that $\mathfrak{n}$ is nilpotent, since $\mathfrak{g}$ is solvable. Hence there exists $r > 0$ such that $\mathfrak{n}^{(r)} \neq \{0\}$ and $\mathfrak{n}^{(r+1)} = \{0\}$.

Claim 6.2. $J\mathfrak{n}^{(r)} \subset \mathfrak{a}$.

Proof. We first note that a derivation D_H of $\mathfrak{n}$ leaves $\mathfrak{n}^{(r)}$ invariant. Let $\lambda_{r,1}, \ldots, \lambda_{r,s}$ be the eigenvalues of $D_H|_{\mathfrak{n}^{(r)}}$ and $\mathfrak{n}_i^{(r)}$ be the eigenspace associated with $\lambda_{r,i}$ for each $i = 1, \ldots, s$. Then $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{n}^{(r)} = \mathfrak{n}_1^{(r)} \oplus \cdots \oplus \mathfrak{n}_s^{(r)}$, where $\mathfrak{n}_i^{(r)}$ is orthogonal to $\mathfrak{n}_j^{(r)}$ for $i \neq j$.

For each i, let $Z \in \mathfrak{n}_i^{(r)}$ be an arbitrary vector in $\mathfrak{n}_i^{(r)}$, and let $X \in \mathfrak{n}$ be any vector in $\mathfrak{n}$. It follows from Condition (K3) together with (1) of Claim 6.1 with X, Z and H that

$$0 = \langle [X, Z], JH \rangle + \langle [Z, H], JX \rangle + \langle [H, X], JZ \rangle$$

$$= -\langle \lambda_{r,i} Z + S_H Z, JX \rangle + \langle (D_H + S_H) X, JZ \rangle$$
$$= \langle \lambda_{r,i} JZ + JS_H Z + D_H JZ - S_H JZ, X \rangle$$
$$= \langle (\lambda_{r,i} \operatorname{id} + D_H) JZ, X \rangle,$$

where id denotes the identity map of $\mathfrak{g}$.

As already remarked, D_H is positive definite on $\mathfrak{n}$. Hence $(\lambda_{r,i} \operatorname{id} + D_H)|_{\mathfrak{n}}$ is nondegenerate. This implies that JZ is orthogonal to $\mathfrak{n}$, so that $J\mathfrak{n}_i^{(r)} \subset \mathfrak{a}$. Since i is arbitrary, we have $J\mathfrak{n}^{(r)} \subset \mathfrak{a}$. □

Claim 6.3. *If $r = 1$, then $\mathfrak{g}$ is decomposed into a direct sum*

$$\mathfrak{g} = \mathbb{R}\{H_1\} \oplus \cdots \oplus \mathbb{R}\{H_s\} \oplus \mathbb{R}\{JH_1\} \oplus \cdots \oplus \mathbb{R}\{JH_s\}$$

satisfying (*a*) *of Proposition* 6.1*.*

Proof. Claim 6.2 shows that $\mathfrak{a}$ contains $J\mathfrak{n}$. If there exists some $A_0 \in \mathfrak{a}$ which is perpendicular to $J\mathfrak{n}$, then by (3) of Claim 6.1 we have

$$D_{A_0} JB = D_B JA_0 = 0 \quad \text{for all} \quad B \in J\mathfrak{n},$$

implying that $D_{A_0} = 0$. This contradicts that D_A is nonvanishing for all $A \in \mathfrak{a}$. Hence we have $J\mathfrak{n} = \mathfrak{a}$.

Since $\{D_A \mid A \in \mathfrak{a}\}$ is a commutative family of derivations on $\mathfrak{n}$, there exist linear functions $\lambda_1, \ldots, \lambda_s \colon \mathfrak{a} \to \mathbb{R}$ satisfying

$$\mathfrak{n}_i = \{X \in \mathfrak{n} \mid D_A X = \lambda_i(A) X \quad \text{for all } A \in \mathfrak{a}\} \neq \{0\}.$$

Then we have a direct sum decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_s$ of $\mathfrak{n}$. Note that for $i \neq j$, $\mathfrak{n}_i$ and $\mathfrak{n}_j$ are perpendicular to each other with respect to $\langle\ ,\ \rangle$. Since $\{D_A \mid A \in \mathfrak{a}\}$ is commutative, D_A leaves $\mathfrak{n}_i$ invariant for all $A \in \mathfrak{a}$ and $i = 1, \ldots, s$.

Setting $\mathfrak{a}_i = J\mathfrak{n}_i$, we get a direct sum decomposition $\mathfrak{a} = \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_s$ of $\mathfrak{a}$. Accordingly, we write $H = H_1 + \cdots + H_s$, where $H_i \in \mathfrak{a}_i$ for $i = 1, \ldots, s$.

We shall show that if $i \neq j$, then $D_{A_i} \mathfrak{a}_j = 0$ for any $A_i \in \mathfrak{a}_i$. Let $A_i \in \mathfrak{a}_i$ and $B_j \in \mathfrak{a}_j$, respectively. By (3) of Claim 6.1 we have

$$D_{A_i} JB_j = D_{B_j} JA_i = 0,$$

proving the assertion. In particular, substituting H_j for B_j in the above equation, we obtain that $D_{H_j} JA_i = \lambda_i(H_j) JA_i = 0$, and hence $\lambda_i(H_j) = 0$ for $i \neq j$. This implies that

$$\lambda_i(H) = \lambda_i(H_1 + \cdots + H_s) = \lambda_i(H_i) > 0,$$

since D_H is positive definite.

Finally, to prove that $\dim \mathfrak{n}_i = 1$ for each $i = 1, \ldots, s$, let A_i be an arbitrary vector in $\mathfrak{a}_i$. Applying (3) of Claim 6.1 to A_i and H_i, we see that

$$\lambda_i(A_i) JH_i = D_{A_i} JH_i = D_{H_i} JA_i = \lambda_i(H_i) JA_i,$$

which implies that $JA_i \in \mathbb{R}\{JH_i\}$. Hence we obtain $\mathfrak{n}_i = \mathbb{R}\{JH_i\}$. □

Claim 6.3 proves Proposition 6.1 in the case where $r = 1$. Hence, from now on, we assume that $r \geq 2$.

Let $\mathfrak{a}^{(r)}$ denote $J\mathfrak{n}^{(r)}$, and let $\mathfrak{a}_r$ be the orthogonal complement of $\mathfrak{a}^{(r)}$ in $\mathfrak{a}$, so that $\mathfrak{a}$ is decomposed into a direct sum $\mathfrak{a} = \mathfrak{a}^{(r)} \oplus \mathfrak{a}_r$. Then $H \in \mathfrak{a}$ can be uniquely written as $H = H^{(r)} + H_r$, where $H^{(r)} \in \mathfrak{a}^{(r)}$ and $H_r \in \mathfrak{a}_r$.

Let $\mathfrak{n}_r$ be the orthogonal complement of $\mathfrak{n}^{(r)}$ in $\mathfrak{n}$, that is, $\mathfrak{n} = \mathfrak{n}_r \oplus \mathfrak{n}^{(r)}$. Since D_A is a derivation of $\mathfrak{n}$, it leaves $\mathfrak{n}^{(r)}$ invariant for all $A \in \mathfrak{a}$, and hence also $\mathfrak{n}_r$. Let $A^{(r)}$ and A_r be arbitrary vectors in $\mathfrak{a}^{(r)}$ and $\mathfrak{a}_r$, respectively. Since $JA_r \in \mathfrak{a}_r \oplus \mathfrak{n}_r$ by the definition of $\mathfrak{a}_r$ and $\mathfrak{n}_r$, we get $D_{A^{(r)}} JA_r \in \mathfrak{n}_r$. Similarly, we have $D_{A_r} JA^{(r)} \in \mathfrak{n}^{(r)}$. Hence, setting $A = A^{(r)}$ and $B = A_r$ in (3) of Claim 6.1, we obtain

$$D_{A^{(r)}} JA_r = D_{A_r} JA^{(r)} = 0. \tag{1.4}$$

For any linear function $\lambda \colon \mathfrak{a}^{(r)} \to \mathbb{R}$, we define the subspace $(\mathfrak{n}_r)_\lambda$ of $\mathfrak{n}_r$ by

$$(\mathfrak{n}_r)_\lambda = \left\{ X \in \mathfrak{n}_r \;\middle|\; D_{A^{(r)}} X = \frac{1}{2} \lambda \left(A^{(r)} \right) X \text{ for all } A^{(r)} \in \mathfrak{a}^{(r)} \right\}.$$

Since $\{D_A \,|\, A \in \mathfrak{a}\}$ is abelian, there exists a linear functional $\lambda \colon \mathfrak{a}^{(r)} \to \mathbb{R}$ such that $(\mathfrak{n}_r)_\lambda \neq \{0\}$. Let $\lambda_{r,0} = 0, \lambda_{r,1}, \ldots, \lambda_{r,s}$ be linear functions such that $(\mathfrak{n}_r)_{\lambda_{r,i}} \neq \{0\}$ for $i = 0, \ldots, s$, and let $\mathfrak{n}_{r,i}$ denote the subspace $(\mathfrak{n}_r)_{\lambda_{r,i}}$ for each $i = 0, 1, \ldots, s$. Then we have a direct sum decomposition of $\mathfrak{n}_r$ as

$$\mathfrak{n}_r = \mathfrak{n}_{r,0} \oplus \mathfrak{n}_{r,1} \oplus \cdots \oplus \mathfrak{n}_{r,s}.$$

It is clear that $\mathfrak{n}_{r,0}, \dots, \mathfrak{n}_{r,s}$ are mutually orthogonal with respect to $\langle\ ,\ \rangle$. We remark that D_A and S_A leave $\mathfrak{n}_{r,i}$ invariant for $i = 0, \dots, s$, since $\{D_A, S_A \mid A \in \mathfrak{a}\}$ is abelian. Moreover, by Equation (1.4) we have $J\mathfrak{a}_r \subset \mathfrak{a}_r \oplus \mathfrak{n}_{r,0}$.

Claim 6.4. (1) $J\mathfrak{n}_{r,i} = \mathfrak{n}_{r,i}$ *for any* $i = 1, \dots, s$.

(2) $[\mathfrak{n}_{r,0}, \mathfrak{n}_{r,k}] \subset \mathfrak{n}_{r,k}$ *for any* $k = 0, 1, \dots, s$.

(3) $\lambda_{r,i}(H^{(r)}) > 0$ *for any* $i = 1, \dots, s$.

(4) $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}] \subset \mathfrak{n}^{(r)}$ *for any* $i = 1, \dots, s$.

(5) $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}] = \{0\}$ *for any* $1 \leq i < j \leq s$.

Proof. (1) For a fixed $1 \leq i \leq s$, let $X \in \mathfrak{n}_{r,i}$ and $A^{(r)} \in \mathfrak{a}^{(r)}$. By making use of Condition (K4) for $A^{(r)}$ and JX, and applying (1) of Claim 6.1, we obtain

$$\begin{aligned} 0 &= [JA^{(r)}, J^2X] - J[JA^{(r)}, JX] - J[A^{(r)}, J^2X] - [A^{(r)}, JX] \\ &= J\left(\frac{1}{2}\lambda_{r,i}(A^{(r)})X + S_{A^{(r)}}X\right) - D_{A^{(r)}}JX - S_{A^{(r)}}JX \\ &= \frac{1}{2}\lambda_{r,i}(A^{(r)})JX - D_{A^{(r)}}JX, \end{aligned}$$

which implies that $JX \in \mathfrak{n}_{r,i}$. Hence we have $J\mathfrak{n}_{r,i} \subset \mathfrak{n}_{r,i}$. Since J is non-degenerate, we obtain $J\mathfrak{n}_{r,i} = \mathfrak{n}_{r,i}$.

(2) We first show that $[\mathfrak{n}_{r,0}, \mathfrak{n}_{r,k}]$ is perpendicular to $\mathfrak{n}^{(r)}$. For each $k = 0, 1, \dots, s$, let $X_k \in \mathfrak{n}_{r,k}$. Also, let $Y \in \mathfrak{n}_{r,0}$ and $A^{(r)} \in \mathfrak{a}^{(r)}$. It follows from Condition (K3) and (1) of Claim 6.1 with X_k, Y and $A^{(r)}$ that

$$\begin{aligned} \langle [Y, X_k], JA^{(r)} \rangle &= -\langle [X_k, A^{(r)}], JY \rangle - \langle [A^{(r)}, Y], JX_k \rangle \\ &= \left\langle \frac{1}{2}\lambda_{r_k}(A^{(r)})X_k + S_{A^{(r)}}X_k, JY \right\rangle - \langle S_{A^{(r)}}Y, JX_k \rangle \\ &= \frac{1}{2}\lambda_{r,k}(A^{(r)})\langle JY, X_k \rangle - \langle (S_{A^{(r)}}J - JS_{A^{(r)}})Y, X_k \rangle \\ &= -\frac{1}{2}\lambda_{r,k}(A^{(r)})\langle Y, JX_k \rangle. \end{aligned}$$

When $k \geq 1$, (1) yields $\langle Y, JX_k \rangle = 0$, and hence we have $\langle [Y, X_k], JA^{(r)} \rangle = 0$. If $k = 0$, then we have $\lambda_{r,0}(A^{(r)}) = 0$, which implies that $\langle [Y, X_0], JA^{(r)} \rangle = 0$. As a consequence, for all $k = 0, 1, \dots, s$, we obtain $\langle [Y, X_k], JA^{(r)} \rangle = 0$. This proves the assertion that $[\mathfrak{n}_{r,0}, \mathfrak{n}_{r,k}]$ is perpendicular to $\mathfrak{n}^{(r)}$.

To show that $[\mathfrak{n}_{r,0}, \mathfrak{n}_{r,k}] \subset \mathfrak{n}_{r,k}$, it suffices to see that

$$D_{A^{(r)}}[Y, X_k] = [D_{A^{(r)}}Y, X_k] + [Y, D_{A^{(r)}}X_k] = \frac{1}{2}\lambda_{r,i}(A^{(r)})[Y, X_k],$$

where $X_k \in \mathfrak{n}_{r,k}, Y \in \mathfrak{n}_{r,0}$ and $A^{(r)} \in \mathfrak{a}^{(r)}$.

(3) Let $X_i \in \mathfrak{n}_{r,i}$. By (1), we have

$$D_{A^{(r)}}[X, JX] = \lambda_{r,i}(A^{(r)})[X, JX] \quad \text{for any } A^{(r)} \in \mathfrak{a}^{(r)},$$

and hence $[X, JX]$ is perpendicular to $\mathfrak{n}_{r,0}$. It then follows that $\langle [X, JX], JH_r\rangle = 0$, that is,

$$\langle [X, JX], JH\rangle = \langle [X, JX], JH^{(r)}\rangle.$$

By making use of Condition (K3) with X, JX and H, we have

$$\begin{aligned} \langle [X, JX], JH\rangle &= -\langle [JX, H], JX\rangle - \langle [H, X], J^2X\rangle \\ &= \langle D_H JX, JX\rangle + \langle D_H X, X\rangle > 0. \end{aligned} \tag{1.5}$$

Again, by Condition (K3) for X, JX and H, we have

$$\langle [X, JX], JH^{(r)}\rangle = \langle D_{H^{(r)}} JX, JX\rangle + \langle D_{H^{(r)}} X, X\rangle = \lambda_{r,i}(H^{(r)})\langle X, X\rangle. \tag{1.6}$$

Combining (1.5) and (1.6), we obtain $\lambda_{r,i}(H^{(r)})\langle X, X\rangle > 0$, that is, $\lambda_{r,i}(H^{(r)}) > 0$.

(5) We first prove that $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}]$ is perpendicular to $\mathfrak{n}_{r,0}$ for any $i, j = 1, \ldots, s$. Let $X \in \mathfrak{n}_{r,i}$ and $Y \in \mathfrak{n}_{r,j}$, respectively. Then we have

$$D_{A^{(r)}}[X, Y] = \frac{1}{2}\left(\lambda_{r,i}(A^{(r)}) + \lambda_{r,j}(A^{(r)})\right)[X, Y] \quad \text{for all} \quad A^{(r)} \in \mathfrak{a}^{(r)}.$$

In particular, if $A^{(r)} = H^{(r)}$, then we have $\lambda_{r,i}(H^{(r)}) + \lambda_{r,j}(H^{(r)}) > 0$ by (3). This proves that $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}]$ is perpendicular to $\mathfrak{n}_{r,0}$.

In order to prove that $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}] \subset \mathfrak{n}^{(r)}$ for $i, j = 1, \ldots, s$, let $X \in \mathfrak{n}_{r,i}$ and $Y \in \mathfrak{n}_{r,j}$, respectively. For $k = 1, \ldots, s$, let $W \in \mathfrak{n}_{r,k}$. Applying Condition (K3) to these X, Y, W, we have

$$\langle [X, Y], JW\rangle = -\langle [Y, W], JX\rangle - \langle [W, X], JY\rangle.$$

Assume that there exists some $W \in \mathfrak{n}_{r,k}$ such that $\langle [X, Y], JW\rangle \neq 0$. Then it follows from the above equation that either of the following holds:

$$\begin{cases} \lambda_{r,i} + \lambda_{r,j} = \lambda_{r,k} \\ \lambda_{r,j} + \lambda_{r,k} = \lambda_{r,i} \end{cases} \quad \text{or} \quad \begin{cases} \lambda_{r,i} + \lambda_{r,j} = \lambda_{r,k} \\ \lambda_{r,k} + \lambda_{r,i} = \lambda_{r,j}. \end{cases}$$

Solving the first equations, we have $\lambda_{r,i} = 0$, which contradicts $i = 1, \dots, s$. Similarly, the second equations imply that $\lambda_{r,j} = 0$, contradicting $j = 1, \dots, s$. Hence $[X, Y]$ is orthogonal to $\mathfrak{n}_{r,k}$ for all $k = 1, \dots, s$, which implies that $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}] \subset \mathfrak{n}^{(r)}$ for $i, j = 1, \dots, s$.

Finally, to prove $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}] = \{0\}$ for $1 \leq i < j \leq s$, let $X \in \mathfrak{n}_{r,i}$ and $Y \in \mathfrak{n}_{r,j}$, respectively. Let $A^{(r)}$ be an arbitrary vector in $\mathfrak{a}^{(r)}$. Using Condition (K3) with these X, Y and $A^{(r)}$, and applying (1) and (1) of Claim 6.1, we obtain

$$\begin{aligned}
\langle [X, Y], JA^{(r)} \rangle &= -\langle [Y, A^{(r)}], JX \rangle - \langle [A^{(r)}, X], JY \rangle \\
&= \left\langle \frac{1}{2}\lambda_{r,j}(A^{(r)})Y + S_{A^{(r)}}Y, JX \right\rangle - \left\langle \frac{1}{2}\lambda_{r,i}(A^{(r)})X + S_{A^{(r)}}X, JY \right\rangle \\
&= \frac{1}{2}\left(\lambda_{r,i}(A^{(r)}) + \lambda_{r,j}(A^{(r)})\right) \langle JX, Y \rangle - \langle (S_{A^{(r)}}J - JS_{A^{(r)}})X, Y \rangle \\
&= 0,
\end{aligned}$$

which shows that $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}]$ is perpendicular to $\mathfrak{n}^{(r)}$. It follows from these assertions that $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}] = \{0\}$ for $1 \leq i < j \leq s$.

(4) In proving (5), we have $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}] \subset \mathfrak{n}^{(r)}$ for any $i = 1 \dots s$. Moreover, Equation (1.6) in the proof of (3) yields $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}] \neq \{0\}$. Hence the assertion follows. □

Now, we note that it follows from Claim 6.4 that

$$\begin{aligned}
\mathfrak{n}^{(2)} &= [\mathfrak{n}, \mathfrak{n}] \\
&= [\mathfrak{n}_{r,0} \oplus \mathfrak{n}_{r,1} \oplus \cdots \oplus \mathfrak{n}_{r,s} \oplus \mathfrak{n}^{(r)}, \mathfrak{n}_{r,0} \oplus \mathfrak{n}_{r,1} \oplus \cdots \oplus \mathfrak{n}_{r,s} \oplus \mathfrak{n}^{(r)}] \\
&= \sum_{k=0}^{s} [\mathfrak{n}_{r_0}, \mathfrak{n}_{r,k}] + \sum_{i=1}^{s} [\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}],
\end{aligned}$$

and

$$\sum_{k=0}^{s} [\mathfrak{n}_{r_0}, \mathfrak{n}_{r,k}] \subseteq \mathfrak{n}_r, \quad \sum_{i=1}^{s} [\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}] \subseteq \mathfrak{n}^{(r)}.$$

From the lower central series we have $\mathfrak{n}^{(2)} \supseteq \mathfrak{n}^{(r)}$, so that

$$\sum_{i=1}^{s} [\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}] \supseteq \mathfrak{n}^{(r)},$$

which implies

$$\sum_{i=1}^{s} [\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}] = \mathfrak{n}^{(r)}.$$

For each i, let $\mathfrak{z}_i$ denote the subspace $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,i}]$ of $\mathfrak{n}^{(r)}$. Note that the restriction $D_{A^{(r)}}|_{\mathfrak{z}_i}$ of $D_{A^{(r)}}$ to $\mathfrak{z}_i$ is given by $D_{A^{(r)}}|_{\mathfrak{z}_i} = \lambda_{r,i}(A^{(r)})\,\mathrm{id}$ for any $A^{(r)} \in \mathfrak{a}^{(r)}$ and $i = 1, \ldots, s$. Hence we have the decomposition $\mathfrak{n}^{(r)} = \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_s$ of $\mathfrak{n}^{(r)}$ into a direct sum. Moreover, it is clear that $\mathfrak{z}_i$ and $\mathfrak{z}_j$ are perpendicular to each other with respect to $\langle \ , \ \rangle$ for $i \neq j$. By the definition of $\mathfrak{a}^{(r)}$, it is automatically decomposed into a direct sum $\mathfrak{a}^{(r)} = J\mathfrak{z}_1 \oplus \ldots \oplus J\mathfrak{z}_s$. Hence $H^{(r)} \in \mathfrak{a}^{(r)}$ can be uniquely written as $H^{(r)} = H_{r,1} + \cdots + H_{r,s}$, where $H_{r,i} \in J\mathfrak{z}_i$ for each $i = 1, \ldots s$.

Claim 6.5. $\lambda_{r,i}(H_{r,j}) = \delta_{ij}\lambda_{r,i}(H^{(r)})$.

Proof. Let $X \in \mathfrak{n}_i^{(r)}$. It follows from Condition (K3) with X, JX and $H_{r,j}$ that

$$\begin{aligned}\langle [X, JX], JH_{r,j}\rangle &= -\langle [JX, H_{r,j}], JX\rangle - \langle [H_{r,j}, X], J^2X\rangle \\ &= \langle D_{H_{r,j}}JX, JX\rangle + \langle D_{H_{r,j}}X, X\rangle \\ &= \lambda_{r,i}(H_{r,j})\langle X, X\rangle.\end{aligned}$$

If $i \neq j$, $[X, JX]$ is orthogonal to $JH_{r,j}$, that is, $\langle [X, JX], JH_{r,j}\rangle = 0$, which implies that $\lambda_{r,i}(H_{r,j}) = 0$ for $i \neq j$. Hence we have $\lambda_{r,i}(H^{(r)}) = \lambda_{r,i}(H_{r,1}) + \cdots + \lambda_{r,i}(H_{r,s}) = \lambda_{r,i}(H_{r,i})$. □

As a consequence of Claim 6.5 together with (3) of Claim 6.4, we have $\lambda_{r,i}(H_{r,i}) > 0$.

Claim 6.6. $[X, Y] = \langle JX, Y\rangle \dfrac{\lambda_{r,i}(H_{r,i})}{|H_{r,i}|^2} JH_{r,i}$ *for any* $X, Y \in \mathfrak{n}_{r,i}$.

Proof. In the same way as we proved Claim 6.3, we conclude that $\dim \mathfrak{z}_i = 1$ for each $i = 1, \ldots, s$. Hence we have $\mathfrak{z}_i = \mathbb{R}\{JH_{r,i}\}$.

On the other hand, using Condition (K3) and applying (1) of Claim 6.1 as well as (1) of Claim 6.4, we have for any $X, Y \in \mathfrak{n}_{r,i}$

$$\begin{aligned}[X, Y] &= \frac{1}{|H_{r,i}|^2}\langle [X, Y], JH_{r,i}\rangle JH_{r,i} \\ &= \frac{1}{|H_{r,i}|^2}\left(-\langle [Y, H_{r,i}], JX\rangle - \langle [H_{r,i}, X], JY\rangle\right) JH_{r,i} \\ &= \frac{1}{|H_{r,i}|^2}\big(\langle (D_{H_{r,i}} + S_{H_{r,i}})Y, JX\rangle - \langle (D_{H_{r,i}} + S_{H_{r,i}})X, JY\rangle\big) JH_{r,i} \\ &= \frac{1}{|H_{r,i}|^2}\big(\lambda_{r,i}(H_{r,i})\langle Y, JX\rangle - \langle (S_{H_{r,i}}J - JS_{H_{r,i}})X, Y\rangle\big) JH_{r,i}\end{aligned}$$

$$= \frac{\lambda_{r,i}(H_{r,i})}{|H_{r,i}|^2} \langle Y, JX \rangle JH_{r,i}.$$

□

As already shown before Claim 6.4, D_{A_r} leaves $\mathfrak{n}_{r,i}$ invariant for any $A_r \in \mathfrak{a}_r$ and $i = 1, \ldots, s$. Since $\{D_{A_r} | A_r \in \mathfrak{a}_r\}$ is commutative, there exist linear functions $\mu^1_{r,i}, \ldots, \mu^{t_{r,i}}_{r,i} : \mathfrak{a}_r \to \mathbb{R}$ such that

$$\mathfrak{n}^p_{r,i} = \left\{ X \in \mathfrak{n}_{r,i} \mid D_{A_r} X = \mu^p_{r,i}(A_r) X \quad \text{for all } A_r \in \mathfrak{a}_r \right\} \neq \{0\}$$

for each $p = 1, \ldots, t_{r,i}$. Hence we have a decomposition $\mathfrak{n}_{r,i} = \mathfrak{n}^1_{r,i} \oplus \cdots \oplus \mathfrak{n}^{t_{r,i}}_{r,i}$ of $\mathfrak{n}_{r,i}$ into a direct sum. Moreover, $\mathfrak{n}^1_{r,i}, \ldots, \mathfrak{n}_{r,i}$ are mutually orthogonal with respect to $\langle\ ,\ \rangle$.

Claim 6.7. *For any $p \in \{1, \ldots, t_{r,i}\}$, there exists a unique $q \in \{1, \ldots, t_{r,i}\}$ satisfying*

$$\mu^p_{r,i} + \mu^q_{r,i} = 0, \quad \textit{and} \quad J\mathfrak{n}^p_{r,i} = \mathfrak{n}^q_{r,i}.$$

Proof. Let $X \in \mathfrak{n}^p_{r,i}$ and $A_r \in \mathfrak{a}_r$. Since JX belongs to $\mathfrak{n}_{r,i}$ (cf. (1) of Claim 6.4), there exists $q \in \{1, \ldots, t_{r,i}\}$ such that $\langle JX, Y \rangle \neq 0$ for some $Y \in \mathfrak{n}^q_{r,i}$. It follows from (4) of Claim 6.4 that $[X, Y]$ is perpendicular to JA_r. Then, applying Condition (K3) to X, Y and A_r and using (1) of Claim 6.1, we have

$$\begin{aligned} 0 &= \langle [X, Y], JA_r \rangle + \langle [Y, A_r], JX \rangle + \langle [A_r, X], JY \rangle \\ &= -\langle \mu^q_{r,i}(A_r) Y + S_{A_r} Y, JX \rangle + \langle \mu^p_{r,i}(A_r) X + S_{A_r} X, JY \rangle \\ &= -\left(\mu^p_{r,i}(A_r) + \mu^q_{r,i}(A_r) \right) \langle JX, Y \rangle, \end{aligned}$$

which implies that $\mu^p_{r,i}(A_r) + \mu^q_{r,i}(A_r) = 0$. Since A_r is arbitrary, we have $\mu^p_{r,i} + \mu^q_{r,i} = 0$, which also shows that q is uniquely determined.

Now we prove that if $\mu^p_{r,i} + \mu^q_{r,i} = 0$, then $J\mathfrak{n}^p_{r,i} = \mathfrak{n}^q_{r,i}$. To this end, let $X \in \mathfrak{n}^p_{r,i}$, and $Y' \in \mathfrak{n}^{q'}_{r,i}$ for $q' \neq q$. Since $\mu^p_{r,i} + \mu^{q'}_{r,i} \neq 0$, there exists $A_r \in \mathfrak{a}_r$ such that $\mu^p_{r,i}(A_r) + \mu^{q'}_{r,i}(A_r) \neq 0$. By Condition (K3) with these X, Y' and A_r, and using (1) of Claim 6.1 together with (4) of Claim 6.4, we have

$$\begin{aligned} 0 &= \langle [X, Y'], JA_r \rangle + \langle [Y', A_r], JX \rangle + \langle [A_r, X], JY' \rangle \\ &= \langle \mu^{q'}_{r,i}(A_r) Y' + S_{A_r} Y', JX \rangle - \langle \mu^p_{r,i}(A_r) X + S_{A_r} X, JY' \rangle \\ &= \left(\mu^p_{r,i}(A_r) + \mu^{q'}_{r,i}(A_r) \right) \langle JX, Y' \rangle, \end{aligned}$$

which implies that $\langle JX, Y' \rangle = 0$, and hence $J\mathfrak{n}^p_{r,i}$ is perpendicular to $\mathfrak{n}^{q'}_{r,i}$. It follows from the choice of q' that $J\mathfrak{n}^p_{r,i} \subseteq \mathfrak{n}^q_{r,i}$. Interchanging p and q, we also obtain $J\mathfrak{n}^q_{r,i} \subseteq \mathfrak{n}^p_{r,i}$. Consequently, $J\mathfrak{n}^p_{r,i} = \mathfrak{n}^q_{r,i}$. $\square$

For each $i = 1, \ldots, s$, we extend $\lambda_{r,i}$ to a linear function on $\mathfrak{a}$ by setting

$$\lambda_{r,i}(A) = \begin{cases} \lambda_{r,i}(A) & \text{if } A \in \mathfrak{a}^{(r)}, \\ 0 & \text{if } A \in \mathfrak{a}_r. \end{cases}$$

Similarly, we extend $\mu^p_{r,i}$ to a linear function on $\mathfrak{a}$ by setting

$$\mu^p_{r,i}(A) = \begin{cases} 0 & \text{if } A \in \mathfrak{a}^{(r)}, \\ \mu^p_{r,i}(A) & \text{if } A \in \mathfrak{a}_r \end{cases}$$

for any $p = 1, \ldots, t_{r,i}$ and $i = 1, \ldots, s$. Then the subspace $\mathfrak{n}^p_{r,i}$ can be expressed as

$$\mathfrak{n}^p_{r,i} = \{X \in \mathfrak{n} \mid D_A X = ((1/2)\,\lambda_{r,i} + \mu^p_{r,i})(A)X \quad \text{for all } A \in \mathfrak{a}\}.$$

In what follows, we call $((1/2)\,\lambda_{r,i} + \mu^p_{r,i})$ a *root* of $\mathfrak{a}$ in $\mathfrak{n}$, and call $\mathfrak{n}^p_{r,i}$ the *root space* associated with a root $((1/2)\,\lambda_{r,i} + \mu^p_{r,i})$ for each $i = 1, \ldots, s$ and $p = 1, \ldots, t_{r,i}$. Note that it follows from Claims 6.6, 6.1 that $[A, JH_{r,i}] = \lambda_{r,i}(A)JH_{r,i}$ for $A \in \mathfrak{a}$.

We next consider $\mathfrak{a}_r \oplus \mathfrak{n}_{r,0}$. Obviously, $\mathfrak{a}_r$ is abelian, and $\mathfrak{n}_{r,0}$ is nilpotent. Also, it is easy to see that the restriction $D_{H_r}|_{\mathfrak{n}_{r,0}}$ of D_{H_r} to $\mathfrak{n}_{r,0}$ is positive definite and $\mathfrak{a}_r \oplus \mathfrak{n}_{r,0}$ is invariant by J. Then we can repeat the above argument with $\mathfrak{a} \oplus \mathfrak{n}$ replaced by $\mathfrak{a}_r \oplus \mathfrak{n}_{r,0}$. In consequence, we obtain a decomposition

$$\mathfrak{a}_r \oplus \mathfrak{n}_{r,0} = \mathfrak{a}_{r_2} \oplus \mathfrak{n}_{r_2,0} \oplus \bigoplus_{i=1}^{s_2} \Big(\mathbb{R}\{H_{r_2,i}\} \oplus \mathfrak{n}^1_{r_2,i} \oplus \cdots \oplus \mathfrak{n}^{t_{r_2,i}}_{r_2,i} \oplus \mathbb{R}\{JH_{r_2,i}\} \Big)$$

of $\mathfrak{a}_r \oplus \mathfrak{n}_{r,0}$ into a direct sum.

For each $H_{r_2,i}$, there exists a linear function $\lambda_{r_2,i} \colon \mathfrak{a}_r \to \mathbb{R}$ satisfying $[A, JH_{r_2,i}] = \lambda_{r_2,i}(A)JH_{r_2,i}$ for any $A \in \mathfrak{a}_r$. Moreover, for each $\mathfrak{n}^p_{r_2,i}$, there exists a linear function $\mu^p_{r_2,i} \colon \mathfrak{a}_r \to \mathbb{R}$ such that $((1/2)\,\lambda_{r_2,i} + \mu^p_{r_2,i})$ is a root of $\mathfrak{a}_r$ in $\mathfrak{n}_{r,0}$. Then $\mathfrak{n}^p_{r_2,i}$ can be expressed as

$$\mathfrak{n}^p_{r_2,i} = \{X \in \mathfrak{n}_{r,0} \mid D_A X = ((1/2)\,\lambda_{r_2,i} + \mu^p_{r_2,i})(A)X \quad \text{for all } A_r \in \mathfrak{a}_r\}.$$

Now we extend $\lambda_{r_2,i}$ to a linear function on $\mathfrak{a}$ by setting

$$\lambda_{r_2,i}(A) = \begin{cases} \lambda_{r_2,i}(A) & \text{if } A \in \mathfrak{a}_r, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we extend $\mu^p_{r_2,i}$ to a linear function on $\mathfrak{a}$ by setting

$$\mu^p_{r_2,i}(A) = \begin{cases} \mu^p_{r_2,i}(A) & \text{if } A \in \mathfrak{a}_r, \\ 0 & \text{otherwise} \end{cases}$$

for any $p = 1, \ldots, t_{r,i}$ and $i = 1, \ldots, s$. Then the subspace $\mathfrak{n}^p_{r_2,i}$ can be expressed as

$$\mathfrak{n}^p_{r_2,i} = \{X \in \mathfrak{n} \mid D_A X = ((1/2)\,\lambda_{r_2,i} + \mu^p_{r_2,i})(A)X \quad \text{for all } A \in \mathfrak{a}\}.$$

As a consequence, setting $r_1 = r$, we obtain a direct sum decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a}_{r_2} \oplus \mathfrak{n}_{r_2,0} \oplus \bigoplus_{\alpha=1}^{2} \bigoplus_{i=1}^{s_\alpha} \left(\mathbb{R}\{H_{r_\alpha,i}\} \oplus \mathfrak{n}^1_{r_\alpha,i} \oplus \cdots \oplus \mathfrak{n}^{t_{r_\alpha,i}}_{r_\alpha,i} \oplus \mathbb{R}\{JH_{r_\alpha,i}\} \right),$$

where $\mathfrak{n}^p_{r_\alpha,i}$ is given by

$$\mathfrak{n}^p_{r_\alpha,i} = \{X \in \mathfrak{n} \mid D_A X = ((1/2)\,\lambda_{r_\alpha,i} + \mu^p_{r_\alpha,i})(A)X \quad \text{for all } A \in \mathfrak{a}\}$$

for any $p = 1, \ldots, t_{r,i}$, $i = 1, \ldots, s$ and $\alpha = 1, 2$. Again, we call $((1/2)\,\lambda_{r_\alpha,i} + \mu^p_{r_\alpha,i})$ is a root of $\mathfrak{a}$ in $\mathfrak{n}$ for any $p = 1, \ldots, t_{r,i}$, $i = 1, \ldots, s$ and $\alpha = 1, 2$.

Iterating the same argument, we finally obtain the direct sum decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a}_{r_m} \oplus \bigoplus_{\alpha=1}^{m} \bigoplus_{i=1}^{s_\alpha} \left(\mathbb{R}\{H_{r_\alpha,i}\} \oplus \mathfrak{n}^1_{r_\alpha,i} \oplus \cdots \oplus \mathfrak{n}^{t_{r_\alpha,i}}_{r_\alpha,i} \oplus \mathbb{R}\{JH_{r_\alpha,i}\} \right).$$

Note that for each $H_{r_\alpha,i}$, there exists a linear function $\lambda_{r_\alpha,i} \colon \mathfrak{a} \to \mathbb{R}$ satisfying $[A, JH_{r_\alpha,i}] = \lambda_{r_\alpha,i}(A)JH_{r_\alpha,i}$ for any $A \in \mathfrak{a}$. Moreover, for any $\mathfrak{n}^p_{r_\alpha,i}$, there exists linear function $\mu^p_{r_\alpha,i} \colon \mathfrak{a} \to \mathbb{R}$ such that $((1/2)\,\lambda_{r_\alpha,i} + \mu^p_{r_\alpha,i})$ is a root of $\mathfrak{a}$ in $\mathfrak{n}$. Also, $\mathfrak{n}^p_{r_\alpha,i}$ can be written as

$$\mathfrak{n}^p_{r_\alpha,i} = \{X \in \mathfrak{n} \mid D_A X = ((1/2)\,\lambda_{r_\alpha,i} + \mu^p_{r_\alpha,i})(A)X \quad \text{for all } A \in \mathfrak{a}\}.$$

Obviously, we have $J\mathfrak{a}_{r_m} = \mathfrak{a}_{r_m}$. To prove $\mathfrak{a}_{r_m} = \{0\}$, let $A \in \mathfrak{a}_{r_m}$ and $X \in \mathfrak{n}^p_{r_\alpha,i}$ for any α, i and p. Applying Condition (K4) to A and X together with Claim 6.1 and Claim 6.7, we obtain

$$\begin{aligned} 0 &= \langle [JA, JX] - J[JA, X] - J[A, JX] - [A, X], X \rangle \\ &= \langle -\mu^p_{r_\alpha,i}(JA)JX + S_{JA}JX - J(\mu^p_{r_\alpha,i}(JA)X + S_{JA}X) \\ &\qquad - J(-\mu^p_{r_\alpha,i}(A)JX + S_A JX) - \mu^p_{r_\alpha,i}(A)X - S_A X, X \rangle \\ &= \left\langle -2\mu^p_{r_\alpha,i}(JA)JX - 2\mu^p_{r_\alpha,i}(A)X, X \right\rangle \end{aligned}$$

$$= -2\mu^p_{r_\alpha,i}(A)\langle X, X\rangle,$$

which implies $\mu^p_{r_\alpha,i}(A) = 0$. Since r_α, i, p are arbitrary, we have $D_A = 0$. This contradicts that $D_A \neq 0$ for all $A \in \mathfrak{a}$, and hence we have $\mathfrak{a}_{r_m} = \{0\}$, completing the proof of (a).

We now prove the first assertion of (b). Let $\mathfrak{n}_{r_\alpha,i} = \mathfrak{n}^1_{r_\alpha,i} \oplus \cdots \oplus \mathfrak{n}^{t_{r_\alpha,i}}_{r_\alpha,i}$. Recall that Claim 6.3 asserts that if $r_\alpha = 1$, then $\mathfrak{n}_{1,i} = \{0\}$ for any i. Moreover, it follows from Claim 6.4 and Claim 6.6 that if $r_\alpha = 2$, then $[JH_{1,j}, \mathfrak{n}_{2,i}] = 0$ for any i, j. Hence we may assume $r_\alpha \geq 3$, and consider $\mathfrak{n}_{r_\alpha,i}$.

Claim 6.8. *For $r_\alpha \geq 3$, there exists some $r_\beta < r_\alpha$ and $j \in \{1, \ldots, s_\beta\}$ such that the restriction $D_{H_{r_\beta,j}}|_{\mathfrak{n}_{r_\alpha,i}}$ of $D_{H_{r_\beta,j}}$ does not vanish identically.*

Proof. We fix $r_\alpha \geq 3$ and $i \in \{1, \ldots, s_\alpha\}$. Assume that $D_{H_{r_\beta,j}}|_{\mathfrak{n}_{r_\alpha,i}} = 0$ for all $r_\beta < r_\alpha$ and $j = 1, \ldots s_\beta$. We first prove that for any $r_\beta < r_\alpha$ and $j = 1, \ldots, s_\beta$

$$[\mathfrak{n}_{r_\beta,j}, \mathfrak{n}_{r_\alpha,i}] = \{0\}, \quad \text{and} \quad [JH_{r_\beta,j}, \mathfrak{n}_{r_\alpha,i}] = \{0\}. \tag{1.7}$$

Indeed, let $X \in \mathfrak{n}_{r_\alpha,i}$, and $Y \in \mathfrak{n}_{r_\beta,j}$ for $r_\beta < r_\alpha$. Since $D_{H_{r_\beta,j}}$ is a derivation of $\mathfrak{n}$, we have

$$D_{H_{r_\beta,j}}[Y, X] = \frac{1}{2}\lambda_{r_\beta,j}(H_{r_\beta,j})[Y, X],$$
$$D_{H_{r_\beta,j}}[JH_{r_\beta,j}, X] = \lambda_{r_\beta,j}(H_{r_\beta,j})[JH_{r_\beta,j}, X].$$

Since $\lambda_{r_\beta,j}(H_{r_\beta,j}) > 0$, these equations show that $[Y, X]$ and $[JH_{r_\beta,j}, X]$ are perpendicular to $\mathfrak{n}_{r_\alpha,i}$. On the other hand, it follows from Claim 6.4 that $\mathfrak{n}_{r_\alpha,i}$ contains $[Y, X]$ and $[JH_{r_\beta,j}, X]$. Consequently, we have $[Y, X] = 0$ and $[JH_{r_\beta,j}, X] = 0$, which implies that $[\mathfrak{n}_{r_\beta,j}, \mathfrak{n}_{r_\alpha,i}] = \{0\}$ and $[JH_{r_\beta,j}, \mathfrak{n}_{r_\alpha,i}] = \{0\}$ for any $j = 1, \ldots, s_\beta$.

We now define subspaces $\mathfrak{b}_\alpha$ and $\mathfrak{c}_{r_\alpha,i}$ of $\mathfrak{n}$ respectively by

$$\mathfrak{b}_\alpha = \bigoplus_{r_\beta < r_\alpha} \bigoplus_{j=1}^{s_\beta} \left(\mathfrak{n}_{r_\beta,j} \oplus \mathbb{R}\{JH_{r_\beta,j}\}\right), \quad \mathfrak{c}_{r_\alpha,i} = \mathfrak{b}_\alpha \oplus \mathfrak{n}_{r_\alpha,i} \oplus \mathbb{R}\{JH_{r_\alpha,i}\}.$$

The lower central series of $\mathfrak{c}_{r_\alpha,i}$, is then given by

$$\mathfrak{c}^{(1)}_{r_\alpha,i} = \mathfrak{c}_{r_\alpha,i} \supseteq \mathfrak{c}^{(2)}_{r_\alpha,i} = [\mathfrak{c}_{r_\alpha,i}, \mathfrak{c}^{(1)}_{r_\alpha,i}] \supseteq \cdots \supseteq \mathfrak{c}^{(k+1)}_{r_\alpha,i} = [\mathfrak{c}_{r_\alpha,i}, \mathfrak{c}^{(k)}_{r_\alpha,i}] \supseteq \cdots.$$

By a direct calculation using (1.7), we have $\mathfrak{c}^{(3)}_{r_\alpha,i} = [\mathfrak{b}_\alpha, [\mathfrak{b}_\alpha, \mathfrak{b}_\alpha]] \subseteq \mathfrak{b}_\alpha$. On the other hand, by the definition of $JH_{r_\alpha,i}$, $\mathfrak{c}^{(r_\alpha)}_{r_\alpha,i} = \mathbb{R}\{JH_{r_\alpha,i}\}$ must hold. Since we assume that

$r_\alpha \geq 3$, the lower central series yields $\mathfrak{c}^{(3)}_{r_\alpha,i} \supseteq \mathfrak{c}^{(r)}_{r_\alpha,i}$, which implies that $\mathfrak{b}_\alpha \supseteq \mathbb{R}\{JH_{r_\alpha,i}\}$. This contradicts that $\mathfrak{b}_\alpha$ is orthogonal to $\mathbb{R}\{JH_{r_\alpha,i}\}$. Therefore, there exist some $r_\beta < r_\alpha$ and j such that $D_{H_{r_\beta,j}}|_{\mathfrak{n}_{r_\alpha,i}} \neq 0$. □

Note that if $D_{H_{r_\beta,j}}|_{\mathfrak{n}_{r_\alpha,i}} \neq 0$ for some $r_\beta < r_\alpha$ and $j \in \{1,\dots,s_\beta\}$, then there exists $p \in \{1,\dots,t_{r_\alpha,i}\}$ such that $\mu^p_{r_\alpha,i}(H_{r_\beta,j}) \neq 0$.

Claim 6.9. *If there exists $r_\beta(< r_\alpha)$ and $j \in \{1,\dots,s_\beta\}$ such that $\mu^p_{r_\alpha,i}(H_{r_\beta,j}) \neq 0$, then $\mu^p_{r_\alpha,i}$ is given by $\mu^p_{r_\alpha,i} = \pm(1/2)\lambda_{r_\beta,j}$.*

Proof. Let $X \in \mathfrak{n}^p_{r_\alpha,i}$. Applying Condition (K4) to $H_{r_\beta,j}$ and X, and combining Claims 6.1 and 6.7, we have

$$\begin{aligned} 0 &= [JH_{r_\beta,j}, JX] - J[JH_{r_\beta,j}, X] - J[H_{r_\beta,j}, JX] - [H_{r_\beta,j}, X] \\ &= [JH_{r_\beta,j}, JX] - J[JH_{r_\beta,j}, X] \\ &\qquad - J\left(-\mu^p_{r_\alpha,i}(H_{r_\beta,j})JX + S_{H_{r_\beta,j}}JX\right) - \left(\mu^p_{r_\alpha,i}(H_{r_\beta,j})X + S_{H_{r_\beta,j}}X\right) \\ &= [JH_{r_\beta,j}, JX] - J[JH_{r_\beta,j}, X] - 2\mu^p_{r_\alpha,i}(H_{r_\beta,j})X. \end{aligned} \tag{1.8}$$

We now look at the eigenvalues of $D_{H_{r_\gamma,k}}$ associated with $[JH_{r_\beta,j}, JX]$, $J[JH_{r_\beta,j}, X]$ and X for $r_\gamma < r_\alpha$ and $k = 1,\dots,s_\gamma$. Then we obtain

	$D_{H_{r_\beta,j}}$	$D_{H_{r_\gamma,k}}$
$[JH_{r_\beta,j}, JX]$	$\lambda_{r_\beta,j}(H_{r_\beta,j}) - \mu^p_{r_\alpha,i}(H_{r_\beta,j})$	$-\mu^p_{r_\alpha,i}(H_{r_\gamma,k})$
$J[JH_{r_\beta,j}, X]$	$-\lambda_{r_\beta,j}(H_{r_\beta,j}) - \mu^p_{r_\alpha,i}(H_{r_\beta,j})$	$-\mu^p_{r_\alpha,i}(H_{r_\gamma,k})$
X	$\mu^p_{r_\alpha,i}(H_{r_\beta,j})$	$\mu^p_{r_\alpha,i}(H_{r_\gamma,k})$

(1.9)

where $(r_\gamma, k) \neq (r_\beta, j)$.

If $[JH_{r_\beta,j}, JX]$ and $J[JH_{r_\beta,j}, X]$ belong to the same eigenspace of $D_{H_{r_\beta,j}}$, then it follows from (1.9) that $\lambda_{r_\beta,j}(H_{r_\beta,j}) = 0$, which contradicts $\lambda_{r_\beta,j}(H_{r_\beta,j}) > 0$. Hence (1.8) implies that either of the following holds:

$$\begin{aligned} [JH_{r_\beta,j}, JX] &= 2\mu^p_{r_\alpha,i}(H_{r_\beta,j})X, \\ J[JH_{r_\beta,j}, X] &= -2\mu^p_{r_\alpha,i}(H_{r_\beta,j})X. \end{aligned}$$

If $[JH_{r_\beta,j}, JX] = 2\mu_p(H_{r_\beta,j})X$ holds, then we see from (1.9) that

$$\mu^p_{r_\alpha,i}(H_{r_\beta,j}) = \frac{1}{2}\lambda_{r_\beta,j}(H_{r_\beta,j}), \quad \mu^p_{r_\alpha,i}(H_{r_\gamma,k}) = 0$$

for $r_\gamma < r_\alpha$ and $(r_\gamma, k) \neq (r_\beta, j)$. This shows that $\mu^p_{r_\alpha,i} = (1/2)\lambda_{r_\beta,j}$, and hence we have $[JH_{r_\beta,j}, JX] = \lambda_{r_\beta,j}(H_{r_\beta,j})X$.

If $J[JH_{r_\beta,j}, X] = -2\mu_p(H_{r_\beta,j})X$ holds, then we see from (1.9) that

$$\mu^p_{r_\alpha,i}(H_{r_\beta,j}) = -\frac{1}{2}\lambda_{r_\beta,j}(H_{r_\beta,j}), \quad \mu^p_{r_\alpha,i}(H_{r_\gamma,k}) = 0$$

for $r_\gamma < r_\alpha$ and $(r_\gamma, k) \neq (r_\beta, j)$. This shows that $\mu^p_{r_\alpha,i} = -(1/2)\lambda_{r_\beta,j}$, and hence we have $[JH_{r_\beta,j}, X] = -\lambda_{r_\beta,j}(H_{r_\beta,j})X$. This completes the proof. □

When $\mu^p_{r_\alpha,i} = \pm(1/2)\lambda_{r_\beta,j}$, the subspace $\mathfrak{n}^p_{r_\alpha,i}$ of $\mathfrak{n}$ is denoted by $\mathfrak{n}^{\pm r_\beta,j}_{r_\alpha,i}$. If $\mu^p_{r_\alpha,i}$ vanishes identically, we denote by $\mathfrak{n}^0_{r_\alpha,i}$ the subspace $\mathfrak{n}^p_{r_\alpha,i}$ of $\mathfrak{n}$. Summing up,we prove the first assertion of (b). Combining Claim 6.7 with (1) of Claim 6.4, we complete proving (i) of (b). Claim 6.6 proves (ii) of (b). (iii) of (b) is verified in the proof of Claim 6.9.

We are now going to prove that $[\mathfrak{n}^{+r_\beta,j}_{r_\alpha,i}, \mathfrak{n}^{+r_\delta,l}_{r_\gamma,k}] = \{0\}$ for any $\alpha, \beta, \gamma, \delta$ and i, j, k, l. Let $X \in \mathfrak{n}^{+r_\gamma,k}_{r_\alpha,i}$ and $Y \in \mathfrak{n}^{+r_\delta,l}_{r_\beta,j}$. Since D_A is a derivation of $\mathfrak{n}$ for any $A \in \mathfrak{a}$, we have

$$D_A[X,Y] = \frac{1}{2}\left(\lambda_{r_\alpha,i}(A) + \lambda_{r_\beta,j}(A) + \lambda_{r_\gamma,k}(A) + \lambda_{r_\delta,l}(A)\right)[X,Y].$$

On the other hand, from Claim 6.9 we see that no subspace $\mathfrak{n}^p_{r_\alpha,i}$ is expressed as

$$\mathfrak{n}^p_{r_\alpha,i} = \left\{X \in \mathfrak{n} \mid D_A X = (1/2)\left(\lambda_{r_\alpha,i} + \lambda_{r_\beta,j} + \lambda_{r_\gamma,k} + \lambda_{r_\delta,l}\right)(A)X \quad \text{for all } A \in \mathfrak{a}\right\}.$$

Hence we obtain $[X, Y] = 0$, that is, $[\mathfrak{n}^{+r_\beta,j}_{r_\alpha,i}, \mathfrak{n}^{+r_\delta,l}_{r_\gamma,k}] = \{0\}$ for any α, β, γ and δ.

Similarly, we obtain that

$$\begin{aligned}
[\mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}, \mathfrak{n}^{-r_\delta,l}_{r_\gamma,k}] &= \{0\} \quad \text{for} \quad (r_\alpha, i) \neq (r_\delta, l) \text{ and } (r_\beta, j) \neq (r_\gamma, k),\\
[\mathfrak{n}^{+r_\beta,j}_{r_\alpha,i}, \mathfrak{n}^{-r_\delta,l}_{r_\gamma,k}] &= \{0\} \quad \text{for} \quad (r_\alpha, i) \neq (r_\delta, l) \text{ and } (r_\beta, j) \neq (r_\delta, l),\\
[\mathfrak{n}^0_{r_\alpha,i}, \mathfrak{n}^{-r_\delta,l}_{r_\gamma,k}] &= \{0\} \quad \text{for} \quad (r_\alpha, i) \neq (r_\delta, l),\\
[\mathfrak{n}^0_{r_\alpha,i}, \mathfrak{n}^{+r_\delta,l}_{r_\gamma,k}] &= \{0\}.
\end{aligned}$$

Claim 6.10. (1) $[Y, X] = -J[JY, X]$ *for* $X \in \mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}, Y \in \mathfrak{n}_{r_\beta,j}$.

(2) $[Y, X] = [JY, JX]$ *for* $X \in \mathfrak{n}^{\mp r_\gamma,k}_{r_\alpha,i}, Y \in \mathfrak{n}^{\pm r_\gamma,k}_{r_\beta,j}$.

(3) $[Y, X] = [JY, JX]$ *for* $X \in \mathfrak{n}^0_{r_\alpha,i}, Y \in \mathfrak{n}^0_{r_\beta,j}$.

Proof. (1) Let $X \in \mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ and $Y \in \mathfrak{n}_{r_\beta,j}$. Applying Condition (K4) to X and Y, we have

$$0 = [JY, JX] - J[JY, X] - J[Y, JX] - [X, Y] = -J[JY, X] - [Y, X].$$

This shows that $[Y, X] = -J[JY, X]$.

(2) It follows from Condition (K4) with $X \in \mathfrak{n}_{r_\alpha,j}^{\pm r_\gamma,k}$ and $Y \in \mathfrak{n}_{r_\beta,j}^{\mp r_\gamma,k}$ that

$$0 = [JY, JX] - J[JY, X] - J[Y, JX] - [X, Y] = [JY, JX] - [X, Y],$$

that is, $[Y, X] = [JY, JX]$.

(3) For any $X \in \mathfrak{n}_{r_\alpha,j}^0$ and $Y \in \mathfrak{n}_{r_\beta,j}^0$, Condition (K4) yields

$$0 = [JY, JX] - J[JY, X] - J[Y, JX] - [X, Y].$$

It is easy to see that $[JY, JX], [X, Y] \in \mathfrak{n}_{r_\alpha,i}^{+r_\beta,j}$ and $J[JY, X], J[Y, JX] \in \mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$. Hence we have $[Y, X] = [JY, JX]$. □

From Claim 6.10 we obtain the first identities in (iv), (v) and (vi) of (b).

Claim 6.11. *Let $X \in \mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ and $Y \in \mathfrak{n}_{r_\beta,i}$. Then we have*

$$|[Y, X]|^2 = \frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2}|Y|^2|X|^2.$$

Proof. Let $X \in \mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ and $Y \in \mathfrak{n}_{r_\beta,i}$. On account of Claim 6.10, we have

$$\begin{aligned}[][[Y, JY], X] &= [Y, [JY, X]] - [JY, [Y, X]] \\ &= [Y, [JY, X]] + [JY, J[JY, X]] = 2[Y, [JY, X]].\end{aligned} \tag{1.10}$$

This together with (ii) and (iii) of (b) implies that for X, Y

$$\begin{aligned}[][[Y, X], J[Y, X]] &= -[[Y, X], [JY, X]] = [Y, [X, [JY, X]]] - [X, [Y, [JY, X]]] \\ &= -\frac{1}{2}[X, [[Y, JY], X]] \\ &= -\frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})}{|H_{r_\beta,j}|^2}|Y|^2[X, [JH_{r_\beta,j}, X]] \\ &= \frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2}|Y|^2[X, JX]\end{aligned}$$

$$= \frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2}|Y|^2\frac{\lambda_{r_\alpha,i}(H_{r_\alpha,i})}{|H_{r_\alpha,i}|^2}|X|^2 JH_{r_\alpha,i}.$$

On the other hand, by (ii) of (b), we have

$$[[Y,X], J[Y,X]] = \frac{\lambda_{r_\alpha,i}(H_{r_\alpha,i})}{|H_{r_\alpha,i}|^2}|[Y,X]|^2 JH_{r_\alpha,i}.$$

Combining these two equations, we finally obtain

$$|[Y,X]|^2 = \frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2}|Y|^2|X|^2.$$

□

We have thus completed the proof of (iv) of (b).

Claim 6.12. (1) *Let $X, X' \in \mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ be orthogonal vectors, and let $Y \in \mathfrak{n}_{r_\beta,j}$. Then $\langle [Y,X],[Y,X'] \rangle$ vanishes identically.*

(2) *Let $\{E_1,\dots,E_n\}$ be an orthonormal basis of $\mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ with respect to $\langle\ ,\ \rangle$, and $Y \in \mathfrak{n}_{r_\beta,j}^{\pm r_\gamma,k}$ be a non-zero vector in $\mathfrak{n}_{r_\beta,j}^{\pm r_\gamma,k}$. Set $e_i = (1/|[Y,E_i]|)[Y,E_i]$ for each $i = 1,\dots,n$. Then $\{e_1,\dots,e_n\}$ is an orthonormal basis of $\mathfrak{n}_{r_\alpha,i}^{\pm r_\gamma,k}$ with respect to $\langle\ ,\ \rangle$.*

Proof. (1) Let $X, X' \in \mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ be orthogonal vectors, and let Y be a non-zero vector in $\mathfrak{n}_{r_\beta,j}$. Then it follows from Claim 6.11 that

$$\begin{aligned}|[Y,X+X']|^2 &= \frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2}|Y|^2(|X+X'|^2)\\ &= \frac{1}{2}\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2}|Y|^2(|X|^2+|X'|^2)\\ &= |[Y,X]|^2 + |[Y,X']|^2.\end{aligned}$$

On the other hand, a direct calculation shows that

$$|[Y,X+X']|^2 = |[Y,X]|^2 + 2\langle [Y,X],[Y,X'] \rangle + |[Y,X']|^2 .$$

Combining these two identities, we obtain $\langle [Y,X],[Y,X'] \rangle = 0$. This completes the proof of (1).

(2) Let $\{E_1,\dots,E_n\}$ be an orthonormal basis of $\mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ with respect to $\langle\ ,\ \rangle$, and let $Y \in \mathfrak{n}_{r_\beta,j}^{+r_\gamma,k}$. Claim 6.11 shows that $[Y,E_i]$ does not vanish for $i = 1,\dots,n$. Moreover,

it follows from (1) that $[Y,E_1],\dots,[Y,E_n]$ are perpendicular to each other with respect to $\langle\ ,\ \rangle$.

To prove that $\{[Y,E_1],\dots,[Y,E_n]\}$ is an orthogonal basis of $\mathfrak{n}_{r_\alpha,i}^{+r_\gamma,k}$, assume that there exists $X\in\mathfrak{n}_{r_\alpha,i}^{+r_\gamma,k}$ which is perpendicular to $[Y,E_i]$ for all $i=1,\dots,n$. We remark here that, by virtue of (i) of (b), $\{JE_1,\dots,JE_n\}$ is also an orthonormal basis of $\mathfrak{n}_{r_\alpha,i}^{+r_\beta,j}$. Then, using Condition (K3), we have

$$[JX,Y]=\sum_{i=1}^{n}\langle[JX,Y],JE_i\rangle JE_i=\sum_{i=1}^{n}\left(\langle[Y,E_i],X\rangle-\langle[E_i,JX],JY\rangle\right)JE_i=0.$$

Hence Claim 4.2 yields that

$$\begin{aligned}\langle R(JX,Y)Y,JX\rangle&=|U(JX,Y)|^2-\langle U(JX,JX),U(Y,Y)\rangle-\frac{3}{4}\left|[JX,Y]\right|^2\\&\qquad-\frac{1}{2}\langle[JX,[JX,Y]],Y\rangle-\frac{1}{2}\langle[Y,[Y,JX]],JX\rangle,\\&=-\langle U(JX,JX),U(Y,Y)\rangle\\&=-\frac{1}{|H_{r_\gamma,k}|^2}\langle U(JX,JX),H_{r_\gamma,k}\rangle\langle U(Y,Y),H_{r_\gamma,k}\rangle\\&=-\frac{1}{|H_{r_\gamma,k}|^2}\langle JX,D_{H_{r_\gamma,k}}JX\rangle\langle Y,D_{H_{r_\gamma,k}}Y\rangle\\&=-\frac{1}{|H_{r_\gamma,k}|^2}\left(-\lambda_{r_\gamma,k}(H_{r_\gamma,k})\right)|JX|^2\left(+\lambda_{r_\gamma,k}(H_{r_\gamma,k})\right)|Y|^2\\&=\frac{\lambda_{r_\gamma,k}(H_{r_\gamma,k})^2}{4|H_{r_\gamma,k}|^2}|X|^2|Y|^2>0,\end{aligned}$$

which contradicts $(\mathfrak{g},\langle\ ,\ \rangle)$ having non-positive sectional curvature $K\le 0$. Consequently, we conclude that $\{[Y,E_1],\dots,[Y,E_n]\}$ is an orthogonal basis of $\mathfrak{n}_{r_\alpha,i}^{+r_\gamma,k}$. In the same argument, we can also prove that for $Y\in\mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ $\{[Y,E_1],\dots,[Y,E_n]\}$ is an orthogonal basis of $\mathfrak{n}_{r_\alpha,i}^{-r_\gamma,k}$. This completes the proof of (2). □

To verify the second identities in (v) of (b), let $X\in\mathfrak{n}_{r_\alpha,i}^{+r_\gamma,k}$ and $Y\in\mathfrak{n}_{r_\beta,j}^{-r_\gamma,k}$. Suppose that $r_\alpha>r_\beta$. Let $\{E_1,\dots,E_n\}$ be an orthonormal basis of $\mathfrak{n}_{r_\alpha,i}^{-r_\beta,j}$ with respect to $\langle\ ,\ \rangle$. By virtue of Claim 6.12, setting $e_i=(1/|[Y,E_i]|)[Y,E_i]$ for $i=1,\dots,n$, an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathfrak{n}_{r_\alpha,i}^{-r_\gamma,k}$. Then, by making use of Condition (K3), we have

$$|[X,Y]|^2=\left|\sum_{i=1}^{n}\langle[X,Y],JE_i\rangle JE_i\right|^2=\sum_{i=1}^{n}\langle[X,Y],JE_i\rangle^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (-\langle [Y, E_i], JX\rangle - \langle [E_i, X], JY\rangle)^2 \\
&= \sum_{i=1}^{n} \langle |[Y, E_i]|\, e_i, JZ\rangle^2 \\
&= \sum_{i=1}^{n} \frac{\lambda_{r_\alpha,i}(H_{r_\alpha,i})^2}{2|H_{r_\alpha,i}|^2} |Y|^2 \langle e_i, JX\rangle^2 \\
&= \frac{\lambda_{r_\alpha,i}(H_{r_\alpha,i})^2}{2|H_{r_\alpha,i}|^2} |Y|^2 |X|^2.
\end{aligned}
$$

This completes the proof of (v) of (b).

To prove the second identity in (vi) of (b), let $Y \in \mathfrak{n}^0_{r_\beta,j}$. We now consider the restriction $\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}} : \mathfrak{n}^0_{r_\alpha,j} \to \mathfrak{n}^{+r_\beta,j}_{r_\alpha,j}$ of the derivation $\operatorname{ad} Y$ to $\mathfrak{n}^0_{r_\alpha,j}$.

Claim 6.13. (1) $\operatorname{Im}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right) = \mathfrak{n}^{+r_\beta,j}_{r_\alpha,j}$.

(2) $\mathfrak{n}^0_{r_\alpha,i}$ *is decomposed into a direct sum* $\mathfrak{n}^0_{r_\alpha,i} = \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right) \oplus \left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$.

(3) $\operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$ *and* $\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$ *are invariant by* J.

Proof. (1) Let $Z \in \mathfrak{n}^{+r_\beta,j}_{r_\alpha,i}$. Note that Claim 6.11 implies $[JY, JZ] \neq 0$. We then define a non-zero vector $X \in \mathfrak{n}^0_{r_\alpha,i}$ by

$$
X = \frac{2|H_{r_\beta,j}|^2}{\lambda_{r_\beta,j}(H_{r_\beta,j})^2 |Y|^2} [JY, JZ].
$$

Using Equation (1.10) in the proof of Claim 6.11, (ii) and (iii) of (b), we have

$$
\begin{aligned}
\operatorname{ad} Y(X) &= \frac{2|H_{r_\beta,j}|^2}{\lambda_{r_\beta,j}(H_{r_\beta,j})^2 |Y|^2} [Y, [JY, JZ]] \\
&= \frac{|H_{r_\beta,j}|^2}{\lambda_{r_\beta,j}(H_{r_\beta,j})^2 |Y|^2} [[Y, JY], JZ] \\
&= \frac{|H_{r_\beta,j}|^2}{\lambda_{r_\beta,j}(H_{r_\beta,j})^2 |Y|^2} \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})}{|H_{r_\beta,j}|^2} |Y|^2 [JH_{r_\beta,j}, JZ] \\
&= -\frac{1}{\lambda_{r_\beta,j}(H_{r_\beta,j})} \lambda_{r_\beta,j}(H_{r_\beta,j}) J^2 Z = Z.
\end{aligned}
$$

This implies that $\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}$ is surjective, that is, $\operatorname{Im}(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}) = \mathfrak{n}^{+r_\beta,j}_{r_\alpha,j}$.

(2) We first prove that $\operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$ is perpendicular to $\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$. Let $X \in \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$ and let $W \in \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}$. Applying Condition (K3) to JX, JY and W, and then using the first identity in (vi) of (b), we have

$$\begin{aligned}
0 =& \langle J^2X, [JY, W]\rangle + \langle J^2Y, [W, JX]\rangle + \langle JW, [JX, JY]\rangle \\
=& -\langle X, [JY, W]\rangle + \langle JW, [X, Y]\rangle \\
=& -\langle X, [JY, W]\rangle.
\end{aligned}$$

Since X and W are arbitrary, the assertion follows.

We now remark that by (1) of Claim 6.12 together with Claim 6.11, the subspace $\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}\right]$ has the same dimension as that of $\mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}$. This combined with (1) and (i) of (b) implies that the dimension of $\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}\right]$ is equal to the dimension of $\operatorname{Im}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$.

Since $\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}$ is a linear operator, we have

$$\dim \mathfrak{n}^0_{r_\alpha,i} - \dim \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right) = \dim \operatorname{Im}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right) = \dim\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right].$$

Hence we obtain a decomposition $\mathfrak{n}^0_{r_\alpha,i} = \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right) \oplus \left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$ of $\mathfrak{n}^0_{r_\alpha,i}$ into a direct sum.

(3) Let $X \in \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$, and let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $\mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}$ with respect to $\langle\ ,\ \rangle$. Applying Claim 4.2 to X and Y and using Condition (K3), we obtain

$$\begin{aligned}
\langle R(X,Y)Y, X\rangle &= |U(X,Y)|^2 - \langle U(X,X), U(Y,Y)\rangle - \frac{3}{4}|[X,Y]|^2 \\
&\qquad - \frac{1}{2}\langle [X,[X,Y]], Y\rangle - \frac{1}{2}\langle [Y,[Y,X]], X\rangle, \\
&= |U(X,Y)|^2 = \sum_{i=1}^{n} \langle U(X,Y), E_i\rangle^2 \\
&= \sum_{i=1}^{n} \frac{1}{4}\langle X, [Y, E_i]\rangle^2 \\
&= \sum_{i=1}^{n} \frac{1}{4}(\langle JY, [E_i, JX]\rangle + \langle JE_i, [JX, Y]\rangle)^2 \\
&= |[JX, Y]|^2.
\end{aligned}$$

Since the sectional curvature K is nonpositive, we have $[JX, Y] = 0$, that is, $JX \in \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$. This yields that $J \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right) = \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$. By virtue of (2), it is now immediate to see that $\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$ is invariant by J. □

Let $X \in \mathfrak{n}^0_{r_\alpha,i}$. Then Claim 6.13 shows that X can be uniquely written as $X = X_1 + X_2$, where $X_1 \in \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$ and $X_2 \in \left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$.

Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $\mathfrak{n}^{-r_\beta,j}_{r_\alpha,i}$ with respect to $\langle\ ,\ \rangle$. Note that by (1) of Claim 6.12, $[JY, E_1], \ldots, [JY, E_n]$ are perpendicular to each other. Setting $e_i = (1/|[JY, E_i]|)[JY, E_i]$ for $i = 1, \ldots, n$, we obtain an orthonormal basis $\{e_1, \ldots, e_n\}$ of the subspace $\left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$ with respect to $\langle\ ,\ \rangle$. Then, it follows from Condition (K3) together with the first identities of (vi), (v) and (vi) of (b) that

$$\begin{aligned}
|[X, Y]|^2 &= |[X_1 + X_2, Y]|^2 = |[X_2, Y]|^2 \\
&= \sum_{i=1}^{n} \langle [X_2, Y], JE_i \rangle^2 = \sum_{i=1}^{n} \left(-\langle [Y, E_i], JX_2\rangle - \langle [E_i, X_2], Y\rangle\right)^2 \\
&= \sum_{i=1}^{n} \langle [Y, E_i], JX_2 \rangle^2 = \sum_{i=1}^{n} \langle [JY, E_i], X_2 \rangle^2 \\
&= \sum_{i=1}^{n} |[JY, E_i]|^2 \langle e_i, X_2 \rangle^2 \\
&= \frac{1}{2} \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2} |Y|^2 \sum_{i=1}^{n} \langle e_i, X_2 \rangle^2 \\
&= \frac{1}{2} \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2} |Y|^2 |X_2|^2.
\end{aligned}$$

On the other hand, we now consider $JX = JX_1 + JX_2$. We remark here that by (3) of Claim 6.13, $JX_1 \in \operatorname{Ker}\left(\operatorname{ad} Y|_{\mathfrak{n}^0_{r_\alpha,j}}\right)$ and $JX_2 \in \left[JY, \mathfrak{n}^{-r_\beta,j}_{r_\alpha,j}\right]$. Then, substituting JX for X in the above equation, we obtain

$$|[JX, Y]|^2 = \frac{1}{2} \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2} |Y|^2 |JX_2|^2 = \frac{1}{2} \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{|H_{r_\beta,j}|^2} |Y|^2 |X_2|^2.$$

Combining these two identities, we have $|[X, Y]|^2 = |[JX, Y]|^2$. This completes the proof of (vi) of (b).

Claim 6.14. *If $\mathfrak{n}^{\pm r_\beta,j}_{r_\alpha,i} \neq \{0\}$, then $\mathfrak{n}^{\pm r_\beta,j}_{r_\alpha,k} = \{0\}$ for $i \neq k$.*

Proof. Let $X \in \mathfrak{n}_{r_\alpha,i}^{+r_\beta,j}$. If there exists a non-zero vector $Y \in \mathfrak{n}_{r_\alpha,k}^{-r_\beta,j}$ for $i \neq k$, then it follows from Claim 4.2 with X and Y that

$$\begin{aligned}
\langle R(X,Y)Y,X\rangle &= |U(X,Y)|^2 - \langle U(X,X),U(Y,Y)\rangle - \frac{3}{4}\,|[X,Y]|^2 \\
&\quad - \frac{1}{2}\langle [X,[X,Y]],Y\rangle - \frac{1}{2}\langle [Y,[Y,X]],X\rangle \\
&= \frac{1}{|H_{r_\beta,j}|^2}\langle U(X,Y),H_{r_\beta,j}\rangle^2 - \frac{1}{|H_{r_\beta,j}|^2}\langle U(X,X),H_{r_\beta,j}\rangle\langle U(Y,Y),H_{r_\beta,j}\rangle \\
&= -\frac{1}{|H_{r_\beta,j}|^2}\langle D_{H_{r_\beta,j}}X,Y\rangle^2 - \frac{1}{|H_{r_\beta,j}|^2}\langle D_{H_{r_\beta,j}}X,X\rangle\langle D_{H_{r_\beta,j}}Y,Y\rangle \\
&= -\frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{4|H_{r_\beta,j}|^2}\langle X,Y\rangle^2 + \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{4|H_{r_\beta,j}|^2}\langle X,X\rangle\langle Y,Y\rangle \\
&= \frac{\lambda_{r_\beta,j}(H_{r_\beta,j})^2}{4|H_{r_\beta,j}|^2}\langle X,X\rangle\langle Y,Y\rangle > 0,
\end{aligned}$$

which contradicts that $(\mathfrak{g},\langle\ ,\ \rangle)$ has nonpositive sectional curvature $K \leq 0$. Hence we have $\mathfrak{n}_{r_\alpha,k}^{-r_\beta,j} = \{0\}$. ☐

Claim 6.14 proves (vii) of (b). This completes the proof of Proposition 6.1.

Let $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ be a solvable Lie algebra under the assumption of Proposition 6.1. We prepare a few Remarks for the properties of $(\mathfrak{g}, J, \langle\ ,\ \rangle)$.

Remark 6.1. *The Levi-Civita connection ∇ and the curvature tensor R of $\mathfrak{g}$ have the following properties*:

(1) $\nabla_X JY = J\nabla_X Y$ *for all* $X, Y \in \mathfrak{g}$.

(2) $R(X,Y)JZ = JR(X,Y)Z$ *and* $R(JX,JY) = R(X,Y)$ *for all* $X,Y,Z \in \mathfrak{g}$.

Proof. (1) Using Condition (K1)–(K4), we have

$$\begin{aligned}
2\langle \nabla_X JY, Z\rangle &= \langle [X,JY],Z\rangle - \langle JY,[X,Z]\rangle - \langle X,[JY,Z]\rangle \\
&= \langle [JY,JZ],JX\rangle - \langle [JZ,X],Y\rangle + \langle JX,[Z,Y]\rangle + \langle JZ,[Y,X]\rangle \\
&\qquad + \langle X,[Y,JZ]\rangle + \langle JX,[Y,Z]\rangle - \langle JX,[JY,JZ]\rangle \\
&= -\langle [JZ,X],Y\rangle + \langle JZ,[Y,X]\rangle + \langle X,[Y,JZ]\rangle \\
&= -2\langle \nabla_X Y, JZ\rangle \\
&= 2\langle J\nabla_X Y, Z\rangle
\end{aligned}$$

for any $X, Y, Z \in \mathfrak{g}$, thereby, proving (1).

(2) It follows from (1) that $R(X,Y)JZ = JR(X,Y)Z$ for any $X, Y, Z \in \mathfrak{g}$. Hence the symmetry property of R yields

$$\langle R(JX, JY)Z, W\rangle = \langle R(Z, W)JX, JY\rangle = \langle R(Z, W)X, Y\rangle = \langle R(X, Y)Z, W\rangle$$

for $X, Y, Z, W \in \mathfrak{g}$, which implies that $R(JX, JY) = R(X, Y)$. □

Let $\{(E_a^{\pm b})_1, \ldots, (E_a^{\pm b})_n\}$ denote an orthonormal basis of $\mathfrak{n}_a^{\pm b}$ with respect to $\langle\ ,\ \rangle$, respectively. Also, let $\{(E_a^0)_1, \ldots, (E_a^0)_n\}$ denote an orthonormal basis of $\mathfrak{n}_a^0$ with respect to $\langle\ ,\ \rangle$.

Remark 6.2. Assume that $\lambda_a(H) > \lambda_b(H)$. The Levi-Civita connection ∇ is given by the following formulas:

(1) $\nabla_A B = 0 \quad$ for $A, B \in \mathfrak{a}$.

(2) $\nabla_A X = S_A X \quad$ for $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$.

(3) $\nabla_X A = -D_A X \quad$ for $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$.

(4) $\nabla_X Y = \nabla_Y X = \frac{1}{2}[JX, JY] \quad$ for $X \in \mathfrak{n}_a^{+b}, Y \in \mathfrak{n}_b$.

(5) $\nabla_X Y = -\nabla_Y X = \frac{1}{2}[X, Y] \quad$ for $X \in \mathfrak{n}_a^{-b}, Y \in \mathfrak{n}_b$.

(6) $\nabla_X Y = \nabla_Y X = -\frac{1}{2}J[X, JY] \quad$ for $X \in \mathfrak{n}_a^{\pm d}, Y \in \mathfrak{n}_b^{\pm d}$.

(7) $\nabla_X Y = \frac{1}{2}[X, Y] - \frac{1}{2}J[X, JY] \quad$ for $X \in \mathfrak{n}_a^0, Y \in \mathfrak{n}_b^0$.

(8) $\nabla_{JH_a} X = -\frac{1}{2}\lambda_a(H_a)JX \quad$ for $X \in \mathfrak{n}_a$.

(9) $\nabla_{JH_b} X = -\frac{1}{2}\lambda_b(H_b)JX \quad$ for $X \in \mathfrak{n}_a^{+b}$.

(10) $\nabla_X Y = \frac{\lambda_a(H_a)}{2|H_a|^2}\langle X, Y\rangle H_a + \frac{\lambda_b(H_b)}{2|H_b|^2}\langle X, Y\rangle H_b \quad$ for $X, Y \in \mathfrak{n}_a^{+b}$.

(11) If $\lambda_b(H) > \lambda_d(H)$, then we have

$$\nabla_X Y = \nabla_Y X = -\frac{1}{2}\sum_{p=1}^{n}\langle X, [Y, (E_b^{-d})_p]\rangle (E_b^{-d})_p \quad \text{for } X \in \mathfrak{n}_a^{+b}, Y \in \mathfrak{n}_a^{+d}.$$

(12) $\nabla_X Y = \nabla_Y X = -\dfrac{1}{2}\sum_{p=1}^{m}\langle X, [Y, (E_b^0)_p]\rangle (E_b^0)_p \quad$ for $X \in \mathfrak{n}_a^{+b}, Y \in \mathfrak{n}_a^0$.

(13) $\nabla_X Y = \dfrac{\lambda_a(H_a)}{2|H_a|^2}\left(\langle X, Y\rangle H_a + \langle JX, Y\rangle JH_a\right) \quad$ for $X, Y \in \mathfrak{n}_a^0$.

(14) If $a \neq c, d$ and $b \neq c, d$, then $\nabla_X Y = 0$ for $X \in \mathfrak{n}_a^{+b}$ and $Y \in \mathfrak{n}_c^{\pm d}$.

Proof. We first remark that for any $A \in \mathfrak{a}$ and $X, Y \in \mathfrak{n}$,

$$\begin{aligned}\langle U(X,Y), A\rangle &= -\frac{1}{2}\left(\langle Y, [X, A]\rangle + \langle X, [Y, A]\rangle\right) \\ &= \frac{1}{2}\left(\langle Y, \operatorname{ad} A(X)\rangle + \langle \operatorname{ad} A^*(X), Y\rangle\right) \\ &= \langle D_A X, Y\rangle.\end{aligned} \tag{1.11}$$

To prove (1), (2) and (3), let $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$. It is easy to see that $(\operatorname{ad} Y)^* A = 0$ for any $Y \in \mathfrak{g}$. It then follows from Claim 4.1 that

$$\begin{aligned}\nabla_A B &= \frac{1}{2}[A, B] + U(A, B) = -\frac{1}{2}\left((\operatorname{ad} A)^* B + (\operatorname{ad} B)^* A\right) = 0, \\ \nabla_A X &= \frac{1}{2}\operatorname{ad} AX - \frac{1}{2}\left((\operatorname{ad} A)^* X + (\operatorname{ad} X)^* A\right) = \frac{1}{2}\left(\operatorname{ad} A - (\operatorname{ad} A)^*\right) X = S_A X, \\ \nabla_X A &= \frac{1}{2}[X, A] - \frac{1}{2}\left((\operatorname{ad} X)^* A + (\operatorname{ad} A)^* X\right) = -\frac{1}{2}\left(\operatorname{ad} A + (\operatorname{ad} A)^*\right) X = -D_A X.\end{aligned}$$

This completes the proof of (1), (2) and (3).

(4) It follows from Condition (K3) with $X \in \mathfrak{n}_a^{+b}, Y \in \mathfrak{n}_b$ and $Z \in \mathfrak{g}$ that

$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= \frac{1}{2}\left(\langle [X, Y], Z\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle\right) = -\frac{1}{2}\langle X, [Y, Z]\rangle \\ &= -\frac{1}{2}\left(\langle JY, [Z, JX]\rangle + \langle JZ, [JX, Y]\right) \\ &= \frac{1}{2}\langle JZ, J[JX, JY]\rangle \\ &= \frac{1}{2}\langle Z, [JX, JY]\rangle,\end{aligned}$$

which implies that $\nabla_X Y = (1/2)[JX, JY]$.

(5) For any $X \in \mathfrak{n}_a^{-b}$ and $Y \in \mathfrak{n}_b$, it is easy to see that $U(X, Y) = 0$, and hence we obtain $\nabla_X Y = (1/2)[X, Y]$.

(6) and (7) are proved in a way similar to that for (4).

(8) Let $X \in \mathfrak{n}_a^{\pm b}$. Then it follows from Remark 6.1 together with (3) that

$$\nabla_{JH_a} X = [JH_a, X] + \nabla_X JH_a = J\nabla_X H_a = -\frac{1}{2}\lambda_a(H_a)JX.$$

In a similar manner, we can also prove (9).

(10) For any $X, Y \in \mathfrak{n}_a^{+b}$ and $W \in \mathfrak{n}$, we have

$$\langle U(X,Y), W\rangle = -\frac{1}{2}\left(\langle Y, [X,W]\rangle + \langle X, [Y,W]\rangle\right) = 0.$$

It then follows from (1.11) that $U(X,Y) \in \mathbb{R}\{H_a\} \oplus \mathbb{R}\{H_b\}$. Hence we have

$$\begin{aligned}
\nabla_X Y &= \frac{1}{2}[X,Y] + U(X,Y) \\
&= \frac{1}{|H_a|^2}\langle U(X,Y), H_a\rangle H_a + \frac{1}{|H_b|^2}\langle U(X,Y), H_b\rangle H_b \\
&= \frac{1}{|H_a|^2}\langle D_{H_a}X, Y\rangle H_a + \frac{1}{|H_b|^2}\langle D_{H_b}X, Y\rangle H_b \\
&= \frac{\lambda_a(H_a)}{2|H_a|^2}\langle X, Y\rangle H_a + \frac{\lambda_b(H_b)}{2|H_b|^2}\langle X, Y\rangle H_b.
\end{aligned}$$

The proof of (11) and (12) are done by straightforward computations using Claim 4.1.

(13) For $X, Y \in \mathfrak{n}_a^0$ and $W \in \mathfrak{n}$, we have

$$\langle U(X,Y), W\rangle = -\frac{1}{2}\left(\langle Y, [X,W]\rangle + \langle Y, [X,W]\rangle\right) = 0.$$

Moreover, using (1.11), it is easy to see that $U(X,Y) \in \mathbb{R}\{H_a\}$. Hence we have

$$\begin{aligned}
\nabla_X Y &= \frac{1}{2}[X,Y] + U(X,Y) \\
&= \frac{\lambda_a(H_a)}{2|H_a|^2}\langle JX, Y\rangle JH_a + \frac{1}{|H_a|^2}\langle U(X,Y), H_a\rangle H_a \\
&= \frac{\lambda_a(H_a)}{2|H_a|^2}\langle JX, Y\rangle JH_a + \frac{\lambda_a(H_a)}{|H_a|^2}\langle X, Y\rangle H_a.
\end{aligned}$$

(14) can be seen by easy computations. □

Remark 6.3. *If S_A is a derivation of $\mathfrak{g}$ for $A \in \mathfrak{a}$, then the following hold:*

(1) $S_A \nabla_X Y = \nabla_{S_A X} Y + \nabla_X S_A Y$,

(2) $R(A,X)Y = -\nabla_{D_A X} Y$

for any $X, Y \in \mathfrak{g}$.

Proof. (1) Let $A \in \mathfrak{a}$ and $X, Y, Z \in \mathfrak{g}$. Since S_A is a skew-symmetric derivation, it follows from Claim 4.1 with X, Y, Z and A that

$$\begin{aligned}
\langle S_A \nabla_X Y, Z\rangle &= -\langle \nabla_X Y, S_A Z\rangle \\
&= \frac{1}{2}(-\langle [X,Y], S_A Z\rangle + \langle Y, [X, S_A Z]\rangle + \langle X, [Y, S_A Z]\rangle) \\
&= \frac{1}{2}(\langle S_A[X,Y], Z\rangle + \langle Y, S_A[X,Z] - [S_A X, Z]\rangle \\
&\qquad + \langle X, S_A[Y,Z] - [S_A Y, Z]\rangle) \\
&= \frac{1}{2}(\langle [S_A X, Y], Z\rangle + \langle [X, S_A Y], Z\rangle - \langle S_A Y, [X,Z]\rangle - \langle Y, [S_A X, Z]\rangle \\
&\qquad - \langle S_A X, [Y,Z]\rangle - \langle X, [S_A Y, Z]\rangle) \\
&= \langle \nabla_{S_A X} Y, Z\rangle + \langle \nabla_X S_A Y, Z\rangle
\end{aligned}$$

which implies that $S_A \nabla_X Y = \nabla_{S_A X} Y + \nabla_X S_A Y$.

(2) Let $A \in \mathfrak{a}$ and $X, Y \in \mathfrak{g}$. By making use of Remark 6.2 and (1), the curvature tensor $R(A,X)Y$ is given by

$$\begin{aligned}
R(A,X)Y &= \nabla_A \nabla_X Y - \nabla_X \nabla_A Y - \nabla_{[A,X]} Y \\
&= S_A \nabla_X Y - \nabla_X S_A Y - \nabla_{D_A X + S_A X} Y \\
&= \nabla_{S_A X} Y + \nabla_X S_A Y - \nabla_X S_A Y - \nabla_{D_A X} Y - \nabla_{S_A X} Y \\
&= -\nabla_{D_A X} Y.
\end{aligned}$$

□

Remark 6.4. If $\mathfrak{n}_a^{+b} \neq \{0\}$, then the following holds:

$$\frac{\lambda_b(H_b)^2}{2|H_b|^2} \leq \frac{\lambda_a(H_a)^2}{|H_a|^2}.$$

Proof. Let X be a non-zero vector in $\mathfrak{n}_a^{+b}$. Then, using Remarks 6.1 and 6.2, we have

$$\begin{aligned}
&\langle R(JX,X)X, JX\rangle \\
&\quad = \langle \nabla_{JX}\nabla_X X - \nabla_X \nabla_{JX} X - \nabla_{[JX,X]} X, JX\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \bigg\langle \nabla_{JX}\left(\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 H_a + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 H_b\right) \\
&\qquad - \nabla_X\left(-\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 JH_a + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 JH_b\right) + \frac{\lambda_a(H_a)}{|H_a|^2}|X|^2\nabla_{JH_a}X, JX\bigg\rangle \\
&= \bigg\langle -\frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)^2}{4|H_b|^2}|X|^2 JX \\
&\qquad - \frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)^2}{4|H_b|^2}|X|^2 JX - \frac{\lambda_a(H_a)^2}{2|H_a|^2}|X|^2 JX, JX\bigg\rangle \\
&= -\frac{\lambda_a(H_a)^2}{|H_a|^2}|X|^4 + \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^4.
\end{aligned}
$$

The nonpositivity of the sectional curvature K then implies that

$$
\frac{\lambda_b(H_b)^2}{2|H_b|^2} \le \frac{\lambda_a(H_a)^2}{|H_a|^2}.
$$

□

Remark 6.5. (1) Let $X, X' \in \mathfrak{n}_a^{-b}$ and $Y, Y' \in \mathfrak{n}_b$. Then we have

$$
\begin{aligned}
\langle [Y,X],[Y',X]\rangle &= \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^2\langle Y,Y'\rangle, \\
\langle [Y,X],[Y,X']\rangle &= \frac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2\langle X,X'\rangle, \\
(\langle [Y,X],[Y',X']\rangle + \langle [Y',X],[Y,X']\rangle) &= \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle X,X'\rangle\langle Y,Y'\rangle.
\end{aligned}
$$

(2) Let $X, X' \in \mathfrak{n}_a^{\pm c}$ and $Y, Y' \in \mathfrak{n}_b^{\mp c}$ for $\lambda_b(H) < \lambda_a(H)$. Then we have

$$
\begin{aligned}
\langle [Y,X],[Y',X]\rangle &= \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^2\langle Y,Y'\rangle, \\
\langle [Y,X],[Y,X']\rangle &= \frac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2\langle X,X'\rangle, \\
(\langle [Y,X],[Y',X']\rangle + \langle [Y',X],[Y,X']\rangle) &= \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle X,X'\rangle\langle Y,Y'\rangle.
\end{aligned}
$$

(3) Let $X, X' \in \mathfrak{n}_a^0$ and $Y, Y' \in \mathfrak{n}_b^0$ for $\lambda_b(H) < \lambda_a(H)$. Then we have

$$
\begin{aligned}
&\langle [Y,X],[Y',X']\rangle + \langle [Y,X'],[Y'X]\rangle \\
&\quad - \langle [Y,JX],[Y',JX']\rangle - \langle [Y,JX'],[Y',JX]\rangle = 0.
\end{aligned}
$$

Proof. (1) Let $X, X' \in \mathfrak{n}_a^{-b}$ and $Y, Y' \in \mathfrak{n}_b$. Applying (iv) of (b) in Proposition 6.1 to X and $Y + Y'$, we have

$$|[Y + Y', X]|^2 = \frac{\lambda_b(H_b)^2}{2|H_b|^2} |Y + Y'|^2 |X|^2 .$$

Here we remark that each side of this equation may be written as follows:

$$\begin{aligned} \text{LHS} &= |[Y, X]|^2 + 2\langle [Y, X], [Y', X]\rangle + |[Y', X]|^2 \\ &= \frac{\lambda_b(H_b)^2}{2|H_b|^2}|Y|^2|X|^2 + 2\langle [Y, X], [Y', X]\rangle + \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^2|Y'|^2. \\ \text{RHS} &= \frac{\lambda_b(H_b)^2}{2|H_b|^2}\left(|Y|^2 + 2\langle Y, Y'\rangle + |Y'|^2\right)|X|^2 , \end{aligned}$$

Hence we have

$$2\langle [Y, X], [Y', X]\rangle = \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle Y, Y'\rangle\, |X|^2 , \tag{1.12}$$

which proves the first identity in (1).

Similarly, from (iv) of (b) in Proposition 6.1, we obtain

$$2\langle [Y, X], [Y, X']\rangle = \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle X, X'\rangle |Y|^2,$$

which is the second identity in (1).

We now prove the third identity in (1). Substituting $X + X'$ for X in (1.12), we have

$$2\langle [Y, X + X'], [Y', X + X']\rangle = \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle Y, Y'\rangle\, |X + X'|^2 .$$

The left hand side of this equation is given by

$$\begin{aligned} 2\langle [Y, X + X'],&[Y', X + X']\rangle \\ &= 2\langle [Y, X], [Y', X]\rangle + 2\langle [Y, X], [Y', X']\rangle \\ &\qquad + 2\langle [Y, X'], [Y', X]\rangle + 2\langle [Y, X'], [Y', X']\rangle. \end{aligned}$$

On the other hand, the right hand side is equal to

$$\begin{aligned} \frac{\lambda_b(H_b)^2}{|H_b|^2}&\langle Y, Y'\rangle\, |X + X'|^2 \\ &= \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle Y, Y'\rangle \left(|X|^2 + 2\langle X, X'\rangle + |X'|^2\right) \end{aligned}$$

$$= 2\langle [Y, X], [Y', X]\rangle + 2\frac{\lambda_b(H_b)^2}{|H_b|^2}\langle Y, Y'\rangle\langle X, X'\rangle + 2\langle [Y, X'], [Y', X']\rangle.$$

Hence we obtain

$$\langle [Y, X], [Y', X']\rangle + \langle [Y, X'], [Y', X]\rangle = \frac{\lambda_b(H_b)^2}{|H_b|^2}\langle Y, Y'\rangle\langle X, X'\rangle,$$

which completes the proof of (1).

(2) This is proved in a similar manner to (1).

(3) Let $X, X' \in \mathfrak{n}_a^0$ and $Y, Y' \in \mathfrak{n}_b^0$. Applying (vi) of (b) in Proposition 6.1 to X and $Y + Y'$, we have

$$|[Y + Y', X]|^2 = |[Y + Y', JX]|^2.$$

Again, by (vi) of (b) in Proposition 6.1, we also see that

$$\begin{aligned} \text{LHS} &= |[Y, X]|^2 + 2\langle [Y, X], [Y', X]\rangle + |[Y', X]|^2 \\ &= |[Y, JX]|^2 + 2\langle [Y, X], [Y', X]\rangle + |[Y', JX]|^2, \\ \text{RHS} &= |[Y, JX]|^2 + 2\langle [Y, JX], [Y', JX]\rangle + |[Y', JX]|^2. \end{aligned}$$

Hence we obtain

$$\langle [Y, X], [Y', X]\rangle = \langle [Y, JX], [Y', JX]\rangle. \tag{1.13}$$

Substituting $X + X'$ for X in (1.13), we have

$$\langle [Y, X + X'], [Y', X + X']\rangle = \langle [Y, JX + JX'], [Y', JX + JX']\rangle.$$

Again, by (1.13), we see that the left hand side of this equation is given by

$$\begin{aligned} &\langle [Y, X + X'], [Y', X + X']\rangle \\ &\quad = \langle [Y, X], [Y', X]\rangle + \langle [Y, X], [Y', X']\rangle \\ &\qquad + \langle [Y, X'], [Y', X]\rangle + \langle [Y, X'], [Y', X']\rangle \\ &\quad = \langle [Y, JX], [Y', JX]\rangle + \langle [Y, X], [Y', X']\rangle \\ &\qquad + \langle [Y, X'], [Y', X]\rangle + \langle [Y, JX'], [Y', JX']\rangle. \end{aligned}$$

On the other hand, the right hand side is given by

$$\langle [Y, JX + JX'], [Y', JX + JX']\rangle$$

$$= \langle [Y, JX], [Y', JX] \rangle + \langle [Y, JX], [Y', JX'] \rangle$$
$$+ \langle [Y, JX'], [Y', JX] \rangle + \langle [Y, JX'], [Y', JX'] \rangle.$$

Hence we have

$$\langle [Y, X], [Y', X'] \rangle + \langle [Y, X'], [Y', X] \rangle = \langle [Y, JX], [Y', JX'] \rangle + \langle [Y, JX'], [Y', JX] \rangle.$$

This completes the proof of Claim 6.5.

Remark 6.6. Assume that $\lambda_a(H) > \lambda_b(H)$ for $a, b \in \Lambda_c$. Then the following conditions hold:

(1) $\dim \mathfrak{n}_a^{\pm c} \geq \dim \mathfrak{n}_b^{\pm c}$.

(2) $\dim \mathfrak{n}_a^{\pm b} = \dim \mathfrak{n}_a^{\pm c}$.

(3) $\dim \mathfrak{n}_a^{\pm b} \geq \dim \mathfrak{n}_b^{\pm c}$.

(4) $\dim \mathfrak{n}_a^{0} \geq \dim \mathfrak{n}_b^{0}$.

Proof. (1) Let $\{E_1, \ldots, E_t\}$ be an orthonormal basis of $\mathfrak{n}_b^{+c}$ with respect to $\langle\ ,\ \rangle$, and let Y be a non-zero vector in $\mathfrak{n}_a^{-b}$. By (iv) of (b) of Proposition 6.1, we have $[Y, E_i] \neq 0$ for each $i = 1, \ldots, n$. Moreover, it follows from (1) of Remark 6.5 that $[Y, E_1], \ldots, [Y, E_t]$ are perpendicular to each other. Hence $\dim \mathfrak{n}_a^{\pm c} \geq \dim \mathfrak{n}_b^{\pm c}$.

(2) is proved by (2) of Claim 6.12. (3) follows from (1) and (2). The proof of (4) is similar to that of (1). □

7 Necessary and sufficient condition

Let (M, J, g) be a connected, simply connected homogneouns Einstein Kähler manifold with non-positive curvature. In this section, we shall give the necessary and sufficient condition for M to be a Riemannian symmetric space.

Recall that by Theorem 3.1, (M, J, g) is identified with a simply connected solvable Lie group G with left invariant almost complex structure J and a left invariant Kähler Einstein metric $\langle\ ,\ \rangle$. Also, since G is simply connected, G is determined by its Lie algebra $\mathfrak{g}$ up to isomorphism.

Now, let $\mathfrak{g}$ be a Lie algebra of G. Then the left invariant almost complex structure J on G induces an endomorphism on $\mathfrak{g}$, denoted also by J, and the left invariant Kähler

Einstein metric $\langle\ ,\ \rangle$ on G induces a inner product on $\mathfrak{g}$, denoted also by $\langle\ ,\ \rangle$. From Lemma 3.1 we see that $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ satisfies Condition (K1)–(K4). As remarked at the beginning of §4, the Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of $\mathfrak{g}$ is defined by the Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of G, respectively.

Since $\langle\ ,\ \rangle$ is a Einstein with nonpositive curvature, the Ricci curvature Ric of $(\mathfrak{g}, \langle\ ,\ \rangle)$ is either strictly negative or zero. If Ric vanishes, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is flat (cf. Lemma 5.2) and we have done. Therefore, it suffices to investigate the case where Ric is strictly negative.

From now on we assume that $(\mathfrak{g}, \langle\ ,\ \rangle)$ is not Ricci flat. Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle$. For any $A \in \mathfrak{a}$ let D_A (resp. S_A) denote the symmetric (resp. skew-symmetric) part of $\operatorname{ad} A$ for $A \in \mathfrak{a}$.

Applying Proposition 6.1 to $(\mathfrak{g}, J, \langle\ ,\ \rangle)$, there exists an orthogonal basis $\{H_a\}_{a\in\Lambda}$ of $\mathfrak{a}$ with respect to $\langle\ ,\ \rangle$ such that

$$\begin{aligned} &[H_a, JH_a] = \lambda_a JH_a && \text{for some } \lambda_a > 0, \\ &[H_b, JH_a] = 0 && \text{if } a \neq b. \end{aligned}$$

We define a linear function $\lambda_a\colon \mathfrak{a} \to \mathbb{R}$ by $\lambda_a(H_b) = \delta_{ab}\lambda_a$ for any $b \in \Lambda$. Also, let $\mathfrak{n}_a^{\pm b}$ and $\mathfrak{n}_a^0$ be subspaces of $\mathfrak{n}$ defined by

$$\begin{aligned} \mathfrak{n}_a^{\pm b} &= \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\left(\lambda_a(A) \pm \lambda_b(A)\right) X \quad \text{for any } A \in \mathfrak{a} \right\}, \\ \mathfrak{n}_a^0 &= \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\lambda_a(A) X \quad \text{for any } A \in \mathfrak{a} \right\}, \end{aligned}$$

where $\lambda_b(H_b) < \lambda_a(H_a)$. Setting

$$\mathfrak{n}_a = \bigoplus_{\lambda_b(H_b) < \lambda_a(H_a)} \left(\mathfrak{n}_a^{+b} \oplus \mathfrak{n}_a^{-b}\right) \oplus \mathfrak{n}_a^0,$$

we have a direct sum decomposition $\mathfrak{g} = \bigoplus_a \mathbb{R}\{H_a\} \oplus \mathfrak{n}_a \oplus \mathbb{R}\{JH_a\}$ of $\mathfrak{g}$.

Proposition 7.1. *Let* $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ *be a solvable Lie algebra which has as a direct sum decomposition* $\mathfrak{g} = \bigoplus_{a\in\Lambda} \mathbb{R}\{H_a\} \oplus \mathfrak{n}_a \oplus \mathbb{R}\{JH_a\}$ *as in Proposition* 6.1. *Then the following conditions are equivalent*:

(a) $\nabla R \equiv 0$.

(b) *For each* $c \in \Lambda$*, let* Λ_c *denote a subset* $\{a \in \Lambda \mid \mathfrak{n}_a^{\pm c} \neq \{0\}\} \cup \{c\}$ *of* Λ*. Then there exists a subset* $\{a_1, \dots, a_m\}$ *of* Λ *satisfying that* $\Lambda_{a_1} \cup \dots \cup \Lambda_{a_m} = \Lambda$ *and that* $\Lambda_{a_i} \cap \Lambda_{a_j} = \{0\}$ *if* $i \neq j$*. Moreover, the following hold:*

(i) *If there exists* a_i *such that* $\mathfrak{n}_{a_i}^0 = \{0\}$*, then* $\mathfrak{n}_b^0 = \{0\}$ *for any* $b \in \Lambda_{a_i}$*.*

(ii) $\dfrac{\lambda_b(H_b)}{|H_b|} = \dfrac{\lambda_c(H_c)}{|H_c|}$ *for any* $b, c \in \mathfrak{n}_{a_i}$*.*

Proof. Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle$. It is known by Azencott and Wilson [1] that $\mathfrak{a}$ is abelian, since $\mathfrak{g}$ has nonpositive sectional curvature $K \leq 0$. By Lemma 5.3, $\{D_A, S_A \mid A \in \mathfrak{a}\}$ is a commuting family of derivations of $\mathfrak{g}$ which vanishes on $\mathfrak{a}$, where D_A and S_A denote the symmetric and skew-symmetric parts of $\operatorname{ad} A$ for $A \in \mathfrak{a}$. Also, by Lemma 5.3, D_A is a nonzero operator for any $A \in \mathfrak{a}$, and D_H is positive definite on $\mathfrak{n}$.

We first prove (a) $\Rightarrow$ (b).

Claim 7.1. *If* b *and* c *are elements in* Λ *such that* $b \notin \Lambda_c$ *and* $c \notin \Lambda_b$*, then* $\Lambda_c \cap \Lambda_b = \{0\}$*.*

Proof. Assume that $\Lambda_b \cap \Lambda_c \neq \{0\}$. For any $a \in \Lambda_b \cap \Lambda_c$ let $X \in \mathfrak{n}_a^{+b}$, and $Y \in \mathfrak{n}_a^{+c}$. Since $b \notin \Lambda_c$ and $c \notin \Lambda_b$, it is easy to see that $\nabla_X Y = 0$ by Claim 4.1. Then, using Remarks 6.2 and 6.3, $(\nabla_X R)(H_b, Y, Y)$ is given by

$$\begin{aligned}
&(\nabla_X R)(H_b, Y, Y) \\
&\quad = \nabla X(R(H_b, Y)Y) - R(\nabla_X H_b, Y)Y - R(H_b, \nabla_X Y)Y - R(H_b, Y)\nabla_X Y \\
&\quad = \frac{1}{2}\lambda_b(H_b)R(X, Y)Y = \frac{1}{2}\lambda_b(H_b)\nabla_X \nabla_Y Y \\
&\quad = \frac{1}{2}\lambda_b(H_b)\nabla_X \left(\frac{\lambda_a(H_a)}{2|H_a|^2}|Y|^2 H_a + \frac{\lambda_c(H_c)}{2|H_c|^2}|Y|^2 H_c\right) \\
&\quad = -\frac{1}{2}\lambda_b(H_b)\frac{\lambda_a(H_a)^2}{4|H_a|^2}|Y|^2 X.
\end{aligned}$$

This contradicts $\nabla R = 0$, and hence we conclude $\Lambda_b \cap \Lambda_c = \{0\}$. □

Claim 7.1 implies that there exists a subset $\{a_1, \dots, a_m\}$ of Λ such that $\Lambda_{a_1} \cup \dots \cup \Lambda_{a_m} = \Lambda$ and $\Lambda_{a_i} \cap \Lambda_{a_j} = \{0\}$ for $i \neq j$. We set $\Lambda_{a_i} = \{a_i = i_1, \dots, i_{m_i}\}$ for each $i = 1, \dots, m$. Without loss of generality, we may suppose $\lambda_{i_1}(H) < \dots < \lambda_{i_{m_i}}(H)$. Then $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \bigoplus_{\alpha=1}^{m_i} (\mathbb{R}\{H_{i_\alpha}\} \oplus \mathfrak{n}_{i_\alpha} \oplus \mathbb{R}\{JH_{i_\alpha}\})$$

where $\mathfrak{n}_{i_\alpha}$ is given by

$$\mathfrak{n}_{i_\alpha} = \bigoplus_{\beta=1}^{\alpha-1} \left(\mathfrak{n}_{i_\alpha}^{+i_\beta} \oplus \mathfrak{n}_{i_\alpha}^{-i_\beta} \right) \oplus \mathfrak{n}_{i_\alpha}^{0}.$$

Claim 7.2. *If $\mathfrak{n}_{i_1} = \{0\}$, then $\mathfrak{n}_{i_\alpha}^0 = \{0\}$ for all $\alpha = 1, \ldots, m_i$.*

Proof. Assume that $\mathfrak{n}_{i_1} = \{0\}$ and there exists $\alpha \in \{1, \ldots, m_i\}$ such that $\mathfrak{n}_{i_\alpha}^0 \neq \{0\}$. Let $X \in \mathfrak{n}_{i_\alpha}^{+i_1}$ and $Y \in \mathfrak{n}_{i_\alpha}^0$. It follows from the assumption $\mathfrak{n}_{i_1} = \{0\}$ that $\nabla_X Y = 0$. Then, by making use of Remark 6.2, we see that $(\nabla_X R)(X, Y, Y)$ is given by

$$\begin{aligned}
&(\nabla_X R)(X, Y, Y) \\
&\quad = \nabla_X (R(X, Y)Y) - R(\nabla_X X, Y)Y - R(X, \nabla_X Y)Y - R(X, Y)\nabla_X Y \\
&\quad = \nabla_X \nabla_X \nabla_Y Y - R\left(\frac{\lambda_{i_\alpha}(H_{i_\alpha})}{2|H_{i_\alpha}|^2} |X|^2 H_{i_\alpha} + \frac{\lambda_{i_1}(H_{i_1})}{2|H_{i_1}|^2} |X|^2 H_{i_1}, Y \right) Y \\
&\quad = \nabla_X \nabla_X \left(\frac{\lambda_{\alpha,i}(H_{\alpha,i})}{2|H_{i_\alpha}|^2} |Y|^2 H_{i_\alpha} \right) + \frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{2|H_{i_\alpha}|^2} |X|^2 \nabla_Y Y \\
&\quad = -\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{2|H_{i_\alpha}|^2} |Y|^2 \nabla_X X + \frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{2|H_{i_\alpha}|^2} |X|^2 \nabla_Y Y \\
&\quad = -\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{2|H_{i_\alpha}|^2} |X|^2 |Y|^2 \left(\frac{\lambda_{i_\alpha}(H_{i_\alpha})}{2|H_{i_\alpha}|^2} H_{i_\alpha} + \frac{\lambda_{i_1}(H_{i_1})}{2|H_{i_1}|^2} H_{i_1} - \frac{\lambda_{i_\alpha}(H_{i_\alpha})}{2|H_{i_\alpha}|^2} H_{i_\alpha} \right) \\
&\quad = -\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{2|H_{i_\alpha}|^2} \frac{\lambda_{i_1}(H_{i_1})}{2|H_{i_1}|^2} |X|^2 |Y|^2 H_{i_1}.
\end{aligned}$$

This contradicts $\nabla R = 0$. Hence, $\mathfrak{n}_{i_\alpha}^0 = \{0\}$. □

Claim 7.2 proves (i) of (b). Next we prove

Claim 7.3. *If $\mathfrak{n}_a^{\pm b} \neq \{0\}$, then*

$$\frac{\lambda_a(H_a)}{|H_a|} = \frac{\lambda_b(H_b)}{|H_b|}.$$

Proof. Assume $\mathfrak{n}_a^{\pm b} \neq \{0\}$, and let $X \in \mathfrak{n}_a^{-b}$. It follows from Remark 6.2 together with Remark 6.1 that for X and H_b, we have

$$\begin{aligned}
&(\nabla_{JX} R)(H_b, X, X) \\
&\quad = \nabla_{JX}(R(H_b, X)X) - R(\nabla_{JX} H_b, X)X - R(H_b, \nabla_{JX} X)X - R(H_b, X)\nabla_{JX} X \\
&\quad = \frac{1}{2}\lambda_b(H_b) \nabla_{JX} \nabla_X X + \frac{1}{2}\lambda_b(H_b) R(JX, X)X
\end{aligned}$$

$$
\begin{aligned}
&- R\left(H_b, -\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 JH_a - \frac{\lambda_a(H_b)}{2|H_b|^2}|X|^2 JH_b\right) X - \frac{1}{2}\lambda_b(H_b)\nabla_X\nabla_{JX}X \\
=&\lambda_b(H_b)\left(\nabla_{JX}\nabla_X X - \nabla_X\nabla_{JX}X\right) - \frac{1}{2}\lambda_b(H_b)\nabla_{[JX,X]}X - \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^2\nabla_{JH_b}X \\
=&\lambda_b(H_b)\nabla_{JX}\left(\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 H_a - \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 H_b\right) \\
&- \lambda_b(H_b)\nabla_X\left(\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 JH_a + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 JH_b\right) \\
&+ \lambda_b(H_b)\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2\nabla_{JH_a}X - \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^2\nabla_{JH_b}X \\
=&\lambda_b(H_b)\left(-\frac{\lambda_a(H_a)^2}{2|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 JX\right) \\
&- \lambda_b(H_b)\left(-\frac{\lambda_a(H_a)^2}{2|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 JX\right) \\
&- \lambda_b(H_b)\frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)^3}{4|H_b|^2}|X|^2 JX \\
=&- \lambda_b(H_b)\frac{1}{4}\left(\frac{\lambda_a(H_a)^2}{|H_a|^2} - \frac{\lambda_b(H_b)^2}{|H_b|^2}\right)|X|^2 JX.
\end{aligned}
$$

Since $\nabla R = 0$, this implies that

$$
\frac{\lambda_a(H_a)}{|H_a|} = \frac{\lambda_b(H_b)}{|H_b|}.
$$

□

This completes the proof that (a) implies (b).

We now prove the converse (b) ⇒ (a). Assume that (b) holds, and let $\{a_1, \ldots, a_m\}$ be a subset of Λ satisfying that $\Lambda_{a_1} \cup \ldots \cup \Lambda_{a_m} = \Lambda$ and that $\Lambda_{a_i} \cap \Lambda_{a_j} = \{0\}$ if $i \neq j$. By rearranging the indices, we may assume that $\mathfrak{n}^0_{a_k} \neq \{0\}$ for $1 \leq k \leq n$ and $\mathfrak{n}^0_{a_l} = \{0\}$ for $n < l \leq m$, where $1 \leq n \leq m$. For $k \in \{1, \ldots, n\}$, we define a subspace $\mathfrak{g}_k$ of $\mathfrak{g}$ by

$$
\mathfrak{g}_k = \mathbb{R}\{H_{a_k}\} \oplus \mathbb{R}\{JH_{a_k}\} \oplus \bigoplus_{a_k \neq \alpha \in \Lambda_{a_k}} \left(\mathbb{R}\{H_\alpha\} \oplus \mathfrak{n}_\alpha \oplus \mathbb{R}\{JH_\alpha\}\right),
$$

where $\mathfrak{n}_\alpha$ is given by

$$
\mathfrak{n}_\alpha = \bigoplus_{\lambda_\beta(H) < \lambda_\alpha(H)} \left(\mathfrak{n}^{+\beta}_\alpha \oplus \mathfrak{n}^{-\beta}_\alpha\right).
$$

Moreover, for $l \in \{n+1, \dots, m\}$, we define a subspace $\mathfrak{g}_l$ of $\mathfrak{g}$ by

$$\mathfrak{g}_l = \bigoplus_{\gamma \in \Lambda_{a_l}} \left(\mathbb{R}\{H_\gamma\} \oplus \mathfrak{n}_\gamma \oplus \mathbb{R}\{JH_\gamma\} \right),$$

where $\mathfrak{n}_\gamma$ is given by

$$\mathfrak{n}_\gamma = \bigoplus_{\lambda_\delta(H) < \lambda_\gamma(H)} \left(\mathfrak{n}_\gamma^{+\delta} \oplus \mathfrak{n}_\gamma^{-\delta} \right) \oplus \mathfrak{n}_\gamma^0.$$

Then $\mathfrak{g}$ is decomposed into a direct sum $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$. Obviously, we have $[\mathfrak{g}_i, \mathfrak{g}_{i'}] = \{0\}$ for $1 \le i < i' \le m$.

We first investigate the case where $\mathfrak{n}_{a_k}^0 \neq \{0\}$ for $1 \le k \le n$. Fix $k \in \{1, \dots, n\}$. For the sake of convenience, we set $\Lambda_{a_k} = \{1(= a_k), \dots, m_k\}$. Without loss of generality, we may assume that $\lambda_1(H) < \cdots < \lambda_{m_k}(H)$. Moreover, unless otherwise stated, Greek indices $\alpha, \beta, \dots$ run from 1 to m_k.

Claim 7.4. (1) $\dim \mathfrak{n}_\alpha^{\pm\beta} = \dim \mathfrak{n}_{m_k}^{\pm 1}$ *for any* $\beta < \alpha$.

(2) $\dim \mathfrak{n}_\alpha^0 = \dim \mathfrak{n}_1$ *for any* α.

Proof. Since $\langle \ , \ \rangle$ is Einstein, we have

$$\frac{1}{|H_\alpha|^2} \operatorname{Ric}(H_\alpha, H_\alpha) = \frac{1}{|H_{m_k}|^2} \operatorname{Ric}(H_{m_k}, H_{m_k}).$$

By making use of Remarks 6.2 and 6.3, the Ricci curvature in the directions H_α and H_{m_k} are given respectively by

$$\operatorname{Ric}(H_\alpha, H_\alpha) = -\frac{1}{4} \lambda_\alpha(H_\alpha)^2 \left(\dim \mathfrak{n}_\alpha + 4 + \sum_{\beta=\alpha+1}^{m} 2 \dim \mathfrak{n}_\beta^{+\alpha} \right),$$
$$\operatorname{Ric}(H_{m_k}, H_{m_k}) = -\frac{1}{4} \lambda_{m_k}(H_{m_k})^2 \left(\dim \mathfrak{n}_{m_k} + 4 \right).$$

Hence we have

$$-\frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2} \left(\dim \mathfrak{n}_\alpha + 4 + \sum_{\beta=\alpha+1}^{m_k} 2 \dim \mathfrak{n}_\beta^{+\alpha} \right) = -\frac{\lambda_{m_k}(H_{m_k})^2}{4|H_{m_k}|^2} \left(\dim \mathfrak{n}_{m_k} + 4 \right).$$

This combined with (ii) of (b) implies that

$$\dim \mathfrak{n}_\alpha + \sum_{\beta=\alpha+1}^{m_k} 2 \dim \mathfrak{n}_\beta^{+\alpha} = \dim \mathfrak{n}_{m_k},$$

and hence

$$\begin{aligned}
0 &= \dim \mathfrak{n}_{m_k} - \left(\dim \mathfrak{n}_\alpha + \sum_{\beta=\alpha+1}^{m} 2 \dim \mathfrak{n}_\beta^{+\alpha} \right) \\
&= 2 \sum_{\beta=1}^{m_k-1} \dim \mathfrak{n}_{m_k}^{+\beta} + \dim \mathfrak{n}_{m_k}^0 - \left(2 \sum_{\beta=1}^{\alpha-1} \dim \mathfrak{n}_\alpha^{+\beta} + \dim \mathfrak{n}_\alpha^0 + 2 \sum_{\gamma=\alpha+1}^{m_k} \dim \mathfrak{n}_\gamma^{+\alpha} \right) \\
&= 2 \sum_{\beta=1}^{\alpha-1} \left(\dim \mathfrak{n}_{m_k}^{+\beta} - \dim \mathfrak{n}_\alpha^{+\beta} \right) + \dim \mathfrak{n}_{m_k}^0 - \dim \mathfrak{n}_\alpha^0 + 2 \sum_{\gamma=\alpha+1}^{m_k} \left(\dim \mathfrak{n}_{m_k}^{+\gamma} - \dim \mathfrak{n}_\gamma^{+\alpha} \right).
\end{aligned}$$

Recall that it is proved in Remark 6.6 that for $\beta < \alpha < \gamma < m_k$

$$\begin{aligned}
&\dim \mathfrak{n}_{m_k}^{+\beta} - \dim \mathfrak{n}_\alpha^{+\beta} \geq 0, \\
&\dim \mathfrak{n}_{m_k}^0 - \dim \mathfrak{n}_\alpha^0 \geq 0, \\
&\dim \mathfrak{n}_{m_k}^{+\gamma} - \dim \mathfrak{n}_\gamma^{+\alpha} \geq 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\dim \mathfrak{n}_{m_k}^{+\beta} - \dim \mathfrak{n}_\alpha^{+\beta} = 0, \\
&\dim \mathfrak{n}_{m_k}^0 - \dim \mathfrak{n}_\alpha^0 = 0, \\
&\dim \mathfrak{n}_{m_k}^{+\gamma} - \dim \mathfrak{n}_\gamma^{+\alpha} = 0.
\end{aligned}$$

Since, by (2) of Remark 6.6, we have $\dim \mathfrak{n}_{m_k}^{+1} = \dim \mathfrak{n}_{m_k}^{+\beta}$ for $\beta < m_k$, we obtain $\dim \mathfrak{n}_{m_k}^{+1} = \dim \mathfrak{n}_\alpha^{+\beta}$ for $\beta < \alpha$. This proves (1).

We remark here that $\mathfrak{n}_1^0 = \mathfrak{n}_1$. Since α is arbitrary, we obtain

$$\dim \mathfrak{n}_{m_k}^0 = \cdots = \dim \mathfrak{n}_1^0 = \dim \mathfrak{n}_1,$$

which proves (2). This completes the proof of Claim 7.4.

We define a lexicographic order on a set $\mathcal{L}_k = \{(\alpha, \beta) \mid m_k \geq \alpha > \beta \geq 0\}$ by

$$(\alpha, \beta) > (\alpha', \beta') \quad \Longleftrightarrow \quad \begin{cases} \alpha > \alpha'. \\ \text{If } \alpha = \alpha', \text{ then } \beta > \beta'. \end{cases}$$

By virtue of Claim 7.4, we set

$$\begin{aligned}
s_k &= \dim \mathfrak{n}_\alpha^{\pm\beta} \quad \text{for } \alpha > \beta > 0, \\
t_k &= \dim \mathfrak{n}_\alpha^{\pm\beta} \quad \text{for } \alpha > 0.
\end{aligned}$$

Let $\{(E_\alpha^{\pm\beta})_1, \ldots, (E_\alpha^{\pm\beta})_{s_k}\}$ denote an orthonormal basis of $\mathfrak{n}_\alpha^{\pm\beta}$ with respect to $\langle\ ,\ \rangle$ for $\alpha > \beta > 0$, respectively. Also, let $\{(E_\alpha^0)_1, \ldots, (E_\alpha^0)_{t_k}\}$ denote an orthonormal basis of $\mathfrak{n}_\alpha^0$ with respect to $\langle\ ,\ \rangle$ for $\alpha > 0$.

Claim 7.5. (1) *Let $(\alpha, \beta), (\gamma, \delta) \in \mathcal{L}_k$, and suppose $(\alpha, \beta) > (\gamma, \delta)$. Let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Y \in \mathfrak{n}_\gamma^{\pm\delta} \oplus \mathfrak{n}_\alpha^{-\beta}$. Then, for any $Z \in \mathfrak{g}$, we have*

$$R(X,Y)Z = -\nabla_{\nabla_X Y} Z.$$

(2) *Let $\alpha > \beta > \gamma > 0$. Then, for $X, Y \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\beta^{+\gamma}$, we have*

$$R(X,Y)Z = -\frac{1}{4}\sum_{p=1}^{s_k} \left(\langle [Z, JY], [(E_\beta^{+\gamma})_p, JX]\rangle - \langle [Z, JX], [(E_\beta^{+\gamma})_p, JY]\rangle\right)(E_\beta^{+\gamma})_p.$$

(3) *Let $\alpha > \gamma > \beta > 0$. Then, for $X, Y \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\gamma^{+\beta}$, we have*

$$R(X,Y)Z = -\frac{1}{4}\sum_{p=1}^{s_k} \left(\langle [Z, JY], [(E_\gamma^{+\beta})_p, JX]\rangle - \langle [Z, JX], [(E_\gamma^{+\beta})_p, JY]\rangle\right)(E_\gamma^{+\beta})_p.$$

(4) *Let $\alpha > \beta > \gamma \geq 0$. Then, for $X, Y \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\alpha^{+\gamma}$, we have*

$$\begin{aligned} R(X,Y)Z = &-\frac{1}{4}\sum_{p=1}^{s_k}\langle Y, [Z, (E_\beta^{-\gamma})_p]\rangle [JX, J(E_\beta^{-\gamma})_p] \\ &+\frac{1}{4}\sum_{p=1}^{s_k}\langle X, [Z, (E_\beta^{-\gamma})_p]\rangle [JY, J(E_\beta^{-\gamma})_p]. \end{aligned}$$

(5) *Let $\alpha > \beta > 0$. Then, for $X, Y, Z \in \mathfrak{n}_\alpha^{+\beta}$, we have*

$$R(X,Y)Z = -\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\left(\langle Y, Z\rangle X - \langle X, Z\rangle Y\right).$$

(6) *Let $\alpha > \gamma > \beta > 0$. Then, for $X, Y \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\alpha^{+\gamma}$, we have*

$$R(X,Y)Z = -\frac{1}{4}\sum_{p=1}^{s_k}\langle Z, [Y, (E_\gamma^{-\beta})_p]\rangle [X, (E_\gamma^{-\beta})_p] - \langle Z, [X, (E_\gamma^{-\beta})_p]\rangle [Y, (E_\gamma^{-\beta})_p].$$

(7) *Let $\gamma > \alpha > \beta > 0$. Then, for $X, Y \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\gamma^{+\beta}$, we have*

$$R(X,Y)Z = \frac{1}{4}[J[Z, JY], X] - \frac{1}{4}[J[Z, JX], Y].$$

(8) *Let $\gamma > \alpha > \beta > 0$. Then, for $X, Y \in \mathfrak{n}_{\alpha}^{+\beta}$ and $Z \in \mathfrak{n}_{\gamma}^{+\alpha}$, we have*

$$R(X,Y)Z = \frac{1}{4}[X,[JY,JZ]] - \frac{1}{4}[Y,[JX,JZ]].$$

(9) *Let $\alpha > \gamma > 0$. Then, for $X, Y \in \mathfrak{n}_{\alpha}^{0}$ and $Z \in \mathfrak{n}_{\gamma}^{0}$, we have*

$$R(X,Y)Z = \frac{1}{2}\sum_{p=1}^{t_k} \big(\langle [X,Z],[Y,(E_c^0)_p]\rangle - \langle [JY,Z],[JX,(E_c^0)_p]\rangle\big)(E_c^0)_p.$$

(10) *Let $\alpha > \gamma > 0$. Then, for $X, Y \in \mathfrak{n}_{\alpha}^{0}$ and $Z \in \mathfrak{n}_{\alpha}^{+\gamma}$, we have*

$$R(X,Y)Z = -\frac{1}{4}\sum_{p=1}^{t_k} \big(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p] - \langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\big)$$
$$+ \frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\langle JX,Y\rangle JZ.$$

(11) *Let $\alpha > 0$. Then, for $X, Y, Z \in \mathfrak{n}_{\alpha}^{0}$, we have*

$$R(X,Y)Z = \frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\left(-\langle Y,Z\rangle X - \langle JY,Z\rangle JX \right.$$
$$\left. + \langle X,Z\rangle Y + \langle JX,Z\rangle JY + 2\langle JX,Y\rangle JZ\right).$$

(12) *Let $\gamma > \alpha > 0$. Then, for $X, Y \in \mathfrak{n}_{\alpha}^{0}$ and $Z \in \mathfrak{n}_{\gamma}^{+\alpha}$, we have*

$$R(X,Y)Z = \frac{1}{2}[[X,JZ],JY] - \frac{\lambda_a(H_a)^2}{4|H_a|^2}\langle X,Y\rangle Z + \frac{\lambda_a(H_a)^2}{4|H_a|^2}\langle JX,Y\rangle JZ.$$

(13) *Let $\gamma > \alpha > 0$. Then, for $X, Y \in \mathfrak{n}_{\alpha}^{0}$ and $Z \in \mathfrak{n}_{\gamma}^{0}$, we have*

$$R(X,Y)Z = \frac{1}{4}([J[Y,Z],JX] - [X,J[Z,JY]] - [J[X,Z],JY] + [Y,J[Z,JX]]).$$

Proof. (1) Suppose that $\alpha > \beta > \gamma > \delta > 0$. We first prove the case where $X \in \mathfrak{n}_{\alpha}^{+\gamma}, Y \in \mathfrak{n}_{\beta}^{-\gamma}$ and $Z \in \mathfrak{n}_{\beta}^{+\delta}$. Let $\{(E_\gamma^{+\delta})_1, \ldots, (E_\gamma^{+\delta})_{s_k}\}$ be an orthonormal basis of $\mathfrak{n}_{\beta}^{+\gamma}$f with respect to $\langle\ ,\ \rangle$. By virtue of Remark 6.5, $[JX, J(E_\gamma^{+\delta})_1], \ldots, [JX, J(E_\gamma^{+\delta})_{s_k}]$ are mutually perpendicular in $\mathfrak{n}_{\alpha}^{-\gamma}$. This combined with Claim 7.4 implies that, setting

$$e_p = \frac{1}{|[JX, J(E_\gamma^{+\delta})_p]|}[JX, J(E_\gamma^{+\delta})_p],$$

we obtain an orthonormal basis $\{e_1, \ldots, e_{s_k}\}$ of $\mathfrak{n}_\alpha^{-\gamma}$.

It follows from Remarks 6.2 and 6.5 together with Condition (K3) that

$$
\begin{aligned}
R(X,Y)Z =& \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
=& \nabla_X \left(-\frac{1}{2} \sum_{p=1}^{s_k} \langle Z, [Y, (E_\gamma^{+\delta})_p] \rangle (E_\gamma^{+\delta})_p \right) - \frac{1}{2} [J[X,Y], JZ] \\
=& -\frac{1}{4} \sum_{p=1}^{s_k} \langle Z, [Y, (E_\gamma^{+\delta})_p] \rangle [JX, J(E_\gamma^{+\delta})_p] - \frac{1}{2} [J[X,Y], JZ] \\
=& -\frac{1}{4} \sum_{p=1}^{s_k} \langle JY, [(E_\gamma^{+\delta})_p, JZ] \rangle [JX, J(E_\gamma^{+\delta})_p] - \frac{1}{2} [J[X,Y], JZ] \\
=& -\frac{1}{4} \sum_{p=1}^{s_k} \frac{2|H_\beta|^2}{\lambda_\beta(H_\beta)|X|^2} \langle [JY, JX], [[(E_\gamma^{+\delta})_p, JZ], JX] \rangle [JX, J(E_\gamma^{+\delta})_p] \\
& - \frac{1}{2} [J[X,Y], JZ] \\
=& -\frac{1}{4} \sum_{p=1}^{s_k} \frac{2|H_\beta|^2}{\lambda_\beta(H_\beta)|X|^2} \langle [JY, JX], [[JX, J(E_\gamma^{+\delta})_p], Z \rangle [JX, J(E_\gamma^{+\delta})_p] \\
& - \frac{1}{2} [J[X,Y], JZ] \\
=& -\frac{1}{4} \sum_{p=1}^{s_k} \frac{2|H_\beta|^2 |[JX, J(E_\gamma^{+\delta})_p]|^2}{\lambda_\beta(H_\beta)|X|^2} \langle [JY, JX], [e_p, Z] \rangle e_p - \frac{1}{2} [J[X,Y], JZ] \\
=& \frac{1}{4} \sum_{p=1}^{s_k} \langle e_p, [JZ, J[Y,X]] \rangle e_p - \frac{1}{2} [J[X,Y], JZ] \\
=& -\frac{1}{4} [J[X,Y], JZ] \\
=& -\nabla_{\nabla_X Y} Z.
\end{aligned}
$$

It remains to show that $R(X,Y)Z = -\nabla_{\nabla_X Y} Z$ in the other cases of (1). However, since the proof is quite similar, we omit the detail.

The proof of (2) (resp. (3), (4), (5), (6), (7) and (8)) is done by a straightforward calculation, using Remark 6.2 together with (iv), (v) and (vi) of (b) in Proposition 6.1. Hence we omit the detail of the caluculation.

(9) For $\alpha > \gamma > 0$, let $X, Y \in \mathfrak{n}_\alpha^0$ and $Z \in \mathfrak{n}_\gamma^0$. Let $\{(E_\gamma^0)_1, \ldots, (E_\gamma^0)_{t_k}\}$ be an orthonormal basis of $\mathfrak{n}_\gamma^0$. By making use of Remarks 6.2 and 6.5, we have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$
\begin{aligned}
=&\nabla_X\left(\frac{1}{2}[Y,Z]-\frac{1}{2}J[Z,JY]\right)-\nabla_Y\left(\frac{1}{2}[X,Z]-\frac{1}{2}J[Z,JX]\right)\\
&-\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}\langle JX,Y\rangle\nabla_{JH_\alpha}Z\\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\langle[Y,Z],[X,(E^0_\gamma)_p]\rangle(E^0_\gamma)_p+\frac{1}{4}\sum_{p=1}^{t_k}\langle X,[J[Z,JY],(E^0_\gamma)_p]\rangle(E^0_\gamma)_p\\
&+\frac{1}{4}\sum_{p=1}^{t_k}\langle[X,Z],[Y,(E^0_\gamma)_p]\rangle(E^0_\gamma)_p-\frac{1}{4}\sum_{p=1}^{t_k}\langle Y,[J[Z,JX],(E^0_\gamma)_p]\rangle(E^0_\gamma)_p\\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\langle[Y,Z],[X,(E^0_\gamma)_p]\rangle(E^0_\gamma)_p-\frac{1}{4}\sum_{p=1}^{t_k}\langle[Z,JY],[(E^0_\gamma)_p,JX]\rangle(E^0_\gamma)_p\\
&+\frac{1}{4}\sum_{p=1}^{t_k}\langle[X,Z],[Y,(E^0_\gamma)_p]\rangle(E^0_\gamma)_p+\frac{1}{4}\sum_{p=1}^{t_k}\langle[Z,JX],[(E^0_\gamma)_p,JY]\rangle(E^0_\gamma)_p\\
=&\frac{1}{2}\sum_{p=1}^{t_k}\left(\langle[X,Z],[Y,(E^0_\gamma)_p]\rangle-\langle[JY,Z],[JX,(E^0_\gamma)_p]\rangle\right)(E^0_\gamma)_p.
\end{aligned}
$$

(10) For $\alpha>\gamma>0$, let $X,Y\in\mathfrak{n}^0_\alpha$ and $Z\in\mathfrak{n}^{+\gamma}_\alpha$. Let $\{(E^0_\gamma)_1,\dots,(E^0_\gamma)_{t_k}\}$ be an orthonormal basis of $\mathfrak{n}^0_\gamma$, and $\{(E^{-\gamma}_\alpha)_1,\dots,(E^{-\gamma}_\alpha)_{s_k}\}$ an orthonormal basis of $\mathfrak{n}^{-\gamma}_\alpha$. It follows from Remark 6.5 that $[(E^0_\gamma)_1,(E^{-\gamma}_\alpha)_q],\dots,[(E^0_\gamma)_{t_k},(E^{-\gamma}_\alpha)_q]$ are mutually orthogonal for each $q=1,\dots,s_k$. Combining this with Claim 7.4, and setting

$$
(e_q)_p=\frac{1}{|[(E^0_\gamma)_p,(E^{-\gamma}_\alpha)_q]|}[(E^0_\gamma)_p,(E^{-\gamma}_\alpha)_q]\quad\text{for } p=1,\dots,t_k,
$$

we obtain an orthonormal basis $\{(e_q)_1,\dots,(e_q)_{t_k}\}$ of $\mathfrak{n}^0_\alpha$.

Then, by making use of Remarks 6.2 and 6.5, we have

$$
\begin{aligned}
R(X,Y)Z=&\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z\\
=&\nabla_X\left(-\frac{1}{2}\sum_{p=1}^{t_k}\langle Z,[Y,(E^0_\gamma)_p]\rangle(E^0_\gamma)_p\right)\\
&-\nabla_Y\left(-\frac{1}{2}\sum_{p=1}^{t_k}\langle Z,[X,(E^0_\gamma)_p]\rangle(E^0_\gamma)_p\right)\\
&-\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}\langle JX,Y\rangle\nabla_{JH_a}Z\\
=&-\frac{1}{2}\sum_{p=1}^{t_k}\langle Z,[Y,(E^0_\gamma)_p]\rangle\left(\frac{1}{2}[X,(E^0_\gamma)_p]-\frac{1}{2}J[X,J(E^0_\gamma)_p]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{p=1}^{t_k}\langle Z,[X,(E_\gamma^0)_p]\rangle\left(\frac{1}{2}[Y,(E_\gamma^0)_p]-\frac{1}{2}J[Y,J(E_\gamma^0)_p]\right)\\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ\\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]+\frac{1}{4}\sum_{p=1}^{t_k}\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\\
&+\frac{1}{4}\sum_{p=1}^{t_k}\sum_{q=1}^{s_k}\langle Z,[Y,(E_\gamma^0)_p]\rangle\langle J[X,J(E_\gamma^0)_p],(E_\alpha^{-\gamma})_q\rangle(E_\alpha^{-\gamma})_q\\
&-\frac{1}{4}\sum_{p=1}^{t_k}\langle Z,[X,J(E_\gamma^0)_p]\rangle J[Y,J^2(E_\gamma^0)_p]\\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ\\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\left(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]-\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\right)\\
&+\frac{1}{4}\sum_{p=1}^{t_k}\sum_{q=1}^{s_k}\langle Z,[Y,(E_\gamma^0)_p]\rangle\langle[(E_\gamma^0)_p,(E_\alpha^{-\gamma})_q],X\rangle(E_\alpha^{-\gamma})_q\\
&-\frac{1}{4}\sum_{p=1}^{t_k}\sum_{q=1}^{s_k}\langle Z,[JX,(E_\gamma^0)_p]\rangle\langle J[Y,(E_\gamma^0)_p],(E_\alpha^{-\gamma})_q\rangle(E_\alpha^{-\gamma})_q\\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ\\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\left(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]-\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\right)\\
&+\frac{1}{4}\sum_{p=1}^{t_k}\sum_{q=1}^{s_k}\langle JY,[(E_\gamma^0)_p,JZ]\rangle\langle[(E_\gamma^0)_p,(E_\alpha^{-\gamma})_q],X\rangle(E_\alpha^{-\gamma})_q\\
&+\frac{1}{4}\sum_{p=1}^{t_k}\sum_{q=1}^{s_k}\langle X,[(E_\gamma^0)_p,JZ]\rangle\langle[(E_\gamma^0)_p,(E_\alpha^{-\gamma})_q],JY\rangle(E_\alpha^{-\gamma})_q\\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ\\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\left(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]-\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{p,r=1}^{t_k}\sum_{q=1}^{s_k}|[(E_\gamma^0)_p,(E_\alpha^{-\gamma})_q]|\langle[(E_\gamma^0)_p,JZ],(e_q)_r\rangle \\
&\qquad\qquad\times\langle JY,(e_q)_r\rangle\langle(e_q)_p,X\rangle(E_\alpha^{-\gamma})_q \\
&+\frac{1}{4}\sum_{p,r=1}^{t_k}\sum_{q=1}^{s_k}|[(E_\gamma^0)_p,(E_\alpha^{-\gamma})_q]|\langle[(E_\gamma^0)_p,JZ],(e_q)_r\rangle \\
&\qquad\qquad\times\langle X,(e_q)_r\rangle\langle(e_q)_p,JY\rangle(E_\alpha^{-\gamma})_q \\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ \\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\big(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]-\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\big) \\
&+\frac{1}{4}\sum_{p,r=1}^{t_k}\sum_{q=1}^{s_k}\langle[(E_\gamma^0)_p,JZ],[(E_\gamma^0)_r,(E_\alpha^{-\gamma})_q]\rangle\langle JY,(e_q)_r\rangle\langle(e_q)_p,X\rangle(E_\alpha^{-\gamma})_q \\
&+\frac{1}{4}\sum_{p,r=1}^{t_k}\sum_{q=1}^{s_k}\langle[(E_\gamma^0)_p,JZ],[(E_\gamma^0)_r,(E_\alpha^{-\gamma})_q]\rangle\langle X,(e_q)_r\rangle\langle(e_q)_p,JY\rangle(E_\alpha^{-\gamma})_q \\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ \\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\big(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]-\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\big) \\
&+\frac{1}{4}\sum_{p,r=1}^{t_k}\sum_{q=1}^{s_k}\langle JY,(e_q)_r\rangle\langle(e_q)_p,X\rangle \\
&\qquad\qquad\times\big(\langle[(E_\gamma^0)_p,JZ],[(E_\gamma^0)_r,(E_\alpha^{-\gamma})_q]\rangle \\
&\qquad\qquad\quad+\langle[(E_\gamma^0)_r,JZ],[(E_\gamma^0)_p,(E_\alpha^{-\gamma})_q]\rangle\big)(E_\alpha^{-\gamma})_q \\
&+\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX,Y\rangle JZ \\
=&-\frac{1}{4}\sum_{p=1}^{t_k}\big(\langle Z,[Y,(E_\gamma^0)_p]\rangle[X,(E_\gamma^0)_p]-\langle Z,[X,(E_\gamma^0)_p]\rangle[Y,(E_\gamma^0)_p]\big) \\
&+\frac{1}{4}\sum_{p,r=1}^{t_k}\sum_{q=1}^{s_k}\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}\langle(E_\gamma^0)_p,(E_\gamma^0)_r\rangle\langle JZ,(E_\alpha^{-\gamma})_q\rangle \\
&\qquad\qquad\times\langle JY,(e_q)_r\rangle\langle(e_q)_p,X\rangle(E_\alpha^{-\gamma})_q
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX, Y\rangle JZ \\
=& -\frac{1}{4}\sum_{p=1}^{t_k}\big(\langle Z, [Y, (E^0_\gamma)_p]\rangle [X, (E^0_\gamma)_p] - \langle Z, [X, (E^0_\gamma)_p]\rangle [Y, (E^0_\gamma)_p]\big) \\
&+ \frac{1}{4}\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha a|^2}\langle JY, X\rangle JZ + \frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle JX, Y\rangle JZ \\
=& -\frac{1}{4}\sum_{p=1}^{t_k}\big(\langle Z, [Y, (E^0_\gamma)_p]\rangle [X, (E^0_\gamma)_p] - \langle Z, [X, (E^0_\gamma)_p]\rangle [Y, (E^0_\gamma)_p]\big) \\
&+ \frac{1}{4}\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}\langle JX, Y\rangle JZ.
\end{aligned}
$$

The verification of (11), (12) and (13) is quite similar to that of (10). This completes the proof of Claim 7.5.

Claim 7.6. *Let $A \in \mathfrak{a}$. Then we have*

$$\nabla_A R = 0.$$

Proof. Let $A \in \mathfrak{a}$, and let $X, Y, Z \in \mathfrak{g}$. It follows from Remark 6.3 that

$$
\begin{aligned}
S_A(R(X,Y)Z) =& S_A(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\
=& \nabla_{S_AX}\nabla_Y Z + \nabla_X\nabla_{S_AY} Z + \nabla_X\nabla_Y S_A Z \\
&- \nabla_{S_AY}\nabla_X Z - \nabla_Y\nabla_{S_AX} Z - \nabla_Y\nabla_X S_A Z \\
&- \nabla_{[S_AX,Y]}Z - \nabla_{[X,S_AY]}Z - \nabla_{[X,Y]}S_A Z \\
=& R(S_AX,Y)Z + R(X,S_AY)Z + R(X,Y)S_AZ,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(\nabla_A R)(X,Y,Z) &= \nabla_A(R(X,Y)Z) - R(\nabla_A X, Y)Z - R(X, \nabla_A Y)Z - R(X,Y)\nabla_A Z \\
&= S_A(R(X,Y)Z) - R(S_AX,Y)Z - R(X,S_AY)Z - R(X,Y)S_AZ \\
&= S_A(R(X,Y)Z) - S_A(R(X,Y)Z) \\
&= 0.
\end{aligned}
$$

Since X, Y and Z are arbitrary, we have $(\nabla_A R) = 0$. ☐

Claim 7.7. *Let $X \in \mathfrak{n}_\alpha^{+\beta}$. Then we have*

$$\nabla_X R = 0.$$

Proof. Assume that $\alpha > \beta > \gamma > \delta > 0$. Let $A, B \in \mathfrak{a}$ and $W \in \mathfrak{g}$. To prove this Claim, we need to check the following cases:

(1) $(\nabla_X R)(A, B, W) = 0$ for any $X \in \mathfrak{g}$.

(2) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(3) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\gamma^{0}$.

(4) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(5) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\beta^{0}$.

(6) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}$ and $Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(7) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}$ and $Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(8) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}$ and $Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(9) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}$ and $Z \in \mathfrak{n}_\beta^{0}$.

(10) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(11) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(12) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}$ and $Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(13) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\alpha^{0}$.

(14) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}$ and $Z \in \mathfrak{n}_\alpha^{\pm\delta}$.

(15) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}$ and $Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(16) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\beta^{+\delta}$ and $Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(17) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}$ and $Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(18) $(\nabla_X R)(A, Z, W) = 0$ for $X \in \mathfrak{n}_\gamma^{+\delta}$ and $Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(19) $(\nabla_X R)(A, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\beta^{+\gamma}$ and $Z \in \mathfrak{n}_\alpha^0$.

(20) $(\nabla_X R)(A, JH_\beta, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}$.

(21) $(\nabla_X R)(A, JH_\alpha, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}$.

(22) $(\nabla_X R)(A, JH_\epsilon, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}$ and $\epsilon \neq \alpha, \beta$.

(23) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(24) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\gamma^0$.

(25) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^{+\delta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(26) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(27) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(28) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^0$.

(29) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^0, Z \in \mathfrak{n}_\gamma^0$.

(30) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\beta^0, Z \in \mathfrak{n}_\beta^0$.

(31) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^{+\delta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(32) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(33) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\gamma^0$.

(34) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\beta^{+\delta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(35) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(36) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(37) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^0$.

(38) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(39) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\beta^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(40) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\beta^0, Z \in \mathfrak{n}_\gamma^0$.

(41) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(42) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\gamma^{0}$.

(43) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(44) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(45) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(46) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(47) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{0}$.

(48) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(49) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(50) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\alpha^{\pm\delta}$.

(51) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(52) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\alpha^{0}$.

(53) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(54) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{0}$.

(55) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(56) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(57) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(58) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{0}$.

(59) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(60) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\gamma^{0}$.

(61) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(62) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(63) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(64) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(65) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{0}$.

(66) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(67) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(68) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\delta}$.

(69) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(70) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(71) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(72) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{0}$.

(73) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{0}, Z \in \mathfrak{n}_\gamma^{0}$.

(74) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{0}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(75) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{0}, Z \in \mathfrak{n}_\beta^{0}$.

(76) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{0}, Z \in \mathfrak{n}_\beta^{\pm\gamma}$.

(77) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\gamma}, Y \in \mathfrak{n}_\alpha^{0}, Z \in \mathfrak{n}_\beta^{0}$.

(78) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\alpha^{0}, Z \in \mathfrak{n}_\alpha^{0}$.

(79) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(80) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(81) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(82) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\gamma^{0}$.

(83) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\beta^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\delta}, Z \in \mathfrak{n}_\gamma^{\pm\delta}$.

(84) $(\nabla_X R)(Y,Z,W) = 0$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\gamma}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(85) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(86) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\delta}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(87) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\delta}, Z\in\mathfrak{n}_\beta^{0}$.

(88) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\gamma^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\delta}, Z\in\mathfrak{n}_\beta^{\pm\delta}$.

(89) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\gamma^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\delta}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(90) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\alpha^{\pm\delta}$.

(91) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\alpha^{\pm\gamma}$.

(92) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\alpha^{0}$.

(93) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\gamma^{\pm\delta}$.

(94) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(95) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\alpha^{\pm\delta}$.

(96) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\gamma^{\pm\delta}$.

(97) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\gamma^{0}$.

(98) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\gamma^{\pm\delta}$.

(99) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\beta^{\pm\delta}$.

(100) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(101) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(102) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\beta^{0}$.

(103) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\gamma^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\beta^{\pm\delta}$.

(104) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\gamma^{+\delta}, Y\in\mathfrak{n}_\alpha^{+\gamma}, Z\in\mathfrak{n}_\beta^{\pm\gamma}$.

(105) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\alpha^{\pm\delta}$.

(106) $(\nabla_X R)(Y,Z,W)=0 \quad$ for $X\in\mathfrak{n}_\beta^{+\gamma}, Y\in\mathfrak{n}_\alpha^{+\beta}, Z\in\mathfrak{n}_\alpha^{\pm\gamma}$.

(107) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\beta^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

(108) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\beta}$.

(109) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\beta^{+\gamma}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{0}$.

(110) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\gamma^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\beta^{\pm\delta}$.

(111) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\gamma^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\delta}$.

(112) $(\nabla_X R)(Y, Z, W) = 0 \quad$ for $X \in \mathfrak{n}_\gamma^{+\delta}, Y \in \mathfrak{n}_\alpha^{+\beta}, Z \in \mathfrak{n}_\alpha^{\pm\gamma}$.

Let $\alpha, \beta \in \Lambda_{a_i}$ such that $\alpha > \beta > 0$. Also, let $(\gamma, \delta), (\epsilon, \zeta) \in \mathcal{L}$ and suppose $(\gamma, \delta) \geq (\epsilon, \zeta)$.

(113) If $\alpha \neq \gamma, \delta, \epsilon, \zeta$ and $\beta \neq \gamma, \delta, \epsilon, \zeta$, then $(\nabla_X R)(Y, Z, W) = 0$ for any $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\gamma^{+\delta}$ and $Z \in \mathfrak{n}_\epsilon^{\pm\zeta}$.

(114) If $\gamma \neq \alpha, \beta, \epsilon, \zeta$ and $\delta \neq \alpha, \beta, \epsilon, \zeta$, then $(\nabla_X R)(Y, Z, W) = 0$ for any $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\gamma^{+\delta}$ and $Z \in \mathfrak{n}_\epsilon^{\pm\zeta}$.

(115) If $\epsilon \neq \alpha, \beta, \gamma, \delta$ and $f \neq \alpha, \beta, \gamma, \delta$, then $(\nabla_X R)(Y, Z, W) = 0$ for any $X \in \mathfrak{n}_\alpha^{+\beta}, Y \in \mathfrak{n}_\gamma^{+\delta}$ and $Z \in \mathfrak{n}_\epsilon^{\pm\zeta}$.

Now, we are going to investigate each case as follows.

(1) It follows from Bianchi's second identity combined with Claim 7.6 that

$$(\nabla_X R)(A, B, W) = -(\nabla_A R)(B, X, W) - (\nabla_B R)(X, A, W) = 0$$

for any $X, W \in \mathfrak{g}$.

(2) Let $X \in \mathfrak{n}_\alpha^\beta$ and $Z \in \mathfrak{n}_\gamma^{\pm\delta}$. Then it is easy to see that $\nabla_X Z = 0$ and $[X, Z] = 0$. This together with Remarks 6.2, 6.3 and Claim 7.5 implies that

$$\begin{aligned}
&(\nabla_X R)(A, Z, W) \\
&\quad = \nabla_X (R(A, Z)W) - R(\nabla_X A, Z)W - R(A, \nabla_X Z)W - R(A, Z)\nabla_X W \\
&\quad = -\frac{1}{2}\lambda_\gamma^{\pm\delta}(A)\nabla_X \nabla_Z W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X, Z)W + \frac{1}{2}\lambda_\gamma^{\pm\delta}(A)\nabla_Z \nabla_X W \\
&\quad = -\frac{1}{2}\lambda_\gamma^{\pm\delta}(A)R(X, Z)W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X, Z)W \\
&\quad = \left(\frac{1}{2}\lambda_\gamma^{\pm\delta}(A) - \frac{1}{2}\lambda_\alpha^{+\beta}(A)\right)\nabla_{\nabla_X Z} W
\end{aligned}$$

$$= 0.$$

We now note that in the cases of (3), (6), (8), (9), (14), (16), (18) and (19), we also have $\nabla_X Z = 0$ and $[X, Z] = 0$. Hence in these cases the proof goes in a quite similar fashion to that of (2).

(4) Let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Z \in \mathfrak{n}_\beta^{\pm\gamma}$. It follows from Remark 6.3 and Claim 7.5 with $[X, Z] = 0$ that

$$\begin{aligned}
&(\nabla_X R)(A, Z, W) \\
&= \nabla_X(R(A, Z)W) - R(\nabla_X A, Z)W - R(A, \nabla_X Z)W - R(A, Z)\nabla_X W \\
&= -\frac{1}{2}\lambda_\beta^{\pm\gamma}(A)\nabla_X\nabla_Z W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X, Z)W - \frac{1}{2}R(A, [JX, JZ])W \\
&\quad + \frac{1}{2}\lambda_\beta^{\pm\gamma}(A)\nabla_Z\nabla_X W \\
&= -\frac{1}{2}\lambda_\beta^{\pm\gamma}(A)R(X, Z)W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X, Z)W + \frac{1}{4}\lambda_\alpha^{\mp\gamma}(A)\nabla_{[JX,JZ]}W \\
&= \frac{1}{2}\lambda_\alpha^{\mp\gamma}(A)R(X, Z)W + \frac{1}{4}\lambda_\alpha^{\mp\gamma}(A)\nabla_{[JX,JZ]}W \\
&= -\frac{1}{2}\lambda_\alpha^{\mp\gamma}(A)\nabla_{\nabla_X Z}W + \frac{1}{2}\lambda_\alpha^{\mp\gamma}(A)\nabla_{\nabla_X Z}W \\
&= 0.
\end{aligned}$$

Regarding the cases of (5), (10), (12) and (13), we still have $[X, Z] = 0$, so that the proof of these cases goes similarly to that of (4). Hence we omit the proof.

(7) Let $X \in \mathfrak{n}_\alpha^{+\gamma}$. We first deal with the case where $Z \in \mathfrak{n}_\beta^{+\gamma}$. Since we have $[X, Z] = 0$, the proof of this case is similar to that of (4).

Next, let $Z \in \mathfrak{n}_\beta^{-\gamma}$. It follows from Remarks 6.2 and 6.3 together with Claim 7.5 that

$$\begin{aligned}
&(\nabla_X R)(A, Z, W) \\
&= \nabla_X(R(A, Z)W) - R(\nabla_X A, Z)W - R(A, \nabla_X Z)W - R(A, Z)\nabla_X W \\
&= -\frac{1}{2}\lambda_\beta^{-\gamma}(A)\nabla_X\nabla_Z W + \frac{1}{2}\lambda_\alpha^{+\gamma}R(X, Z)W - \frac{1}{2}R(A, [X, Z])W \\
&\quad + \frac{1}{2}\lambda_\beta^{-\gamma}(A)\nabla_Z\nabla_X W \\
&= -\frac{1}{2}\lambda_\beta^{-\gamma}(A)R(X, Z)W - \frac{1}{2}\lambda_\beta^{-\gamma}(A)\nabla_{[X,Z]}W + \frac{1}{2}\lambda_\alpha^{+\gamma}R(X, Z)W \\
&\quad + \frac{1}{4}\lambda_\alpha^{+\beta}\nabla_{[X,Z]}W
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\lambda_\alpha - \lambda_\beta + 2\lambda_\gamma\right)(A) R(X,Z)W + \frac{1}{4}\left(\lambda_\alpha - \lambda_\beta + 2\lambda_\gamma\right)(A)\nabla_{[X,Z]}W \\
&= -\frac{1}{2}\left(\lambda_\alpha - \lambda_\beta + 2\lambda_\gamma\right)(A)\nabla_{\nabla_X Z}W + \frac{1}{2}\left(\lambda_\alpha - \lambda_\beta + 2\lambda_\gamma\right)(A)\nabla_{\nabla_X Z}W \\
&= 0.
\end{aligned}
$$

The proof of (15) and (17) are similar to (7), since $\nabla_X Z \neq 0$ and $[X, Z + JZ] \neq 0$ in these cases. Hence we omit the proof.

(11) Let $X \in \mathfrak{n}_\alpha^{+\beta}$. We first prove the case where $Z \in \mathfrak{n}_\alpha^{+\beta}$. It follows from Remarks 6.2 and 6.3 that

$$
\begin{aligned}
&(\nabla_X R)(A, Z, W) \\
&\quad = \nabla_X(R(A,Z)W) - R(\nabla_X A, Z)W - R(A, \nabla_X Z)W - R(A,Z)\nabla_X W \\
&\quad = -\frac{1}{2}\lambda_\alpha^{+\beta}(A)\nabla_X\nabla_Z W + \frac{1}{2}\lambda_\alpha^{+\beta} R(X,Z)W \\
&\qquad - R\left(A, \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, Y\rangle H_\alpha + \frac{\lambda_\gamma(H_\beta)}{2|H_\beta|^2}\langle X, Y\rangle H_\beta\right)W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)\nabla_Z\nabla_X W \\
&\quad = -\frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X,Z)W + \frac{1}{2}\lambda_\alpha^{+\beta}R(X,Z)W \\
&\quad = 0.
\end{aligned}
$$

Next, we prove the case where $Z \in \mathfrak{n}_\alpha^{-\beta}$. It follows from Remarks 6.2 and 6.3 together with Claim 7.5 that

$$
\begin{aligned}
&(\nabla_X R)(A, Z, W) \\
&\quad = \nabla_X(R(A,Z)W) - R(\nabla_X A, Z)W - R(A, \nabla_X Z)W - R(A,Z)\nabla_X W \\
&\quad = -\frac{1}{2}\lambda_\alpha^{-\beta}(A)\nabla_X\nabla_Z W + \frac{1}{2}\lambda_\alpha^{+\beta} R(X,Z)W \\
&\qquad - R\left(A, \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle JX, Y\rangle JH_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle JX, Y\rangle JH_\beta\right)W \\
&\qquad + \frac{1}{2}\lambda_\alpha^{-\beta}(A)\nabla_Z\nabla_X W \\
&\quad = -\frac{1}{2}\lambda_\alpha^{-\beta}(A)R(X,Z)W - \frac{1}{2}\lambda_\alpha^{-\beta}(A)\nabla_{[X,Z]}W + \frac{1}{2}\lambda_\alpha^{+\beta}R(X,Z)W \\
&\qquad + \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\lambda_\alpha(A)\langle JX, Y\rangle\nabla_{JH_\alpha}W + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\lambda_\beta(A)\langle JX, Y\rangle\nabla_{JH_\beta}W \\
&\quad = -\lambda_\beta(A)\nabla_{\nabla_X Z}W - (\lambda_\alpha - \lambda_\beta)(A)\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle JX, Z\rangle\nabla_{JH_\alpha}W
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\lambda_\alpha(A)\langle JX, Y\rangle \nabla_{JH_\alpha} W + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\lambda_\beta(A)\langle JX, Y\rangle \nabla_{JH_\beta} W \\
= &- \lambda_\beta(A)\left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle JX, Z\rangle \nabla_{JH_\alpha} W + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle JX, Z\rangle \nabla_{JH_\beta} W\right) \\
&+ \lambda_\beta(A)\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle JX, Z\rangle \nabla_{JH_\alpha} W + \frac{\lambda_\gamma(H_\beta)}{2|H_\beta|^2}\lambda_\beta(A)\langle JX, Y\rangle \nabla_{JH_\beta} W \\
= &\ 0.
\end{aligned}
$$

(20) It follows from Remarks 6.2 and 6.3 with $X \in \mathfrak{n}_\alpha^{+\beta}$ that

$$
\begin{aligned}
&(\nabla_X R)(A, JH_\beta, W) \\
&= \nabla_X(R(A, JH_\beta)W) - R(\nabla_X A, JH_\beta)W \\
&\quad - R(A, \nabla_X JH_\beta)W - R(A, JH_\beta)\nabla_X W \\
&= - \lambda_\beta(A)\nabla_X \nabla_{JH_\beta} W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X, JH_\beta)W \\
&\quad + \frac{1}{2}\lambda_\beta(H_\beta)R(A, JX)W + \lambda_\beta(A)\nabla_{JH_\beta}\nabla_X W \\
&= - \lambda_\beta(A)R(X, JH_\beta)W + \frac{1}{2}\lambda_\alpha^{+\beta}(A)R(X, JH_\beta)W - \frac{1}{4}\lambda_\beta(H_\beta)\lambda_\alpha^{-\beta}(A)\nabla_{JX} W \\
&= \frac{1}{2}\lambda_\alpha^{-\beta}(A)R(H_\beta, JX)W - \frac{1}{4}\lambda_\beta(H_\beta)\lambda_a^{-\beta}(A)\nabla_{JX} W \\
&= \frac{1}{4}\lambda_\alpha^{-\beta}(A)\lambda_\beta(H_\beta)\nabla_{JX} W - \frac{1}{4}\lambda_\beta(H_\beta)\lambda_\alpha^{-\beta}(A)\nabla_{JX} W \\
&= 0.
\end{aligned}
$$

The proof of (21) and (22) can be done in a quite similar manner to that of (20).

To prove (23) and (24), let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Y \in \mathfrak{n}_\beta^{+\gamma}$. If $Z \in \mathfrak{n}_\gamma$, then it is easy to see that $\nabla_X Z = 0$ and $[X, \nabla_Y Z] = 0$. Then, making use of Remark 6.2 combined with Claim 7.5, we have

$$
\begin{aligned}
&(\nabla_X R)(Y, Z, W) \\
&\quad = \nabla_X(R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W \\
&\quad = -\nabla_X \nabla_{\nabla_Y Z} W - \frac{1}{2}R([JX, JY], Z)W + \nabla_{\nabla_Y Z}\nabla X W \\
&\quad = -\frac{1}{2}\nabla_X \nabla_{[JY, JZ]} W - \frac{1}{2}R([JX, JY], Z)W + \frac{1}{2}\nabla_{[JY, JZ]}\nabla_X W \\
&\quad = -\frac{1}{2}R(X, [JY, JZ])W - \frac{1}{2}R(J[JX, JY], JZ)W \\
&\quad = \frac{1}{2}\nabla_{\nabla_X [JY, JZ]} W + \frac{1}{2}\nabla_{\nabla_{J[JX, JY]} JZ} W
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\nabla_{[JX,J[JY,JZ]]}W + \frac{1}{4}\nabla_{[J^2[JX,JY],J^2Z]}W \\
&= -\frac{1}{4}\nabla_{[JX,[JY,Z]]}W + \frac{1}{4}\nabla_{[[JX,JY],Z]}W \\
&= -\frac{1}{4}\nabla_{[JX,[JY,Z]]}W + \frac{1}{4}\nabla_{[JX,[JY,Z]]}W \\
&= 0.
\end{aligned}
$$

We now note that in the cases of (24), (25), (29), (35), (37), (41), (42), (43), (48), (62), (64), (65), (67), (73), (77), (90), (92), (105), (107) and (109), we also have $\nabla_X Z = 0$ and $[X, \nabla_Y Z] = 0$. Hence the proof of these cases goes similarly to that of (23).

(26) Let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Y \in \mathfrak{n}_\beta^{+\gamma}$. It follows from Remark 6.2 with $Z \in \mathfrak{n}_\beta^{\pm\delta}$ that $[X, \nabla_Y Z] = 0$ and $\nabla_X \nabla_Y Z = 0$.

In the case where $Z \in \mathfrak{n}_\beta^{+\delta}$, it follows from Claim 7.5 together with these formulas that

$$
\begin{aligned}
&(\nabla_X R)(Y, Z, W) \\
&\quad = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W \\
&\quad = -\nabla_X \nabla_{\nabla_Y Z} W - \frac{1}{2}R([JX,JY],Z)W - \frac{1}{2}R(Y,[JX,JZ])W + \nabla_{\nabla_Y Z}\nabla_X W \\
&\quad = -R(X, \nabla_Y Z)W - \frac{1}{2}R([JX,JY],Z)W - \frac{1}{2}R(Y,[JX,JZ])W \\
&\quad = \nabla_{\nabla_X \nabla_Y Z} W - \frac{1}{2}R(J[JX,JY],JZ)W + \frac{1}{2}R(J[JX,JZ],JY)W \\
&\quad = \frac{1}{2}\nabla_{\nabla_{J[JX,JY]}JZ} W - \frac{1}{2}\nabla_{\nabla_{J[JX,JZ]}JY} W \\
&\quad = 0.
\end{aligned}
$$

The proof for the case of $Z \in \mathfrak{n}_\beta^{-\delta}$ is quite similar to that for $Z \in \mathfrak{n}_\beta^{+\delta}$. Hence we omit the detail.

In the cases of (28), (32), (33), (34), (50), (52), (68), (70), (72), (81), (82), (83), (85), (88), (99), (101), (102) and (104), we also have $[X, \nabla_Y Z] = 0$ and $\nabla_X \nabla_Y Z = 0$, so that the proof is quite similar to that of (26).

(27) Let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Y \in \mathfrak{n}_\beta^{+\gamma}$. In the case where $Z \in \mathfrak{n}_\beta^{+\gamma}$, by a direct calculation using Remark 6.2 and Claim 7.5, we see that $(\nabla_X R)(Y, Z, W) = 0$ for $W \in \mathfrak{n}$, except the case where W belongs to $\mathfrak{n}_\beta^{+\gamma}$ or $\mathfrak{n}_\alpha^{+\gamma}$. To prove the case where $W \in \mathfrak{n}_\beta^{+\gamma}$, let $\{(E_\alpha^{-\gamma})_1, \ldots, (E_\alpha^{-\gamma})_{s_k}\}$ be an orthonormal basis of $\mathfrak{n}_\alpha^{-\gamma}$ with respect to $\langle\ ,\ \rangle$. It follows

from Remarks 6.2 and 6.5 together with Claim 7.5 that

$$
\begin{aligned}
&(\nabla_X R)(Y,Z,W)\\
&= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W\\
&= \nabla_X\left(-\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\left(\langle Z,W\rangle Y - \langle Y,W\rangle Z\right)\right)\\
&\quad - \frac{1}{2}R([JX,JY],Z)W - \frac{1}{2}R(Y,[JX,JZ])W - \frac{1}{2}R(Y,Z)[JX,JW]\\
&= -\frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\left(\langle Z,W\rangle[JX,JY] - \langle Y,W\rangle[JX,JZ]\right)\\
&\quad + \frac{1}{2}R([JX,Y],JZ)W - \frac{1}{2}R([JX,Z],JY)W - \frac{1}{2}JR(Y,Z)[JX,W]\\
&= -\frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\left(\langle Z,W\rangle[JX,JY] - \langle Y,W\rangle[JX,JZ]\right)\\
&\quad - \frac{1}{2}\nabla_{\nabla_{[JX,Y]}JZ}W + \frac{1}{2}\nabla_{\nabla_{[JX,Z]}JY}W\\
&\quad - \frac{1}{8}J[J[[JX,W],JZ],Y] + \frac{1}{8}J[J[[JX,W],JY],Z]\\
&= -\frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\left(\langle Z,W\rangle[JX,JY] - \langle Y,W\rangle[JX,JZ]\right)\\
&\quad - \frac{1}{4}\nabla_{[[JX,Y],JZ]}W + \frac{1}{4}\nabla_{[[JX,Z],JY]}W\\
&\quad - \frac{1}{8}(J[J[[JX,JZ],W],Y] + J[J[JX,[W,JZ]],Y])\\
&\quad + \frac{1}{8}(J[J[[JX,JY],W],Z] + J[J[JX,[W,JY]],Z])\\
&= -\frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\left(\langle Z,W\rangle J[JX,Y] - \langle Y,W\rangle J[JX,Z]\right)\\
&\quad - \frac{1}{8}[J[[JX,Y],JZ],JW] + \frac{1}{8}[J[[JX,Z],JY],JW]\\
&\quad - \frac{1}{8}J[J[[JX,JZ],W],Y] + \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JW,JZ\rangle J[JX,Y]\\
&\quad + \frac{1}{8}J[J[[JX,JY],W],Z] - \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JW,JY\rangle J[JX,Z]\\
&= -\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\left(\langle Z,W\rangle J[JX,Y] - \langle Y,W\rangle J[JX,Z]\right)\\
&\quad - \frac{1}{8}\sum_{p=1}^{s_k}\langle [J[[JX,Y],JZ],JW], (E_\alpha^{-\gamma})_p\rangle (E_\alpha^{-\gamma})_p
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle [J[[JX,Z],JY],JW],(E_\alpha^{-\gamma})_p\rangle (E_\alpha^{-\gamma})_p\\
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle [J[[JX,Z],JW],JY],(E_\alpha^{-\gamma})_p\rangle (E_\alpha^{-\gamma})_p\\
&-\frac{1}{8}\sum_{p=1}^{s_k}\langle [J[[JX,Y],JW],JZ],(E_\alpha^{-\gamma})_p\rangle (E_\alpha^{-\gamma})_p\\
=&-\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\left(\langle Z,W\rangle J[JX,Y]-\langle Y,W\rangle J[JX,Z]\right)\\
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle [JW,J(E_\alpha^{-\gamma})_p],[[JX,Y],JZ]\rangle (E_\alpha^{-\gamma})_p\\
&-\frac{1}{8}\sum_{p=1}^{s_k}\langle [JW,J(E_\alpha^{-\gamma})_p],[[JX,Z],JY]\rangle (E_\alpha^{-\gamma})_p\\
&-\frac{1}{8}\sum_{p=1}^{s_k}\langle [JY,J(E_\alpha^{-\gamma})_p],[[JX,Z],JW]\rangle (E_\alpha^{-\gamma})_p\\
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle [JZ,J(E_\alpha^{-\gamma})_p],[[JX,Y],JW]\rangle (E_\alpha^{-\gamma})_p\\
=&-\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\left(\langle Z,W\rangle J[JX,Y]-\langle Y,W\rangle J[JX,Z]\right)\\
&-\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\sum_{p=1}^{s_k}\langle JW,JZ\rangle\langle J(E_\alpha^{-\gamma})_p,[JX,Y]\rangle (E_\alpha^{-\gamma})_p\\
&+\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\sum_{p=1}^{s_k}\langle JW,JY\rangle\langle J(E_\alpha^{-\gamma})_p,[JX,Z]\rangle (E_\alpha^{-\gamma})_p\\
=&-\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\left(\langle Z,W\rangle J[JX,Y]-\langle Y,W\rangle J[JX,Z]\right)\\
&+\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\langle W,Z\rangle J[JX,Y]-\frac{\lambda_\alpha(H_\alpha)^2}{8|H_\alpha|^2}\langle W,Y\rangle J[JX,Z]\\
=&\,0.
\end{aligned}
$$

Next, to prove the case where $W\in\mathfrak{n}_\alpha^{+\gamma}$, let $\{(E_\beta^{-\gamma})_1,\ldots,(E_\beta^{-\gamma})_{s_k}\}$ be an orthonormal basis of $\mathfrak{n}_\beta^{-\gamma}$. By making use of Remarks 6.2 and 6.5 together with Claim 7.5, we have

$$(\nabla_X R)(Y,Z,W)$$

$$
\begin{aligned}
&= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W \\
&= \nabla_X\left(\frac{1}{4}[J[W,JZ],Y] - \frac{1}{4}[J[W,JY],Z]\right) \\
&\quad - \frac{1}{2}R([JX,JY],Z)W - \frac{1}{2}R(Y,[JX,JZ])W \\
&\quad - R(Y,Z)\left(-\frac{1}{2}\sum_{p=1}^{s_k}\langle X,[W,(E_\beta^{-\gamma})_p]\rangle (E_\beta^{-\gamma})_p\right) \\
&= -\frac{1}{8}\sum_{p=1}^{s_k}\langle X,[[J[W,JZ],Y],(E_\beta^{-\gamma})_p]\rangle (E_\beta^{-\gamma})_p \\
&\quad + \frac{1}{8}\sum_{p=1}^{s_k}\langle X,[[J[W,JY],Z],(E_\beta^{-\gamma})_p]\rangle (E_\beta^{-\gamma})_p \\
&\quad - \frac{1}{2}R(J[JX,JY],JZ)W + \frac{1}{2}R(J[JX,JZ],JY)W \\
&\quad - \frac{1}{2}\sum_{p=1}^{s_k}\langle X,[W,(E_\beta^{-\gamma})_p]\rangle JR(Y,Z)J(E_\beta^{-\gamma})_p \\
&= -\frac{1}{8}\sum_{p=1}^{s_k}\langle X,[[J[W,JZ],(E_\beta^{-\gamma})_p],Y] + [J[W,JZ],[Y,(E_\beta^{-\gamma})_p]]\rangle (E_\beta^{-\gamma})_p \\
&\quad + \frac{1}{8}\sum_{p=1}^{s_k}\langle X,[[J[W,JY],(E_\beta^{-\gamma})_p],Z] + [J[W,JY],[Z,(E_\beta^{-\gamma})_p]]\rangle (E_\beta^{-\gamma})_p \\
&\quad + \frac{1}{2}\nabla_{\nabla_{J[JX,JY]}JZ}W - \frac{1}{2}\nabla_{\nabla_{J[JX,JZ]}JY}W \\
&\quad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\sum_{p=1}^{s_k}\langle X,[W,(E_\beta^{-\gamma})_p]\rangle J\left(\langle Z,J(E_\beta^{-\gamma})_p\rangle Y - \langle Y,J(E_\beta^{-\gamma})_p\rangle Z\right) \\
&= -\frac{1}{8}\sum_{p=1}^{s_k}\langle J[J[W,JZ],(E_\beta^{-\gamma})_p],[Y,JX]\rangle (E_\beta^{-\gamma})_p \\
&\quad - \frac{1}{8}\sum_{p=1}^{s_k}\frac{\lambda_\beta(H_\beta)^2}{|H_\beta|^2}\langle JY,(E_\beta^{-\gamma})_p\rangle\langle X,J^2[W,JZ]\rangle (E_\beta^{-\gamma})_p \\
&\quad + \frac{1}{8}\sum_{p=1}^{s_k}\langle J[J[W,JY],(E_\beta^{-\gamma})_p],[Z,JX]\rangle (E_\beta^{-\gamma})_p \\
&\quad + \frac{1}{8}\sum_{p=1}^{s_k}\frac{\lambda_\beta(H_\beta)^2}{|H_\beta|^2}\langle JZ,(E_\beta^{-\gamma})_p\rangle\langle X,J^2[W,JY]\rangle (E_\beta^{-\gamma})_p
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\nabla_{[J[JX,JY],JZ]}W-\frac{1}{4}\nabla_{[J[JX,JZ],JY]}W \\
&-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X,[W,JZ]\rangle JY+\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X,[W,JY]\rangle JZ \\
=&-\frac{1}{8}\sum_{p=1}^{s_k}\langle[(E_\beta^{-\gamma})_p,[Y,JX]],J^2[W,JZ]\rangle(E_\beta^{-\gamma})_p+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle X,[W,JZ]\rangle JY \\
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle[(E_\beta^{-\gamma})_p,[Z,JX]],J^2[W,JY]\rangle(E_\beta^{-\gamma})_p-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle X,[W,JY]\rangle JZ \\
&-\frac{1}{8}\sum_{p=1}^{s_k}\langle[J[JX,JY],JZ],[W,(E_\beta^{-\gamma})_p]\rangle(E_\beta^{-\gamma})_p \\
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle[J[JX,JZ],JY],[W,(E_\beta^{-\gamma})_p]\rangle(E_\beta^{-\gamma})_p \\
&-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X,[W,JZ]\rangle JY+\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X,[W,JY]\rangle JZ \\
=&\ \frac{1}{8}\sum_{p=1}^{s_k}\langle[[JX,Y],(E_\beta^{-\gamma})_p],[W,JZ]\rangle(E_\beta^{-\gamma})_p \\
&-\frac{1}{8}\sum_{p=1}^{s_k}\langle[[JX,Z],(E_\beta^{-\gamma})_p],[W,JY]\rangle(E_\beta^{-\gamma})_p \\
&+\frac{1}{8}\sum_{p=1}^{s_k}\langle[[JX,Y],JZ],[W,(E_\beta^{-\gamma})_p]\rangle(E_\beta^{-\gamma})_p \\
&-\frac{1}{8}\sum_{p=1}^{s_k}\langle[[JX,Z],JY],[W,(E_\beta^{-\gamma})_p]\rangle(E_\beta^{-\gamma})_p \\
&-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle X,[W,JZ]\rangle JY+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle X,[W,JY]\rangle JZ \\
=&\ \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\sum_{p=1}^{s_k}\langle[JX,Y],W\rangle\langle(E_\beta^{-\gamma})_p,JZ\rangle(E_\beta^{-\gamma})_p \\
&-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\sum_{p=1}^{s_k}\langle[JX,Z],W\rangle\langle(E_\beta^{-\gamma})_p,JY\rangle(E_\beta^{-\gamma})_p \\
&-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JW,[JZ,JX]\rangle JY+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JW,[JY,JX]\rangle JZ \\
=&\ \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle[JX,Y],W\rangle JZ-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle[JX,Z],W\rangle JY
\end{aligned}
$$

$$+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle W,[JX,Z]\rangle JY-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle W,[JX,Y]\rangle JZ$$
$$=0.$$

In the case where $Z\in\mathfrak{n}_\beta^{-\gamma}$, a direct calculation using Claim 7.5 and Remark 6.2 shows that $(\nabla_X R)(Y,Z,W)=0$ for any $W\in\mathfrak{g}$. This completes the proof of (27).

By a similar argument, we can also prove the cases of (36), (51), (91) and (108).

(30) Let $X\in\mathfrak{n}_\alpha^{+\beta}$ and $Y,Z\in\mathfrak{n}_\beta^0$. By a direct calculation making use of Remark 6.2 and Claim 7.5, we have $(\nabla_X R)(Y,Z,W)=0$ for $W\in\mathfrak{g}$, except the case where W belongs either to $\mathfrak{n}_\beta^0$, or $\mathfrak{n}_\alpha^{+\beta}$ or $\mathfrak{n}_\alpha^0$.

Now, to prove the case where $W\in\mathfrak{n}_\beta^0$, let $\{(E_\alpha^0)_1,\ldots,(E_\alpha^0)_{t_k}\}$ be an orthonormal basis of $\mathfrak{n}_\alpha^0$. It follows from Remarks 6.2 and 6.5 together with Claim 7.5 that

$$\begin{aligned}
&(\nabla_X R)(Y,Z,W)\\
&=\nabla_X(R(Y,Z)W)-R(\nabla_X Y,Z)W-R(Y,\nabla_X Z)W-R(Y,Z)\nabla_X W\\
&=\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\nabla_X\,(-\langle Z,W\rangle Y-\langle JZ,W\rangle JY\\
&\qquad\qquad+\langle Y,W\rangle Z+\langle JY,W\rangle JZ+2\langle JY,Z\rangle JW)\\
&\quad-\frac{1}{2}R([JX,JY],Z)W-\frac{1}{2}R(Y,[JX,JZ])W-\frac{1}{2}R(Y,Z)[JX,JW]\\
&=-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Z,W\rangle[JX,JY]+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JZ,W\rangle[JX,Y]\\
&\quad+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Y,W\rangle[JX,JZ]-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JY,W\rangle[JX,Z]\\
&\quad-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY,Z\rangle[JX,W]+\frac{1}{2}\nabla_{\nabla_{[JX,JY]}Z}W-\frac{1}{2}\nabla_{\nabla_{[JX,JZ]}Y}W\\
&\quad-\frac{1}{8}([J[Z,[JX,JW]],JY]-[Y,J[[JX,JW],JZ]]\\
&\qquad\quad-[J[Y,[JX,JW]],JZ]+[Z,J[[JX,JW],JY]])\\
&=-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Z,W\rangle[JX,JY]+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JZ,W\rangle[JX,Y]\\
&\quad+\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Y,W\rangle[JX,JZ]-\frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JY,W\rangle[JX,Z]\\
&\quad-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY,Z\rangle[JX,W]\\
&\quad+\frac{1}{4}\nabla_{[[JX,JY],Z]-J[[JX,JY],JZ]}W-\frac{1}{4}\nabla_{[[JX,JZ],Y]-J[[JX,JZ],JY]}W
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8}([J[Z,[JX,JW]],JY] - [Y,J[[JX,JW],JZ]] \\
&\qquad - [J[Y,[JX,JW]],JZ] + [Z,J[[JX,JW],JY]]) \\
= &- \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Z,W\rangle[JX,JY] + \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JZ,W\rangle[JX,Y] \\
&+ \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Y,W\rangle[JX,JZ] - \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JY,W\rangle[JX,Z] \\
&- \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY,Z\rangle[JX,W] \\
&+ \frac{1}{8}[J[[JX,JY],Z],JW] - \frac{1}{8}[J[[JX,JY],JZ],W] \\
&- \frac{1}{8}[J[[JX,JZ],Y],JW] + \frac{1}{8}[J[[JX,JZ],JY],W] \\
&- \frac{1}{8}([J[Z,[JX,JW]],JY] - [Y,J[[JX,JW],JZ]] \\
&\qquad - [J[Y,[JX,JW]],JZ] + [Z,J[[JX,JW],JY]]) \\
= &- \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Z,W\rangle[JX,JY] + \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JZ,W\rangle[JX,Y] \\
&+ \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Y,W\rangle[JX,JZ] - \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JY,W\rangle[JX,Z] \\
&- \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY,Z\rangle[JX,W] \\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [J[[JX,JY],Z],JW],(E_\alpha^0)_p\right\rangle (E_\alpha^0)_p \\
&- \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [J[[JX,JY],JZ],W],(E_\alpha^0)_p\right\rangle (E_\alpha^0)_p \\
&- \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [J[[JX,JZ],Y],JW],(E_\alpha^0)_p\right\rangle (E_\alpha^0)_p \\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [J[[JX,JZ],JY],W],(E_\alpha^0)_p\right\rangle (E_\alpha^0)_p \\
&- \frac{1}{8}[J[[Z,JX],JW] + J[JX,[Z,JW]],JY] \\
&+ \frac{1}{8}[Y,J[[JX,JZ],JW] + J[JX,[JW,JZ]]] \\
&+ \frac{1}{8}[J[[Y,JX],JW] + J[JX,[Y,JW]],JZ]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{8}[Z, J[[JX, JY], JW] + J[JX, [JW, JY]]]\\
= &- \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Z, W\rangle[JX, JY] + \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JZ, W\rangle[JX, Y]\\
&+ \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Y, W\rangle[JX, JZ] - \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle JY, W\rangle[JX, Z]\\
&- \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle[JX, W]\\
&- \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [JW, J(E_\alpha^0)_p], [[JX, JY], Z]\right\rangle (E_\alpha^0)_p\\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [[JX, JY], JZ]\right\rangle (E_\alpha^0)_p\\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [JW, J(E_\alpha^0)_p], [[JX, JZ], Y]\right\rangle (E_\alpha^0)_p\\
&- \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [[JX, JZ], JY]\right\rangle (E_\alpha^0)_p\\
&- \frac{1}{8}\sum_{p=1}^{t_k}\langle [J[[Z, JX], JW], JY], (E_\alpha^0)_p\rangle (E_\alpha^0)_p + \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Z, W\rangle[JX, JY]\\
&- \frac{1}{8}\sum_{p=1}^{t_k}\langle [J[[JX, JZ], JW], Y], (E_\alpha^0)_p\rangle (E_\alpha^0)_p + \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle W, JZ\rangle[Y, JX]\\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\langle [J[[Y, JX], JW], JZ], (E_\alpha^0)_p\rangle (E_\alpha^0)_p - \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle Y, W\rangle[JX, JZ]\\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\langle [J[[JX, JY], JW], Z], (E_\alpha^0)_p\rangle (E_\alpha^0)_p - \frac{\lambda_\beta(H_\beta)^2}{8|H_\beta|^2}\langle W, JY\rangle[Z, JX]\\
= &- \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle[JX, W]\\
&- \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [JW, J(E_\alpha^0)_p], [[JX, JY], Z]\right\rangle (E_\alpha^0)_p\\
&+ \frac{1}{8}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [[JX, JY], JZ]\right\rangle (E_\alpha^0)_p
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\sum_{p=1}^{t_k}\left\langle [JW, J(E_\alpha^0)_p], [[JX, JZ], Y]\right\rangle (E_\alpha^0)_p \\
&-\frac{1}{8}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [[JX, JZ], JY]\right\rangle (E_\alpha^0)_p \\
&+\frac{1}{8}\sum_{p=1}^{t_k}\langle [JY, J(E_\alpha^0)_p], [[JX, JZ], W]\rangle (E_\alpha^0)_p \\
&+\frac{1}{8}\sum_{p=1}^{t_k}\langle [Y, J(E_\alpha^0)_p], [[JX, JZ], JW]]\rangle (E_\alpha^0)_p \\
&-\frac{1}{8}\sum_{p=1}^{t_k}\langle [JZ, J(E_\alpha^0)_p], [[JX, JY], W]\rangle (E_\alpha^0)_p \\
&-\frac{1}{8}\sum_{p=1}^{t_k}\langle [Z, J(E_\alpha^0)_p], [[JX, JY], JW]]\rangle (E_\alpha^0)_p \\
=&-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle [JX, W] \\
&+\frac{1}{4}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [[JX, JY], JZ]\right\rangle (E_\alpha^0)_p \\
&-\frac{1}{4}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [[JX, JZ], JY]\right\rangle (E_\alpha^0)_p \\
=&-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle [JX, W] + \frac{1}{4}\sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], [JX, [JY, JZ]]\right\rangle (E_\alpha^0)_p \\
=&-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle [JX, W] - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle \sum_{p=1}^{t_k}\left\langle [W, J(E_\alpha^0)_p], X\right\rangle (E_\alpha^0)_p \\
=&-\frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle [JX, W] + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle \sum_{p=1}^{t_k}\left\langle [JX, W], (E_\alpha^0)_p\right\rangle (E_\alpha^0)_p \\
=&\ 0.
\end{aligned}
$$

On the other hand, if $W \in \mathfrak{n}_\alpha^{+\beta}$, then Remark 6.2 and Claim 7.5 together with Remark 6.5 implies that

$$
\begin{aligned}
&(\nabla_X R)(Y, Z, W) \\
&\quad = \nabla_X(R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W
\end{aligned}
$$

$$
\begin{aligned}
&= \nabla_X \left(\frac{1}{2}[[Y, JW], JZ] - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle Y, Z\rangle W + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle JW \right) \\
&\quad - \frac{1}{2}R([JX, JY], Z)W - \frac{1}{2}R(Y, [JX, JZ])W \\
&\quad - R(Y, Z)\left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, W\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, W\rangle H_\beta \right) \\
&= \frac{\lambda_\alpha(H_\alpha)}{4|H_\alpha|^2}\langle X, [[Y, JW], JZ]\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{4|H_\beta|^2}\langle X, [[Y, JW], JZ]\rangle H_\beta \\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle Y, Z\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, W\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, W\rangle H_\beta \right) \\
&\quad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, W\rangle JH_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, W\rangle JH_\beta \right) \\
&\quad + \frac{1}{4}\nabla_{[[JX,JY],Z]-J[[JX,JY],JZ]}W - \frac{1}{4}\nabla_{[[JX,JZ],Y]-J[[JX,JZ],JY]}W \\
&\quad + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, W\rangle \left(R(Z, H_\beta)Y + R(H_\beta, Y)Z \right) \\
&= \frac{\lambda_\alpha(H_\alpha)}{4|H_\alpha|^2}\langle X, [[Y, JW], JZ]\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{4|H_\beta|^2}\langle X, [[Y, JW], JZ]\rangle H_\beta \\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle Y, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}H_\beta \right) \\
&\quad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}JH_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}JH_\beta \right) \\
&\quad + \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[[JX, JY], Z], W\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[[JX, JY], Z], W\rangle H_\beta \\
&\quad + \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[[JX, JY], JZ], W\rangle JH_\alpha - \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[[JX, JY], JZ], W\rangle JH_\beta \\
&\quad - \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[[JX, JZ], Y], W\rangle H_\alpha - \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[[JX, JZ], Y], W\rangle H_\beta \\
&\quad - \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[[JX, JZ], JY], W\rangle JH_\alpha + \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[[JX, JZ], JY], W\rangle JH_\beta \\
&\quad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, W\rangle \nabla_Z Y - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, W\rangle \nabla_Y Z \\
&= \frac{\lambda_\alpha(H_\alpha)}{4|H_\alpha|^2}\langle[JY, JW], [JZ, JX]\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{4|H_\beta|^2}\langle[JY, JW], [JZ, JX]\rangle H_\beta \\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle Y, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}H_\beta \right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}JH_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}JH_\beta\right)\\
&\quad - \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[JZ, JW], [JX, JY]\rangle H_\alpha - \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[JZ, JW], [JX, JY]\rangle H_\beta\\
&\quad + \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[JX, [JY, JZ]], W\rangle JH_\alpha - \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[JX, [JY, JZ]], W\rangle JH_\beta\\
&\quad + \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\langle[JY, JW], [JX, JZ]\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\langle[JY, JW], [JX, JZ]\rangle H_\beta\\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, W\rangle[Y, Z]\\
&= \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}(\langle[JY, JW], [JZ, JX]\rangle + \langle[JZ, JW], [JY, JX]\rangle)H_\alpha\\
&\quad + \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}(\langle[JY, JW], [JZ, JX]\rangle + \langle[JZ, JW], [JY, JX]\rangle)H_\beta\\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle Y, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}H_\beta\right)\\
&\quad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle JY, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}JH_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}JH_\beta\right)\\
&\quad - \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\frac{\lambda_\beta(H_\beta)^2}{|H_\beta|^2}\langle JY, Z\rangle\langle X, W\rangle JH_\alpha + \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\frac{\lambda_\beta(H_\beta)^2}{|H_\beta|^2}\langle JY, Z\rangle\langle X, W\rangle JH_\beta\\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\frac{\lambda_\beta(H_\beta)}{|H_\beta|^2}\langle X, W\rangle\langle JY, Z\rangle JH_\beta\\
&= \frac{\lambda_\alpha(H_\alpha)}{8|H_\alpha|^2}\frac{\lambda_\beta(H_\beta)^2}{|H_\beta|^2}\langle JY, JZ\rangle\langle JW, JX\rangle H_\alpha\\
&\quad + \frac{\lambda_\beta(H_\beta)}{8|H_\beta|^2}\frac{\lambda_\beta(H_\beta)^2}{|H_\beta|^2}\langle JY, JZ\rangle\langle JW, JX\rangle)H_\beta\\
&\quad - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle Y, Z\rangle\langle X, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}H_\beta\right)\\
&= 0.
\end{aligned}
$$

Finally, we remark that the proof for the case of $W \in \mathfrak{n}_\alpha^0$ goes in a similar fashion to that of $W \in \mathfrak{n}_\beta^0$, so that we omit the detail. This completes the proof of (30).

In the same way as we proved (30), we can verify the case of (78).

(31) Let $X \in \mathfrak{n}_\alpha^{+\gamma}$ and $Y \in \mathfrak{n}_\beta^{+\delta}$. It follows from Remark 6.2 that $\nabla_X Y = 0$. We first look at the case where $Z \in \mathfrak{n}_\gamma^{+\delta}$. By making use of Remark 6.2 and Claim 7.5, for

any $W \in \mathfrak{g}$ we have

$$\begin{aligned}
&(\nabla_X R)(Y, Z, W) \\
&= \nabla_X (R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W \\
&= -\nabla_X \nabla_{\nabla_Y Z} W - \frac{1}{2} R(Y, [JX, JZ])W + \nabla_{\nabla_Y Z} \nabla_X W \\
&= \frac{1}{2} \nabla_X \nabla_{J[Y,JZ]} W + \frac{1}{2} R(J[JX, JZ], JY)W - \frac{1}{2} \nabla_{J[Y,JZ]} \nabla_X W \\
&= \frac{1}{2} R(X, J[Y, JZ])W + \frac{1}{2} \nabla_{[X,J[Y,JZ]]} W - \frac{1}{2} \nabla_{\nabla_{J[JX,JZ]} JY} W \\
&= -\frac{1}{2} \nabla_{\nabla_X J[Y,JZ]} W + \frac{1}{2} \nabla_{[X,J[Y,JZ]]} W - \frac{1}{4} \nabla_{[J[JX,JZ],JY]} W \\
&= -\frac{1}{4} \nabla_{[X,J[Y,JZ]]} W + \frac{1}{2} \nabla_{[X,J[Y,JZ]]} W - \frac{1}{4} \nabla_{[[JX,JZ],Y]} W \\
&= -\frac{1}{4} \nabla_{[JX,[Y,JZ]]} W - \frac{1}{4} \nabla_{[JX,[JZ,Y]]} W \\
&= 0.
\end{aligned}$$

Next, we prove the case where $Z \in \mathfrak{n}_\gamma^{-\delta}$. It follows from Remark 6.2 and Claim 7.5 that

$$\begin{aligned}
&(\nabla_X R)(Y, Z, W) \\
&= \nabla_X (R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W \\
&= -\nabla_X \nabla_{\nabla_Y Z} W - \frac{1}{2} R(Y, [JX, JZ])W + \nabla_{\nabla_Y Z} \nabla_X W \\
&= -\frac{1}{2} \nabla_X \nabla_{[Y,Z]} W + \frac{1}{2} R([JX, JZ], Y)W + \frac{1}{2} \nabla_{[Y,Z]} \nabla_X W \\
&= -\frac{1}{2} R(X, [Y, Z])W + \frac{1}{2} R([JX, JZ], Y)W \\
&= \frac{1}{2} \nabla_{\nabla_X [Y,Z]} W - \frac{1}{2} \nabla_{\nabla_{[JX,JZ]} Y} W \\
&= -\frac{1}{4} \nabla_{J[X,J[Y,Z]]} W + \frac{1}{4} \nabla_{J[[JX,JZ],JY]} W \\
&= \frac{1}{4} \nabla_{J[JX,[JY,JZ]]} W + \frac{1}{4} \nabla_{J[JX,[JZ,JY]]} W \\
&= 0,
\end{aligned}$$

which completes the proof of (31).

(38), (39), (40), (79), (80), (93), (94), (95), (97), (98), (110), (111) and (112) are proved in a similar way to the proof of (31). Hence we omit the detail.

(44) Let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Y \in \mathfrak{n}_\alpha^{+\gamma}$. If $Z \in \mathfrak{n}_\beta^{\pm\delta}$, then we have $\nabla_Y Z = 0$, and hence $R(Y,Z) = 0$. Let $\{(E_\beta^{-\gamma})_1, \ldots, (E_\beta^{-\gamma})_{s_k}\}$ be an orthonormal basis of $\mathfrak{n}_\beta^{-\gamma}$.

First, we study the case where $Z \in \mathfrak{n}_\beta^{+\delta}$. Let $\{(E_\gamma^{-\delta})_1, \ldots, (E_\gamma^{-\delta})_{s_k}\}$ be an orthonormal basis of $\mathfrak{n}_\gamma^{-\delta}$. Then, using Remark 6.2 and Claim 7.5, we obtain

$$
\begin{aligned}
&(\nabla_X R)(Y,Z,W) \\
&= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W \\
&= \frac{1}{2}\sum_{p=1}^{s_k} \langle X, [Y, (E_\beta^{-\gamma})_p]\rangle R((E_\beta^{-\gamma})_p, Z)W - \frac{1}{2}R(Y,[JX,JZ])W \\
&= \frac{1}{2}\sum_{p=1}^{s_k} \langle X, [Y, (E_\beta^{-\gamma})_p]\rangle R(J(E_\beta^{-\gamma})_p, JZ)W - \frac{1}{2}R(Y,J[JX,Z])W \\
&= -\frac{1}{2}\sum_{p=1}^{s_k} \langle X, [Y, (E_\beta^{-\gamma})_p]\rangle \nabla_{\nabla_{J(E_\beta^{-\gamma})_p} JZ} W + \frac{1}{2}\nabla_{\nabla_Y J[JX,Z]} W \\
&= \frac{1}{4}\sum_{p=1}^{s_k}\sum_{q=1}^{s_k} \langle X, [Y, (E_\beta^{-\gamma})_p]\rangle \langle J(E_\beta^{-\gamma})_p, [Z, (E_\gamma^{-\delta})_q]\rangle \nabla_{J(E_\gamma^{-\delta})_q} W \\
&\quad - \frac{1}{4}\sum_{p=1}^{s_k} \langle Y, [[JX,Z], (E_\gamma^{-\delta})_p]\rangle \nabla_{J(E_\gamma^{-\delta})_p} W \\
&= \frac{1}{4}\sum_{q=1}^{s_k} \langle X, [JY, [Z, (E_\gamma^{-\delta})_q]]\rangle \nabla_{J(E_\gamma^{-\delta})_q} W \\
&\quad - \frac{1}{4}\sum_{p=1}^{s_k} \langle Y, [JX, [Z, (E_\gamma^{-\delta})_p]]\rangle \nabla_{J(E_\gamma^{-\delta})_p} W \\
&= \frac{1}{4}\sum_{q=1}^{s_k} \langle J^2 Y, [[Z, (E_\gamma^{-\delta})_q], JX]\rangle \nabla_{J(E_\gamma^{-\delta})_q} W \\
&\quad - \frac{1}{4}\sum_{p=1}^{s_k} \langle Y, [JX, [Z, (E_\gamma^{-\delta})_p]]\rangle \nabla_{J(E_\gamma^{-\delta})_p} W \\
&= 0.
\end{aligned}
$$

Similarly, the case of $Z \in \mathfrak{n}_\gamma^{-\delta}$ is proved.

In the cases of (46), (47), (49), (59), (60), (61), (66), (74), (76), (84), (86), (87), (89), (96) and (96), we have $\nabla_Y Z = 0$. Hence the proof for these cases goes similarly to that of (44).

(45) Let $X \in \mathfrak{n}_\alpha^{+\beta}$ and $Y \in \mathfrak{n}_\alpha^{+\gamma}$. First, to prove the case where $Z \in \mathfrak{n}_\beta^{+\gamma}$, let

$\{(E_\beta^{-\gamma})_1, \ldots, (E_\beta^{-\gamma})_{s_k}\}$ be an orthonormal basis of $\mathfrak{n}_\beta^{-\gamma}$. By making use of Remark 6.2 and Claim 7.5, we have

$$
\begin{aligned}
&(\nabla_X R)(Y,Z,W)\\
&\quad= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W\\
&\quad= -\nabla_X\nabla_{\nabla_Y Z}W + \frac{1}{2}\sum_{p=1}^{s_k}\langle X,[Y,(E_\beta^{-\gamma})_p]\rangle R((E_\beta^{-\gamma})_p, Z)W\\
&\qquad\quad - \frac{1}{2}R(Y,[JX,JZ])W + \nabla_{\nabla_Y Z}\nabla_X W\\
&\quad= \frac{1}{2}\nabla_X\nabla_{J[Y,JZ]}W - \frac{1}{2}\sum_{p=1}^{s_k}\langle X,[Y,(E_\beta^{-\gamma})_p]\rangle R(Z,(E_\beta^{-\gamma})_p)W\\
&\qquad\quad - \frac{1}{2}R(Y,[JX,JZ])W - \frac{1}{2}\nabla_{J[Y,JZ]}\nabla_X W\\
&\quad= \frac{1}{2}R(X,J[Y,JZ])W + \frac{1}{2}\nabla_{[X,J[Y,JZ]]}W + \frac{1}{2}\sum_{p=1}^{s_k}\langle X,[Y,(E_\beta^{-\gamma})_p]\rangle \nabla_{\nabla_Z (E_\beta^{-\gamma})_p}W\\
&\qquad\quad + \frac{1}{2}\nabla_{\nabla_Y [JX,JZ]}W\\
&\quad= -\frac{1}{2}\nabla_{\nabla_X J[Y,JZ]}W + \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle JX, J[Y,JZ]\rangle\nabla_{JH_\alpha}W\\
&\qquad\quad + \frac{1}{2}\sum_p\langle X,[Y,(E_\beta^{-\gamma})_p]\rangle\langle JZ,(E_\beta^{-\gamma})_p\rangle\nabla_{\frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}JH_\beta + \frac{\lambda_\gamma(H_\gamma)}{2|H_\gamma|^2}JH_\gamma}W\\
&\qquad\quad + \frac{1}{4}\langle JY,[JX,JZ]\rangle\nabla_{\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}JH_\alpha + \frac{\lambda_\gamma(H_\gamma)}{|H_\gamma|^2}JH_\gamma}W\\
&\quad= -\frac{1}{4}\langle X,[Y,JZ]\rangle\nabla_{\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}JH_\alpha + \frac{\lambda_\beta(H_\beta)}{|H_\beta|^2}JH_\beta}W + \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle JX, J[Y,JZ]\rangle\nabla_{JH_\alpha}W\\
&\qquad\quad + \frac{1}{4}\sum_p\langle X,[Y,JZ]\rangle\nabla_{\frac{\lambda_\beta(H_\beta)}{|H_\beta|^2}JH_\beta + \frac{\lambda_\gamma(H_\gamma)}{|H_\gamma|^2}JH_\gamma}W\\
&\qquad\quad + \frac{1}{4}\langle JX,[JZ,Y]\rangle\nabla_{\frac{\lambda_\alpha(H_\alpha)}{|H_\alpha|^2}JH_\alpha + \frac{\lambda_\gamma(H_\gamma)}{|H_\gamma|^2}JH_\gamma}W\\
&\quad= 0.
\end{aligned}
$$

One can prove the case of $Z \in \mathfrak{n}_\beta^{-\gamma}$ in a quite similar way to that of the case $Z \in \mathfrak{n}_\beta^{+\gamma}$. Hence we omit the proof.

Also, the proof of (63), (75) and (106) can be done in a similar way to that for (45).

(57) Let $X, Y \in \mathfrak{n}_\alpha^{+\beta}$. If $Z \in \mathfrak{n}_\alpha^{+\beta}$, by Bianchi's second identity, we have $(\nabla_X R)(Y,Z,W) = 0$ for $W \in \mathfrak{n}$, except the case where $W \in \mathfrak{n}_\alpha^{+\beta}$. Hence it follows

from Remark 6.2 and Claim 7.5 that

$$
\begin{aligned}
&(\nabla_X R)(Y, Z, W) \\
&\quad = \nabla_X (R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W \\
&\quad = \nabla_X \left(-\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2} (\langle Z, W\rangle Y - \langle Y, W\rangle Z) \right) \\
&\qquad - R\left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, Y\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, Y\rangle H_\beta, Z \right) W \\
&\qquad - R\left(Y, \frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, Z\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, Z\rangle H_\beta \right) W \\
&\qquad - (\nabla_Y \nabla_Z - \nabla_Z \nabla_Y - \nabla_{[Y,Z]}) \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, W\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, W\rangle H_\beta \right) \\
&\quad = -\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle Z, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, Y\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, Y\rangle H_\beta \right) \\
&\qquad - \frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle Y, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle X, Z\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle X, Z\rangle H_\beta \right) \\
&\qquad + \frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\langle X, Y\rangle \nabla_Z W + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, Y\rangle \nabla_Z W \\
&\qquad - \frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\langle X, Z\rangle \nabla_Y W - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, Z\rangle \nabla_Y W \\
&\qquad + \frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\langle X, W\rangle \nabla_Y Z - \frac{\lambda_\alpha(H_\alpha)^2}{4|H_\alpha|^2}\langle X, W\rangle \nabla_Z Y \\
&\qquad + \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, W\rangle \nabla_Y Z - \frac{\lambda_\beta(H_\beta)^2}{4|H_\beta|^2}\langle X, W\rangle \nabla_Z Y \\
&\quad = -\frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle X, Y\rangle\langle Z, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2} H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2} H_\beta \right) \\
&\qquad - \frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle X, Z\rangle\langle Y, W\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2} H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2} H_\beta \right) \\
&\qquad + \frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle X, Y\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle Z, W\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle Z, W\rangle H_\beta \right) \\
&\qquad - \frac{\lambda_\alpha(H_\alpha)^2}{2|H_\alpha|^2}\langle X, Z\rangle \left(\frac{\lambda_\alpha(H_\alpha)}{2|H_\alpha|^2}\langle Y, W\rangle H_\alpha + \frac{\lambda_\beta(H_\beta)}{2|H_\beta|^2}\langle Y, W\rangle H_\beta \right) \\
&\quad = 0.
\end{aligned}
$$

On the other hand, in the case where $Z \in \mathfrak{n}_\alpha^{-\beta}$, it is immediate by a straightforward computation that $(\nabla_X R)(Y, Z, W) = 0$ for $W \in \mathfrak{n}$.

The proof of the case of (53), (54), (55), (56), (58) and (100) goes in a similar fashion to that of (57), so that we omit the detail.

The proof of Claim 7.7 in now complete. □

By a similar argument, we can also prove the following, for which we omit the proof.

Claim 7.8. $\nabla_X R = 0$ *for* $X \in \mathfrak{n}_\alpha^{-\beta}$.

Claim 7.9. $\nabla_X R = 0$ *for* $X \in \mathfrak{n}_\alpha^{0}$.

Claim 7.10. $\nabla_X R = 0$ *for* $A \in \mathfrak{a}$.

Claim 7.7 through Claim 7.10 imply that $\nabla R = 0$. This completes the proof of Proposition 7.1 in the case where $\mathfrak{n}_{a_k} \neq \{0\}$ for $1 \leq k \leq n$.

In the same way as we have proved just for the case where $\mathfrak{n}_{a_k} \neq \{0\}$ for $1 \leq k \leq n$, we can also prove $\nabla R = 0$ for the case where $\mathfrak{n}_{a_l} = \{0\}$ for $n < l \leq m$.

This completes the proof of Proposition 7.1.

8 Proof of Theorem

Let $M = (M, J, g)$ be a simply connected homogeneous Kähler Einstein manifold with nonpositive curvature operator $\hat{R} \leq 0$. Recall that, as mentioned in Remark 1.1, M has nonpositive sectional curvature. By Theorem 3.1, M is identified with a simply connected solvable Lie group G with a left invariant complex structure J and a left invariant Kähler Einstein metric $\langle\ ,\ \rangle$ on G. Moreover, the Lie algebra $\mathfrak{g}$ of G admits an endomorphism J and an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfying Conditions (K1)–(K4) in Section 2. The Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of $\mathfrak{g}$ is defined by the Levi-Civita connection ∇, the curvature tensor R and the sectional curvature K of G, respectively.

Recall that if Ricci tensor Ric vanishes, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is flat. Hence it suffices to see the case where $(\mathfrak{g}, \langle\ ,\ \rangle)$ is not Ricci flat, that is, $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ satisfies the assumption of Proposition 6.1.

Proposition 8.1. *Let* $(\mathfrak{g}, J, \langle\ ,\ \rangle)$ *be as in Proposition* 6.1. *Assume that* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *has nonpositive curvature operator. Then,* $\nabla R = 0$.

Proof. Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle$. Recall that by a result of Azencott and Wilson [1], the nonpositivity of sectional curvature implies that $\mathfrak{a}$ is abelian. Let D_A and S_A denote the symmetric and skew-symmetric parts of $\operatorname{ad} A$ for $A \in \mathfrak{a}$. Moreover, recall that $\{D_A, S_A \mid A \in \mathfrak{a}\}$ is a commuting family of derivations of $\mathfrak{g}$, and that D_A is a nonzero operator vanishing on $\mathfrak{a}$ for any $A \in \mathfrak{a}$.

It is proved in Proposition 6.1 that there exists an orthogonal basis $\{H_a\}_{a\in\Lambda}$ of $\mathfrak{a}$ with respect to $\langle\ ,\ \rangle$ such that

$$\begin{aligned} &[H_a, JH_a] = \lambda_a JH_a && \text{for some } \lambda_a > 0,\\ &[H_b, JH_a] = 0 && \text{if } a \neq b. \end{aligned}$$

Moreover, setting $H = \sum_{a\in\Lambda} H_a$, we have $\langle H, X\rangle = \operatorname{tr}\operatorname{ad} X$ for any $X \in \mathfrak{g}$.

Define a linear function $\lambda_a : \mathfrak{a} \to \mathbb{R}$ by $\lambda_a(H_b) = \delta_{ab}\lambda_a$ for any $b \in \Lambda$. Let $\mathfrak{n}_a^{\pm b}$ and $\mathfrak{n}_a^0$ be subspaces of $\mathfrak{n}$ defined by

$$\begin{aligned} \mathfrak{n}_a^{\pm b} &= \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\left(\lambda_a(A) \pm \lambda_b(A)\right) X \quad \text{for any } A \in \mathfrak{a} \right\},\\ \mathfrak{n}_a^{0} &= \left\{ X \in \mathfrak{n} \;\middle|\; D_A X = \frac{1}{2}\lambda_a(A) X \quad \text{for any } A \in \mathfrak{a} \right\}, \end{aligned}$$

where $\lambda_b(H) < \lambda_a(H)$, and set

$$\mathfrak{n}_a = \bigoplus_{\lambda_b(H)<\lambda_a(H)} \left(\mathfrak{n}_a^{+b} \oplus \mathfrak{n}_a^{-b}\right) \oplus \mathfrak{n}_a^0.$$

Then $\mathfrak{g}$ is decomposed into a direct sum $\mathfrak{g} = \bigoplus_a \mathbb{R}\{H_a\} \oplus \mathfrak{n}_a \oplus \mathbb{R}\{JH_a\}$.

We now remark the following identities.

Claim 8.1. (1) $\langle R(A, JH_a)JH_a, A\rangle = -\lambda_a(A)^2|H_a|^2$ *for* $A \in \mathfrak{a}$.

(2) $\langle R(A, X)X, A\rangle = -\dfrac{1}{4}\left(\lambda_a(A) \pm \lambda_b(A)\right)^2 |X|^2$ *for* $A \in \mathfrak{a}$ *and* $X \in \mathfrak{n}_a^{\pm b}$.

(3) $\langle R(JX, X)X, JX\rangle = -\dfrac{\lambda_a(H_a)^2}{|H_a|^2}|X|^4 + \dfrac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^4$ *for any* $X \in \mathfrak{n}_a^{+b}$.

(4) $\langle R(JX, X)X, JX\rangle = -\dfrac{\lambda_a(H_a)^2}{|H_a|^2}|X|^4$ *for any* $X \in \mathfrak{n}_a^0$.

(5) $\operatorname{Ric}(H_a, H_a) = -\dfrac{1}{4}\lambda_a(H_a)^2 \left(\dim \mathfrak{n}_a + 4 + \displaystyle\sum_{d\in\Lambda_a} 2\dim \mathfrak{n}_d^{+a}\right)$.

Proof. Since S_A is a derivation of $\mathfrak{g}$ for $A \in \mathfrak{a}$, we can apply Claim 6.3, to get $R(A,X)Y = -\nabla_{D_A X} Y$. Then we obtain the following.

(1) For any $A \in \mathfrak{a}$, we have

$$\begin{aligned}\langle R(A, JH_a)JH_a, A\rangle &= -\lambda_a(A)\langle \nabla_{JH_a} JH_a, A\rangle \\ &= \lambda_a(A)\langle JH_a, \nabla_{JH_a} A\rangle \\ &= -\lambda_a(A)^2 |H_a|^2.\end{aligned}$$

(2) Similarly, for any $X \in \mathfrak{n}_a^{+b}$, we have

$$\begin{aligned}\langle R(A,X)X, A\rangle =& -\frac{1}{2}(\lambda_a(A)+\lambda_b(A))\langle \nabla_X X, A\rangle \\ =& \frac{1}{2}(\lambda_a(A)+\lambda_b(A))\langle X, \nabla_X A\rangle \\ =& -\frac{1}{4}(\lambda_a(A)+\lambda_b(A))^2 |X|^2.\end{aligned}$$

(3) Let $X \in \mathfrak{n}_a^{+b}$. It follows from Remark 6.2 that

$$\begin{aligned}R(JX,X)X &= \nabla_{JX}\nabla_X X - \nabla_X \nabla_{JX} X - \nabla_{[JX,X]} X \\ &= \nabla_{JX}\left(\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 H_a + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 H_b\right) \\ &\quad - \nabla_X\left(-\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2 JH_a + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^2 JH_b\right) \\ &\quad + \frac{\lambda_a(H_a)}{|H_a|^2}|X|^2 \nabla_{JH_a} X \\ &= -\frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)^2}{4|H_b|^2}|X|^2 JX \\ &\quad - \frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)^2}{4|H_b|^2}|X|^2 JX - \frac{\lambda_a(H_a)^2}{2|H_a|^2}|X|^2 JX \\ &= -\frac{\lambda_a(H_a)^2}{|H_a|^2}|X|^2 JX + \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^2 JX.\end{aligned}$$

Hence we have

$$\langle R(JX,X)X, JX\rangle = -\frac{\lambda_a(H_a)^2}{|H_a|^2}|X|^4 + \frac{\lambda_b(H_b)^2}{2|H_b|^2}|X|^4.$$

(4) Let $X \in \mathfrak{n}_a^0$. By using Remark 6.2, we have

$$R(JX,X)X = \nabla_{JX}\nabla_X X - \nabla_X \nabla_{JX} X - \nabla_{[JX,X]} X$$

$$
\begin{aligned}
&=\nabla_{JX}\left(\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2H_a\right)+\nabla_X\left(\frac{\lambda_a(H_a)}{2|H_a|^2}|X|^2JH_a\right)\\
&\quad+\frac{\lambda_a(H_a)}{|H_a|^2}|X|^2\nabla_{JH_a}X\\
&=-\frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2JX-\frac{\lambda_a(H_a)^2}{4|H_a|^2}|X|^2JX-\frac{\lambda_a(H_a)^2}{2|H_a|^2}|X|^2JX\\
&=-\frac{\lambda_a(H_a)^2}{|H_a|^2}|X|^2JX,
\end{aligned}
$$

so that

$$
\langle R(JX,X)X,JX\rangle=-\frac{\lambda_a(H_a)^2}{|H_a|^2}|X|^4.
$$

(5) For $a,b\in\Lambda$, let $t(a,b)=\dim\mathfrak{n}_a^{+b}$, and $s(a)=\dim\mathfrak{n}_a^0$. Let $\{(E_a^0)_1,\ldots,(E_a^0)_{s(a)}\}$ be an orthonormal basis of $\mathfrak{n}_a^0$, and let $\{(E_a^{\pm b})_1,\ldots,(E_a^{\pm b})_{t(a,b)}\}$ be an orthonormal basis of $\mathfrak{n}_a^{\pm b}$ with respect to $\langle\ ,\ \rangle$, respectively. Applying (1) and (2), we have

$$
\begin{aligned}
&\mathrm{Ric}(H_a,H_a)\\
&=\sum_{b\in\Lambda}\frac{1}{|H_b|^2}\langle R(H_a,H_b)H_b,H_a\rangle+\sum_{b,c\in\Lambda}\sum_{p=1}^{t(b,c)}\langle R(H_a,(E_b^{+c})_p)(E_b^{+c})_p,H_a\rangle\\
&\quad+\sum_{b,c\in\Lambda}\sum_{p=1}^{t(b,c)}\langle R(H_a,(E_b^{-c})_p)(E_b^{-c})_p,H_a\rangle+\sum_{b\in\Lambda}\sum_{p=1}^{s(b)}\langle R(H_a,(E_b^{0})_p)(E_b^{0})_p,H_a\rangle\\
&\quad+\sum_{b\in\Lambda}\frac{1}{|H_b|^2}\langle R(H_a,JH_b)JH_b,H_a\rangle\\
&=\sum_{a\in\Lambda_c}\sum_{p=1}^{t(a,c)}\left(-\frac{1}{4}\lambda_a(H_a)^2|(E_a^{+c})_p|^2\right)+\sum_{b\in\Lambda_a}\sum_{p=1}^{t(b,a)}\left(-\frac{1}{4}\lambda_a(H_a)^2|(E_b^{+a})_p|^2\right)\\
&\quad+\sum_{a\in\Lambda_c}\sum_{p=1}^{t(a,c)}\left(-\frac{1}{4}\lambda_a(H_a)^2|(E_a^{-c})_p|^2\right)+\sum_{b\in\Lambda_a}\sum_{p=1}^{t(b,a)}\left(-\frac{1}{4}\lambda_a(H_a)^2|(E_b^{-a})_p|^2\right)\\
&\quad+\sum_{p=1}^{s(a)}\left(-\frac{1}{4}\lambda_a(H_a)^2|(E_a^{0})_p|^2\right)-\lambda_a(H_a)^2|H_a|^2\frac{1}{|H_a|^2}\\
&=-\frac{1}{4}\lambda_a(H_a)^2\sum_{a\in\Lambda_c}\dim\mathfrak{n}_a^{+c}-\frac{1}{4}\lambda_a(H_a)^2\sum_{b\in\Lambda_a}\dim\mathfrak{n}_b^{+a}-\frac{1}{4}\lambda_a(H_a)^2\sum_{a\in\Lambda_c}\dim\mathfrak{n}_a^{-c}\\
&\quad-\frac{1}{4}\lambda_a(H_a)^2\sum_{b\in\Lambda_a}\dim\mathfrak{n}_b^{+a}-\frac{1}{4}\lambda_a(H_a)^2\dim\mathfrak{n}_a^{0}-\lambda_a(H_a)^2
\end{aligned}
$$

$$= -\frac{1}{4}\lambda_a(H_a)^2 \sum_{a\in\Lambda_c} \dim \mathfrak{n}_a - \frac{1}{4}\lambda_a(H_a)^2 \sum_{b\in\Lambda_a} 2\dim \mathfrak{n}_b^{+a} - \lambda_a(H_a)^2.$$

This completes the proof of Claim 8.1.

As defined in Proposition 6.1, for each $c \in \Lambda$, let Λ_c denote a subset $\{a \in \Lambda \mid \mathfrak{n}_a^{\pm c} \neq \{0\}\} \cup \{c\}$ of Λ. Then the following holds.

Claim 8.2. *Let b and c be elements in Λ such that $b \notin \Lambda_c$ and $c \notin \Lambda_b$. Then we have $\Lambda_c \cap \Lambda_b = \{0\}$.*

Proof. Let b and c be elements in Λ such that $b \notin \Lambda_c$ and $c \notin \Lambda_b$. Assume that $\Lambda_b \cap \Lambda_c \neq \{0\}$, and let $a \in \Lambda_b \cap \Lambda_c$.

Let $X \in \mathfrak{n}_a^{+b}$. For any $d \in \Lambda_a$, let $Y \in \mathfrak{n}_d^{-a}$. By (iv) of Proposition 6.1, we have $[X, Y] \neq 0$. It follows from $[X, Y] \in \mathfrak{n}_d^{+b}$ that $d \in \Lambda_b$. Since $d \in \Lambda_a$ is arbitrary, Λ_b contains Λ_a. Applying Remark 6.6 then yields

$$\begin{aligned}
&\dim \mathfrak{n}_a + \sum_{d\in\Lambda_a} 2\dim \mathfrak{n}_d^{+a} - \left(\dim \mathfrak{n}_b + \sum_{d\in\Lambda_b} 2\dim \mathfrak{n}_d^{+b}\right) \\
&\quad = \sum_{\lambda_e<\lambda_a} 2\dim \mathfrak{n}_a^{+e} + \dim \mathfrak{n}_a^0 + \sum_{d\in\Lambda_a} 2\dim \mathfrak{n}_d^{+a} \\
&\qquad\qquad - \sum_{\lambda_e<\lambda_b} 2\dim \mathfrak{n}_b^{+e} - \dim \mathfrak{n}_b^0 - \sum_{d\in\Lambda_b} 2\dim \mathfrak{n}_d^{+b} \\
&\quad = \sum_{\lambda_e<\lambda_b} 2(\dim \mathfrak{n}_a^{+e} - \dim \mathfrak{n}_b^{+e}) + \sum_{\lambda_e=\lambda_b} 2\dim \mathfrak{n}_a^{+e} + 2\dim \mathfrak{n}_a^{+c} \\
&\qquad\qquad + \sum_{d\in(\Lambda_b-\Lambda_a)} 2(\dim \mathfrak{n}_a^{+d} - \dim \mathfrak{n}_d^{+b}) + \dim \mathfrak{n}_a^0 - \dim \mathfrak{n}_b^0 \\
&\quad > 0,
\end{aligned}$$

where we set $\lambda_p = \lambda_p(H_p)$ for any p. This implies that

$$\dim \mathfrak{n}_a + \sum_{d\in\Lambda_a} 2\dim \mathfrak{n}_d^{+a} > \dim \mathfrak{n}_b + \sum_{d\in\Lambda_b} 2\dim \mathfrak{n}_d^{+b}. \tag{1.14}$$

Since $\langle\ ,\ \rangle$ is assumed Einstein, we have

$$\frac{1}{|H_a|^2}\operatorname{Ric}(H_a, H_a) = \frac{1}{|H_b|^2}\operatorname{Ric}(H_b, H_b).$$

Hence it follows from Claim 8.1 that

$$-\frac{\lambda_a(H_a)^2}{4|H_a|^2}\left(\dim \mathfrak{n}_a + 4 + \sum_{d\in\Lambda_a} 2\dim \mathfrak{n}_d^{+a}\right) = -\frac{\lambda_b(H_b)^2}{4|H_b|^2}\left(\dim \mathfrak{n}_b + 4 + \sum_{d\in\Lambda_b} 2\dim \mathfrak{n}_d^{+b}\right).$$

This combined with (1.14) implies that

$$\frac{\lambda_a(H_a)^2}{|H_a|^2} < \frac{\lambda_b(H_b)^2}{|H_b|^2}.$$

Since $\mathfrak{n}_a^{+b} \neq \{0\}$, we can apply Remark 6.4 to get

$$\frac{1}{2}\frac{\lambda_b(H_b)^2}{|H_b|^2} \leq \frac{\lambda_a(H_a)^2}{|H_a|^2} < \frac{\lambda_b(H_b)^2}{|H_b|^2}.$$

By the same argument as above, we also have

$$\frac{1}{2}\frac{\lambda_c(H_c)^2}{|H_c|^2} \leq \frac{\lambda_a(H_a)^2}{|H_a|^2} < \frac{\lambda_c(H_c)^2}{|H_c|^2}.$$

It follows from these inequalities that

$$\frac{1}{2}\frac{\lambda_b(H_b)^2}{|H_b|^2} \leq \frac{\lambda_a(H_a)^2}{|H_a|^2} < \frac{\lambda_c(H_c)^2}{|H_c|^2} \quad \text{or} \quad \frac{1}{2}\frac{\lambda_c(H_c)^2}{|H_c|^2} \leq \frac{\lambda_a(H_a)^2}{|H_a|^2} < \frac{\lambda_b(H_b)^2}{|H_b|^2}. \tag{1.15}$$

Now, let $X \in \mathfrak{n}_a^{+b}$ and $Y \in \mathfrak{n}_a^{+c}$. Note that it follows from Claim 4.1 that $\nabla_Y X = \nabla_{JY} X = 0$. Then we define a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \langle\!\langle \hat{R}(xX \wedge JX + Y \wedge JY), xX \wedge JX + Y \wedge JY \rangle\!\rangle.$$

By Remark 6.2 and Claim 8.1, f is represented as

$$\begin{aligned}
f(x) &= x^2\langle\!\langle \hat{R}(X \wedge JX), X \wedge JX\rangle\!\rangle + \langle\!\langle \hat{R}(Y \wedge JY), Y \wedge JY\rangle\!\rangle + 2x\langle\!\langle \hat{R}(X \wedge JX), Y \wedge JY\rangle\!\rangle \\
&= x^2\langle R(X, JX)JX, JX\rangle + \langle R(Y, JY)JY, Y\rangle + 2x\langle R(Y, JY)JX, X\rangle \\
&= x^2\left(-\frac{\lambda_a(H_a)}{|H_a|^2}|X|^4 + \frac{\lambda_b(H_b)}{2|H_b|^2}|X|^4\right) - \frac{\lambda_a(H_a)^2}{|H_a|^2}|Y|^4 + \frac{\lambda_c(H_c)^2}{2|H_c|^2}|Y|^4 \\
&\quad + 2x\langle \nabla_Y\nabla_{JY}JX - \nabla_{JY}\nabla_Y JX - \nabla_{[Y,JY]}JX, X\rangle \\
&= -x^2\left(\frac{\lambda_a(H_a)^2}{|H_a|^2} - \frac{\lambda_b(H_b)^2}{2|H_b|^2}\right)|X|^4 - \left(\frac{\lambda_a(H_a^2)}{|H_a|^2} - \frac{\lambda_c(H_c)^2}{2|H_c|^2}\right)|Y|^4 \\
&\quad - 2x\frac{\lambda_a(H_a)}{|H_a|^2}|Y|^2\langle \nabla_{JH_a}JX, X\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -x^2\left(\frac{\lambda_a(H_a)^2}{|H_a|^2}-\frac{\lambda_b(H_b)^2}{2|H_b|^2}\right)|X|^4-\left(\frac{\lambda_a(H_a^2)}{|H_a|^2}-\frac{\lambda_c(H_c)^2}{2|H_c|^2}\right)|Y|^4\\
&\qquad - x\frac{\lambda_a(H_a)^2}{|H_a|^2}|Y|^2|X|^2.
\end{aligned}
$$

We note that the discriminant $\mathcal{D}$ of f is given by

$$
\begin{aligned}
\mathcal{D} &= \frac{\lambda_a(H_a)^4}{|H_a|^4}|Y|^4|X|^4-4\left(\frac{\lambda_a(H_a)^2}{|H_a|^2}-\frac{\lambda_b(H_b)^2}{2|H_b|^2}\right)|X|^4\left(\frac{\lambda_a(H_a^2)}{|H_a|^2}-\frac{\lambda_c(H_c)^2}{2|H_c|^2}\right)|Y|^4\\
&= |X|^4|Y|^4\left\{\frac{\lambda_a(H_a)^4}{|H_a|^4}-4\frac{\lambda_a(H_a)^4}{|H_a|^4}\right.\\
&\qquad\qquad \left.+2\frac{\lambda_a(H_a)^2}{|H_a|^2}\left(\frac{\lambda_b(H_b)^2}{|H_b|^2}+\frac{\lambda_c(H_c)^2}{|H_c|^2}\right)-\frac{\lambda_b(H_b)^2}{|H_b|^2}\frac{\lambda_c(H_c)^2}{|H_c|^2}\right\}\\
&= |X|^4|Y|^4\left\{-3\frac{\lambda_a(H_a)^4}{|H_a|^4}+2\frac{\lambda_a(H_a)^2}{|H_a|^2}\left(\frac{\lambda_b(H_b)^2}{|H_b|^2}+\frac{\lambda_c(H_c)^2}{|H_c|^2}\right)-\frac{\lambda_b(H_b)^2}{|H_b|^2}\frac{\lambda_c(H_c)^2}{|H_c|^2}\right\}.
\end{aligned}
$$

It is easy to see from (1.15) that $\mathcal{D} > 0$. This implies that there exist different solutions $\eta_1 < \eta_2$ of f. Then, $f(x) > 0$ holds for $\eta_1 < x < \eta_2$, contradicting the nonpositivity of $\hat{R}$. Hence $\Lambda_b \cap \Lambda_c = \{0\}$. □

Claim 8.2 implies that there exists a subset $\{a_1, \dots, a_m\}$ of Λ such that $\Lambda_{a_1} \cup \cdots \cup \Lambda_{a_m} = \Lambda$ and $\Lambda_{a_i} \cap \Lambda_{a_j} = \{0\}$ for $i \neq j$. We set $\Lambda_{a_i} = \{a_i = i_1, \dots, i_{m_i}\}$ for each $i = 1, \dots, m$. Without loss of generality, we may suppose $\lambda_{i_1}(H) < \cdots < \lambda_{i_{m_i}}(H)$. Then $\mathfrak{g}$ can be written as

$$
\mathfrak{g} = \bigoplus_{i=1}^{m}\bigoplus_{\alpha=1}^{m_i}\left(\mathbb{R}\{H_{i_\alpha}\}\oplus\mathfrak{n}_{i_\alpha}\oplus\mathbb{R}\{JH_{i_\alpha}\}\right),
$$

where $\mathfrak{n}_{i_\alpha}$ is given by

$$
\mathfrak{n}_{i_\alpha} = \bigoplus_{\beta=1}^{\alpha-1}\left(\mathfrak{n}_{i_\alpha}^{+i_\beta}\oplus\mathfrak{n}_{i_\alpha}^{-i_\beta}\right)\oplus\mathfrak{n}_{i_\alpha}^{0}.
$$

Claim 8.3. *If $\mathfrak{n}_{i_1} = \{0\}$, then $\mathfrak{n}^0_{i_\alpha} = \{0\}$ for all $\alpha = 1, \dots, m_i$.*

Proof. Suppose that $\mathfrak{n}_{i_1} = \{0\}$. It follows from Remark 6.6 that $\dim\mathfrak{n}^0_{i_{m_i}} = 0$ implies that $\dim\mathfrak{n}^0_{i_\alpha} = 0$. Hence it suffices to show that $\mathfrak{n}^0_{i_{m_i}} = \{0\}$.

Assume that $\mathfrak{n}^0_{i_{m_i}} \neq \{0\}$. Then, by Remark 6.6, we have

$$
\dim\mathfrak{n}_{i_{m_i}} - 2\sum_{\alpha=2}^{m_i}\dim\mathfrak{n}_{i_\alpha}^{+i_1} = 2\sum_{\alpha=1}^{m_i-1}\dim\mathfrak{n}_{i_{m_i}}^{+i_\alpha}+\dim\mathfrak{n}^0_{i_{m_i}}-2\sum_{\alpha=2}^{m_i}\dim\mathfrak{n}_{i_\alpha}^{+i_1}
$$

$$= 2\sum_{\alpha=2}^{m_i-1}\left(\dim \mathfrak{n}_{i_{m_i}}^{+i_\alpha} - \dim \mathfrak{n}_{i_\alpha}^{+i_1}\right) + \dim \mathfrak{n}_{i_{m_i}}^{0}$$
$$> 0,$$

which implies that

$$\dim \mathfrak{n}_{i_{m_i}} > 2\sum_{\alpha=2}^{m_i} \dim \mathfrak{n}_{i_\alpha}^{+i_1}. \tag{1.16}$$

Since $\langle\ ,\ \rangle$ is assumed Einstein, we have

$$\frac{1}{|H_{i_{m_i}}|^2}\,\mathrm{Ric}(H_{i_{m_i}}, H_{i_{m_i}}) = \frac{1}{|H_{i_1}|^2}\,\mathrm{Ric}(H_{i_1}, H_{i_1}).$$

Hence it follows from Claim 8.1 that

$$-\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{4|H_{i_{m_i}}|^2}\left(\dim \mathfrak{n}_{i_{m_i}} + 4\right) = -\frac{\lambda_{i_1}(H_{i_1})^2}{4|H_{i_1}|^2}\left(4 + 2\sum_{\alpha=2}^{m_i} \dim \mathfrak{n}_{i_\alpha}^{+i_1}\right).$$

This combined with (1.16) then implies that

$$\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} < \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2}. \tag{1.17}$$

Let $X \in \mathfrak{n}_{i_{m_i}}^{i_1}$ and $Y \in \mathfrak{n}_{i_{m_i}}^{0}$. Then it follows from Claim 4.2 that $\nabla_X Y = 0$.

Now, we set

$$\omega = \frac{1}{|H_{i_1}|^2} H_{i_1} \wedge JH_{i_1} + \frac{1}{|H_{i_{m_i}}|^2} H_{i_{m_i}} \wedge JH_{i_{m_i}} - 2X \wedge JX + Y \wedge JY.$$

Then, Remark 6.2 and Claim 8.1, we obtain

$$\begin{aligned}
\langle\!\langle \hat{R}(\omega), \omega \rangle\!\rangle = {} & \frac{1}{|H_{i_1}|^4}\langle\!\langle \hat{R}(H_{i_1} \wedge JH_{i_1}), H_{i_1} \wedge JH_{i_1} \rangle\!\rangle \\
& + \frac{1}{|H_{i_{m_i}}|^4}\langle\!\langle \hat{R}(H_{i_{m_i}} \wedge JH_{i_{m_i}}), H_{i_{m_i}} \wedge JH_{i_{m_i}} \rangle\!\rangle \\
& + 4\langle\!\langle \hat{R}(X \wedge JX), X \wedge JX \rangle\!\rangle + \langle\!\langle \hat{R}(Y \wedge JY), Y \wedge JY \rangle\!\rangle \\
& + \frac{2}{|H_{i_1}|^2|H_{i_{m_i}}|^2}\langle\!\langle \hat{R}(H_{i_1} \wedge JH_{i_1}), H_{i_{m_i}} \wedge JH_{i_{m_i}} \rangle\!\rangle \\
& - \frac{4}{|H_{i_1}|^2}\langle\!\langle \hat{R}(H_{i_1} \wedge JH_{i_1}), X \wedge JX \rangle\!\rangle + \frac{2}{|H_{i_1}|^2}\langle\!\langle \hat{R}(H_{i_1} \wedge JH_{i_1}), Y \wedge JY \rangle\!\rangle \\
& - \frac{4}{|H_{i_{m_i}}|^2}\langle\!\langle \hat{R}(H_{i_{m_i}} \wedge JH_{i_{m_i}}), X \wedge JX \rangle\!\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2}{|H_{i_{m_i}}|^2} \langle\!\langle \hat{R}(H_{i_{m_i}} \wedge JH_{i_{m_i}}), Y \wedge JY \rangle\!\rangle - 4 \langle\!\langle \hat{R}(X \wedge JX), Y \wedge JY \rangle\!\rangle \\
&= \frac{1}{|H_{i_1}|^4} \langle R(H_{i_1}, JH_{i_1}) JH_{i_1}, H_{i_1} \rangle + \frac{1}{|H_{i_{m_i}}|^4} \langle R(H_{i_{m_i}}, JH_{i_{m_i}}) JH_{i_{m_i}}, H_{i_{m_i}} \rangle \\
&+ 4 \langle R(X, JX) JX, X \rangle + \langle R(Y, JY) JY, Y \rangle \\
&+ \frac{2}{|H_{i_1}|^2 |H_{i_{m_i}}|^2} \langle R(H_{i_{m_i}}, JH_{i_{m_i}}) JH_{i_1}, H_{i_1} \rangle \\
&- \frac{4}{|H_{i_1}|^2} \langle R(X, JX) JH_{i_1}, H_{i_1} \rangle + \frac{2}{|H_{i_1}|^2} \langle R(Y, JY) JH_{i_1}, H_{i_1} \rangle \\
&- \frac{4}{|H_{i_{m_i}}|^2} \langle R(X, JX) JH_{i_{m_i}}, H_{i_{m_i}} \rangle \\
&+ \frac{2}{|H_{i_{m_i}}|^2} \langle R(Y, JY) JH_{i_{m_i}}, H_{i_{m_i}} \rangle - 4 \langle R(Y, JY) JX, X \rangle \\
&= - \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^4} |H_{i_1}|^2 - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^4} |H_{i_{m_i}}|^2 + 4 \left(- \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} + \frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2} \right) \\
&- \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} + 4 \frac{\lambda_{i_1}(H_{i_1})}{|H_{i_1}|^2} \langle \nabla_{JH_{i_1}} JX, X \rangle + 4 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})}{|H_{i_{m_i}}|^2} \langle \nabla_{JH_{i_{m_i}}} JX, X \rangle \\
&- 2 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})}{|H_{i_{m_i}}|^2} \langle \nabla_{JH_{i_{m_i}}} JY, Y \rangle \\
&- 4 \langle \nabla_Y \nabla_{JY} JX - \nabla_{JY} \nabla_Y JX - \nabla_{[Y, JY]} JX, X \rangle \\
&= \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2} - 6 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^4} + 2 \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2} + 2 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} \\
&- \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} + 4 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})}{|H_{i_{m_i}}|^2} \langle \nabla_{JH_{i_{m_i}}} JX, X \rangle \\
&= 3 \frac{\lambda_{i_i}(H_{i_1})^2}{|H_{i_1}|^4} |H_{i_1}|^2 - 5 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^4} |H_{i_{m_i}}|^2 + 2 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} \\
&= 3 \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2} - 3 \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}.
\end{aligned}
$$

The nonpositivity of $\hat{R}$ then implies that $\langle\!\langle \hat{R}(\omega), \omega \rangle\!\rangle \leq 0$, that is,

$$
\frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^4} |H_{i_1}|^2 \leq \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^4} |H_{i_{m_i}}|^2.
$$

This contradicts (1.17), and hence $\mathfrak{n}^0_{i_{m_i}} = \{0\}$. This completes the proof of Claim 8.3.

Claim 8.4. $\dfrac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} \leq \cdots \leq \dfrac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2}$ *for any* i.

Proof. First we consider the case where $\mathfrak{n}_{i_1} \neq \{0\}$. Suppose $\alpha, \beta \in \{1, \ldots, m_i\}$ satisfy $\lambda_{i_\alpha}(H) > \lambda_{i_\beta}(H)$. It then follows from Remark 6.6 that

$$\begin{aligned}
&\dim \mathfrak{n}_{i_\alpha} + 2\sum_{\gamma=\alpha+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\alpha} - \dim \mathfrak{n}_{i_\beta} - 2\sum_{\gamma=\beta+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\beta} \\
&= 2\sum_{\gamma=1}^{\alpha-1} \dim \mathfrak{n}_{i_\alpha}^{+i_\gamma} + \dim \mathfrak{n}_{i_\alpha}^{0} + 2\sum_{\gamma=\alpha+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\alpha} \\
&\quad - 2\sum_{\gamma=1}^{\beta-1} \dim \mathfrak{n}_{i_\beta}^{+i_\gamma} - \dim \mathfrak{n}_{i_\beta}^{0} - 2\sum_{\gamma=\beta+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\beta} \\
&= 2\sum_{\gamma=1}^{\beta-1} \left(\dim \mathfrak{n}_{i_\alpha}^{+i_\gamma} - \dim \mathfrak{n}_{i_\beta}^{+i_\gamma}\right) + \dim \mathfrak{n}_{i_\alpha}^{0} - \dim \mathfrak{n}_{i_\beta}^{0} \\
&\quad + 2\sum_{\gamma=\beta+1}^{\alpha-1} \left(\dim \mathfrak{n}_{i_\alpha}^{+i_\gamma} - \dim \mathfrak{n}_{i_\gamma}^{+i_\beta}\right) \\
&\geq 0,
\end{aligned}$$

which implies that

$$\dim \mathfrak{n}_{i_\alpha} + 2\sum_{\gamma=\alpha+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\alpha} \geq \dim \mathfrak{n}_{i_\beta} + 2\sum_{\gamma=\beta+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\beta}. \tag{1.18}$$

Since $\langle\ ,\ \rangle$ is Einstein, we have

$$\frac{1}{|H_{i_\alpha}|^2} \operatorname{Ric}(H_{i_\alpha}, H_{i_\alpha}) = \frac{1}{|H_{i_\beta}|^2} \operatorname{Ric}(H_{i_\beta}, H_{i_\beta}).$$

Hence, by making use of Claim 8.1, we obtain

$$\begin{aligned}
&-\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{4|H_{i_\alpha}|^2} \left(\dim \mathfrak{n}_{i_\alpha} + 4 + 2\sum_{\gamma=\alpha+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\alpha}\right) \\
&\qquad = -\frac{\lambda_{i_\beta}(H_{i_\beta})^2}{4|H_{i_\beta}|^2} \left(\dim \mathfrak{n}_{i_\beta} + 4 + 2\sum_{\gamma=\beta+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\beta}\right).
\end{aligned}$$

This together with (1.18) then yields

$$\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{|H_{i_\alpha}|^2} \leq \frac{\lambda_{i_\beta}(H_{i_\beta})^2}{|H_{i_\beta}|^2}.$$

Consequently, recall that $\lambda_{i_1}(H) < \cdots < \lambda_{i_{m_i}}(H)$, we have

$$\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} \leq \cdots \leq \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2}.$$

The proof for the case where $\mathfrak{n}_{i_1} = \{0\}$ can be done in a similar way to that of the case where $\mathfrak{n}_{i_1} \neq \{0\}$. $\square$

Claim 8.5. $\dfrac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{|H_{i_\alpha}|^2} = \dfrac{\lambda_{i_\beta}(H_{i_\beta})^2}{|H_{i_\beta}|^2}$ *for* $\alpha, \beta \in \Lambda_{a_i}$.

Proof. First, we consider the case where $\mathfrak{n}_{i_1} = 0$ and $\dim \mathfrak{n}_{i_{m_i}}^{+i_1} = 1$. Then Remark 6.6 shows that $\dim \mathfrak{n}_{i_{m_i}}^{+i_\beta} = \dim \mathfrak{n}_{i_{m_i}}^{+i_1} = 1$ for any β. Moreover, it follows from Remark 6.6 that

$$1 = \dim \mathfrak{n}_{i_{m_i}}^{+i_\beta} \geq \dim \mathfrak{n}_{i_\alpha}^{+i_\beta} > 0,$$

which implies that $\dim \mathfrak{n}_{i_\alpha}^{+i_\beta} = 1$ for any α, β. Hence the Ricci curvatures in the directions H_{i_α} and H_{i_β} are given respectively by

$$\begin{aligned}
\mathrm{Ric}(H_{i_\alpha}, H_{i_\alpha}) &= -\frac{1}{4}\lambda_{i_\alpha}(H_{i_\alpha})^2 \left(\dim \mathfrak{n}_{i_\alpha} + 4 + 2\sum_{\gamma=\alpha+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\alpha}\right) \\
&= -\frac{1}{4}\lambda_{i_\alpha}(H_{i_\alpha})^2 \left(2\sum_{\gamma=1}^{\alpha-1} \dim \mathfrak{n}_{i_\alpha}^{+i_\gamma} + 4 + 2\sum_{\gamma=\alpha+1}^{m_i} \dim \mathfrak{n}_{i_\gamma}^{+i_\alpha}\right) \\
&= -\frac{1}{4}\lambda_{i_\alpha}(H_{i_\alpha})^2 \left(2(\alpha-1) + 4 + 2(m_i - \alpha)\right) \\
&= -\frac{1}{4}\lambda_{i_\alpha}(H_{i_\alpha})^2 \left(2m_i + 2\right), \\
\mathrm{Ric}(H_{i_\beta}, H_{i_\beta}) &= -\frac{1}{4}\lambda_{i_\alpha}(H_{i_\alpha})^2 \left(2m_i + 2\right).
\end{aligned}$$

Since $\langle\ ,\ \rangle$ is Einstein, we have

$$-\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{4|H_{i_\alpha}|^2}\left(2m_i + 2\right) = -\frac{\lambda_{i_\beta}(H_{i_\beta})^2}{4|H_{i_\beta}|^2}\left(2m_i + 2\right),$$

which implies that

$$\frac{\lambda_{i_\alpha}(H_{i_\alpha})^2}{|H_{i_\alpha}|^2} = \frac{\lambda_{i_\beta}(H_{i_\beta})^2}{|H_{i_\beta}|^2}.$$

Next, we consider the case that $\mathfrak{n}_{i_1} = \{0\}$ and $\dim \mathfrak{n}_{i_{m_i}} \geq 2$. Note that there exist non-zero vectors $X \in \mathfrak{n}_{i_{m_i}}^{+i_1}$ and $Y \in \mathfrak{n}_{i_{m_i}}^{-i_1}$ satisfying $\langle JX, Y\rangle = 0$. Then it is easy to see that $[X, Y] = 0$ and $\nabla_Y X = 0$. This implies that

$$\begin{aligned}\langle R(Y,X)X,Y\rangle &= \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X - \nabla_{[X,Y]} X, Y\rangle \\ &= \left\langle \nabla_Y \left(\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})}{2|H_{i_{m_i}}|^2} |X|^2 H_{i_{m_i}} + \frac{\lambda_{i_1}(H_{i_1})}{2|H_{i_1}|^2} |X|^2 H_{i_1} \right), Y \right\rangle \\ &= \left\langle -\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{4|H_{i_{m_i}}|^2} |X|^2 Y + \frac{\lambda_{i_1}(H_{i_1})^2}{4|H_{i_1}|^2} |X|^2 Y, Y \right\rangle \\ &= -\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{4|H_{i_{m_i}}|^2} |X|^2|Y|^2 + \frac{\lambda_{i_1}(H_{i_1})^2}{4|H_{i_1}|^2} |X|^2|Y|^2.\end{aligned}$$

Then the nonpositivity of $\hat{R}$ implies that

$$\frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2} \leq \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}.$$

This together with Claim 8.4 then yields

$$\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2} = \cdots = \frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^2}.$$

Finally, we consider the case that $\mathfrak{n}_{i_1} \neq \{0\}$. Let $\{(E^0_{i_1})_1, \ldots, (E^0_{i_1})_{t_i}\}$ be an orthonormal basis of $\mathfrak{n}^0_{i_1}$. Let $X \in \mathfrak{n}^0_{i_{m_i}}$ and $Y \in \mathfrak{n}^{+i_1}_{i_{m_i}}$, and let $Z \in \mathfrak{n}_{i_1}$. We now define a quadratic function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \langle\!\langle \hat{R}(xX \wedge Y + H_{i_1} \wedge Z), xX \wedge Y + H_{i_1} \wedge Z \rangle\!\rangle.$$

Then, using Claim 4.2 and Remark 6.2, we see that f is represented as

$$\begin{aligned} f(x) &= x^2 \langle\!\langle \hat{R}(X \wedge Y), X \wedge Y\rangle\!\rangle + 2x \langle\!\langle \hat{R}(X \wedge Y), H_{i_1} \wedge Z\rangle\!\rangle + \langle\!\langle \hat{R}(H_{i_1} \wedge Z), H_{i_1} \wedge Z\rangle\!\rangle \\ &= x^2 \langle R(Y,X)X,Y\rangle + 2x\langle R(Z,H_{i_1})Y,X\rangle + \langle R(Z,H_{i_1})H_{i_1}, Z\rangle \\ &= x^2 \Bigg(|U(X,Y)|^2 - \langle U(X,X), U(Y,Y)\rangle - \frac{3}{4}\,|[X,Y]|^2 \\ &\qquad - \frac{1}{2}\langle [X,[X,Y]], Y\rangle - \frac{1}{2}\langle [Y,[Y,X]], X\rangle \Bigg) \\ &\quad + 2x\langle R(H_{i_1}, Z)X, Y\rangle + \langle R(H_{i_1}, Z)Z, H_{i_1}\rangle \\ &= x^2 \left(\sum_{p=1}^{t_i} \langle U(X,Y), (E^0_{i_1})_p\rangle^2 - \frac{1}{|H_{i_{m_i}}|^2} \langle U(X,X), H_{i_{m_i}}\rangle \langle U(Y,Y), H_{i_{m_i}}\rangle \right)\end{aligned}$$

$$
\begin{aligned}
&- x\frac{1}{2}\lambda_{i_1}(H_{i_1})\langle \nabla_Z X, Y\rangle - \frac{1}{2}\lambda_{i_1}(H_{i_1})\langle \nabla_Z Z, H_{i_1}\rangle \\
&= x^2\left(\frac{1}{4}\sum_{p=1}^{t_i}\langle Y, [X, (E^0_{i_1})_p]\rangle^2 - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{4|H_{i_{m_i}}|^2}|X|^2|Y|^2\right) \\
&- x\frac{1}{2}\lambda_{i_1}(H_{i_1})\langle Y, [Z, X]\rangle - \frac{1}{4}\lambda_{i_1}(H_{i_1})^2|Z|^2.
\end{aligned}
$$

The discriminant $\mathcal{D}$ of f is then given by

$$
\begin{aligned}
\mathcal{D} &= \frac{1}{4}\lambda_{i_1}(H_{i_1})^2\langle Y, [Z, X]\rangle^2 \\
&+ 4\left(\frac{1}{4}\sum_{p=1}^{t_i}\langle Y, [X, (E^0_{i_1})_p]\rangle^2 - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{4|H_{i_{m_i}}|^2}|X|^2|Y|^2\right)\frac{1}{4}\lambda_{i_1}(H_{i_1})^2|Z|^2 \\
&= \frac{1}{4}\lambda_{i_1}(H_{i_1})^2\left(\langle Y, [Z, X]\rangle^2 + |Z|^2\sum_{p=1}^{t_i}\langle Y, [X, (E^0_{i_1})_p]\rangle^2 - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}|X|^2|Y|^2|Z|^2\right).
\end{aligned}
$$

We now set $X = [JY, JZ]$. Note that X does not vanish by (iv) of Proposition 6.1. Setting $e_p = 1/|[J(E^0_{i_1})_p, JY]|\,[J(E^0_{i_1})_p, JY]$, it follows from (iv) of Proposition 6.1 that $e_1, \ldots, e_{t_i}$ are perpendicular to each other in $\mathfrak{n}^0_{i_{m_i}}$. Hence the subspace $[\mathfrak{n}^0_{i_1}, JY]$ of $\mathfrak{n}^0_{i_{m_i}}$ is spanned by $\{e_1, \ldots, e_{t_i}\}$.

It follows from Condition (K3) together with Claim 6.12 that

$$
\begin{aligned}
\mathcal{D} &= \frac{1}{4}\lambda_{i_1}(H_{i_1})^2\left(\langle Y, [Z, [JY, JZ]]\rangle^2 + |Z|^2\sum_{p=1}^{t_i}\langle Y, [[JY, JZ], (E^0_{i_1})_p]\rangle^2\right. \\
&\qquad\left. - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}|[JY, JZ]|^2|Y|^2|Z|^2\right) \\
&= \frac{1}{4}\lambda_{i_1}(H_{i_1})^2\left(\frac{1}{4}\langle Y, [[Z, JZ], JY]\rangle^2 + |Z|^2\sum_{p=1}^{t_i}\langle [JY, JZ], [J(E^0_{i_1})_p, JY]\rangle^2\right. \\
&\qquad\left. - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2}|Y|^2|Z|^2|Y|^2|Z|^2\right) \\
&= \frac{1}{4}\lambda_{i_1}(H_{i_1})^2\left(\frac{1}{4}\frac{\lambda_{i_1}(H_{i_1})^2}{|H_{i_1}|^4}|Z|^4\langle Y, [JH_{i_1}, JY]\rangle^2\right. \\
&\qquad + |Z|^2\sum_{p=1}^{t_i}|[J(E^0_{i_1})_p, JY]|^2\langle [JY, JZ], e_p\rangle^2 \\
&\qquad\left. - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2}|Y|^2|Z|^2|Y|^2|Z|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\lambda_{i_1}(H_{i_1})^2 \left(\frac{1}{4}\frac{\lambda_{i_1}(H_{i_1})^4}{|H_{i_1}|^4}|Z|^4|Y|^4 + |Z|^2\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2}|Y|^2|[JY,JZ]|^2 \right.\\
&\qquad\qquad \left. -\frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{|H_{i_{m_i}}|^2}\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2}|Y|^2|Z|^2|Y|^2|Z|^2 \right)\\
&= \frac{\lambda_{i_1}(H_{i_1})^4}{4|H_{i_1}|^2}|Z|^4|Y|^4 \left(\frac{\lambda_{i_1}(H_{i_1})^2}{4|H_{i_1}|^2} + \frac{\lambda_{i_1}(H_{i_1})^2}{4|H_{i_1}|^2} - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{2|H_{i_{m_i}}|^2} \right)\\
&= \frac{\lambda_{i_1}(H_{i_1})^4}{4|H_{i_1}|^2}|Z|^4|Y|^4 \left(\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2} - \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{2|H_{i_{m_i}}|^2} \right).
\end{aligned}
$$

The nonpositivity of $\hat{R}$ then implies that $f(x) \leq 0$ for all $x \in \mathbb{R}$. Hence the discriminant $\mathcal{D}$ of f is nonpositive, that is,

$$
\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2} \leq \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{2|H_{i_{m_i}}|^2}.
$$

This combined with Claim 8.4 then shows that

$$
\frac{\lambda_{i_1}(H_{i_1})^2}{2|H_{i_1}|^2} = \frac{\lambda_{i_{m_i}}(H_{i_{m_i}})^2}{2|H_{i_{m_i}}|^2}.
$$

This completes the proof of Claim 8.5.

Summing up the above argument, we obtain Proposition 8.1.

Proposition 8.1 shows that a connected, simply connected homogeneous Kähler Einstein manifold M of nonpositive curvature operator is a Riemannian symmetric space if it is not Ricci flat. On the other hand, if M is Ricci flat, then it is flat, and hence is also a Riemannian symmetric space. In consequence, we obtain our Main Theorem.

Chapter 2

Examples

It is well-known that all irreducible symmetric Kähler manifolds of non-compact type have nonpositive sectional curvature. In this chapter, we study the curvature operator of the following *classical type* irreducible symmetric Kähler manifolds of noncompact type.

1 Type $I_{m,n}$

Let M be an open subset $D^I_{m,n} = \left\{\zeta \in M(m,n;\mathbb{C}) \,\middle|\, I_n - {}^t\bar{\zeta}\zeta > 0\right\}$ of $\mathbb{C}^{mn}$, where I_n is the $n \times n$ identity matrix, ${}^t\zeta$ is the transpose of $\zeta \in M$ and $\overline{\zeta}$ denotes the complex conjugate of $\zeta \in M$.

Let $\zeta = (z_{ip}) \in M$ with $z_{ip} = x_{ip} + \sqrt{-1}\, y_{ip}$, $i = 1, \ldots, m$ and $p = 1, \ldots, n$ be the canonical complex coordinate system of M. Let Φ be a real-valued function in a coordinate neighborhood U at the origin 0 of M defined by

$$\begin{aligned}\Phi(\zeta) &= \log\det\left(I_n - {}^t\bar{\zeta}\zeta\right)^{-1} \\ &= \sum |z_{ip}|^2 + \frac{1}{2}\sum \bar{z}_{ip} z_{iq} \bar{z}_{jq} z_{jp} + \text{(higher order terms)}\end{aligned}$$

for any $\zeta \in U$.

Unless otherwise stated, Greek indices $\alpha, \beta, \ldots$ denote all subscripts appearing as pairs $\{11, 12, \ldots, mn\}$, while Latin capitals $A, B, \ldots$ denote $\{11, 12, \ldots, mn, \overline{11}, \ldots, \overline{mn}\}$. We set

$$Z_\alpha = \frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - \sqrt{-1}\,\frac{\partial}{\partial y_\alpha}\right),$$

$$Z_{\overline{\alpha}} = \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + \sqrt{-1}\,\frac{\partial}{\partial y_\alpha}\right).$$

The Kähler metric g of M is given, at the origin 0, by

$$\begin{aligned} g_{\alpha\beta} &= \frac{\partial^2\Phi}{\partial z_\alpha \partial z_\beta}(0), \\ g_{\alpha\bar{\beta}} &= \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}(0) \ \left(= g_{\overline{\beta}\alpha}\right), \\ g_{\overline{\alpha}\overline{\beta}} &= \frac{\partial^2\Phi}{\partial \bar{z}_\alpha \partial \bar{z}_\beta}(0), \end{aligned}$$

where $g_{AB} = g(Z_A, Z_B)(0)$. Note that we have

$$g_{ip\,\overline{jq}} = g_{\overline{jq}\,ip} = \delta_{ij}\delta_{pq}, \quad g_{\alpha\beta} = g_{\overline{\alpha}\overline{\beta}} = 0.$$

Let R be the curvature tensor of the Kähler manifold (M, g), and define $R_{ABCD} = g(R(Z_C, Z_D)Z_B, Z_A)(0)$. Then we obtain

$$\begin{aligned} R_{ip\,\overline{jq}\,kr\,\overline{ls}} &= \left.\frac{\partial^4\Phi}{\partial z_{ip}\partial \bar{z}_{jq}\partial z_{kr}\partial \bar{z}_{ls}}(\zeta)\right|_{\zeta=0} \\ &= \delta_{ij}\delta_{kl}\delta_{ps}\delta_{qr} + \delta_{il}\delta_{jk}\delta_{pq}\delta_{rs}. \end{aligned}$$

By the symmetry properties of R, it is easy to see that $R_{ABCD} = R_{CDAB} = -R_{BACD}$ and $R_{\alpha\beta CD} = R_{\overline{\alpha}\overline{\beta}CD} = 0$.

Then it is immediate to see that the curvature operator $\hat{R}$ of (M, g) is given by

$$\begin{aligned} \hat{R}(Z_{ip} \wedge Z_{\overline{jq}}) &= -\delta_{ij} \sum_{k=1}^{m} Z_{kp} \wedge Z_{\overline{kq}} - \delta_{pq} \sum_{r=1}^{n} Z_{ir} \wedge Z_{\overline{jr}}, \\ \hat{R}(Z_\alpha \wedge Z_\beta) &= \hat{R}(Z_{\overline{\alpha}} \wedge Z_{\overline{\beta}}) = 0. \end{aligned}$$

We divide our investigation into the following four cases: (1) $m = n = 1$; (2) $m = n = 2$; (3) $m = n \geq 3$; (4) $m \neq n$.

(1) In this case, $\hat{R}$ has two eigenvalues 0 and -2, whose eigenvectors are the following:

$$\begin{aligned} 0\ ;& \quad Z_{11} \wedge Z_{11}, \quad Z_{\overline{11}} \wedge Z_{\overline{11}}, \\ -2\ ;& \quad Z_{11} \wedge Z_{\overline{11}}. \end{aligned}$$

(2) In this case, $\hat{R}$ has three eigenvalues $0, -2$ and -4, whose eigenvectors are given as follows:

$$
\begin{aligned}
0\ ; \quad & Z_{11} \wedge Z_{\overline{11}} - Z_{12} \wedge Z_{\overline{12}} - Z_{21} \wedge Z_{\overline{21}} + Z_{22} \wedge Z_{\overline{22}}, \\
& Z_{11} \wedge Z_{\overline{21}} - Z_{12} \wedge Z_{\overline{22}}, \quad Z_{21} \wedge Z_{\overline{11}} - Z_{22} \wedge Z_{\overline{12}}, \\
& Z_{11} \wedge Z_{\overline{12}} - Z_{21} \wedge Z_{\overline{22}}, \quad Z_{12} \wedge Z_{\overline{11}} - Z_{22} \wedge Z_{\overline{21}}, \\
& Z_{11} \wedge Z_{\overline{22}}, \quad Z_{12} \wedge Z_{\overline{21}}, \quad Z_{21} \wedge Z_{\overline{12}}, \quad Z_{22} \wedge Z_{\overline{11}}, \\
& Z_{\alpha} \wedge Z_{\beta}, \quad Z_{\overline{\alpha}} \wedge Z_{\overline{\beta}}, \quad \text{for any} \quad \alpha, \beta,
\end{aligned}
$$

$$
\begin{aligned}
-2\ ; \quad & Z_{11} \wedge Z_{\overline{11}} - Z_{22} \wedge Z_{\overline{22}}, \quad Z_{12} \wedge Z_{\overline{12}} - Z_{21} \wedge Z_{\overline{21}}, \\
& Z_{11} \wedge Z_{\overline{21}} + Z_{12} \wedge Z_{\overline{22}}, \quad Z_{21} \wedge Z_{\overline{11}} + Z_{22} \wedge Z_{\overline{12}}, \\
& Z_{11} \wedge Z_{\overline{12}} + Z_{21} \wedge Z_{\overline{22}}, \quad Z_{12} \wedge Z_{\overline{11}} + Z_{22} \wedge Z_{\overline{21}},
\end{aligned}
$$

$$
-4\ ; \quad Z_{11} \wedge Z_{\overline{11}} + Z_{12} \wedge Z_{\overline{12}} + Z_{21} \wedge Z_{\overline{21}} + Z_{22} \wedge Z_{\overline{22}}.
$$

(3) In this case, $\hat{R}$ has three eigenvalues $0, -m$ and $-2m$, whose eigenvectors are given respectively by

$$
\begin{aligned}
0\ ; \quad & Z_{ip} \wedge Z_{\overline{ip}} - Z_{im} \wedge Z_{\overline{im}} - Z_{mp} \wedge Z_{\overline{mp}} - Z_{mm} \wedge Z_{\overline{mm}} \quad \text{for} \quad i, p \neq m, \\
& Z_{kp} \wedge Z_{\overline{kq}} - Z_{mp} \wedge Z_{\overline{mq}} \quad \text{for} \quad k \neq m \quad \text{and} \quad p \neq q, \\
& Z_{ir} \wedge Z_{\overline{jr}} - Z_{im} \wedge Z_{\overline{jm}} \quad \text{for} \quad r \neq m \quad \text{and} \quad j \neq i, \\
& Z_{ip} \wedge Z_{\overline{jq}} \quad \text{for} \quad i \neq j \quad \text{and} \quad p \neq q, \\
& Z_{\alpha} \wedge Z_{\beta}, Z_{\overline{\alpha}} \wedge Z_{\overline{\beta}} \quad \text{for any} \quad \alpha, \beta,
\end{aligned}
$$

$$
\begin{aligned}
-m\ ; \quad & \sum_{r=1}^{m} Z_{ir} \wedge Z_{\overline{ir}} - \frac{1}{m-2}\left(\sum_{k=1}^{m-1}\sum_{r=1}^{m-1} Z_{kr} \wedge Z_{\overline{kr}} + Z_{mm} \wedge Z_{\overline{mm}}\right) \quad \text{for} \quad i \neq m, \\
& \sum_{k=1}^{m} Z_{kp} \wedge Z_{\overline{kp}} - \frac{1}{m-2}\left(\sum_{k=1}^{m}\sum_{r=1}^{m} Z_{kr} \wedge Z_{\overline{kr}} + Z_{mm} \wedge Z_{\overline{mm}}\right) \quad \text{for} \quad p \neq m, \\
& \sum_{k=1}^{m} Z_{kp} \wedge Z_{\overline{kq}} \quad \text{for} \quad p \neq q, \\
& \sum_{r=1}^{m} Z_{ir} \wedge Z_{\overline{jr}} \quad \text{for} \quad i \neq j,
\end{aligned}
$$

$$-2m\ ; \qquad \sum_{i,p=1}^{m} Z_{ip}\wedge Z_{\overline{ip}}.$$

(4) In this case, $\hat{R}$ has four eigenvalues $0, -m, -n$ and $-(m+n)$, whose eigenvectors are given respectively by

$$\begin{aligned}
0\ ; \quad & Z_{ip}\wedge Z_{\overline{ip}} - Z_{in}\wedge Z_{\overline{in}} - Z_{mp}\wedge Z_{\overline{mp}} + Z_{mn}\wedge Z_{\overline{mn}} \quad \text{for} \quad i\neq m \quad \text{and} \quad p\neq n,\\
& Z_{ir}\wedge Z_{\overline{jr}} - Z_{in}\wedge Z_{\overline{jn}} \quad \text{for} \quad i\neq j \quad \text{and} \quad r\neq n,\\
& Z_{kp}\wedge Z_{\overline{kq}} - Z_{mp}\wedge Z_{\overline{mq}} \quad \text{for} \quad k\neq m \quad \text{and} \quad p\neq q,\\
& Z_{ip}\wedge Z_{\overline{jq}} \quad \text{for} \quad i\neq j \quad \text{and} \quad p\neq q,\\
& Z_{\alpha}\wedge Z_{\beta}, Z_{\overline{\alpha}}\wedge Z_{\overline{\beta}} \quad \text{for any} \quad \alpha,\beta,
\end{aligned}$$

$$\begin{aligned}
-m\ ; \quad & \sum_{k=1}^{m}\big(Z_{kp}\wedge Z_{\overline{kp}} - Z_{kn}\wedge Z_{\overline{kp}}\big) \quad \text{for} \quad p\neq n\\
& \sum_{k=1}^{m} Z_{kp}\wedge Z_{\overline{kq}} \quad \text{for} \quad p\neq q,
\end{aligned}$$

$$\begin{aligned}
-n\ ; \quad & \sum_{r=1}^{n}(Z_{ir}\wedge Z_{\overline{ir}} - Z_{mr}\wedge Z_{\overline{mr}}) \quad \text{for} \quad i\neq m,\\
& \sum_{r=1}^{n} Z_{ir}\wedge Z_{\overline{jr}} \quad \text{for} \quad i\neq j,
\end{aligned}$$

$$-(m+n)\ ; \quad \sum_{i,p} Z_{ip}\wedge Z_{\overline{ip}}.$$

2 Type II_m

Let M be an open set $D^I_{m,n} = \{\zeta\in M(n;\mathbb{C})\,|\,{}^t\zeta = -\zeta, I_n - {}^t\bar{\zeta}\zeta > 0\}$ of $\mathbb{C}^{n(n-1)/2}$, where I_n is the $n\times n$ identity matrix, ${}^t\zeta$ is the transpose of $\zeta\in M$ and $\bar{\zeta}$ denotes the complex conjugate of $\zeta\in M$. Note that M is a subset of $D^I_{n,m}$.

Denote by $\zeta = (z_{ij}) \in M$ with $z_{ij} = x_{ij} + \sqrt{-1}\,y_{ij}$, $i,j = 1,\dots,n$. Since ζ is a skew-symmetric matrix for all $\zeta\in M$, we have $z_{ij} = -z_{ji}$ and hence the components

z_{ij} for $i < j$ is the canonical complex coordinate system of M. Let Φ be a real-valued function in a coordinate neighborhood U at the origin 0 of M defined by

$$\begin{aligned}\Phi(\zeta) &= \frac{1}{2}\log\det\left(I_n - {}^t\bar{\zeta}\zeta\right)^{-1}\\ &= \sum_{i<j}|z_{ij}|^2 + \frac{1}{4}\sum \bar{z}_{ik}z_{il}\bar{z}_{jl}z_{jk} + (\text{higher order terms})\end{aligned}$$

for any $\zeta \in U$.

Now, Greek indices $\alpha, \beta, \ldots$ denote all subscripts appearing as pairs $\{12, \ldots, 1n, 23, \ldots, (n-1)n\}$, while Latin capitals $A, B, \ldots$ denote $\{12, \ldots, (n-1)n, \overline{12}, \ldots, \overline{(n-1)n}\}$, and we set

$$\begin{aligned}Z_\alpha &= \frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - \sqrt{-1}\frac{\partial}{\partial y_\alpha}\right),\\ Z_{\overline{\alpha}} &= \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + \sqrt{-1}\frac{\partial}{\partial y_\alpha}\right).\end{aligned}$$

Then the complexification of the tangent space T_0M at the origin 0 of M is represented as

$$T_0^{\mathbb{C}}M = \mathrm{span}_{\mathbb{C}}\left\{Z_{ij}, Z_{\overline{ij}} \mid 1 \leq i < j \leq n\right\}.$$

The Kähler metric g of M is given, at the origin 0, by

$$\begin{aligned}g_{\alpha\beta} &= \frac{\partial^2\Phi}{\partial z_\alpha \partial z_\beta}(0),\\ g_{\alpha\overline{\beta}} &= \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}(0) \ \left(= g_{\overline{\beta}\alpha}\right),\\ g_{\overline{\alpha}\,\overline{\beta}} &= \frac{\partial^2\Phi}{\partial \bar{z}_\alpha \partial \bar{z}_\beta}(0),\end{aligned}$$

where $g_{AB} = g(Z_A, Z_B)(0)$. Thus we have

$$g_{ij\,\overline{kl}} = g_{\overline{kl}\,ij} = \delta_{ik}\delta_{jl}, \quad g_{ij\,kl} = g_{\overline{ij}\,\overline{kl}} = 0.$$

Let R be the curvature tensor of the Kähler manifold (M, g), and define $R_{ABCD} = g(R(Z_C, Z_D)Z_B, Z_A)(0)$. Then we obtain

$$R_{ij\,\overline{kl}\,pq\,\overline{rs}} = \left.\frac{\partial^4\Phi}{\partial z_{ij}\partial \bar{z}_{kl}\partial z_{pq}\partial \bar{z}_{rs}}(\zeta)\right|_{\zeta=0}$$

$$= \delta^{rl}_{pq}\delta^{ks}_{ij} - \delta^{rk}_{pq}\delta^{ls}_{ij} - \delta^{sl}_{pq}\delta^{kr}_{ij} + \delta^{sk}_{pq}\delta^{lr}_{ij},$$

where $\delta^{kl}_{ij} = \partial z_{kl}/\partial z_{ij} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. By the symmetry properties of R, we also have $R_{ABCD} = R_{CDAB} = -R_{BACD}$ and $R_{\alpha\beta CD} = R_{\overline{\alpha}\overline{\beta}CD} = 0$.

Then it is immediate to see that the curvature operator $\hat{R}$ of (M, g) is given by

$$\hat{R}(Z_{ij} \wedge Z_{\overline{kl}}) = \sum_{p,q,r,s} \left(-\delta^{rl}_{pq}\delta^{ks}_{ij} + \delta^{rk}_{pq}\delta^{ls}_{ij} + \delta^{sl}_{pq}\delta^{kr}_{ij} - \delta^{sk}_{pq}\delta^{lr}_{ij}\right) Z_{rs} \wedge Z_{\overline{pq}},$$
$$\hat{R}(Z_\alpha \wedge Z_\beta) = \hat{R}(Z_{\overline{\alpha}} \wedge Z_{\overline{\beta}}) = 0.$$

We divide our investigation into the following two cases: (1) $n = 2$; (2) $n \geq 3$.

(1) In this case, $\hat{R}$ has two eigenvalues 0 and -2, whose eigenvectors are the following:

$$\begin{aligned} 0\ ; &\quad Z_{12} \wedge Z_{12}, \quad Z_{\overline{12}} \wedge Z_{\overline{12}}, \\ -2\ ; &\quad Z_{12} \wedge Z_{\overline{12}}. \end{aligned}$$

(2) In this case, $\hat{R}$ has three eigenvalues $0, -(n-2)$ and $-2(n-1)$, whose eigenvectors are given respectively by

$$\begin{aligned}
0\ ; \quad & Z_{ij} \wedge Z_{\overline{ij}} - Z_{in} \wedge Z_{\overline{in}} \\
& \quad - Z_{jn} \wedge Z_{\overline{jn}} - Z_{(n-2)(n-1)} \wedge Z_{\overline{(n-2)(n-1)}} \\
& \qquad + Z_{(n-2)n} \wedge Z_{\overline{(n-2)n}} + Z_{(n-1)n} \wedge Z_{\overline{(n-1)n}} \quad \text{for} \quad 1 \leq i < j \leq n-2, \\
& Z_{i(n-1)} \wedge Z_{\overline{i(n-1)}} - Z_{(n-1)n} \wedge Z_{\overline{(n-1)n}} \\
& \quad - Z_{in} \wedge Z_{\overline{in}} - Z_{(n-2)(n-1)} \wedge Z_{\overline{(n-2)(n-1)}} \\
& \qquad + Z_{(n-1)n} \wedge Z_{\overline{(n-1)n}} + Z_{(n-2)n} \wedge Z_{\overline{(n-2)n}} \quad \text{for} \quad 1 \leq i \leq n-3, \\
& Z_{ij} \wedge Z_{\overline{ik}} - Z_{jn} \wedge Z_{\overline{kn}} \quad \text{for} \quad 1 \leq i < j < k \leq n-1, \\
& Z_{ij} \wedge Z_{\overline{jk}} + Z_{in} \wedge Z_{\overline{kn}} \quad \text{for} \quad 1 \leq i < j < k \leq n-1, \\
& Z_{ik} \wedge Z_{\overline{jk}} - Z_{in} \wedge Z_{\overline{jn}} \quad \text{for} \quad 1 \leq i < j < k \leq n-1, \\
& Z_{ik} \wedge Z_{\overline{ij}} - Z_{kn} \wedge Z_{\overline{jn}} \quad \text{for} \quad 1 \leq i < j < k \leq n-1, \\
& Z_{jk} \wedge Z_{\overline{ij}} + Z_{kn} \wedge Z_{\overline{in}} \quad \text{for} \quad 1 \leq i < j < k \leq n-1, \\
& Z_{jk} \wedge Z_{\overline{ik}} - Z_{jn} \wedge Z_{\overline{in}} \quad \text{for} \quad 1 \leq i < j < k \leq n-1, \\
& Z_{ij} \wedge Z_{\overline{in}} + Z_{j(n-1)} \wedge Z_{\overline{(n-1)n}} \quad \text{for} \quad 1 \leq i < j \leq n-2, \\
& Z_{ij} \wedge Z_{\overline{jn}} - Z_{i(n-1)} \wedge Z_{\overline{(n-1)n}} \quad \text{for} \quad 1 \leq i < j \leq n-2,
\end{aligned}$$

$$Z_{in} \wedge Z_{\overline{ij}} + Z_{(n-1)n} \wedge Z_{\overline{j(n-1)}} \quad \text{for} \quad 1 \leq i < j \leq n-2,$$
$$Z_{jn} \wedge Z_{\overline{ij}} - Z_{(n-1)n} \wedge Z_{\overline{i(n-1)}} \quad \text{for} \quad 1 \leq i < j \leq n-2,$$
$$Z_{i(n-1)} \wedge Z_{\overline{in}} - Z_{(n-2)(n-1)} \wedge Z_{\overline{(n-2)n}} \quad \text{for} \quad 1 \leq i \leq n-3,$$
$$Z_{in} \wedge Z_{\overline{i(n-1)}} - Z_{(n-2)n} \wedge Z_{\overline{(n-2)(n-1)}} \quad \text{for} \quad 1 \leq i \leq n-3,$$
$$Z_{ij} \wedge Z_{\overline{kl}} \quad \text{for} \quad i \neq k, l \quad \text{and} \quad j \neq k, l,$$
$$Z_{\alpha} \wedge Z_{\beta}, \quad Z_{\overline{\alpha}} \wedge Z_{\overline{\beta}} \quad \text{for any} \quad \alpha, \beta,$$

$$-(n-2) \ ; \qquad -\frac{2}{n-2}\sum_{r=2}^{i-1} Z_{ri} \wedge Z_{\overline{ri}} + \sum_{r<i} Z_{ri} \wedge Z_{\overline{ri}} + \sum_{i<r} Z_{ir} \wedge Z_{\overline{ir}} \quad \text{for} \quad 2 \leq i \leq n,$$
$$\sum_{r=1}^{i-1} Z_{ri} \wedge Z_{\overline{rj}} - \sum_{r=i+1}^{j-1} Z_{ir} \wedge Z_{\overline{rj}} + \sum_{r=j+1}^{n} Z_{ir} \wedge Z_{\overline{jr}} \quad \text{for} \quad 1 \leq i < j \leq n,$$
$$\sum_{r=1}^{i-1} Z_{rj} \wedge Z_{\overline{ri}} - \sum_{r=i+1}^{j-1} Z_{rj} \wedge Z_{\overline{ir}} + \sum_{r=j+1}^{n} Z_{jr} \wedge Z_{\overline{ir}} \quad \text{for} \quad 1 \leq i < j \leq n,$$

$$-2(n-1) \ ; \qquad \sum_{i<j} Z_{ij} \wedge Z_{\overline{ij}}.$$

3 Type III_m

Let M be an open subset $D_n^{III} = \{\zeta \in M(n;\mathbb{C}) \,|\, {}^t\zeta = \zeta, I_n - {}^t\bar{\zeta}\zeta > 0\}$ of $\mathbb{C}^{n(n+1)/2}$, where I_n is the $n \times n$ identity matrix, ${}^t\zeta$ is the transpose of $\zeta \in M$ and $\bar{\zeta}$ denotes the complex conjugate of $\zeta \in M$.

Denote by $\zeta = (z_{ij}) \in M$ with $z_{ij} = x_{ij} + \sqrt{-1}\, y_{ij}$, $i, j = 1, \ldots, n$. Since ζ is a symmetric matrix for all $\zeta \in M$, we have $z_{ij} = z_{ji}$, so that the components z_{ij} for $i \leq j$ is the canonical complex coordinate system of M. Let Φ be a real-valued function in a coordinate neighborhood U at the origin 0 of M defined by

$$\begin{aligned}\Phi(\zeta) &= \frac{1}{2}\log\det\left(I_n - {}^t\bar{\zeta}\zeta\right)^{-1} \\ &= \frac{1}{2}\sum_{i,j=1}^{n} |z_{ij}|^2 + \frac{1}{4}\sum \bar{z}_{ik} z_{il} \bar{z}_{jl} z_{jk} + (\text{higher order terms})\end{aligned}$$

for any $\zeta \in U$.

In this case, Greek indices $\alpha, \beta, \ldots$ denote all subscripts appearing as pairs $\{11, 12, \ldots, nn\}$, while Latin capitals $A, B, \ldots$ denote $\{11, 12, \ldots, mn, \overline{11}, \ldots, \overline{nn}\}$, and we set

$$Z_\alpha = \frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - \sqrt{-1}\frac{\partial}{\partial y_\alpha}\right),$$
$$Z_{\overline{\alpha}} = \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + \sqrt{-1}\frac{\partial}{\partial y_\alpha}\right).$$

Then the complexfication of the tangent space T_0M at the origin 0 of M is represented as

$$T_0^{\mathbb{C}}M = \mathrm{span}_{\mathbb{C}}\left\{Z_{ij}, Z_{\overline{ji}} \,|\, 1 \le j \le j \le n\right\}.$$

The Kähler metric g of M is given, at the origin 0, by

$$g_{ij\,kl} = \frac{\partial^2\Phi}{\partial z_{ij}\partial z_{kl}}(0),$$
$$g_{ij\,\overline{kl}} = \frac{\partial^2\Phi}{\partial z_{ij}\partial \bar{z}_{kl}}(0) \;\left(= g_{\overline{kl}\,ij}\right),$$
$$g_{\overline{ij}\,\overline{kl}} = \frac{\partial^2\Phi}{\partial \bar{z}_{ij}\partial \bar{z}_{kl}}(0),$$

where $g_{AB} = g(Z_A, Z_B)(0)$. Then we have

$$g_{ij\,\overline{kl}} = g_{\overline{kl}\,ij} = e_{ij}^{kl} \quad \text{for} \quad i \ne j, k \ne l,$$
$$g_{ii\,\overline{kk}} = g_{\overline{ii}\,kk} = \frac{1}{2}\delta_{ik},$$

where $e_{ij}^{kl} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}$.

Let R be the curvature tensor of the Kähler manifold (M, g), and define $R_{ABCD} = g(R(Z_C, Z_D)Z_B, Z_A)(0)$. Then we obtain

$$R_{ij\,\overline{kl}\,pq\,\overline{rs}} = \left.\frac{\partial^4\Phi}{\partial z_{ij}\partial \bar{z}_{kl}\partial z_{pq}\partial \bar{z}_{rs}}(\zeta)\right|_{\zeta=0}$$

so that

$$R_{ii\,\overline{kk}\,pp\,\overline{rr}} = \delta_{rp}\delta_{ri}\delta_{pk}\delta_{ki},$$
$$R_{ij\,\overline{kl}\,pp\,\overline{rr}} = \delta_{rp}\delta_{lp}e_{ij}^{kr} + \delta_{rp}\delta_{kp}e_{ij}^{lr},$$
$$R_{ij\,\overline{kk}\,pq\,\overline{rr}} = e_{ij}^{kr}e_{pq}^{kr},$$

$$
\begin{aligned}
R_{ij\,\overline{kk}\,pp\,\overline{rs}} &= \delta_{ik}\delta_{pk}e^{pj}_{rs} + \delta_{jk}\delta_{pk}e^{pi}_{rs},\\
R_{ii\,\overline{kl}\,pp\,\overline{rs}} &= e^{pi}_{rs}e^{pi}_{kl},\\
R_{ii\,\overline{kl}\,pq\,\overline{rs}} &= e^{ip}_{kl}e^{iq}_{rs} + e^{iq}_{kl}e^{ip}_{rs},\\
R_{ij\,\overline{kk}\,pq\,\overline{rs}} &= e^{kr}_{pq}e^{ks}_{ij} + e^{ks}_{pq}e^{kr}_{ij},\\
R_{ij\,\overline{kl}\,pq\,\overline{rs}} &= e^{kr}_{pq}e^{ls}_{ij} + e^{ks}_{pq}e^{lr}_{ij} + e^{lr}_{pq}e^{ks}_{ij} + e^{ls}_{pq}e^{kr}_{ij},
\end{aligned}
$$

where $i < j$, $k < l$, $p < q$ and $r < s$. By the symmetry properties of R, it is easy to see that $R_{ABCD} = R_{CDAB} = -R_{BACD}$ and $R_{\alpha\beta CD} = R_{\overline{\alpha}\overline{\beta}CD} = 0$.

It then is immediate to see that the curvature operator $\hat{R}$ of (M, g) is given by

$$
\begin{aligned}
\hat{R}(Z_{ii} \wedge Z_{\overline{kk}}) =& -4\sum_{r,p} \delta_{rp}\delta_{ri}\delta_{kp}\delta_{ik}Z_{rr} \wedge Z_{\overline{pp}} - \sum_{p,q,r,s} \left(\delta_{ik}\delta_{is}e^{kr}_{pq} + \delta_{ik}\delta_{ir}e^{sk}_{pq}\right) Z_{rs} \wedge Z_{\overline{pq}},\\
\hat{R}(Z_{ij} \wedge Z_{\overline{kk}}) =& -2\sum_{r,p,q} e^{kr}_{ij}e^{kr}_{pq}Z_{rr} \wedge Z_{\overline{pq}} - 2\sum_{p,r,s} \left(\delta_{ik}\delta_{pk}e^{pj}_{rs} + \delta_{jk}\delta_{pk}e^{pi}_{rs}\right) Z_{rs} \wedge Z_{\overline{pp}}\\
& - \sum_{p,q,r,s} \left(e^{kr}_{pq}e^{ks}_{ij} + e^{ks}_{pq}e^{kr}_{ij}\right) Z_{rs} \wedge Z_{\overline{pq}},\\
\hat{R}(Z_{ii} \wedge Z_{\overline{kl}}) =& -2\sum_{p,q,r} \left(\delta_{pr}\delta_{ir}e^{iq}_{kl} + \delta_{qr}\delta_{ir}e^{ip}_{kl}\right) Z_{rr} \wedge Z_{\overline{pq}} - 2\sum_{p,r,s} e^{pi}_{rs}e^{pi}_{kl}Z_{rs} \wedge Z_{\overline{pp}}\\
& - \sum_{p,q,r,s} \left(e^{ip}_{kl}e^{iq}_{rs} + e^{iq}_{kl}e^{ip}_{rs}\right) Z_{rs} \wedge Z_{\overline{pq}},\\
\hat{R}(Z_{ij} \wedge Z_{\overline{kl}}) =& -4\sum_{p,r} \left(\delta_{rp}\delta_{lp}e^{kr}_{ij} + \delta_{rp}\delta_{kp}e^{lr}_{ij}\right) Z_{rr} \wedge Z_{\overline{pp}}\\
& - 2\sum_{p,q,r} \left(e^{rk}_{ij}e^{rl}_{pq} + e^{rl}_{ij}e^{rk}_{pq}\right) Z_{rr} \wedge Z_{\overline{pq}}\\
& - 2\sum_{p,r,s} \left(e^{pi}_{rs}e^{pj}_{kl} + e^{pj}_{rs}e^{pi}_{kl}\right) Z_{rs} \wedge Z_{\overline{pp}}\\
& - \sum_{p,q,r,s} \left(e^{kr}_{pq}e^{ls}_{ij} + e^{ks}_{pq}e^{lr}_{ij} + e^{lr}_{pq}e^{ks}_{ij} + e^{ls}_{pq}e^{kr}_{ij}\right) Z_{rs} \wedge Z_{\overline{pq}}.
\end{aligned}
$$

Consequently, the curvature operator $\hat{R}$ has three eigenvalues $0, -(n+2)$ and $-2(n+1)$, whose eigenvectors are given respectively by

$$
\begin{aligned}
0\ ; \quad & Z_{ij} \wedge Z_{\overline{ij}} - Z_{in} \wedge Z_{\overline{in}} - Z_{jn} \wedge Z_{\overline{jn}} + 2Z_{nn} \wedge Z_{\overline{nn}} \quad \text{for} \quad 1 \le i < j \le n-1,\\
& Z_{ii} \wedge Z_{\overline{ii}} - Z_{in} \wedge Z_{\overline{in}} + Z_{nn} \wedge Z_{\overline{nn}} \quad \text{for} \quad 1 \le i \le n-1,\\
& Z_{ij} \wedge Z_{\overline{ik}} - Z_{jn} \wedge Z_{\overline{kn}} \quad \text{for} \quad 1 \le i \le j < k \le n,\\
& Z_{ij} \wedge Z_{\overline{jk}} - Z_{in} \wedge Z_{\overline{kn}} \quad \text{for} \quad 1 \le i < j \le k \le n-1,
\end{aligned}
$$

$$
\begin{aligned}
&Z_{ij} \wedge Z_{\overline{jn}} - Z_{in} \wedge Z_{\overline{jn}} \quad \text{for} \quad 1 \le i < j \le n-1, \\
&Z_{ik} \wedge Z_{\overline{jk}} - Z_{in} \wedge Z_{\overline{jn}} \quad \text{for} \quad 1 \le i < j < k \le n-1, \\
&Z_{ik} \wedge Z_{\overline{ij}} - Z_{kn} \wedge Z_{\overline{jn}} \quad \text{for} \quad 1 \le i \le j < k \le n, \\
&Z_{jk} \wedge Z_{\overline{ij}} - Z_{kn} \wedge Z_{\overline{in}} \quad \text{for} \quad 1 \le i < j \le k \le n-1, \\
&Z_{jn} \wedge Z_{\overline{ij}} - Z_{nn} \wedge Z_{\overline{in}} \quad \text{for} \quad 1 \le i < j \le n-1, \\
&Z_{jk} \wedge Z_{\overline{ik}} - Z_{jn} \wedge Z_{\overline{in}} \quad \text{for} \quad 1 \le i < j < k \le n-1, \\
&Z_{ii} \wedge Z_{\overline{kk}} \quad \text{for} \quad i \ne k, \\
&Z_{ij} \wedge Z_{\overline{kk}} \quad \text{for} \quad i < j \quad \text{and} \quad i, j \ne k, \\
&Z_{ii} \wedge Z_{\overline{kl}} \quad \text{for} \quad k < l \quad \text{and} \quad i \ne k, l, \\
&Z_{ij} \wedge Z_{\overline{kl}} \quad \text{for} \quad i < j, \quad k < l \quad \text{and} \quad i \ne k, l, \quad j \ne k, l, \\
&Z_{\alpha} \wedge Z_{\beta}, \quad Z_{\overline{\alpha}} \wedge Z_{\overline{\beta}} \quad \text{for any} \quad \alpha, \beta,
\end{aligned}
$$

$$
\begin{aligned}
-(n+2)\ ; \quad & Z_{ii} \wedge Z_{\overline{ii}} + \frac{1}{4}\sum_{r=i+1}^{n} Z_{ir} \wedge Z_{\overline{ir}} + \frac{1}{4}\sum_{r=1}^{i-1} Z_{ri} \wedge Z_{\overline{ri}} \\
& \quad - Z_{nn} \wedge Z_{\overline{nn}} - \frac{1}{4}\sum_{r=1}^{n-1} Z_{rn} \wedge Z_{\overline{rn}} \quad \text{for} \quad 1 \le i \le n-1, \\
& \frac{1}{2}\sum_{r=1}^{i-1} Z_{ri} \wedge Z_{\overline{rk}} + Z_{ii} \wedge Z_{\overline{ik}} + \frac{1}{2}\sum_{r=i+1}^{k-1} Z_{ir} \wedge Z_{\overline{rk}} + Z_{ik} \wedge Z_{\overline{kk}} \\
& \quad + \frac{1}{2}\sum_{r=k+1}^{n} Z_{ir} \wedge Z_{\overline{kr}} \quad \text{for} \quad 1 \le i < k \le n, \\
& \frac{1}{2}\sum_{p=1}^{k-1} Z_{pk} \wedge Z_{\overline{pi}} + Z_{kk} \wedge Z_{\overline{ik}} + \frac{1}{2}\sum_{p=k+1}^{i-1} Z_{kp} \wedge Z_{\overline{pi}} + Z_{ki} \wedge Z_{\overline{ii}} \\
& \quad + \frac{1}{2}\sum_{p=i+1}^{n} Z_{kp} \wedge Z_{\overline{ip}} \quad \text{for} \quad 1 \le i < k \le n,
\end{aligned}
$$

$$
-2(n+1)\ ; \quad 2\sum_{r=1}^{n} Z_{rr} \wedge Z_{\overline{rr}} + \sum_{r<s} Z_{rs} \wedge Z_{\overline{rs}}.
$$

4 Type IV_m

Let M be an open subset $D_n^{IV} = \{\zeta \in M(n,1);\mathbb{C}) \mid 1+|{}^t\zeta\zeta|^2 - 2{}^t\overline{\zeta}\zeta > 0, {}^t\bar{\zeta}\zeta < 1\}$ of $\mathbb{C}^n$, where ${}^t\zeta$ is the transpose of $\zeta \in M$, and $\bar{\zeta}$ denotes the complex conjugate of $\zeta \in M$.

Let $\zeta = {}^t(z_1, \ldots, z_n)$ with $z_i = x_i + \sqrt{-1}\, y_i$, $i = 1, \ldots, n$ be the canonical complex coordinate system of M. Let Φ be a real-valued function in a coordinate neighborhood U at the origin 0 of M defined by

$$\begin{aligned}\Phi(\zeta) &= \log\det\left(1 - 2^t\zeta\zeta + |\zeta|^2\right)^{-1} \\ &= 2\sum_{\alpha=1}^{n}|z_\alpha|^2 + \sum_{\alpha,\beta}(z_\alpha)^2(\bar{z}_\beta)^2 + 2\sum_{\alpha,\beta}|z_\alpha|^2|z_\beta|^2 + (\text{higher order terms})\end{aligned}$$

for any $\zeta \in U$.

Now, Greek indices $\alpha, \beta, \ldots$ run from 1 to n, while Latin capitals $A, B, \ldots$ run through $1, \ldots, n, \bar{1}, \ldots, \bar{n}$, and we set

$$\begin{aligned} Z_\alpha &= \frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - \sqrt{-1}\frac{\partial}{\partial y_\alpha}\right), \\ Z_{\overline{\alpha}} &= \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + \sqrt{-1}\frac{\partial}{\partial y_\alpha}\right). \end{aligned}$$

The Kähler metric g of M is given, at the origin 0, by

$$\begin{aligned} g_{\alpha\beta} &= \frac{\partial^2\Phi}{\partial z_\alpha \partial z_\beta}(0), \\ g_{\alpha\overline{\beta}} &= \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}(0), \ \left(= g_{\overline{\beta}\alpha}\right) \\ g_{\overline{\alpha}\overline{\beta}} &= \frac{\partial^2\Phi}{\partial \bar{z}_\alpha \partial \bar{z}_\beta}(0), \end{aligned}$$

where $g_{AB} = g(Z_A, Z_B)(0)$. Then we have

$$g_{\alpha\overline{\beta}}(0) = g_{\overline{\beta}\alpha}(0) = \delta_{ij}\delta_{\alpha\beta}.$$

Let R be the curvature tensor of the Kähler manifold (M, g), and define $R_{ABCD} = g(R(Z_C, Z_D)Z_B, Z_A)(0)$. Then we have

$$\begin{aligned} R_{\alpha\overline{\beta}\gamma\overline{\delta}}(0) &= \left.\frac{\partial^4\Phi}{\partial z_\alpha \partial \bar{z}_\beta \partial z_\gamma \partial \bar{z}_\delta}(\zeta)\right|_{\zeta=0} \\ &= -4\delta_{\alpha\gamma}\delta_{\beta\delta} + 4\delta_{\gamma\delta}\delta_{\alpha\beta} + 4\delta_{\alpha\delta}\delta_{\beta\gamma}, \end{aligned}$$

and it is easy to see that$R_{ABCD} = R_{CDAB} = -R_{BACD}$ and $R_{\alpha\beta CD} = R_{\overline{\alpha}\overline{\beta}CD} = 0$.

Then it is immediate to see that the curvature operator $\hat{R}$ of (M, g) is given by

$$\hat{R}(Z_\alpha \wedge Z_{\overline{\alpha}}) = -\sum_{\gamma=1}^{n} Z_\gamma \wedge Z_{\overline{\gamma}},$$

$$\hat{R}(Z_\alpha \wedge Z_{\overline{\beta}}) = Z_\beta \wedge Z_{\overline{\alpha}} - Z_\alpha \wedge Z_{\overline{\beta}}, \quad \text{for} \quad \alpha \neq \beta.$$

Then $\hat{R}$ has three eigenvalues $0, -2$ and $-n$, whose eigenvectors are given respectively by

$$0\ ; \qquad Z_\alpha \wedge Z_{\overline{\alpha}} - Z_n \wedge Z_{\overline{n}}, \quad \text{for} \quad 1 \leq \alpha \leq n-1,$$

$$Z_\alpha \wedge Z_{\overline{\beta}} + Z_\beta \wedge Z_{\overline{\alpha}} \quad \text{for} \quad \alpha \neq \beta,$$

$$-2\ ; \qquad Z_\alpha \wedge Z_{\overline{\beta}} - Z_\beta \wedge Z_{\overline{\alpha}}, \quad \text{for} \quad \alpha \neq \beta,$$

$$-n\ ; \qquad \sum_{\alpha=1}^{n} Z_\alpha \wedge Z_{\overline{\alpha}}.$$

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